Antigravity Between Crunch and Bang in a Geodesically Complete Universe

Itzhak Bars
University of Southern California
Talk @ PDF-2011
Providence, August 2011

This talk is about solving cosmological equations analytically, no approximations. Found all the solutions for a specific model, and then discovered model independent phenomena that could not be noticed with approximate solutions. Among them is the notion of geodesic completeness, from which it follows there is a period of antigravity in the history of the universe. Also new general lessons for cosmology.

1) I.B. and S.H. Chen, 1004.0752
2) I.B., and S.H. Chen and Neil Turok, 1105.3606
3) I.B. + Chen + Turok + Steinhardt, to appear (several papers)
Cosmology with a scalar coupled to gravity

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R(g) - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V(\sigma) \right\} + \text{radiation + matter}
\]

\[
ds_E^2 = -dt^2 + \alpha_E^2(t) \left( \frac{dr^2}{1 - k\rho^2/r_0^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right);
\]

FRW

\[
K = \frac{k}{r_0^2}
\]

Friedmann equations

Also anisotropic metrics: Kasner, Bianchi IX.
Two more fields in metric important only near Big Bang

Analytically solved with this \( V \): found ALL solutions

\[
V(\sigma) = \left( \frac{\sqrt{6}}{\kappa} \right)^4 \left[ b \cosh^4 \left( \frac{\kappa \sigma}{\sqrt{6}} \right) + c \sinh^4 \left( \frac{\kappa \sigma}{\sqrt{6}} \right) \right]
\]

Generic solution is geodesically incomplete in Einstein gravity.

There is a subset of geodesically complete solutions only with conditions on initial values and parameters of the model.
For geodesic completeness: a slight extension of Einstein gravity (with gauge degrees of freedom)

Local scaling symmetry (Weyl): allows only conformally coupled scalars (generalization possible)

(Plus gauge bosons, fermions, more conformal scalars, in complete Weyl invariant theory.)

\[ \left( \phi, s \right) \rightarrow \left( \phi, s \right) e^{\lambda(x)}, \quad g_{\mu\nu} \rightarrow g_{\mu\nu} e^{-2\lambda(x)} \]

\[ S = \int d^4x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g^{\mu\nu} \partial_\mu s \partial_\nu s + \frac{1}{12} \left( \phi^2 - s^2 \right) R(g) - \phi^4 f \left( \frac{s}{\phi} \right) \right) \]

A prediction of **2T-gravity in 4+2 dims.**

**Also motivated by colliding branes scenario.**

Fundamental approach: Gauge symmetry in phase space

I.B. 0804.1585, I.B.+Chen 0811.2510

Khury + Seiberg + Steinhardt + Turok

McFadden + Turok 0409122

Weyl symmetry can be gauge fixed in several forms.

**Einstein gauge**

\[ \frac{1}{12} \left( \phi_E^2 - s_E^2 \right) = \frac{1}{2\kappa^2} \phi_E(x), \quad s_E(x), \quad g^{\mu\nu}_E(x) \]

\[ \phi_E(x) = \pm \sqrt{6} \frac{\kappa \sigma(x)}{\kappa} \cosh \left( \frac{\kappa \sigma(x)}{\sqrt{6}} \right), \quad s_E(x) = \sqrt{6} \frac{\kappa \sigma(x)}{\kappa} \sinh \left( \frac{\kappa \sigma(x)}{\sqrt{6}} \right) \]

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R(g_E) - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V(\sigma) \right\} \]

This is not the whole story: Einstein gauge is valid only in a patch of spacetime,
when the gauge invariant quantity \( [1 - s^2(x^\mu) / \phi^2(x^\mu)] \) is positive.

Can dynamics push this factor to negative values? **ANTIGRAVITY** in some regions of spacetime?
\( \gamma \)-gauge \( \phi_{\gamma}, s_{\gamma}, g^{\mu\nu}_{\gamma} \)

Conformal factor of metric =1 for any metric. \( \rightarrow \)

For all t,x dependence.

\[
d s^2_{\gamma} = -d\tau^2 + \frac{dr^2}{1 - kr^2/r_0^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

FRW\( _{\gamma} \)

\[
R(g_{\gamma}) = 6K, \text{ with } K \equiv \frac{k}{r_0^2}, \quad k = 0, \pm 1.
\]

This is equivalent to the 00 Einstein eq. \( G_{00} = T_{00} \)

Nothing singular in \( \gamma \)-gauge

Plus the energy constraint: \( H=0 \)

which compensates for the ghost.

