The Epsilon Expansion via Hypergeometric Functions and Differential Reduction

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including work by V.V. Bytev, B.A. Kniehl, and B.F.L. Ward

Overview

Typically, each type of Feynman diagram (massless, single-scale, multileg, massive, *etc.*) has required a new technique.

In the late 60's Regge suggested that any Feynman Diagram can be considered as a special function. In modern language, the general case is a Gelfand-Kapranov-Zelevinsky (GKZ) function.

The Hypergeometric representation is a powerful tool applicable to many different kinds of diagrams. The universal properties stemming from the existence of this representation should be investigated.

Dimensional regularization creates a need to characterize the coefficients of the expansion of hypergeometric functions about rational values of the parameters.

Outline:

- Integration-by-parts (IBP) techniques
- Hypergeometric representations
- Differential reduction
- Approaches to constructing the epsilon expansion

IBP Relations and Master Integrals

- **Integration by parts** leads to a set of recurrance relations among diagrams of a given topology (and those that can be obtained by deleting any number of lines) but different powers of the propagators.

$$\int d^d k \frac{\partial}{\partial k_\mu} G = 0 \; .$$

- The full set of recurrence relations should be solved by finding how the integral with powers of propagators $(j_1 + j_2 + \cdots + j_k)$ reduced to integrals with powers $(j_1 + j_2 + \cdots + j_k - 1)$
- The method involves taking derivatives of each integral with respect to momenta and reducing it to the original integral.
- The relations found permit a **reduction** to a basis set of **master integrals** in terms of which the diagrams of this class may be expressed.
- For new integrals which may appear within the reduction, the procedure is repeated.
- There is no algebraic, geometric, or topological criterion for the irreducibility of a diagram: Only by applying the IBP algorithm (Laporta) can this be established.

F.V. Tkachov, Phys. Lett. B100 (1981) 65; K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B192 (1981) 159; O.V. Tarasov, Phys. Rev. D54 (1996) 6479; Laporta, Int.J.Mod.Phys. A15 (2000) 5087; Anastasiou, Lazopoulos, JHEP 0407 (2004) 046; Smirnov, JHEP 0810 (2008) 107; Studerus, Comp. Phys. Commun. 181 (2010) 1293

IBP Example

One-loop Feynman integral with an arbitrary power of the propagator:

$$I_n = \int \frac{d^d k}{(k^2 - m^2)^n}.$$

The IBP identity

$$\int d^d k \frac{\partial}{\partial k_\mu} \left[\frac{k_\mu}{(k^2 - m^2)^n} \right] = 0$$

leads to a recurrance relation

$$(d - 2n) I_n - 2nm^2 I_{n+1} = 0$$

with solution

$$I_n = \frac{(-1)^n \Gamma\left(n + 1 - \frac{d}{2}\right)}{(n-1)! m^{2(n-1)} \Gamma\left(1 - \frac{d}{2}\right)} I_1$$

with a single master integral

$$I_1 = -i\pi^{d/2} m^{d-2} \Gamma\left(1 - \frac{d}{2}\right).$$

Hypergeometric Approach

The importance of hypergeometric functions to the study of Feynman diagrams is rooted in the following **conjecture**:

Any Feynman diagram can be written as linear combination of Horn-type hypergeometric functions H_i with rational parameters and rational coefficient functions P_i:

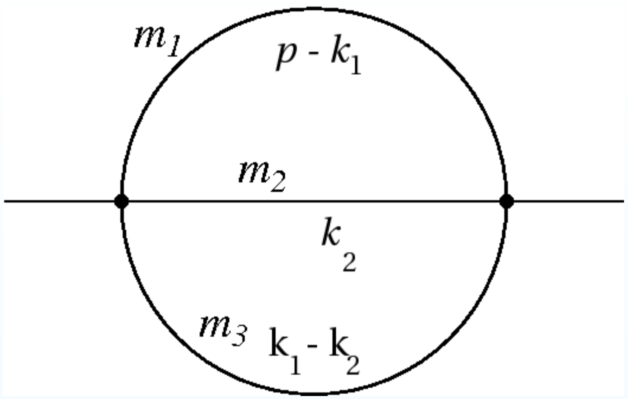
$$F_G(\vec{j}, \vec{m}) \sim \sum_i P_i(\vec{x}, \vec{j}) H_i(\vec{J}, Q_i(\vec{x})) \ .$$

We are not aware of any exceptions to this statement. It always holds at one loop.

We expect the Feynman diagram (left) to share mathematical properties with each hypergeometric function (right):

- The surface of singularities of the Feynman Diagram should coincide with the joint surface of singularities of the Horn-type functions.
- The number of master integrals on the left should be correlated to the number of basis functions on the right.

Example: Sunset Diagram



$$J_{m_1 m_2 m_3} = \int \frac{d^d k_1 d^d k_2}{[(k_1-p)^2 - m_1^2][k_2^2 - m_2^2][(k_1-k_2)^2 - m_3^2]}$$

$$= \int_{-i\infty}^{i\infty} ds_1 ds_2 ds_3 \frac{m_1^{2s_1} m_2^{2s_2} m_3^{2s_3}}{(-p^2)^{s_1+s_2+s_3}} \Gamma(-s_1)\Gamma(-s_2)\Gamma(-s_3)$$

$$\Gamma(3-d+s_1+s_2+s_3) \frac{\Gamma(d/2-1-s_1)\Gamma(d/2-1-s_2)\Gamma(d/2-1-s_3)}{\Gamma(3d/2-3-s_1-s_2-s_3)}$$

$$\sim z_1^{d/2-1} z_2^{d/2-1} F_c^{(3)}(1, d/2, d/2, d/2, d/2; z_1, z_2, z_3)$$
$$- z_1^{d/2-1}\Gamma^2(1-d/2) F_c^{(3)}(1, 2-d/2, d/2, 2-d/2, d/2, z_1, z_2, z_3)$$
$$- z_2^{d/2-1}\Gamma^2(1-d/2) F_c^{(3)}(1, 2-d/2, d/2, 2-d/2, d/2, z_1, z_2, z_3)$$
$$- \Gamma(d/2-1)\Gamma(1-d/2)\Gamma(3-d) F_c^{(3)}(3-d, 2-d/2, 2-d/2, 2-d/2, d/2, z_1, z_2, z_3) \,,$$

in terms of the hypergeometric function (in the case $n = 3$)

$$F_c^{(n)}(a, b; c_1, \cdots, c_n; z_1, \cdots z_n) = \sum_{k_1,\ldots k_n} \frac{(a)_{k_1+\cdots+k_n}(b)_{k_1+\cdots+k_n}}{(c_1)_{k_1}\cdots(c_n)_{k_n}} \frac{z_1^{k_1}\cdots z_n^{k_n}}{k_1!\cdots k_n!}$$

with $z_1 = m_1^2/m_3^2, \quad z_2 = m_2^2/m_3^2, \quad z_3 = p^2/m_3^2, \quad (a)_k = \Gamma(a+k)/\Gamma(a).$

Horn-Type Series Representation

Horn's definition: a Laurent series in r variables,

$$H(\vec{x}) = \sum C(\vec{m})\vec{x}^{m} \equiv \sum_{m_1, m_2, \cdots, m_r} C(m_1, m_2, \cdots, m_r) x_1^{m_1} \cdots x_r^{m_r},$$

is called **hypergeometric** if for each $i = 1, \cdots, r$ the ratio

$$\frac{C(\vec{m} + \vec{e}_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})}.$$

is a rational function in the index of summation: $\vec{m} = (m_1, \cdots, m_r)$, where $\vec{e}_j = (0, \cdots, 0, 1, 0, \cdots, 0)$, is unit vector with unity in the $j^{\rm th}$ place, and P_j, Q_j are polynomials.

