Multidimensional Quantum Tunneling: A Perturbative Approach

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Work in progress with Albion Lawrence (Brandeis) and Cecilia Garraffo (CfA)
Potential landscape with multiple local minima exhibits localized tunneling between them, as bubble formation. (See S. R. Coleman and F. De Luccia, Phys. Rev. D 21, 3305 (1980) and citations therein.)

If we are in such a bubble, slow-roll inflation happens after bubble nucleates.

Quantum state of bubble interior right after tunneling serves as initial state for inflation.

Calculations of inflaton perturbations usually assume Bunch-Davies vacuum, a good assumption only if inflation is very long.

Deviations from BD can manifest in CMB spectrum, bispectrum, etc. at low multipoles.
We want to compute deviations from BD as a function of landscape parameters, and develop qualitative understanding of the connection.

- QFT and gravity add many complications, so we study multidimensional quantum mechanics as a first step.
- Pay careful attention to initial state; we find late-time behavior controlled by resonances.
- Separate directions cleanly; only treat tunneling direction with WKB.
- Isolate two distinct types of cross-couplings: curvature and changes in transverse potential.
- Discuss approximations and their applicability. (Small curvature, rate of variation of transverse potential)
Consider 2-well system for simplicity.

Quantum tunneling between wells is dominated by regions around the “Dominant Escape Path” (DEP).

DEP is Euclidean “bounce” trajectory.

Assume initial state is false ground state; expand wavefunction around DEP.

Previous attempts assume WKB in all directions—\textit{not} a good assumption for states with little kinetic energy in transverse directions.

Consider flat-space 2-d toy model with two-well potential landscape.

Use coordinates \((\tau, \eta)\) that parametrize position along DEP and distance from it respectively.

Metric is dependent on DEP curvature \(\kappa\) (inverse radius of curvature).

Write Hamiltonian in covariant form:
\[
H = -\frac{\hbar^2}{2m} g^{\alpha\beta} \nabla_\alpha \nabla_\beta + V(\tau, \eta).
\]
Write covariant derivatives explicitly and expand potential in \(\eta\).
We want to describe the time evolution of a wavepacket that starts well-localized in the metastable well.

▶ Assume bottom of lower well is very far away for timescales of interest—motivated by need for inflaton to roll and reheat.

▶ Decompose into energy eigenstates and evolve in time; these are approximately scattering states.

▶ Initial wavefunction $|\phi(0)\rangle = \int dE \langle E| |\phi(0)\rangle |E\rangle$ becomes $|\phi(t)\rangle = \int dE \langle E| |\phi(0)\rangle e^{-iEt} |E\rangle$ under time evolution.

▶ Compact support in $E \rightarrow$ extend $E$ integral to entire real line.

▶ For large $t$, closing contour picks up poles below real axis for $x < Et/p$.

▶ Pole closest to real axis dominates at late times; this looks like the false ground state.

Note: this argument follows one in Merzbacher's QM book.
So, we analytically continue $E$ onto the pole. This looks like the time-evolved false ground state up to some value of $\tau$.

- Advantage: we can work in time-independent formalism.
- Resulting state is pure-outgoing in $\tau$ direction, like a good tunneling state.
- Perturbation theory with complex-$E$ states is difficult; not even $\delta$-function normalizable, not in Hilbert space, no probability conservation, etc.
- Solution: do PT (or other approximation) on scattering states, analytically continue at the end.
Explicitly, Hamiltonian is $H = H_\tau + H_\eta + H_i$, with

$$H_\tau = -\frac{\hbar^2}{2m} \partial_\tau^2 + V^{(0)}(\tau)$$

$$H_\eta = -\frac{\hbar^2}{2m} \partial_\eta^2 + \sum_{a=2}^{\infty} \frac{1}{a!} V^{(a)}(\tau) \eta^a$$

$$H_i = -\frac{\hbar^2}{2m} \left( \frac{\kappa \eta (2 - \kappa \eta)}{(1 - \kappa \eta)^2} \partial_\tau^2 + \frac{(\partial_\tau \kappa) \eta}{(1 - \kappa \eta)^3} \partial_\tau - \frac{\kappa}{1 - \kappa \eta} \partial_\eta \right) + V^{(1)}(\tau) \eta$$

$V^{(a)} = \left( \frac{\partial}{\partial \eta} \right)^a V(\tau, \eta)|_{\eta=0}$; $V^{(1)}(\tau) = 2\kappa (V^{(0)}(\tau) - V^{(0)}(0))$.

