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Old Scattering Amplitudes For New Mathematics
These developments including

Many have contributed significantly to
The goals of this research program are:

- to explore the rich mathematical structure of scattering amplitudes
- to exploit that structure to make previously impossible calculations trivial
individual shame!

Command pride triumphs (occasional)

timescale of months not years.

Phase transition often occurred on the
impossible ← hard ← trivial

From 2004 through 2011 (and beyond?) the

The Appeal of Long-Lasing Vitality
Four-dimensional QFT.

Why $N=4$ Super-Yang-Mills Theory?
gauged theory, and hence a cousin of QCD.

But not just some random QFT. It is a

four-dimensional QFT.

A nontrivial but apparently solvable

\[ N=1 \text{ Super-Yang Mills Theory?} \]
\* Moreover, it = quantum gravity in Ads space.

\* But not just some random GFT; it is a four-dimensional GFT.

\* A nontrivial but apparently solvable.

\underline{Why N=4 Super Yang Mills Theory?}
Yes we can!

Most importantly, because it is there and quantum gravity in AdS space.

Moreover, it = gauge theory, and hence a cousin of QCD.

But not just some random QFT, it is a four-dimensional QFT, a nontrivial but apparently solvable\[N=4\] Super Yang Mills Theory?
It is because the past, it is because we have seen further than the giants of S-Matrix Theory.

The spirit behind much recent progress is similar to that of S-Matrix Theory.

S-Matrix Theory Reloading?
the dark jungle looming ahead (GC). Beneath our feet (N=4) another than squarely on the well-manicured lawn.

If we have seen further than the giants of the past, it is because we're focused on the well-manicured lawn.

S-Matrix Theory? Is it similar to that of "S-Matrix Theory?"

S-Matrix Theory Reloaded?
A Completely Different Philosophy

Then, the idea was

(1) Enumerate the principles

(2) Solve!

(locally, analyticity, etc., etc.)
(1) enumerate the principles
(2) means necessary
(3) write down the solution by any

Now, our aim is to

(2) solve

locality, analythically, etc.

(1) enumerate the principles

Then, the idea was

A completely different philosophy
BUCKLE YOUR SAFETY BELTS
Amplitude

color-ordered partial amplitude

\[ \mathcal{A}(1, \ldots, n) = \exists A(e(1), \ldots, e(n)) \in \text{perm} \]

in \( N = 4 \) Yang-Mills theory.

planar, color-stripped partial amplitudes

ill focus almost exclusively on
Amplitude

Free variables

A linear constraint is switched to

\[ s + t + u = 0 \]

They are equal, if you use

\[ \frac{u s}{11} + \frac{u s}{12 + t} + \frac{s}{5} - \frac{s}{22} + \frac{n}{15} - \frac{n}{34 + t} + \frac{n}{5} - \frac{n}{10 + t} + \frac{n}{10 + t} + \frac{n}{32} - \frac{n}{95} + \frac{n}{105} + 20 + \frac{n}{10} \]

\[ A = \frac{10u}{s} \]

And

\[ A = -\frac{s}{1} \]

In massless 3\(^3\) theory, two representations

Representation Representation
Amplitude Representation Spinor Helicity

\[ \langle \psi | + | \phi \rangle = \langle \psi | \phi \rangle + \langle \psi \rangle \langle \phi \rangle \]

Notation

\[ \langle \psi \rangle = e^{a \cdot p} \]

Objects

2-Component

\[ \vec{\psi} \]

Solve \( K = 0 \) via \( \vec{K} = 0 \) via \( \vec{\psi} \)

Helicity variables

For particles with spin we need spinor
A spin-1 particle has 2 physical states with helicity $+1$ or $-1$.

The $n$-particle MHV amplitude involves $n-2$ particles with helicity and $2$ particles with $-helicity$.

$N_{\text{MHV}}: (n-3, 3)$

$N^2_{\text{MHV}}: (n-4, 4)$

etc.
This is spinor helicity. The most important formula in our field.

This is, psychologically, the most interesting.

\[
\langle \phi \rangle \langle \chi_1 \rangle \ldots \langle \chi_{23} \rangle \langle \chi_{24} \rangle = \text{Parke-Taylor, 88, Berns's circle 88}
\]

The tree-level MHV amplitude is...
1. Simplicities do not happen by accident.

The Philosophy of Anthropology
Working in $N=4$ superspace we write

$$A_{\text{MHV}} = \delta^8 \left( \sum_{i=1}^{n} \lambda_i^a \eta_i^4 \right) \frac{1}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

(Nair '88)

supermomentum conservation

$$A = A_{\text{MHV}} \left[ 1 + P_{\text{NMHV}} + P_{\text{NNNMHV}} + \cdots \right]$$

$\Theta(\eta^8)$ $\Theta(\eta^4)$ $\Theta(\eta^8)$ etc.
different (but secretly =) results.

different discrete choices give apparently

There is no single "BCTM representation"

practically — for rapidly generating data.