Horn-type Hypergeometric Functions: Solution

Ore[1930] and Sato[1990] found the general form of the coefficients,

$$C(\vec{m}) = \prod_{i=1}^{r} \lambda_i^{m_i} R(\vec{m}) \left(\frac{\prod_{j=1}^{N} \Gamma(\mu_j(\vec{m}) + \gamma_j)}{\prod_{k=1}^{M} \Gamma(\nu_k(\vec{m}) + \delta_k)} \right) ,$$

where $N, M \geq 0$, $\lambda_j, \delta_j, \gamma_j \in \mathbb{C}$ are arbitrary complex numbers, $\mu_j, \nu_k : \mathbb{Z}^r \to \mathbb{Z}$ are arbitrary integer-valued linear maps, and R is an arbitrary rational function.

The Horn-type hypergeometric function satisfies the following system of equations:

$$Q_j \left(\sum_{k=1}^{r} x_k \frac{\partial}{\partial x_k} \right) \frac{1}{x_j} H(\vec{x}) = P_j \left(\sum_{k=1}^{r} x_k \frac{\partial}{\partial x_k} \right) H(\vec{x}) .$$

Differential Reduction of Horn-Type Functions

Any Horn-type hypergeometric function has a differential reduction (Takayama) of the form

$$H(\vec{a}+\vec{K};\vec{z}) = \sum_{\alpha} R_{\vec{\alpha}} D_{\vec{\alpha}} H(\vec{a};\vec{z})$$

with integers $\vec{K}$, rational functions $R_{\vec{\alpha}}$ and differential operators $D_{\vec{\alpha}} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_r^{\alpha_k}}$.

A differential reduction algorithm is a method for constructing the basis on the right (which is always finite).

- For an irreducible Horn-type hypergeometric function, the number of basis elements in r.h.s. is equal to number of solutions of the system of differential equations satisfied by the l.h.s. above.
- When difference between some of the parameters a are integers, the number of irreducible basis elements is reduced.

New Criteria for Reducibility

Conjecture:

If a Feynman diagram is expressible as a linear combination of Horn-type hypergeometric functions with rational coefficients, then

(i) Each hypergeometric function has the same number of basis elements in the framework of differential reduction; which, for hypergeometric functions of several variables, is equal to the dimension of solution space of corresponding system of partial differential equations (up to a module of rational functions).

(ii) The number of nontrivial master integrals is equal to the number of basis elements of the hypergeometric functions, up to the modules of Feynman integrals expressible in terms of products of algebraic functions and Γ functions.

The conjecture was suggested by analyzing a variety of examples, and (ii) is in general a consequence of (i).

Differential Reduction: Shift Operators

Consider the hypergeometric series

$$H(\vec{\gamma};\vec{\sigma};\vec{x}) = \sum_{m_1,m_2,\cdots,m_r=0}^{\infty} \left(\frac{\prod_{j=1}^{K} \Gamma\left(\sum_{a=1}^{r} \mu_{ja} m_a + \gamma_j\right)}{\prod_{k=1}^{L} \Gamma\left(\sum_{b=1}^{r} \nu_{kb} m_b + \sigma_k\right)} \right) x_1^{m_1} \cdots x_r^{m_r} \ .$$

The lists $\vec{\gamma} = (\gamma_1, \cdots, \gamma_K)$ and $\vec{\sigma} = (\sigma_1, \cdots, \sigma_L)$ are called *upper* and *lower* parameters of the hypergeometric function, respectively.
Two functions with lists of parameters shifted by a unit, $\Phi(\vec{\gamma} + \vec{e_c}; \vec{\sigma}; \vec{x})$ and $\Phi(\vec{\gamma}; \vec{\sigma}; \vec{x})$, are related by a linear differential operator:

$$\begin{aligned} H(\vec{\gamma} + \vec{e_c}; \vec{\sigma}; \vec{x}) &= \left(\sum_{a=1}^{r} \mu_{ca} x_a \frac{\partial}{\partial x_a} + \gamma_c \right) H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \\ H(\vec{\gamma}; \vec{\sigma} - \vec{e_c}; \vec{x}) &= \left(\sum_{b=1}^{r} \nu_{cb} x_b \frac{\partial}{\partial x_b} + \sigma_c - 1 \right) H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \ . \end{aligned}$$

Differential Reduction: Inverse Operators

The inverse differential operators can be constructed:

$$\begin{aligned} H(\vec{\gamma} - \vec{e_c}; \vec{\sigma}; \vec{x}) &= \sum_a S_a(\vec{x}, \vec{\partial_x}) H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \\ H(\vec{\gamma}; \vec{\sigma} + \vec{e_c}; \vec{x}) &= \sum_b L_b(\vec{x}, \vec{\partial_x}) H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \ . \end{aligned}$$

In this way, the Horn-type structure provides an opportunity to reduce hypergeometric functions to a set of basis functions with parameters differing from the original values by integer shifts:

$$P_0(\vec{x}) H(\vec{\gamma} + \vec{k}; \vec{\sigma} + \vec{l}; \vec{x}) = \sum_{m_1, \cdots, m_p = 0} P_{m_1, \cdots, m_r}(\vec{x}) \left(\frac{\partial}{\partial \vec{x}} \right)^{\vec{m}} H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \ ,$$

where $P_0(\vec{x})$ and $P_{m_1, \cdots, m_p}(\vec{x})$ are polynomials with respect to $\vec{\gamma}, \vec{\sigma}$ and $\vec{x}$ and $\vec{k}, \vec{l}$ are lists of integers.

Example: Direct index-shifting operators

The generalized hypergeometric functions have the form

$$ {}_pF_q(\vec{a};\vec{b};z) \equiv {}_pF_q\left(\begin{array}{c} \vec{a} \\ \vec{b} \end{array} \middle| z \right) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\prod_{i=1}^{p}(a_i)_k}{\prod_{j=1}^{q}(b_j)_k} \,, $$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is a Pochhammer symbol. The lists $\vec{a} = (a_1, \cdots, a_p)$ and $\vec{b} = (b_1, \cdots, b_q)$ are the upper and lower parameters of hypergeometric functions, respectively.

