A New Algorithm for Estimating Quark Propagation in Lattice QCD

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Outline

- Introduction
- Hadron Spectroscopy
  - Lattice QCD
  - Stochastic LapH Algorithm
    - in a nut-shell
- Some Results
- Future/Summary
Introduction

Goal:
To determine the low-lying hadron spectrum from first principals Lattice QCD calculations

Require:
- large, dynamical lattices with “light” quarks
  (in this study: $M_{pi} > 240$ MeV, $V < (5 \text{ fm})^3$)
- inclusion of the multi-particle spectrum
  (we can no longer ignore the contributions from disconnected diagrams)

Focus: multi-particle spectrum and disconnected diagrams in addition to the single-particle spectrum obtained in the past couple of years
Introduction

Examples of the single-particle spectrum obtained in the past couple of years: (no longer quenched)


and more ...

Why not multi-particle states?

simple answer: computationally expensive
Nucleon Spectrum

J. Bulava et al

PRD79 2009

\( \text{M}_\pi = 415 \text{ MeV} \ (2.5 \text{fm})^3 \)

thresholds
Isovector Mesons

$24^3 \times 128$ 170 cfigs  $\text{Mpi}=390\text{MeV}$
Lattice Hadron Spectroscopy

Euclidean Correlation Functions

\[ C(t, t_0) = \langle O_\pi(t) O_\pi^\dagger(t_0) \rangle \rightarrow A_0 e^{-M_\pi(t-t_0)} + A_1 e^{-E^{(1)}(t-t_0)} + \ldots \]

\[ C(t, t_0) = \langle \sum \bar{\psi}(\vec{x}_0, t_0) \gamma_5 \psi(\vec{x}_0, t_0) \rangle \]
Lattice Hadron Spectroscopy

Euclidean Correlation Functions

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\[ C(t, t_0) = \left\langle \sum_{\vec{x}_0} \bar{\psi}(\vec{x}_0, t_0) \gamma_5 \psi(\vec{x}_0, t_0) \right\rangle \]

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Lattice Hadron Spectroscopy

Euclidean Correlation Functions

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C(t, t_0) = \langle O_\pi(t) O_\pi^\dagger(t_0) \rangle \rightarrow A_0 e^{-M_\pi(t-t_0)} + A_1 e^{-E_1^{(1)}(t-t_0)} + \ldots
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\]

\[
C(t, t_0) \simeq \frac{1}{N_{\text{cfg}}} \sum \text{Tr} \left[ \gamma_5 M^{-1}(\vec{x}, t; \vec{x}_0, t_0) \gamma_5 M^{-1}(\vec{x}_0, t_0; \vec{x}, t) \right]
\]

Quark Propagator \( M^{-1}(\vec{x}, t; \vec{x}_0, t_0) \)

from matrix inversion \( M \psi(\vec{x}, t) = \rho(\vec{x}_0, t_0) \)
Lattice Hadron Spectroscopy

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Masses can be extracted from the large time behaviour of the correlation function \( C(t, t_0) \) by inverting from a single site \((x_0, t_0)\).
The difficulty with simulating multi-particle states and isosinglet states:

In a nutshell ... “t-to-t” diagrams

Expensive!
Each quark propagator requires the inversion of a $N^2$ matrix where $N = (3 \times 4 \times V_s \times L_t)$

One must do this for all spatial sites on the lattice (all-to-all quark propagators)

Example: $V = 24^3$  $T = 128$

Total Number of inversions $= N = 21,233,664$

Each inversion of the $21,233,664 \times 21,233,664$ matrix becomes expensive as the quarks get lighter

Typical calculation: $3 \times 4 \times 10 = 120$ inversions

Multi-hadron simulations and disconnected diagrams are difficult to evaluate
Stochastic LapH Algorithm
(C. Morningstar et al. PRD 2011)
Exact evaluation of the all-to-all propagators is unnecessary because we are making a Monte-Carlo estimates of the low energy correlation functions which have “gauge noise”
Stochastic LapH Algorithm

( C. Morningstar et al. PRD 2011 )

Exact evaluation of the all-to-all propagators is unnecessary because we are making a Monte-Carlo estimates of the low energy correlation functions which have “gauge noise”