It is important both conceptually and

\[ \frac{z}{2 - v} = \frac{k}{2} \]

\[ \therefore \quad z = k \]

\[ \therefore \quad z = 2 \]

\[ \therefore \quad z = 2 \]

Rearrangion relation; schematically

BCTM is a quadratic on-shell
Example: The amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ may be expressed as either

$$\frac{[4|5+6|1]^3}{[23][34](56)(61)[213+415] \, S_{234}} + \frac{[6|1+2|3]^3}{[61][12]<34)(45)(213+415) \, S_{612}}$$

or

$$\frac{S_{123}}{[12][23](45)(56)[12+314][3|4+5|6]} + \frac{\langle 12\rangle \langle 45\rangle^3}{[34][61][3|4+5|6] \langle 516+12 \rangle \, S_{612}} + \frac{\langle 23\rangle \langle 56\rangle^3}{[34][61][1|2+3|4] \langle 516+12 \rangle \, S_{234}}$$

Equality of these two expressions can (essentially) only be checked by numerical experimentation.

[Richardson's theorem: there is no algorithm for determining if two mathematical expressions are equal.]
This is one of many examples showing that space triangulated in different ways!

Each expression computes the volume of a polytope in complex projective space triangulated in different ways!

Why in this case (2 terms) = (3 terms), found that there is a reason.

A. Hodges followed his nose and
The Philosophy of Amplitudeology

1. Simplifications do not happen by accident.

2. This is an experimental science.
   (Get the answer first, by any means necessary, then analyze it.)
They parameterize $\mathbb{CP}^3$ variables $(x, w)$ leaves for unchanged, \[ A(x, y) = \int_{x, y} e^{ixy} A(x, y) \]

We can go to twistor space by coming out a 1/2-fourier transform.
$$A_{\text{MHV}} = \prod_{i=1}^{n} \exp \left( \sum_{i<j} \int \frac{dx_i x_j}{x_i - x_j} \right)$$

Nothing that momentum conserving delta function
Amplitude over curves of degree $k+1$.

The level $N$MHV amplitude as an integral theory on $\mathbb{P}^3$ which computes the Parke-Taylor superamplitude.

Then invented a topological string measure on the moduli space of lines in $\mathbb{P}^3$.

$$A_{\text{MHV}} = \frac{1}{\prod_{i \neq j} |z_i z_j|} \prod_{i} \exp(\sum_{a} z_i a_a)$$

Nothing that momenta conserving delta function

Twistor String
Twistor Space
A=Volumes
BCFW
Superamplitude
Parke-Taylor
MHV etc.
Spinor Helicity
Representation
Amplitude
Loop Level

To

Moving on
A pattern in this data reveals a suprise...

(Amplitude, Spinor Helicity, etc.

(Bern, Dixon, Kosower, Skinner

\[
\begin{align*}
\text{+} & \quad \text{+} \\
\text{+} & \quad \text{+}
\end{align*}
\]

Parke-Taylor

Superamplitude

Twistor Space

Twistor String

Gen. Unitarity

A=Volumes

BCFW

parke-taylor

MHV etc.

Spinor Helicity

Representation

Amplitude

eg. the four-loop four-particle amplitude

allowed much more data to be collected

recent advances (Bitha, Carrasco, Feng)

(Bern, Dixon, Dunbar, Kosower) with more

union of the unitarity-based method
What is Dual Conformal Symmetry?
A strange non-local symmetry under which scattering amplitudes are co-variant.

What is Dual Conformal Symmetry?
Under conformal transformations on the X:

Then amplitudes are invariant

$K = K_1 - x_1$

$K_2 = K_3 - x_3$

$K_4 = K_2 - x_2$

Parametrize

$K^v = K$ (Drammond, Hen’ n, Korchemsky, Sokatchev)

Dual conformal symmetry

(Przumowski, Konev, Korchemsky, Sokatchev)
A leading singularity is a residue of poles. The integrand (no one of these poles has a pole) has no on-shell propagators when all 8 propagators go off-shell.

The integrand vanishes. The loop momentum poles in $\mathcal{C}_l$ (by definition) only simple function with (by definition) only simple

The $L$-loop integrand is a rational
Yangian Symmetry

Leading Sing.
The Integrand
Dual Conformal Inv.
Twistor String
Twistor Space
A=Volumes
BCFW
Superamplitude
Parke-Taylor
MHV etc.
Spinor Helicity
Representation

Ampplitude

Dual Conformal Transformations,

\{ ordinary conformal transformations, \\
\} dual conformal transformations,

= the closure under commutation of Yangian Symmetry
( = 8-functions Constraining the 5 points in CP^3 to lie inside a common CP^3 subspace.)
Grassmannian Int. R-Invariants Yangian Symmetry Leading Singulars. 
The Integrand Dual Conformal Inv. Twistor String Twistor Space A=Volumes BCFW Superamplitude \text{Parke-Taylor} \text{MHV etc.} 