Direct index-shifting operators may be defined as follows:

$$ \begin{aligned} {}_pF_q(a_1+1,\vec{a};\vec{b};z) &= B^{+}_{a_1}\,{}_pF_q(a_1,\vec{a};\vec{b};z) \equiv \frac{1}{a_1}(\theta + a_1)\,{}_pF_q(a_1,\vec{a};\vec{b};z)\,, \\ {}_pF_q(\vec{a};b_1-1,\vec{b};z) &= H^{-}_{b_1}\,{}_pF_q(\vec{a};b_1,\vec{b};z) \equiv \frac{1}{b_1-1}(\theta + b_1 - 1)\,{}_pF_q(\vec{a};b_1,\vec{b};z)\,, \end{aligned} $$

where

$$ \theta = z\frac{d}{dz}\,. $$

Example: Inverse Operators

For the special case ${}_{p+1}F_p$, inverse shifting operators satisfying

$$\begin{aligned} {}_{p+1}F_p(a_i - 1, \vec{a}; \vec{b}; z) &= B^-_{a_i}\ {}_{p+1}F_p(a_i, \vec{a}; \vec{b}; z)\ , \\ {}_{p+1}F_p(\vec{a}; b_i + 1, \vec{b}; z) &= H^+_{b_i}\ {}_{p+1}F_p(\vec{a}; b_1, \vec{b}; z)\ , \end{aligned}$$

are found to be given by

$$B^-_{a_i} = -\frac{a_i}{c_i}\left[t_i(\theta) - z\prod_{j\neq i}(\theta + a_j)\right]_- \ , \quad H^+_{a_i} = \frac{b_i - 1}{d_i}\left[\frac{d}{dz}\prod_{j\neq i}(\theta + b_j - 1) - s_i(\theta)\right]_+$$

with

$$\begin{aligned} c_i &= -a_i\prod_{j=1}^{p}(b_j - 1 - a_i)\ , \quad t_i(x) = \frac{x\prod_{j=1}^{p}(x + b_j - 1) - c_i}{x + a_i} \\ d_i &= \prod_{j=1}^{p+1}(1 + a_j - b_i)\ , \quad s_i(x) = \frac{\prod_{j=1}^{p+1}(x + a_j) - d_i}{x + b_i - 1}, \end{aligned}$$

and the $\pm$ subscripts on the brackets are shorthand indicating that $a_i \to a_i - 1$, $b_i \to b_i + 1$, inside the respective brackets.

Differential Reduction of ${}_{p+1}F_p$

In this way, any function ${}_{p+1}F_p(\vec{a}+\vec{m};\vec{b}+\vec{k};z)$ is expressible in terms of a single basic function and its first p derivatives:

$$S(a_i,b_j,z)\,{}_{p+1}F_p(\vec{a}+\vec{m};\vec{b}+\vec{k};z) =$$
$$\left\{R_1(a_i,b_j,z)\theta^p + R_2(a_i,b_j,z)\theta^{p-1} + \cdots + R_p(a_i,b_j,z)\theta + R_{p+1}(a_i,b_j,z)\right\}{}_{p+1}F_p(\vec{a};\vec{b};z)\ ,$$

where $\vec{m},\vec{k}$ are lists of integers, and S and R_i are polynomials in the parameters $\{a_i\}, \{b_j\}$ and z.

In the special case some of the upper parameters $\vec{a}$ are integers, the corresponding expression has different form:

$$S(a_i,b_j,z)\,{}_{p+1}F_p(\vec{l};\vec{a}+\vec{m};\vec{b}+\vec{k};z) =$$
$$\left\{Q_1(a_i,b_j,z)\theta^{p-1} + \cdots + Q_{p-1}(a_i,b_j,z)\theta + Q_p(a_i,b_j,z)\right\}{}_{p+1}F_p(\vec{1};\vec{a};\vec{b};z)$$
$$+Q_{p+1}(a_i,b_i,z)\ ,$$

where Q_i are polynomials in the parameters $\{a_i\}, \{b_j\}$ and z.

Invariants of Hypergeometric Representation

Is there a correlation between the number of master integrals for a given Feynman Diagram (via IBP) and number of basis elements in the differential reduction of Hypergeometric Functions?

Consider the standard hypergeometric representation of a Feynman diagram,

$$\Phi(d, \vec{j}; \vec{z}) = \sum_{a=0}^{k} S_a(d, \vec{j}, \vec{z})_{p+1+a}F_{p+a}(\vec{\beta}_a; \vec{\lambda}_a; \vec{\xi})$$

where $\vec{j}$ is a list of the powers of the propagators in the Feynman diagram, d is the space-time dimension, $\vec{\xi}$ are the arguments of the hypergeometric functions, which are related the kinematic invariants of the Feynman diagram, $\{\beta_a, \lambda_a\}$ are linear combinations of $\vec{j}$ and d with polynomial coefficients, and S_a are rational functions of the variables $\vec{z}$ with coefficients depending on d and $\vec{j}$.

Invariants of Hypergeometric Representation

Being a sum of holonomic functions, $\Phi(\vec{j};\vec{z})$ is also holonomic. Thus, the number of basis elements on the r.h.s. of the above equation is equal to the number of master-integrals $\Phi_k(\vec{z})$ that may be derived from the l.h.s. via IBP, giving, symbolically,

$$\Phi(d,\vec{j};\vec{z}) = \sum_{k=1}^{h} B_k(d,\vec{j};z)\Phi_k(d;z) \; .$$

The number h of nontrivial master integrals following from IBP which are not expressible in terms of gamma functions is then equal to the number of basis elements L for each term of r.h.s. of equation. Here, it is understood that diagrams that are expressible in terms of Gamma functions are not counted.

The number of basis elements in the framework of differential reduction is defined to be the highest power of the differential operator θ in

$${}_{p+1}F_p(\vec{A};\vec{B};z) = \sum_{l=0}^{v} P_l(z)\theta^l \, {}_{s+1}F_s(\vec{A}-\vec{I_1};\vec{B}-\vec{I_2};z) \; ,$$

where $\vec{I_1},\vec{I_2}$ are lists of integers and $P_l(z)$ are rational functions.

