Multidimensional Quantum Tunneling: A Perturbative Approach

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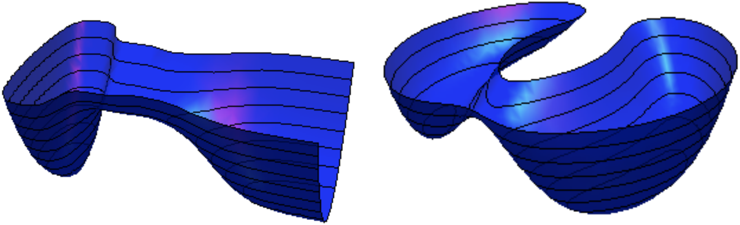
- Potential landscape with multiple local minima exhibits localized tunneling between them, as bubble formation. (See S. R. Coleman and F. De Luccia, Phys. Rev. D **21**, 3305 (1980) and citations therein.)
- If we are in such a bubble, slow-roll inflation happens after bubble nucleates.
- Quantum state of bubble interior right after tunneling serves as initial state for inflation.
- Calculations of inflaton perturbations usually assume Bunch-Davies vacuum, a good assumption only if inflation is very long.
- Deviations from BD can manifest in CMB spectrum, bispectrum, etc. at low multipoles.

We want to compute deviations from BD as a function of landscape parameters, and develop qualitative understanding of the connection.

- QFT and gravity add many complications, so we study multidimensional quantum mechanics as a first step.
- Pay careful attention to initial state; we find late-time behavior controlled by resonances.
- Separate directions cleanly; only treat tunneling direction with WKB.
- Isolate two distinct types of cross-couplings: curvature and changes in transverse potential.
- Discuss approximations and their applicability. (Small curvature, rate of variation of transverse potential)

- Consider 2-well system for simplicity.
- Quantum tunneling between wells is dominated by regions around the “Dominant Escape Path” (DEP).
- DEP is Euclidean “bounce” trajectory.
- Assume initial state is false ground state; expand wavefunction around DEP.
- Previous attempts assume WKB in all directions—*not* a good assumption for states with little kinetic energy in transverse directions.

(See T. Banks, C. M. Bender and T. T. Wu, Phys. Rev. D **8**, 3346 (1973); J. L. Gervais and B. Sakita, Phys. Rev. D **16**, 3507 (1977); T. Tanaka, M. Sasaki and K. Yamamoto, Phys. Rev. D **49**, 1039 (1994); . . .)



- Consider flat-space 2-d toy model with two-well potential landscape.
- Use coordinates (τ, η) that parametrize position along DEP and distance from it respectively.
- Metric is dependent on DEP curvature κ (inverse radius of curvature).
- Write Hamiltonian in covariant form: $H = -\frac{\hbar^2}{2m} g^{\alpha\beta} \nabla_\alpha \nabla_\beta + V(\tau, \eta)$. Write covariant derivatives explicitly and expand potential in η.

We want to describe the time evolution of a wavepacket that starts well-localized in the metastable well.

- Assume bottom of lower well is very far away for timescales of interest—motivated by need for inflaton to roll and reheat.
- Decompose into energy eigenstates and evolve in time; these are approximately scattering states.
- Initial wavefunction $|\phi(0)\rangle = \int dE \, \langle E| \, |\phi(0)\rangle \, |E\rangle$ becomes $|\phi(t)\rangle = \int dE \, \langle E| \, |\phi(0)\rangle \, e^{-iEt} \, |E\rangle$ under time evolution.
- Compact support in $E \to$ extend E integral to entire real line.
- For large t, closing contour picks up poles below real axis for $x < Et/p$.
- Pole closest to real axis dominates at late times; this looks like the false ground state.

Note: this argument follows one in Merzbacher's QM book.

So, we analytically continue E onto the pole. This looks like the time-evolved false ground state up to some value of τ.

- Advantage: we can work in time-independent formalism.
- Resulting state is pure-outgoing in τ direction, like a good tunneling state.
- Perturbation theory with complex-E states is difficult; not even δ-function normalizable, not in Hilbert space, no probability conservation, etc.
- Solution: do PT (or other approximation) on *scattering* states, analytically continue at the end.