1. We are interested in the low-momentum modes of the fields (M. Peardon et al. PRD 2009)

- “smeared fields” $e^{-\vec{p}^2/\Lambda^2} \psi \rightarrow \left(1 + \frac{\sigma}{n} \vec{\nabla}^2\right)^n \psi$
- Laplacian operator (eigenvector rep)
  
  $-\vec{\nabla}^2 = \sum_{i}^{N_{ev}} \frac{1}{\lambda_i} \psi(i) \psi(i)\dagger$
  
  - momentum cutoff $\rightarrow$ Heaviside step function
Distillation (LapH)

- Single-timeslice to all quark propagators
- Can give mesons/baryons momentum

If one only has connected diagrams to evaluate, then this can give phaseshifts, threshold params ...

(J. Dudek et al., S. Prelovsek et al., D. Richards et al.)

$3^{(2.5 \text{ fm})}$

$\text{Mpi}=390 \text{ MeV}$

90 configs

J. Bulava et al. (2010)
2. V scaling problem ...
   \# low modes needed increases with V,
   but we need to make the volume bigger ...

3. Can’t do disconnected diagrams unless one computes the perambulators from all timeslices
   a) disconnected diagrams typically have larger gauge noise
   b) anisotropic dynamical lattices
      fine temporal lattice spacing ... Nt=256
      \[ \frac{1}{\sqrt{N}} \]
      exact evaluation can be wasteful

4. Stochastic method tends to be noisy
   - need many noise sources to average over
Stochastic LapH Algorithm

1. Solve for $N_{ev}$ lowest lying eigenvectors of the Laplacian on each timeslice $v^{(\alpha)}(\vec{x}, t)$

   $$ -\nabla^2 v^{(\alpha)}(\vec{x}, t) = \lambda_\alpha v^{(\alpha)}(\vec{x}, t) \quad \alpha = 1 \ldots N_{ev} $$

2. Solve for the solution of the Dirac equation using each eigenvector as the source $\varphi^{(\alpha)}(\vec{x}, t)$

   $$(\slashed{D} + m) \varphi^{(\alpha)}(\vec{x}, t) = v^{(\alpha)}(\vec{x}, t)$$

3. Introduce a Z4 noise field in the LapH subspace $\varrho^{[\alpha k \tau]}(\vec{x}, t)$

   $\alpha$ eigenvector index
   $k$ spin index
   $\tau$ timeslice index
4. **Dilute** the noise according to some scheme
- eigenvector indices (full: $\alpha = 1 \ldots N_{ev}$)
- spin indices (full: $k = 1 \ldots 4$)
- time indices (full: $\tau = 0 \ldots (N_t - 1)$)

5. Solve the Dirac equation with $\varrho^{[\alpha k \tau]}(\vec{x}_0, t_0)$ as the source

\[(\not{D} + m) \varphi[i] (\vec{x}, t) = \varrho[i] (\vec{x}_0, t_0)\]

6. **Quark propagator:**

\[\ldots \sum_i \varphi[i] (\vec{x}, t) \otimes \varrho[i] (\vec{x}_0, t_0)\]

But now, $t_0$ can be anytime ... “all-to-all”
Now we can evaluate all of these “t-to-t” diagrams ... and with a judicious choice of dilution schemes, we can make the errors due to the stochastic estimation to be $< \text{gauge noise}$.
Now we can evaluate all of these “t-to-t” diagrams
... and with a judicious choice of dilution schemes, we can make the errors due to the stochastic estimation to be $\ll$ gauge noise.
Now we can evaluate all of these “t-to-t” diagrams...

... and with a judicious choice of dilution schemes, we can make the errors due to the stochastic estimation to be \( < \) gauge noise.
Now we can evaluate all of these “t-to-t” diagrams ... and with a judicious choice of dilution schemes, we can make the errors due to the stochastic estimation to be $< \text{gauge noise}$.
Examples:

1. isospin-2 $\pi$-$\pi$ scattering length
2. $\rho$ to $\pi$-$\pi$ decay (mixing diagram)
3. isospin-0 $\pi$-$\pi$ correlation function
4. isoscalar mixing
   - glueball to $\pi$-$\pi$ decay (mixing diagram)
Simulation Parameters

