From Navier-Stokes to Einstein

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1006.1902, 1101.2451 and 1106.3084 (hep-th arXiv)

12 August, 2011

Holography: A gravitational system has a dual description which is a non-gravitational field theory in one fewer dimensions

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- **Gravity**
  - Einstein equations: $G_{\mu\nu} = 0$

- **Fluid Mechanics:** $\left(v_i(t, x^j), P(t, x^j)\right)$ obey
  - Incompressibility: $\nabla_i v^i = 0$
  - Navier-Stokes equation:

$$\partial_t v_i + v^j \nabla_j v_i + \nabla_i P - \eta \left(\nabla^2 v_i + R_{ij} v^j\right) = 0$$

Previous work on fluid/gravity duality

Black hole membrane paradigm (1970s Damour)

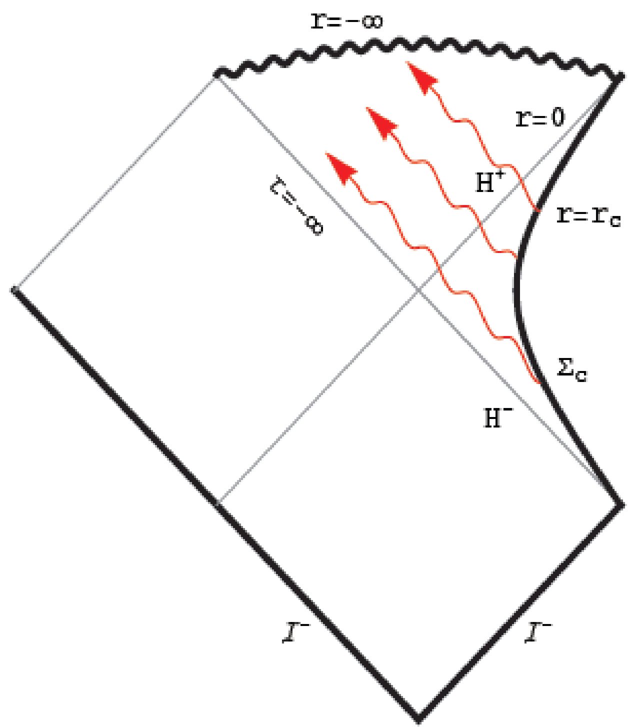
- Dynamics of the fluctuations around the horizon is very similar to that of a viscous fluid
- Governed by the Damour-Navier-Stokes equation

AdS/CFT (2000s Policastro, Son, Starinets etc)

- Deform AdS spacetimes and analyse the low energy limit of the dual field theory at the boundary $r \to \infty$
- Behaviour given by fluid mechanics

We want to relate solutions of the incompressible Navier-Stokes equation to solutions of the Einstein equations

- Consider gravitational fluctuations around a background Einstein solution
- The holographically dual fluid will live on the 'cutoff' surface at $r = r_c = 1$
- We constrain the induced metric on this cutoff surface to be flat
- No singularities in the radial direction



Fluctuations in the near-horizon region

Background: $ds^2 = -RdT^2 + 2dTdR + dx_i dx^i$

Horizon at $R = 0$

$$(T, R) \to (t, r) = (\lambda T, R/\lambda)$$

$\lambda \to 0$ zooms in on the near-horizon region

$$ds^2 = -\frac{r}{\lambda}dt^2 +$$
$$[2dtdr + r = r_c = 1\,(1 - r)v_i dx^i dt + (1 - r)(v^2 + 2P)dt^2] +$$
$$\lambda[(1 - r)v_i v_j dx^i dx^j - 2v_i dx^i dr + (v^2 + 2P)dtdr + (1 - r^2)\partial^2 v_i dx^i dt] + \ldots$$

Induced metric at the cutoff surface $r = r_c = 1$ is flat and there are no singularities in r so this metric obeys our boundary conditions.

Does this deformed geometry satisfy the Einstein equations?

Yes, it satisfies the Einstein equations...

Our deformed metric satisfies the Einstein equations through $\mathcal{O}(\lambda^0)$

if (v_i, P) satisfy precisely

the incompressibility condition: $\nabla_i v^i = 0$

and the Navier-Stokes equation:

$$\partial_t v_i + v^j \nabla_j v_i + \nabla_i P - \eta \left(\nabla^2 v_i + R_{ij} v^j \right) = 0$$

with viscosity $\eta = 1$.

The constraint is the incompressible Navier-Stokes equation on the planar cutoff surface $ds^2 = -dt^2 + dx_i dx^i$; as expected from holography.

- We have made no assumptions about the asymptotics of our spacetime; AdS piece is not necessary.

- By zooming in on the near-horizon region we find a precise mathematical relation between the Einstein and the Navier-Stokes equations.

- Can generalise to spherical Schwarzschild black holes. Zoom in on the region near the horizon at $r = 2m$. (v_i, P) is required to satisfy the incompressible Navier-Stokes equation on a sphere.

- By identifying the near-horizon parameter λ with a low energy parameter we can show that this geometry is the same as one where the cutoff surface is at a finite radius and which perturbatively solves the Einstein equations if the dual low energy theory at the cutoff is fluid mechanics. This is analogous to fluid/gravity duality as seen in AdS/CFT.