Explicitly, Hamiltonian is $H = H_\tau + H_\eta + H_i$, with

$$H_\tau = -\frac{\hbar^2}{2m}\partial_\tau^2 + V^{(0)}(\tau)$$

$$H_\eta = -\frac{\hbar^2}{2m}\partial_\eta^2 + \sum_{a=2}^{\infty}\frac{1}{a!}V^{(a)}(\tau)\eta^a$$

$$H_i = -\frac{\hbar^2}{2m}\left(\frac{\kappa\eta(2-\kappa\eta)}{(1-\kappa\eta)^2}\partial_\tau^2 + \frac{(\partial_\tau\kappa)\eta}{(1-\kappa\eta)^3}\partial_\tau - \frac{\kappa}{1-\kappa\eta}\partial_\eta\right) + V^{(1)}(\tau)\eta$$

- $V^{(a)} = \left(\frac{\partial}{\partial\eta}\right)^a V(\tau,\eta)|_{\eta=0}$; $V^{(1)}(\tau) = 2\kappa(V^{(0)}(\tau) - V^{(0)}(0))$.
- Two types of cross-couplings: through τ-dependence of $V^{(a\geq 2)}$, and through κ.
- Note coordinates are bad for $\eta > \kappa^{-1}$: must have small curvature relative to $\sqrt{\langle\eta^2\rangle}$.

High, relatively smooth potential barrier common to most physically motivated models

- Low tunneling rate.
- Small κ due to large momentum of bounce; leads to perturbation expansion in κ.

WKB: In tunneling direction, high and smooth V is sufficient. But kinetic energy in η direction can be small, so WKB is not good that way; potential varies as fast as wavefunction when one moves transverse to DEP.

For simplicity, we also assume τ and η are uncoupled outside of the barrier.

Suppose $V^{(a>0)}(\epsilon\tau)$ for small ϵ.

- Define $H_\eta \chi_n(\eta, \epsilon\tau) = E_n(\epsilon\tau)\chi_n(\eta, \epsilon\tau)$ as "instantaneous" energy basis states.
- Wavefunction $\Psi(\tau, \eta)$ factorizes: $\Psi(\tau, \eta) = \sum_n \psi_n(\tau) |\chi_n\rangle$.
- In this basis H_i takes the form of a matrix of operators on ψ_n.
- Time-independent Schrödinger equation becomes a set of coupled differential equations for ψ_n.
- We can do perturbation theory in ϵ.
- Take $\psi_n \sim \delta_{n0}$ to zeroth order— $\Psi(\tau, \eta)$ looks like ground state in well.

Is this expansion kosher? Check:

$$\sum_n \left[(H_\tau \psi_n)\chi_n + E_n \psi_n \chi_n - E\psi_n\chi_n - \frac{\hbar^2}{m}(\partial_\tau \psi_n)(\partial_\tau \chi_n) - \frac{\hbar^2}{2m}\psi_n(\partial_\tau^2 \chi_n) \right] = 0$$

- $\frac{\hbar^2}{m}(\partial_\tau \psi_n)(\partial_\tau \chi_n) \sim \frac{\hbar p \epsilon}{m}$ and $\frac{\hbar^2}{2m}\psi_n(\partial_\tau^2 \chi_n) \sim \frac{\hbar^2 \epsilon^2}{m}$.
- p is momentum of ψ_n in τ direction, $\sim \sqrt{2m(V-E)}$.
- Compare to transverse-eigenstate level spacing; when both are small, perturbation theory is okay.
- First term tends to be larger than expected when barrier is high—ϵ expansion is not always good!
- Seen explicitly in sample calculations.

Curvature expansion

- Treating curvature alone, perturbation theory is good.
- To leading order DEP curvature produces excitations primarily through curvature near the classical turning points (where WKB prefactors aren't too small).
- If "adiabatic" approximation is also good, one can do a double expansion.

"Sudden" approximation

- Opposite of "adiabatic": assume $V^{(a)}$ change sharply with τ.

Extend to field theory, add gravity.

- $O(4)$-symmetric Euclidean solution is the DEP, and the field fluctuations around it are the equivalent to η.
- Write
$$\phi(\mathbf{x}) = \phi_{DEP}(\mathbf{x};\tau) + \hat{a}_0 \partial_{\mathbf{x};\tau}\phi_{DEP}(\mathbf{x};\tau) + \sum_k \hat{a}_k \varphi_k(\mathbf{x};\tau).$$
- Redefinition absorbs a_0 into τ, makes quantum variables $(\hat{\tau}, \{\hat{a}_k\})$.
- Choose φ_k so kinetic term is diagonal.
- Coupling to gravity adds nontrivial causal structure