connection between the \( \gamma \)-gauge and the Einstein gauge

\[
a^2_E = \frac{\kappa^2}{6} \left( \phi^2_{\gamma} - s^2_{\gamma} \right)
\]

\[
\sigma = \frac{\sqrt{6}}{\kappa} \frac{1}{2} \ln \left( \left| \frac{\phi_{\gamma} + s_{\gamma}}{\phi_{\gamma} - s_{\gamma}} \right| \right)
\]

Positive region

BB singularity at \( a_e = 0 \) in E-gauge : gauge invariant factor vanishes in \( \gamma \)-gauge, or any gauge!!
Analytic solutions – all of them!!

\[ L = \frac{1}{2} \left( -\dot{\phi}_\gamma^2 + \dot{s}_\gamma^2 \right) - \frac{K}{2} \left( -\phi_\gamma^2 + s_\gamma^2 \right) - \phi^4 f \left( \frac{s}{\phi} \right) \]

Special case: \( \phi^4 f \left( \frac{s}{\phi} \right) = b\phi^4 + cs^4 \)

BCST transform

Friedmann equations become:

\[
\begin{align*}
0 &= \ddot{\phi}_\gamma - 4b\phi_\gamma^3 + K\phi_\gamma, \\
0 &= \ddot{s}_\gamma + 4cs_\gamma^3 + Ks_\gamma, \\
0 &= -\left( \frac{1}{2}\phi_\gamma^2 - b\phi_\gamma^4 + \frac{1}{2}K\phi_\gamma^2 \right) + \left( \frac{1}{2}\dot{s}_\gamma^2 + cs_\gamma^4 + \frac{1}{2}Ks_\gamma^2 \right) + \rho_0
\end{align*}
\]

First integral

\[
\frac{1}{2}\phi_\gamma^2 - b\phi_\gamma^4 + \frac{K}{2}\phi_\gamma^2 = E_\phi, \quad \frac{1}{2}\dot{s}_\gamma^2 + cs_\gamma^4 + \frac{K}{2}s_\gamma^2 = E_s \quad E_s \equiv E, \quad E_\phi = E + \rho_0
\]

Particle in a potential problem, intuitively solved by looking at the plot of the potential.

\[
\begin{align*}
H (\phi) &= \frac{1}{2}\dot{\phi}^2 + V (\phi) & V (\phi) &= \frac{1}{2}K\phi^2 - b\phi^4 \\
H (s) &= \frac{1}{2}\dot{s}^2 + V (s) & V (s) &= \frac{1}{2}K\phi^2 + cs^4
\end{align*}
\]

Generic solution has 6 parameters \( (b, c, K, \rho_0, E, \phi (\tau_0)) \) with \( s_\gamma(\tau_0) = 0 \)
\[ V(s) = \frac{1}{2} K \phi^2 + cs^4 \quad V(\phi) = \frac{1}{2} K \phi^2 - b\phi^4 \]

**K=0 case quartic potentials**

\[ \phi(\tau), s(\tau) \]

\[ = A \times F[sn(z|m), cn(z|m), dn(z|m)] \]

\[ z = (\tau - \tau_0)/T \]

A, m, T depend on b, c, K, \( \rho_0, E \)

For generic initial conditions, the sign of \((\phi^2 - s^2)(\tau)\) changes over time.

Generic solution is geodesically incomplete in the Einstein gauge.

Geodesically complete with the natural extension in \( \phi, s \) space.

There are special solutions that are geodesically complete in the restricted Einstein frame, but must constrain parameter space.
Geodesically complete larger space: $\phi_\gamma, s_\gamma$ plane

Generic solution: $\phi(\tau), s(\tau)$ periodic

- parametric plot using Mathematica
- A smooth curve that spans the various quadrants (not shown)
- Closed curve if periods relatively quantized.

Recall Kruskal-Szekeres versus Schwarzchild; now in field space.

No signature change

$\alpha_E^2 = |z|$

$z \equiv \frac{\kappa^2}{6} \left( \phi_\gamma^2 - s_\gamma^2 \right)$, $\sigma = \frac{\sqrt{6}}{\kappa} \frac{1}{2} \ln \left( \left| \frac{\phi_\gamma + s_\gamma}{\phi_\gamma - s_\gamma} \right| \right)$

$\phi_\gamma = \pm \frac{\sqrt{6}}{\kappa} \sqrt{|z|} \cosh \left( \frac{\kappa \sigma}{\sqrt{6}} \right)$, $s_\gamma = \pm \frac{\sqrt{6}}{\kappa} \sqrt{|z|} \sinh \left( \frac{\kappa \sigma}{\sqrt{6}} \right)$

big bangs or big crunches in spacetime $\leftrightarrow$ at the lightcone in $\phi, s$ field space.

Generic solution is a cyclic universe with antigravity stuck between crunch and bang! Probably true for all $V(\sigma)$. 