Invariants of Hypergeometric Representation

This analysis demonstrates that there is a very simple relation between the number h of nontrivial master integrals found from IBP (which are not expressible in terms of Gamma functions) and the maximal power v of θ generated by the differential reduction, namely

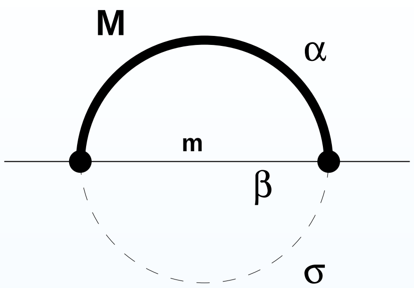
$$h = v + 1 \, .$$

This relation does not depend on the number k of hypergeometric functions entering original equation. This evidence leads to a conjecture:

Regardless of the type of functions in the r.h.s. of this equation, the number of basis elements is the same (up to a module of rational functions).

We are not aware of any exceptions to this statement, but it has not been proved in general.

Example: On-Shell Sunset Diagram



J012

$$J_{012}(\sigma,\beta,\alpha) = \int \frac{d^d k_1 d^d k_2}{[(k_1-p)^2]^\sigma [k_2^2 - m^2]^\beta [(k_1-k_2)^2 - M^2]^\alpha}\Bigg|_{p^2=m^2}$$

$$= \frac{(-M^2)^{d-\sigma-\alpha-\beta}(-1)^{1-d}\pi^d \Gamma(\frac{d}{2}-\sigma)}{\Gamma(\sigma)\Gamma(\alpha)\Gamma(\beta)\Gamma(\frac{d}{2})}$$

$$\times \Bigg[\Gamma\left(\frac{d}{2}-\beta\right)\Gamma\left(\alpha+\beta+\sigma-d\right)\Gamma\left(\beta+\sigma-\frac{d}{2}\right) {}_4F_3\left(\begin{array}{c} \alpha+\beta+\sigma-d, \beta+\sigma-\frac{d}{2}, \frac{\beta}{2}, \frac{1+\beta}{2} \\ 1+\beta-\frac{d}{2}, \beta, \frac{d}{2} \end{array}\Bigg| z\right)$$

$$+ \left(\frac{z}{4}\right)^{d/2-\beta}\Gamma\left(\beta-\frac{d}{2}\right)\Gamma(\sigma)\Gamma\left(\alpha+\sigma-\frac{d}{2}\right) {}_4F_3\left(\begin{array}{c} \sigma, \alpha+\sigma-\frac{d}{2}, \frac{d-\beta}{2}, \frac{1+d-\beta}{2} \\ 1+\frac{d}{2}-\beta, n-\beta, \frac{d}{2} \end{array}\Bigg| z\right)\Bigg]$$

with $z = 4m^2/M^2$.

This diagram contributes to the top quark pole mass at order $\alpha\alpha_s$.

See F. Jegerlehner and M.Yu. Kalmykov, Nucl. Phys. B676 (2004) 365 [arXiv: hep-ph/0308216].

The present analysis appears in M.Yu. Kalmykov and B.A. Kniehl, Phys. Lett. B702 (2011) 268 [arXiv:1105.5319].

On-Shell Sunset Diagram: Differential Reduction

Differential reduction allows the ${}_4F_3$ functions to be reduced to hypergeometric functions ${}_2F_1$ and ${}_3F_2$ and their derivatives:

$$
{}_4F_3\left(\begin{array}{c} \alpha+\beta+\sigma-d, \beta+\sigma-\frac{d}{2}, \frac{\beta}{2}, \frac{1+\beta}{2} \\ 1+\beta-\frac{d}{2}, \beta, \frac{d}{2} \end{array}\middle| z\right)
$$

$$
= (P_1(z) + Q_1(z)\theta)\, {}_2F_1\left(\begin{array}{c} I_1-n, \frac{1}{2}+I_2 \\ \frac{d}{2}+I_3 \end{array}\middle| z\right) + R_1(z) \ ,
$$

$$
{}_4F_3\left(\begin{array}{c} \sigma, \alpha+\sigma-\frac{d}{2}, \frac{d-\beta}{2}, \frac{1+d-\beta}{2} \\ 1+\frac{d}{2}-\beta, d-\beta, \frac{d}{2} \end{array}\middle| z\right)
$$

$$
= (P_2(z) + Q_2(z)\theta)\, {}_3F_2\left(\begin{array}{c} 1, I_1-\frac{d}{2}, \frac{d}{2}+\frac{1}{2}+I_2 \\ d+I_3, \frac{d}{2}+I_4 \end{array}\middle| z\right) + R_2(z) \ ,
$$

where $\theta = zd/dz$, P_i, Q_i, R_i are rational functions and I_i are integers.

Thus, we have a reduction to two master integrals that are not expressible in terms of Γ functions.

On-Shell Sunset Diagram: Integration by Parts

The IBP relations yield three master integrals,

$$J_{012}(1,1,1), J_{012}(1,2,1), J_{012}(1,1,2)$$

Does this contradict our counting?

On-Shell Sunset Diagram: Integration by Parts

The IBP relations yield three master integrals,

$$J_{012}(1,1,1), J_{012}(1,2,1), J_{012}(1,1,2)$$

Does this contradict our counting?

NO! Kalmykov and Kniehl discovered a new relation among the master integrals, derived from the differential reduction technique:

$$\begin{aligned}(3d-8)J_{012}(1,1,1) - 4m^2 J_{012}(1,2,1) - 2M^2 J_{012}(1,1,2) \\ = \quad 2(M^2)^{d-3}\Gamma\left(\frac{d}{2}-1\right)\Gamma(3-d)\Gamma\left(2-\frac{d}{2}\right) \; .\end{aligned}$$

- Thus, there are only two **independent** master integrals that are non-trivial (not expressible in terms of rational functions).
- The Tarasov and Laporta algorithms do not produce such a relation between master integrals.

Epsilon Expansion

- Normally, the dimension is taken to be close to an integer, usually $d = 4 - 2\varepsilon$.
- The Feynman integral may be expressed as a Laurent series in ε, called the **epsilon expansion.**
- The final goal of the evaluation of a Feynman integral is to obtain an expression for all terms in the ε-expanded Feynman Integrals.

A good representation of the ε expansion should satisfy the following conditions:

- Stable numerical evaluation at the arbitrary values of agreements (the values of mass and external momenta).
- Analytical continuation in any region of values of physical parameters;
- Ability to construct the Laurent expansion at an arbitrary complex point;
- Ability to explicitly extract any logarithmic terms.