Two types of cross-couplings: through $\tau$-dependence of $V^{(a \geq 2)}$, and through $\kappa$.

Note coordinates are bad for $\eta > \kappa^{-1}$: must have small curvature relative to $\sqrt{\langle \eta^2 \rangle}$. 
High, relatively smooth potential barrier common to most physically motivated models

- Low tunneling rate.
- Small $\kappa$ due to large momentum of bounce; leads to perturbation expansion in $\kappa$.

WKB: In tunneling direction, high and smooth $V$ is sufficient. But kinetic energy in $\eta$ direction can be small, so WKB is not good that way; potential varies as fast as wavefunction when one moves transverse to DEP.

For simplicity, we also assume $\tau$ and $\eta$ are uncoupled outside of the barrier.
Suppose $V^{(a>0)}(\epsilon \tau)$ for small $\epsilon$.

- Define $H_\eta \chi_n(\eta, \epsilon \tau) = E_n(\epsilon \tau) \chi_n(\eta, \epsilon \tau)$ as “instantaneous” energy basis states.

- Wavefunction $\Psi(\tau, \eta)$ factorizes: $\Psi(\tau, \eta) = \sum_n \psi_n(\tau) |\chi_n\rangle$.

- In this basis $H_i$ takes the form of a matrix of operators on $\psi_n$.

- Time-independent Schrödinger equation becomes a set of coupled differential equations for $\psi_n$.

- We can do perturbation theory in $\epsilon$.

- Take $\psi_n \sim \delta_{n0}$ to zeroth order— $\Psi(\tau, \eta)$ looks like ground state in well.
Is this expansion kosher? Check:

$$\sum_n \left[ (H_{\tau} \psi_n) \chi_n + E_n \psi_n \chi_n - E \psi_n \chi_n - \frac{\hbar^2}{m} (\partial_{\tau} \psi_n) (\partial_{\tau} \chi_n) - \frac{\hbar^2}{2m} \psi_n (\partial_{\tau}^2 \chi_n) \right] = 0$$

- $\frac{\hbar^2}{m} (\partial_{\tau} \psi_n) (\partial_{\tau} \chi_n) \sim \frac{\hbar p \epsilon}{m}$ and $\frac{\hbar^2}{2m} \psi_n (\partial_{\tau}^2 \chi_n) \sim \frac{\hbar^2 \epsilon^2}{m}$.

- $p$ is momentum of $\psi_n$ in $\tau$ direction, $\sim \sqrt{2m(V - E)}$.

- Compare to transverse-eigenstate level spacing; when both are small, perturbation theory is okay.

- First term tends to be larger than expected when barrier is high—$\epsilon$ expansion is not always good!

- Seen explicitly in sample calculations.
Curvature expansion

- Treating curvature alone, perturbation theory is good.
- To leading order DEP curvature produces excitations primarily through curvature near the classical turning points (where WKB prefactors aren’t too small).
- If “adiabatic” approximation is also good, one can do a double expansion.

“Sudden” approximation

- Opposite of “adiabatic”: assume $V^{(a)}$ change sharply with $\tau$. 
Extend to field theory, add gravity.

- $O(4)$-symmetric Euclidean solution is the DEP, and the field fluctuations around it are the equivalent to $\eta$.

- Write
  \[ \phi(x) = \phi_{DEP}(x; \tau) + \hat{a}_0 \partial_{x;\tau} \phi_{DEP}(x; \tau) + \sum_k \hat{a}_k \varphi_k(x; \tau). \]

- Redefinition absorbs $a_0$ into $\tau$, makes quantum variables ($\hat{\tau}, \{\hat{a}_k\}$).

- Choose $\varphi_k$ so kinetic term is diagonal.

- Coupling to gravity adds nontrivial causal structure.