Spinor Helicity Representation Amplitude 

(Araki-Hamad, (Chung, Cheung, Kaplan) 

$$\int [dC]_{K \times n} \left( \begin{array}{c} \ldots \left( K+1 \right) \cdot \ldots \left( n+K-1 \right) \\ C_{1, Z_{1}, \ldots , Z_{n}} \end{array} \right) 8h_{\frac{1}{2}}(C_{1, Z_{1}}) Z_{1} = (X_{1}, \ldots , X_{n})$$ 

Integral over the space of $K \times n$ matrices 

Grassmannian Integrand \int_{\text{Contour}} = A \text{ Amplitude} \n
$$\left( \begin{array}{c} \text{minors of the matrix C} \\
\text{It's residues are R-invariants.} \\
\text{n-particle } \mathcal{N} = 4 \text{ Yangian invariants,} \\
\text{which is a generating function for all} \\
(\text{Araki-Hamad, (Chung, Cheung, Kaplan)} \right)$$
(Albay, Cachazo, Maldecaena)

Hence integrability methods can be brought to bear on the problem.

\[ \text{Wilson loops} = \text{minimal area surfaces} \]

\[ \text{amplitudes are computed as certain} \]

\[ \text{showed that at strong coupling,} \]

\[ \text{using } \text{AdS/CFT,} \text{ Albay's Maldecaena} \]

\[ \text{Strong Coupling} \]

Grassmannian Int.

R-Invariants

Yangian Symmetry

Leading Sings.

The Integrand

Dual Conformal Inv.

Twistor String

Twistor Space

A=Volumes

BCFW

Superamplitude

Parkes-Taylor

MHV etc.

Spinor Helicity

Repr. Representation

Amplitude
Another remarkable feature of $\mathcal{N}=4$ supersymmetric Yang-Mills is the relationship between amplitudes and Wilson loops.
A = Correlation Fns.
A = Wilson Loop
Strong Coupling
Grassmannian Int.
R-Invariants
Yangian Symmetry
Leading Sing.
The Integrand
Dual Conformal Inv.
Twistor String
Twistor Space
A = Volumes
BCFW
Superamplitude
Parke-Taylor
MHV etc.
Spinor Helicity
Representation
Amplitude

Adamo, Bullimore, Mason, Skinner
Eden, Heslop, Korchemskiy, Sokatchev

\[ \frac{1}{2} \text{ BPS operators} \]
lightlike separated
Lagrangian invariants

\[ \langle 0(x_1) \cdots 0(x_n) x'(y_1) \cdots x'(y_l) \rangle \]

of the form

yet another way, by correlation functions

The integrand may in fact be computed
\[ A = \int dp \ldots dp \frac{\partial}{\partial p} \Rightarrow \text{Feynman diagrams} \]

You Say You Want a Revolution

One revolution has come to fruition

\( = \int dp \ldots dp \frac{\partial}{\partial p} (\text{relatively simple integrand}) \)

\[ \text{Exciting new technology} \uparrow \]
But another is desperately needed, still very hard to evaluate!

\[
\int \frac{d^3p}{(2\pi)^3} \rho \rightarrow \int \frac{d^4p}{(2\pi)^4} \text{relatively simple integral}
\]

\[
S \rho \cdots d^4p = \text{Feynman diagrams}
\]

\[
A = \int d^4p \rho \rightarrow \text{Feynman diagrams}
\]

One revolution has come to fruition

You say you want a revolution!
The natural "observables" for loop-level

The natural "observables" for loop-level

Amplitude Representation
Spinor Helicity
MHV etc.
Parke-Taylor
Superamplitude
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MHV etc.
Spinor Helicity
Remainder
A = Correlation Fn.
A = Wilson Loop
Strong Coupling
Grassmannian Int.
R-Invariants
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A = Wilson Loop
The first nontrivial loop amplitude in $\mathcal{N}=4$ SYM is the

2-loop (1-loop is absorbed into BDS subtraction)

6-particle (≠ any cross-ratios for n<6)

MHV remainder function.

Computed analytically in late 2009 by Del Duca, Duhr & Smirnov
If that were the best we could do, we
all ought to quit now, there would be
no hope...
Lin polylogarithm functions, "standard" involving nothing more complicated than the
this expression down to a single line,
a transcendental function to simplify of mathematical magic called the symbol of
Gendarmery, very, Volovich and I used some
no hope...
all ought to quit now, there would be
If that were the best we could do, we
The worth of our formula is not that if it computes the value of the 2-loop 6-point MHV amplitude!
Nobody cares about the value of this amplitude.

The worth of our formula is NOT 2-loop 6-point MHV amplitude. That it computes the value of the
...might really unlock the secrets to all orders... hey, you're really none, that we hope, were previously was none, that we.

Rather, it is important because it provides amplitude.

Nobody cares about the value of this.

2-loop 6-point MHV amplitude.

That if computes the value of the

The worth of our formula is not
The Philosophy of Amplitudeology

1. Simplifications do not happen by accident.

2. This is an experimental science. (Get the answer first, by any means necessary, then analyze it.)

3. Simplicity has to be believed to be seen.