Epsilon Expansion of a Hypergeometric Function

If each parameter list is shifted so that

$$\vec{A} \to \vec{A} + \varepsilon \vec{a}, \qquad \vec{B} \to \vec{B} + \varepsilon \vec{b},$$

then $\quad {}_pF_{p-1}(\vec{A} + \varepsilon\vec{a}; \vec{B} + \varepsilon\vec{b}; z) = {}_pF_{p-1}(\vec{A}; \vec{B}; z)$

$$+ \sum_{m_i, l_j = 1}^{\infty} \prod_{i=1}^{p} \prod_{j=1}^{p-1} \frac{(\varepsilon a_i)^{m_i}}{m_i!} \frac{(\varepsilon b_j)^{l_j}}{l_j!} \left(\frac{\partial}{\partial A_i} \right)^{m_i} \left(\frac{\partial}{\partial B_j} \right)^{l_j} {}_pF_{p-1}(\vec{A}; \vec{B}; z)$$

$$= {}_pF_{p-1}(\vec{A}; \vec{B}; z) + \sum_{m_i, l_j = 1} \prod_{i=1}^{p} \prod_{j=1}^{p-1} (\varepsilon a_i)^{m_i} (\varepsilon b_i)^{l_j} L_{\vec{A}, \vec{B}}(z) \ ,$$

$$= {}_pF_{p-1}(\vec{A}; \vec{B}; z) = {}_pF_{p-1}(\vec{A}; \vec{B}; z) + \sum_{k=1}^{\infty} \varepsilon^k L_{\vec{a}, \vec{b}, k}(z) \equiv \sum_{k=0}^{\infty} \varepsilon^k L_{\vec{a}, \vec{b}, k}(z) \ .$$

The goal of the ε expansion is to completely describe the coefficients $L_{\vec{a},\vec{b},k}(z)$.

Direct Solution of the System of Differential Equations

Horn-type hypergeometric functions satisfy a system of linear differential equations, these can be expanded to obtain equations for the terms in the ε expansion.

In series of papers by Kalmykov, Ward and Yost, it was observed that for special values of the parameters, the original differential equations for the hypergeometric function (second or higher order) can be expressed in diagonal form as series in powers of ϵ. Symbolically:

$$\sum_{i=0}^{p} P_i(x,\varepsilon) \left(\frac{d}{dx} \right)^{p-i} \omega(x) = 0 \; ,$$

where $P_i(x,\varepsilon)$ are polynomials.

$$P_i(x,\varepsilon) = \sum_{k=0}^{\infty} p_i^{(k)}(x) \varepsilon^k \; , \quad \omega(x) = \sum_{k=0}^{\infty} \omega_k(x) \varepsilon^k \; ,$$

and each polynomial can be decomposed into a product of linear terms:

$$p_i^{(k)} = \Pi_{m=1}^{n} (x - b_m) \; , \qquad\qquad (n \leq 4)$$

Direct Solution of the System of Differential Equations

By introducing a set a new functions

$$\phi_i^{(k)} = \rho_i^{(k)}(x)\left(\frac{d}{dx} + c_i\right)\phi_{i-1}^{(k)}\,, \quad i = 2, p-1; \qquad \phi_1^{(k)} = \rho_1(x)\omega_0^{(k)}\,,$$

where c_i are arbitrary rational numbers and $\{\rho_i^{(k)}\}$ are set of auxiliary functions, the original expression can be rewrite as infinite system of block-diagonal form:

$$Q_k^{(p)}(x)\frac{d}{dx}\phi_{p-1}^{(k)} = \sum_{m,j=0} R_p^{(m,j)}(x)\phi_{p-1-j}^{(k-1-m)}\,.$$

When $Q_k^{(p)}(x)$ and $R_p^{(m,j)}(x)$ are functions of the form of the $p_i^{(k)}$ above (products of linear factors), and first few coefficients of ε-expansion are equal to zero, this system can be solved iteratively and the results can be written in terms of Goncharov's polylogarithms.

$$G_{k,\vec{m}}(z;a,\vec{b}) = \underbrace{\int_0^z \frac{dt_k}{t_k}\ldots,\int_0^{t_3}\frac{dt_2}{t_2}}_{k-1}\int_0^{t_2}\frac{dt_1}{t_1-a}G_{\vec{m}}(t_1;\vec{b})$$

For generalized hypergeometric functions, there are only two freedoms: one function and redefinition of variables. No other new functions are relevant.

ε Expansion for Gauss Hypergeometric Functions

Let us see how the equations look for the Gauss hypergeometric function

$$\omega(z) = {}_2F_1\left(\frac{p_1}{q_1} + a_1\varepsilon, \frac{p_2}{q_2} + a_2\varepsilon; 1 - \frac{p_3}{q_3} + c\varepsilon; z\right) .$$

It is the solution of the differential equation

$$\left(z\frac{d}{dz} + \frac{p_1}{q_1} + a_1\varepsilon\right)\left(z\frac{d}{dz} + \frac{p_2}{q_2} + a_2\varepsilon\right)\omega(z) = \frac{d}{dz}\left(z\frac{d}{dz} - \frac{p_3}{q_3} + c\varepsilon\right)\omega(z) ,$$

with boundary conditions $\omega(0) = 1$ and $z\frac{d}{dz}\omega(z)\Big|_{z=0} = 0$.
Due to the analyticity of the Gauss hypergeometric function with respect to its parameters, this equation is valid in each order of ε, i.e. it holds for every coefficient function $\omega_k(z)$ in the expansion

$$\omega(z) = \sum_{k=0}^{\infty} \omega_k(z)\varepsilon^k.$$

ε Expansion for Gauss Hypergeometric Functions

The boundary conditions for the coefficient functions are

$$\begin{aligned} \omega_k(z) &= 0 & \qquad (k < 0)\ , \\ \omega_k(0) &= 0 & \qquad (k \geq 1)\ , \\ z\tfrac{d}{dz}\omega_k(z)\Big|_{z=0} &= 0 & \qquad (k \geq 0)\ . \end{aligned}$$

the original equation can be rewritten in terms of the coefficients functions ω_k as

$$\begin{aligned} &\left[(1-z)\frac{d}{dz} - \left(\frac{p_1}{q_1} + \frac{p_2}{q_2}\right) - \frac{1}{z}\frac{p_3}{q_3}\right] z\frac{d}{dz}\omega_k - \frac{p_1 p_2}{q_1 q_2}\omega_k \\ &= \left(a_1 + a_2 - \frac{c}{z}\right) z\frac{d}{dz}\omega_{k-1} + \left(a_1\frac{p_2}{q_2} + a_2\frac{p_1}{q_1}\right)\omega_{k-1} + a_1 a_2 \omega_{k-2}\ . \end{aligned}$$

ε Expansion for Gauss Hypergeometric Functions

The **main idea** of our approach is to find a new parametrization, through a change of variables $z \to \xi(z)$, and to define new functions $\rho_k(\xi)$, related to the first derivative of the original functions $\omega_k(\xi)$ as

$$\rho_k(\xi) = \sum_j \Gamma_{kj}(\xi) \frac{d}{d\xi} \omega_j(\xi) \; ,$$

so that original equation can be rewritten as a system of first-order linear differential equations with rational coefficients:

$$\begin{aligned} \frac{d}{d\xi}\omega_k(\xi) &= \rho_k(\xi) \sum_j \frac{A_j}{\xi - \alpha_j} \; , \\ \frac{d}{d\xi}\rho_k(\xi) &= \rho_{k-1}(\xi) \sum_j \frac{B_j}{\xi - \beta_j} + \omega_{k-1}(\xi) \sum_j \frac{C_j}{\xi - \gamma_j} + \omega_{k-2}(\xi) \sum_j \frac{D_j}{\xi - \sigma_j} \; , \end{aligned}$$

where $A_j, B_j, C_j, D_j, \alpha_j, \beta_j, \gamma_j, \sigma_j \in \mathbb{C}$. Then the iterative solution of this system can be constructed.