BCST
Transform E-gauge to $\gamma$-gauge
Geodesically complete solutions in the Einstein gauge, without antigravity

Conditions on 6 parameter space:
1. Synchronized initial values
   \[ \phi(0) = s(0) = 0 \]
2. Relative quantization of periods
   \[ P_{\phi}(5 \text{ parameters}) = nP_s(5 \text{ parameters}) \]

Example \( n=5 \) → 5 nodes in picture

Universe expands to infinite size, turnaround at infinite size

b<0

Example n=5 → 5 nodes in picture

Universe expands to infinite size, turnaround at infinite size

b>0

Universe expands to infinite size, turnaround at infinite size

Cyclic universe
Anisotropy

\[ ds^2 = a^2(\tau) \left( -d\tau^2 + ds_3^2 \right) \]  
If \( K \neq 0 \), \( ds_3 = \text{Bianchi IX} \) (Misner)

\[ K \to 0 \quad (ds_3^2)_{\text{Kasner}} = e^{2\alpha_1(\tau)} + 2\sqrt{3}\alpha_2(\tau) \, dx^2 + e^{2\alpha_1(\tau)} - 2\sqrt{3}\alpha_2(\tau) \, dy^2 + e^{-4\alpha_1(\tau)} \, dz^2 \]

In Friedmann equations, 2 more fields \( \alpha_1(\tau), \alpha_2(\tau) \), just like the \( \sigma(\tau) \)

Friedman Eqs: kinetic terms for \( \alpha_1, \alpha_2 \) just like \( \sigma \), plus anisotropy potential if \( K \neq 0 \)

\[ V(\alpha_1, \alpha_2) = \frac{K}{\kappa^2 a^2} \left( e^{-8\alpha_1} + 4e^{\alpha_1} \sinh^2 \left( 2\sqrt{3}\alpha_2 \right) - 4e^{-2\alpha_1} \cosh \left( 2\sqrt{3}\alpha_2 \right) + 3 \right) \]

Free scalars if \( K = 0 \), then canonical conjugate momenta \( p_1, p_2 \) are constants of motion.

Near singularity, kinetic terms dominate, so all potentials, including \( V(\sigma) \) negligible.

Then \( \sigma \) momentum \( q \) is also conserved near the singularity.
For a range of \( q, p_1, p_2 \) mixmaster universe is avoided when \( \sigma \) is present (agree with BKL, etc.)

Without potentials can find all solutions analytically for any (initial) anisotropy momenta \( p_1, p_2 \); or \( \sigma \) momentum \( q \) including the parameters \( K, \rho_0 \).
Antigravity Loop (K=0 case)

\[ \phi + s = \sqrt{T} \left( \sqrt{p^2 + q^2} + \rho \tau \right) \left( \frac{\tau}{T} \right)^2 \left( \frac{\left( \frac{\tau}{T} \right)^2}{\rho \tau + \sqrt{p^2 + q^2}} \right)^{\frac{1}{4}} \left(1 + \frac{q}{\sqrt{p^2 + q^2}}\right) \]

\[ \phi - s = 2 \frac{\tau}{\sqrt{T}} \left( \frac{\left( \frac{\tau}{T} \right)^2}{\rho \tau + \sqrt{p^2 + q^2}} \right)^{-\frac{1}{4}} \left(1 + \frac{q}{\sqrt{p^2 + q^2}}\right) \]

K=0: \( p_1, p_2 \) are conserved throughout motion.

\( q \) changes during the loop because of \( V(\sigma) \).

If small loop, \( \approx \) no change.

Duration of loop: \( (p^2+q^2)^{1/2}/\rho \)

ATTRACTION MECHANISMS!!
if \( p_1 \) or \( p_2 \) is not 0:
both \( \phi, s \rightarrow 0 \) at the big bang
or crunch singularity,
for all initial conditions.

ALWAYS
a period of antigravity
sandwiched between
crunch and bang
What have we learned?

- Found new techniques to solve cosmological equations \textit{analytically}. Found all solutions for several special potentials \(V(\sigma)\). Several model independent general results: geodesic completeness, and an \textit{attractor mechanism to the origin}, \(\phi, s \to 0\), for any initial values.

- Antigravity is very hard to avoid. Anisotropy + radiation + KE \textit{requires} it.

- Studied Wheeler-deWitt equation (quantum) for the same system, can solve some cases exactly, others semi-classically. Same conclusions.

- Will this new insight survive the effects of a full quantum theory (very likely yes). Should be studied in string theory.

- These phenomena are direct predictions of 2T-physics in 4+2 dimensions.

- Open: What are the observational effects today of a past antigravity period? This is an important project. Study of small fluctuations and fitting to current observations of the CMB (under investigation).