Complex Path Integrals & String Glasses

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Integro-Differential Problem

$$\mathscr{Z}[J] = \int_{\mathscr{C}} e^{iS[\varphi]}\, e^{iJ\varphi}\, \mathfrak{D}\varphi < \infty \;:\; \text{Integration Cycles}$$

$$(\delta_\varphi S[\varphi \mapsto -i\,\delta_J])\, \mathscr{Z}[J] = 0 \;:\; \text{Boundary Conditions}$$

- Complexification of the Path Integral.
- Riemann Sphere: Modular symmetry ∼ dualities.
- φ, J: $\mathbb{R}$-, $\mathbb{C}$-, $\mathbb{M}$atrix-, $\mathscr{L}$ie Algebra-valued.
- $\mathbb{M}$atrix models: integrability (Lax pairs), M-theory (BFSS).
- $\mathscr{L}$ie Algebras: Group Field Theory (LQG, CDT, Spin Foams).
- Quantum Phases: structure cnts $\xleftrightarrow{\text{cycles}}$ Matrix ⟷ Lie algebra.

Fredholm Theory

- Schwinger-Dyson operator: $SD = \delta_\varphi S[\varphi \mapsto -i\,\delta_J]$.
- Fredholm Kernel: $K[J,\tilde{J}] = \sum \Omega[J]\,\Omega[\tilde{J}] \sim \delta[J-\tilde{J}]$.
- Fredholm Integral: $G[J] = \int K[J,\tilde{J}]\,F[\tilde{J}]\,\mathfrak{D}\tilde{J}$; $SD[G] = F$.
- "Quantum Phase Coherent State quantization".

Example: Cubic Action

$D0$-brane with $S[\varphi] = \varphi^3/3$.

$$\mathscr{Z}[J] = \int_{\mathscr{C}} e^{i\varphi^3/3}\, e^{iJ\varphi}\, \mathfrak{D}\varphi < \infty \Rightarrow \mathscr{C}\text{: Arg}(\varphi)=\{0, \pm 2\pi/3\}$$

$$(\partial_J^2 - J)\, \mathscr{Z}[J] = 0 \Rightarrow \mathscr{Z}[J] = a\, \text{Ai}[J] + b\, \text{Bi}[J]$$

- Quantum Phases: Ai and Bi.
- $\mathscr{C}$: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -2\pi g \\ \frac{1}{2\pi g} & 0 \end{smallmatrix}\right)$; $F[\varphi] = e^{i\varphi^3/3}$

Example: Quartic Action

$D0$-brane with $S[\varphi] = \varphi^2/2 + g\,\varphi^4/4$, $g = \lambda/\mu^2$.

$$\mathscr{Z}[J] = \int_{\mathscr{C}} e^{i(\varphi^2/2 + g\varphi^4/4)}\, e^{iJ\varphi}\, \mathfrak{D}\varphi < \infty \Rightarrow \mathscr{C}\text{: Arg}(\varphi)=\{0, \pm\pi/2\}$$

$$(g\,\partial_J^3 + \partial_J - J)\, \mathscr{Z}[J] = 0 \Rightarrow \mathscr{Z}[J] = a\,U[g,J] + b\,V[g,J] + c\,W[g,J]$$

- Quantum Phases (ground states wrt g): U, V and W.
- $\mathscr{C}$: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} -g & -2\pi \\ \frac{1-g}{2\pi} & -1 \end{smallmatrix}\right)$; $F[\varphi] = e^{i\varphi^4/4}$

Wall-Crossing

- Wall-Crossing: Stokes Phenomena for SD Op.
- SD Op: meromorphic connection on $\mathbb{C}$-plane.
- SD Eq: isomonodromic transfs of non-linear diff eq in $\mathbb{C} \Rightarrow$ Stokes phenomena.
- Hodge Theory: phase factors of meromorphic connections.
- FPI ⟺ integral discriminant (Lagrangian ∼ Quadrics).
- SUSY: 'det' as 'exp' of Grassmann variables.

Quantum Phases

- Stratification of (super-)Quadrics ∼ Lagrangian: parameter dependent quadratic forms (2-tensor).
- Strata: multiplicities of the eigenvalues.
- Curvature of SD-connection (Jacobi fields, Euler character): jumps at Wall-Crossings, Stokes phenomena.
- Quantum Phases: filtration of the space of SD-operators by multiplicity of vacuum state.

Integral Transform

Linear Canonical Transformation (Modular Group $SL(2,\mathbb{R})$):

$$\mathscr{F}_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}[J] = \sqrt{-i}\; e^{i\pi J^2 d/b} \int_{-\infty}^{\infty} F[\varphi]\, e^{i\pi\phi^2 a/b}\, e^{-2\pi i\varphi J/b}\, \mathfrak{D}\varphi$$

$$\mathscr{F}_{\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)}[J] = \sqrt{d}\; e^{i\pi J^2 c d}\, F[J\,d]$$

- Special Cases:
  - Fourier Transform (FPI): $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$;
  - Fractional FT: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{smallmatrix}\right)$;
  - Fresnel Transform: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & \wedge \\ 0 & 1 \end{smallmatrix}\right)$.
- Range of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$ selects quantum phase.
- Quantum Phase (integration cycle): Stokes phase factor.
- Modular Transform: preserves Riemann Sphere.