Nf=2+1 anisotropic lattices

R. Edwards, B. Joo and H.-W. Lin
Phys. Rev. D78 014505

as/at = 3.5

\[ \begin{align*}
3^3 \times 128 & \quad \text{Mpi} \approx 240 \text{ MeV} \quad \text{mL} \approx 3.4 \\
& \quad \text{112 eigenvectors} \\
& \quad \text{550 configs} \\
3^3 \times 128 & \quad \text{Mpi} \approx 390 \text{ MeV} \quad \text{mL} \approx 5.6 \\
& \quad \text{580 configs} \\
20^3 \times 128 & \quad \text{Mpi} \approx 390 \text{ MeV} \quad \text{mL} \approx 5 \\
2^3 \times 128 & \quad \text{Mpi} \approx 390 \text{ MeV} \quad \text{mL} \approx 3.9
\end{align*} \]
I=2 pi-pi Scattering Length


The scattering length can be obtained from a Euclidean lattice calculation of the energy shift of a two particle state in a finite box.

\[
E_1 = m
\]

\[
E_2 = 2m + \delta E
\]

\[
E_2 \neq 2m
\]
I=2 pi-pi Scattering Length


\[
\begin{align*}
(a_t \delta E) &= -\frac{1}{\xi^2} \frac{4\pi \tilde{a}_0}{(a_t m)} \left( \frac{L}{a_s} \right)^3 
\left[ 1 + c_1 \frac{\tilde{a}_0}{\left( \frac{L}{a_s} \right)} + c_2 \frac{\tilde{a}_0^2}{\left( \frac{L}{a_s} \right)^2} \right] 
\end{align*}
\]

\[
c_1 = -2.837297
\]
\[
c_2 = 6.375183
\]
\[
\tilde{a}_0 = \frac{a_0}{a_s} \quad \frac{L}{a_s} = 24 \quad \xi = \frac{a_s}{a_t} \approx 3.5
\]
\[
a_t m = \text{pion mass}
\]
\[
a_t \delta E = (a_t E_{\pi\pi} - 2m_\pi) = \text{energy shift}
\]
I=2 pi-pi Scattering Length

- Need the “better” \( \gamma_5 \) operators so that we can reduce the number of fitting parameters (running)

\[ (-M_{\text{eff}}) \]

and shifted vertically

\( \gamma_5 \) operator simulation is now running ...
I=1 rho to pi-pi

$24^3 \times 128$  $M_{\pi}=240$ MeV

$E_0 = 0.855(13)$ GeV  $E_1 = 1.168(23)$ GeV

C.H. Wong, Lattice 2011
Despite the inclusion of the box diagram, the errors are comparable to the $I=2$ case.

Inserting finite momenta operators proceeds in the same way (the same building blocks as the $I=2$ case with stochastic LapH propagators).
Glueballs

D. Lenkner : Lattice 2011

$24^3 \times 128$  \hspace{1cm} Mpi=240 MeV

New glueball operator : trace of LapH operator

0 $^{++}$

trace of plaquette
Isoscalar Mixings

D. Lenkner: Lattice 2011

$16^3 \times 128$  \hspace{1cm} $M_{\pi} = 400$ MeV  \hspace{1cm} 100 configs

Diagonalized levels in the isoscalar channel

Mixing of scalar mesons, multiparticles and glueballs!

(preliminary)
Nucleons

Example of how the errors are reduced for a nucleon correlation function (at time $t=5$)
Volume Scaling

The number of inversions required to achieve similar errors on $20^3$ and $16^3$ lattices
more results to come ...

single hadron operator pruning has been done on small volumes ...

(PRD79)
Sigma

$16^3 \times 128 \; 100 \; \text{cfgs}$
Delta

$16^3 \times 128 \; 100 \; \text{cfgs}$

$\text{Mass (GeV)}$

$I = \frac{3}{2}, \; S = 0$

($\Delta$ baryons)
Moving Mesons

Isovector $C_{4v}$ Meson Spectrum, $V=20^3$, $P=(0,0,1)$

Preliminary
Conclusions

- **Stochastic LapH** algorithm works like a charm!
- No volume scaling problem (important for light q)
- Two-particle states
  - Finite momenta operators are easy to implement
  - Phase shifts are accessible (if V is big enough)
- Mixing between single-particle and two-particle
- **t-to-t diagrams** do not appear to be a problem!
- A new glueball operator

Excitation spectra with multi-particle states within reach ...