When $\omega_0(z) = 1$ ($\rho_0(z) = 0$), this solution can be expressed in terms of multiple polylogarithms (MPLs – to be described later) depending on the parameters $\alpha_j, \beta_j, \gamma_j, \sigma_j$, possibly times powers of logarithms.

ε Expansion for Gauss Hypergeometric Functions

These methods have been extended to rational values of the parameters as well, leading to the following result...

Theorem:
If I_1, I_2, I_3 are arbitrary integers, the Laurent expansions of the Gauss hypergeometric functions

$$
\begin{aligned}
&{}_2F_1(I_1 + a\varepsilon, I_2 + b\varepsilon; I_3 + \tfrac{p}{q} + c\varepsilon; z)\ , \\
&{}_2F_1(I_1 + \tfrac{p}{q} + a\varepsilon, I_2 + \tfrac{p}{q} + b\varepsilon; I_3 + \tfrac{p}{q} + c\varepsilon; z)\ , \\
&{}_2F_1(I_1 + \tfrac{p}{q} + a\varepsilon, I_2 + b\varepsilon; I_3 + c\varepsilon; z)\ , \\
&{}_2F_1(I_1 + \tfrac{p}{q} + a\varepsilon, I_2 + b\varepsilon; I_3 + \tfrac{p}{q} + c\varepsilon; z)
\end{aligned}
$$

are expressible in terms of multiple polylogarithms (MPLs) of arguments being powers of q-roots of unity and a new variable, that is an algebraic function of z, with coefficients that are ratios of polynomials.

ε Expansion for ${}_pF_{p-1}$

More complicated example: ${}_pF_{p-1}$ with parameters as shown...

$\omega(z) = {}_pF_{p-1}\left(\vec{a}\varepsilon, A+c\varepsilon; \vec{1}+\vec{b}\varepsilon, B+f\varepsilon; z\right)$ satisfies

$$\left[z\,(\theta+A+c\varepsilon)\prod_{j=1}^{p-1}(\theta+a_j\varepsilon)-\theta\,(\theta+B-1+f\varepsilon)\prod_{k=1}^{p-2}(\theta+b_k\varepsilon)\right]\omega(z)=0\;,$$

Expanding this gives equations for its ε expansion $\omega(z) = 1 + \sum_{j=1}^{\infty} w_k(z)\varepsilon^k$:

$$\left[(1-z)\frac{d}{dz}+\frac{B-1}{z}-A\right]\theta^{p-1}w_m(z) = \left[P_1^{(p)}(\vec{a},c)-\frac{1}{z}P_1^{(p-1)}(\vec{b},f)\right]\theta^{p-1}w_{m-1}(z)$$
$$+\sum_{j=2}^{p-1}\left[P_j^{(p)}(\vec{a},c)-\frac{1}{z}P_j^{(p-1)}(\vec{b},f)\right]\theta^{p-j}w_{m-j}(z)+AP_{p-1}^{(p-1)}(\vec{a})w_{m-p+1}(z)$$
$$+\sum_{k=1}^{p-2}\left[AP_k^{(p-1)}(\vec{a})-\frac{(B-1)}{z}P_k^{(p-2)}(\vec{b})\right]\theta^{p-1-k}w_{m-k}(z)+P_p^{(p)}(\vec{a},c)w_{m-p}(z)\;,$$

where $\theta = zd/dz$ and the polynomials $P_j^{(p)}(r_1,\cdots,r_p)$ are

$$\prod_{k=1}^{p}(z+r_k)=\sum_{j=0}^{p}P_{p-j}^{(p)}(r_1,\cdots,r_p)z^j\equiv\sum_{j=0}^{p}P_{p-j}^{(p)}(\vec{r})z^j\equiv\sum_{j=0}^{p}P_j^{(p)}(\vec{r})z^{p-j}\;.$$

${}_pF_{p-1}$ Solution: Finite Part

- The first non-vanishing term corresponds to $m = p$ if $A = 0$, and to $m = p - 1$ otherwise. In both cases, the main equation reduces to

$$\left[(1-z)\frac{d}{dz}+\frac{B-1}{z}-A\right]\theta^{p-1}w_{p-1+\delta_{A,0}}(z) = (A+c\delta_{A,0})P^{(p-1)}_{p-1}(\vec{a}) \ ,$$

  where $\delta_{A,0}$ is 1 if $A = 0$, and 0 otherwise.

- The solutions can be expressed in terms of multiply iterated integrals, which lead to an expression for MPLs, provided a reparametrization $z \to \xi(z)$ exists such that the following two conditions are fulfilled for some rational functions $Q(\xi)$, $R(\xi)$:

$$\frac{dz}{(1-z)h(z)} = Q(\xi)d\xi \ , \qquad \frac{dz}{z} = R(\xi)d\xi \ .$$

  with $h(z) = (-1)^A z^{1-B}(z-1)^{B-A-1}$ for some rational numbers A, B.

${}_pF_{p-1}$ Solution: ε Parts

To analyze the structure of the highest coefficients of the ε expansions, let us consider the original function $\omega(z)$ and its first $p-1$ derivatives as independent functions, $f^{(k)} = (\omega, \theta\omega, \cdots, \theta^{p-1}\omega)$, $k = 0, \cdots, p-1$.

Each $f^{(k)}$ has a ε expansion $f^{(k)}(z) = \sum_{j=0}^{\infty} f_j^{(k)}(z)\varepsilon^j$ with boundary conditions $f_0^{(0)}(z) = 1$ and $f_j^{(k)}(0) = 0$, $j \geq 1$, $k = 1, \cdots, p-1$. Defining $\theta^{p-1}\omega_k(z) = h(z)\phi_j^{(p-1)}(z)$, we can convert the original equation into a system of first-order differential equations,

$$h(z)(1-z)\frac{d}{dz}\phi_m^{(p-1)}(z) = h(z)\left[P_1^{(p)}(\vec{a},c) - \frac{1}{z}P_1^{(p-1)}(\vec{b},f)\right]\phi_{m-1}^{(p-1)}(z)$$

$$+\sum_{j=2}^{p-1}\left[P_j^{(p)}(\vec{a},c) - \frac{1}{z}P_j^{(p-1)}(\vec{b},f)\right]f_{m-j}^{(p-j)}(z) + AP_{p-1}^{(p-1)}(\vec{a})w_{m-p+1}(z)$$

$$+\sum_{k=1}^{p-2}\left[AP_k^{(p-1)}(\vec{a}) - \frac{(B-1)}{z}P_k^{(p-2)}(\vec{b})\right]f_{m-k}^{(p-1-k)}(z) + P_p^{(p)}(\vec{a},c)w_{m-p}(z)\ ,$$

$$\theta f_m^{(p-2)}(z) = h\phi_m^{(p-1)}(z)\ ,$$

$$\theta f_m^{(j-1)}(z) = f_m^{(j)}(z) \quad \text{for} \qquad j = 1, \cdots, p-2\ .$$

