Fredholm Theory

- Schwinger-Dyson operator: $SD = \delta_{\varphi} S[\varphi] \implies \varphi = -i \delta_{\varphi}$.
- Fredholm Kernel: $K[J, J'] = \int K[J', J] \sim \delta[J - J']$.
- "Quantum Phase Coherent State quantization".

Example: Cubic Action

$D0$-brane with $S[\varphi] = \varphi^3/3$.

$\mathscr{Z}[J] = \int e^{i/3} e^{i J \varphi} \mathcal{D}\varphi < \infty \implies \text{c: Arg(}\varphi) = \{0, \pm 2 \pi/3\}$

$(\partial^2 J) \mathscr{Z}[J] = 0 \Rightarrow \mathscr{Z}[J] = a A[J] + b B[J]$

Quantum Phases: $A$ and $B$.

$c: (a, b, c, d) = (\frac{1}{\pi i g}, -2 \pi g, 0, 0); F[\varphi] = e^{i \varphi^3/3}$

Example: Quartic Action

$D0$-brane with $S[\varphi] = \varphi^2/2 + g \varphi^4/4$, $g = \lambda/\mu^2$.

$\mathscr{Z}[J] = \int e^{i(\varphi^2/2 + g \varphi^4/4)} e^{i J \varphi} \mathcal{D}\varphi < \infty \implies \text{c: Arg(}\varphi) = \{0, \pm \pi/2\}$

$(g \partial^3 \varphi + \partial J) \mathscr{Z}[J] = 0 \Rightarrow \mathscr{Z}[J] = a U[g, J] + b W[g, J] + c W[g, J]$

Quantum Phases (ground states w.r.t $g$): $U$, $V$ and $W$.

$c: (a, b, c, d) = (\frac{-g}{2}, \frac{-2 \pi}{\beta g}, 1); F[\varphi] = e^{i \varphi^4/4}$

Integral Transform

Linear Canonical Transformation (Modular Group $SL(2, \mathbb{R})$):

$\mathscr{L}_{(a, b, c, d)}[f] = \sqrt{d} e^{i \pi d/2} \int \mathcal{D}\varphi e^{i \pi \varphi^2 a/b} e^{-2 \pi i \varphi J/b} f[\varphi]$.

Special Cases:
- Fourier Transform (FPI): $(a, b, c, d) = (0, 1, 0)$;
- Fractional FT: $(a, b, c, d) = (\cos \omega, \sin \omega, \cos \omega, \sin \omega)$;
- Fresnel Transform: $(a, b, c, d) = (\frac{2}{1}, \frac{1}{1})$.

Range of $(a, b, c, d) \in SL(2, \mathbb{R})$ selects quantum phase.

Quantum Phase (integration cycle); Stokes phase factor.

Modular Transform: preserves Riemann Sphere.

Integro-Differential Problem

$\mathscr{Z}[J] = \int e^{i S[\varphi]} e^{i J \varphi} \mathcal{D}\varphi < \infty :\text{ Integration Cycles} (\delta_{\varphi} S[\varphi] \implies -i \delta_{\varphi})$.

$\mathscr{Z}[J] = 0 :\text{ Boundary Conditions}$.

Complexification of the Path Integral.

Riemann Sphere: Modular symmetry $\sim$ dualities.

$\varphi, J$: $\mathbb{R}$-, $\mathbb{C}$-, Matrix-, Lie Algebra-valued.

Matrix models: integrability (Lax pairs), M-theory (BFSS).

Lie Algebras: Group Field Theory (LOG, CDT, Spin Foams).

Quantum Phases: structure cnrs $\implies$ Matrix $\implies$ Lie algebra.

Wall-Crossing

- SD Op: meromorphic connection on $\mathbb{C}$-plane.
- SD Eq: isomonodromic transfs of non-linear diff eq in $\mathbb{C} \Rightarrow$ Stokes phenomena.
- Hodge Theory: phase factors of meromorphic connections.
- FPI $\leftrightarrow$ integral discriminant (Lagrangian $\sim$ Quadrics).
- SUSY: ‘det’ as ‘exp’ of Grassmann variables.

Quantum Phases

- Stratification of (super-)Quadrics $\sim$ Lagrangian: parameter dependent quadratic forms (2-tensor).
- Strata: multiplicities of the eigenvalues.
- Curvature of SD-connection (Jacobi fields, Euler character): jumps at Wall-Crossings, Stokes phenomena.
- Quantum Phases: filtration of the space of SD-operators by multiplicity of vacuum state.

Complex Path Integrals & String Glasses

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