${}_pF_{p-1}$ Solution: ε Parts

- The solution of this system can again be presented as an iterated integral over a rational one-form, if two additional conditions are satisfied:

$$\frac{dz}{z}\frac{1}{h(z)} = P_1(\xi)d\xi \,, \qquad \frac{dz}{z}h(z) = P_2(\xi)d\xi \,.$$

  where P_1 and P_2 are rational functions.

- As a consequence of the universality of MPLs, any iterated integral over a rational function may be expressed again in terms of MPLs. It is easy to show that the two equations are not functionally independent. In fact, we obtain

$$R^2(\xi) = P_1(\xi)P_2(\xi) \,, \quad h(z) = \frac{R(\xi)}{P_1(\xi)} = \frac{P_2(\xi)}{R(\xi)} \,.$$

ε Expansion for ${}_pF_{p-1}$

These methods have been extended to rational values of the parameters as well, leading to the following result...

Theorem:
If $\vec{I}, \vec{K}, \vec{L}$ are arbitrary integers, the Laurent expansions of the hypergeometric functions

$$
\begin{aligned}
&{}_pF_{p-1}(\vec{I}_p + \vec{a}\varepsilon; \vec{K}_{p-2} + \vec{b}\varepsilon; L + \tfrac{p}{q} + c\varepsilon; z)\ , \\
&{}_pF_{p-1}(I + \tfrac{p}{q} + a\varepsilon, \vec{K}_{p-1} + b\varepsilon; \vec{L}_{p-1} + \vec{c}\varepsilon; z)\ , \\
&{}_pF_{p-1}(\vec{I}_{p-1} + \tfrac{p}{q} + \vec{a}\varepsilon, K + b\varepsilon; \vec{L}_{p-1} + \vec{c}\varepsilon; z)\ , \\
&{}_pF_{p-1}(\vec{I}_p + \tfrac{p}{q} + \vec{a}; \vec{L}_{p-1} + \vec{c}\varepsilon; z)\ ,
\end{aligned}
$$

are expressible in terms of MPLs of arguments being powers of q-roots of unity and a new variable, that is an algebraic function of z, with coefficients that are ratios of polynomials.

Multiple Polylogarithms

- The **multiple polylogarithm** (MPL) is important for expressing the sums appearing in the ε expansion. It is defined by power series

$$\mathrm{Li}_{k_1,k_2,\cdots,k_n}(x_1,x_2,\cdots,x_n) = \sum_{m_n>\cdots>m_1>0}^{\infty} \frac{x_1^{m_1}}{m_1^{k_1}} \frac{x_2^{m_2}}{m_2^{k_2}} \cdots \frac{x_n^{m_n}}{m_n^{k_n}} \,,$$

  where **weight** $k = k_1 + k_2 + \cdots + k_n$ and **depth** is equal to n.
  It is defined for $|x_n| < 1$ and admit an analytical continuation.
- The case $x_1 = \cdots = x_n = 1$ corresponds to multiple zeta values.
- A particularly useful case is the "multiple polylogarithm of a square root of unity,"

$$\mathrm{Li}_{\left(\begin{smallmatrix}\sigma_1,\sigma_2,\cdots,\sigma_n\\ s_1,s_2,\cdots,s_n\end{smallmatrix}\right)}(z) = \sum_{m_n>m_{n-1}>\cdots m_1>0} z^{m_n} \frac{\sigma_n^{m_n}\cdots\sigma_1^{m_1}}{m_n^{s_n}\cdots m_1^{s_1}} \,.$$

  where $\vec{s} = (s_1,\cdots s_n)$ and $\vec{\sigma} = (\sigma_1,\cdots,\sigma_n)$ are multi-indices and σ_k belongs to the set of the square roots of unity, $\sigma_k = \pm 1$.

Iterated Integrals

An iterated integral is defined as

$$\begin{aligned} I(z; a_k, a_{k-1} \cdots, a_1) &= \int_0^z \frac{dt}{t - a_k} I(t; a_{k-1}, \cdots, a_1) \\ &= \int_0^z \frac{dt_k}{t_k - a_k} \int_0^{t_k} \frac{dt_{k-1}}{t_{k-1} - a_{k-1}} \cdots \int_0^{t_2} \frac{dt_1}{t_1 - a_1} \end{aligned}$$

An important special case of this integral is the Goncharov polylogarithm

$$\begin{aligned} & G_{m_n, m_{n-1}, \cdots, m_1}(z; x_n, \cdots, x_1) \\ & \equiv I(z; \underbrace{0, \cdots, 0}_{m_n - 1 \text{ times}}, x_n, \underbrace{0, \cdots, 0}_{m_{n-1} - 1 \text{ times}}, x_{n-1}, \cdots, \underbrace{0, \cdots, 0}_{m_1 - 1 \text{ times}}, x_1) \end{aligned}$$

The multiple polylogarithm is a special case of this iterated integral:

$$\begin{aligned} & \mathrm{Li}_{k_1, k_2, \cdots, k_n}(y_1, y_2, \cdots, y_n) \\ & = (-1)^n G_{k_n, k_{n-1}, \cdots, k_2, k_1}\left(1; \frac{1}{y_n}, \frac{1}{y_n y_{n-1}}, \cdots, \frac{1}{y_1 \cdots y_n}\right) . \end{aligned}$$

Result for Multiple Inverse Binomial Sums

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j} S_{a_1}(j-1)\cdots S_{a_k}(j-1)\Bigg|_{u=-\frac{(1-y)^2}{y}} = \frac{1-y}{1+y} \sum_{p,\vec{s}} c_{p,\vec{s}} \ln^p y \mathrm{Li}_{\left(\vec{\sigma} \atop \vec{s}\right)}(y)$$

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j^c} S_{a_1}(j-1)\cdots S_{a_k}(j-1)\Bigg|_{u=-\frac{(1-y)^2}{y}} = \sum_{p,\vec{s}} \tilde{c}_{p,\vec{s}} \ln^p y \mathrm{Li}_{\left(\vec{\sigma} \atop \vec{s}\right)}(y) \,, \quad c \geq 2 \,.$$

where c is a positive integer, $c_{p,\vec{s}}$ and $\tilde{c}_{p,\vec{s}}$ are rational coefficients, $S_a(j-1) = \sum_{i=1}^{j-1} \frac{1}{i^a}$, is harmonic sum and $\mathrm{Li}_{\left(\vec{\sigma} \atop \vec{s}\right)}(z)$ is the multiple polylogarithm of a square root of unity.

The result is the same for more general sums,

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j^c} \sum_{k_1 > k_2 > \cdots k_m > 0}^{j-1} \frac{1}{m_1^{a_1} m_2^{a_2} \cdots m_k^{a_k}}$$

Result for Multiple Binomial Sums

$$\sum_{j=1}^{\infty} \binom{2j}{j} u^j S_{a_1}(j-1)\cdots S_{a_k}(j-1)\bigg|_{u=\frac{\chi}{(1+\chi)^2}} = \sum_{p,\vec{s}} \left[\frac{c_{p,\vec{s}}}{1-\chi} + d_{p,\vec{s}}\right] \ln^p \chi \, \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(\chi) \ ,$$

$$\sum_{j=1}^{\infty} \binom{2j}{j} \frac{u^j}{j^c} S_{a_1}(j-1)\cdots S_{a_k}(j-1)\bigg|_{u=\frac{\chi}{(1+\chi)^2}} = \sum_{p,\vec{s}} \tilde{c}_{p,\vec{s}} \ln^p \chi \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(\chi) \ , \quad c \geq 1$$

where c is a positive integer, $c_{p,\vec{s}}$, $\tilde{c}_{p,\vec{s}}$ and $d_{p,\vec{s}}$ are rational coefficients, $\mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(z)$ is the multiple polylogarithm of a square root of unity and $S_a(j-1) = \sum_{i=1}^{j-1} \frac{1}{i^a}$.

The result is the same for more general sums,

$$\sum_{j=1}^{\infty} \binom{2j}{j} \frac{u^j}{j^c} \sum_{k_1 > k_2 > \cdots k_m > 0}^{j-1} \frac{1}{m_1^{a_1} m_2^{a_2} \cdots m_k^{a_k}}$$

Other Methods: Expansion by Moch, Uwer, Weinzierl

Systematic algorithms for constructing the ε expansion of hypergeometric functions around integer values of the parameters has been developed by Moch, Uwer, and Weinzierl (MUW in the following). These involve sums S_a and Z_a defined via

$$\begin{aligned} S_a(n;x) &= Z_a(n;x) = \sum_{k=1}^{n} \frac{x^k}{k^i} \\ S_{i,\vec{j}}(n;x_1,\cdots,x_l) &= \sum_{k=1}^{n} \frac{x_1^k}{k^i} S_{\vec{j}}(k;x_2,\cdots,x_l)\ , \\ Z_{i,\vec{j}}(n;x_1,\cdots,x_l) &= \sum_{k=1}^{n} \frac{x_1^k}{k^i} Z_{\vec{j}}(k-1;x_2,\cdots,x_l)\ , \end{aligned}$$

These are useful for expanding the gamma function about integer values:

$$\begin{aligned} \Gamma(n+\varepsilon) &= \Gamma(1+\varepsilon)\Gamma(n)\left(1+\varepsilon Z_1(n-1)+\varepsilon^2 Z_{11}(n-1)+\varepsilon^3 Z_{111}(n-1)+\ldots\right)\ , \\ \Gamma(-n+1+\varepsilon) &= \frac{\Gamma(1+\varepsilon)}{\varepsilon}\frac{(-1)^{n-1}}{\Gamma(n)}\left(1+\varepsilon S_1(n-1)+\varepsilon^2 S_{11}(n-1)+\ldots\right)\ . \end{aligned}$$

When These Algorithms Fail...

There are some cases where the MUW algorithm fails or does not apply. This was a primary motivation for seeking other methods.

- Integer values of parameters: Appell function F_4 and its generalizations;

$$F_4(a,b,c_1,c_2;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m(c_2)_n} \frac{x^m}{m!} \frac{y^n}{n!} \quad \text{with } (\alpha)_j \equiv \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)}$$

(Pochhammer symbol)

- With one or more unbalanced rational parameter:

$${}_4F_3\left(\begin{array}{c} 1, \frac{1}{2}+b_1\varepsilon, 1+a_1\varepsilon, 1+a_2\varepsilon \\ \frac{3}{2}+f_1\varepsilon, \frac{3}{2}+f_2\varepsilon \end{array} \middle| z\right)$$

- With rational values of parameters: (2-loop sunset, equal masses)

$${}_2F_1\left(\begin{array}{c} \frac{1}{3}+b_1\varepsilon, \frac{2}{3}+b_2\varepsilon \\ 2+c\varepsilon \end{array} \middle| z\right)$$

Recent extension of Z-sums approach

Jakob Ablinger, Johannes Bl umlein, Carsten Schneider "Harmonic Sums and Polylogarithms Generated by Cyclotomic Polynomials" [arXiv:1105.6063]

Paulo A. Rottmann, Laura Reina "Z-Sum Approach to Loop Integrals using Taylor Expansion" [arXiv:1106.4629]

The Z-sum approach of MUW has been extended to generalize harmonic sums, *e.g.*

$$S(\{a,b,k,z\},\vec{B}) = \sum_{j=1}^{n} \frac{z^j}{(aj+b)^k} S(\vec{B},j-1) \ ,$$

where a, b, k are integers.

From ε-expansion of massive Feynman Diagram we have:

$$\sum_{j=0}^{\infty} \prod_{k=1}^{\infty} \Gamma(\alpha_k j + \beta_k)^{j_k} \times S(\{a,b,\vec{k},z\}) \ ,$$

where α_k, β_k are rational, j_k are any integers (positive or negative).

Open problems:

- After the reduction of multiple series to some basis it is necessary to evaluate the result in terms of analytical functions.
- We need to establish the special functions related to series of this type.
- Analytical continuation to physically interesting regions is needed.

Summary

- Feynman integrals may be expressed as linear combinations of Horn-type hypergeometric functions. Differential reduction can be applied to Horn-type hypergeometric functions with an arbitrary values of parameters.
- We conjecture an invariant property for hypergeometric representations: The number of basis elements for each term of hypergeometric represenation should be the same (up to module of rational functions). The number of nontrivial master integrals is coincides with number of basis elements for differential reduction of hypergeometric functions. This is an example of the “universal” type of result that we expect to arise from the existence of a hypergeometric representation.
- Dimensional regularization requires expansions of the hypergeometric functions about rational values of the parameters. We have developed an algorithm based on differential reduction for the systematic construction of the analytical coefficients of ε-expansion around arbitrary rational numbers. The main drawback of this method is that it is necessary to get a full analytical solution of differential reduction.

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The hypergeometric representation (our methods should allow ε-expansion):

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