QED radiative corrections in Monte Carlo simulations of pion and kaon decays

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Overview

Hadronic parameters from experiment

- QED corrections left out entirely
- some of the relevant terms neglected
- approximative results (leading logs, soft-photon approximation)

Artificial discrepancies between theory and experiment

↪ measured observables or related hadronic parameters may include unsubtracted QED part

QED radiative corrections in the low-energy QCD sector

↪ π^0, $\eta^{(\prime)}$, Σ^0, K^+, ...

↪ direct application to experiment

Outline

- $\pi^0 \to e^+e^-$
- $\pi^0 \to e^+e^-\gamma$
- $K^+ \to \pi^+\ell^+\ell^-$
- $K^+ \to \pi^+e^+e^-e^+e^-$
- $K^+ \to \pi^0e^+\nu\gamma$

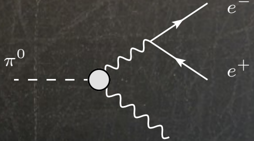
Radiative corrections for π^0 decays

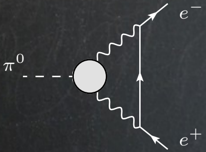
Rare decay of π^0

Introduction

Decay modes of the neutral pion:

Process	Branching ratio
$\pi^0 \to \gamma\gamma$	$(98.823 \pm 0.034)\,\%$
$\pi^0 \to e^+e^-\gamma$	$(1.174 \pm 0.035)\,\%$
$\pi^0 \to e^+e^+e^-e^-$	$(3.34 \pm 0.16) \times 10^{-5}$
$\pi^0 \to e^+e^-$	$(6.46 \pm 0.33) \times 10^{-8}$





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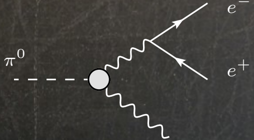
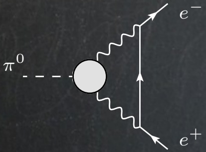




Rare decay $\pi^0 \to e^+e^-$

- interesting way to study low-energy (long-distance) dynamics in the SM
- systematic theoretical treatment dates back to *Drell*, NC (1959)
- suppressed compared to the decay $\pi^0 \to \gamma\gamma$ by a factor of $2\,(\alpha m_e/M_\pi)^2$
  - ↪ one-loop structure + helicity suppressed
  - ↪ may be sensitive to possible effects of new physics

Rare decay of π^0

KTeV measurement

KTeV-E799-II experiment at Fermilab (*Abouzaid et al.*, PRD 75 (2007))

$\hookrightarrow$ precise measurements of branching ratio $\pi^0 \to e^+e^-$ (794 candidates)

$$\frac{\Gamma(\pi^0 \to e^+e^-(\gamma),\, x > 0.95)}{\Gamma(\pi^0 \to e^+e^-\gamma,\, x > 0.232)} = (1.685 \pm 0.064 \pm 0.027) \times 10^{-4}$$

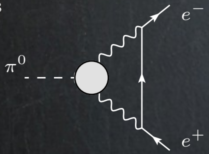
$\hookrightarrow$ extrapolate the Dalitz decay branching ratio to full range of $x \equiv m^2_{e^+e^-}/M^2_{\pi^0}$

$$B_{\text{KTeV}}(\pi^0 \to e^+e^-(\gamma),\, x > 0.95) = (6.44 \pm 0.25 \pm 0.22) \times 10^{-8}$$

$\hookrightarrow$ PDG average value $(6.46 \pm 0.33) \times 10^{-8}$ mainly based on this result

$\hookrightarrow$ extrapolate full radiative tail beyond $x > 0.95$ (*Bergström*, Z.Ph.C 20 (1983))
& scale the result back by the overall radiative corrections

$$B^{\text{no-rad}}_{\text{KTeV}}(\pi^0 \to e^+e^-) = (7.48 \pm 0.29 \pm 0.25) \times 10^{-8}$$



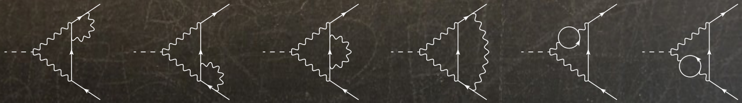
SM prediction (*Dorokhov and Ivanov*, PRD 75 (2007))

$$B^{\text{no-rad}}_{\text{SM}}(\pi^0 \to e^+e^-) = (6.23 \pm 0.09) \times 10^{-8}$$

$\hookrightarrow$ interpreted as 3.3 σ discrepancy between theory and experiment

Radiative corrections for $\pi^0 \to e^+e^-$

Final results



Vaško and Novotný, JHEP 1110 (2011)

Size of the NLO radiative corrections (exact calculation)

$$\delta^{\text{NLO}}(0.95) \equiv \delta^{\text{virt.}} + \delta^{\text{BS}}(0.95) = (-5.5 \pm 0.2)\,\%$$

↪ differs significantly from earlier approximate calculations

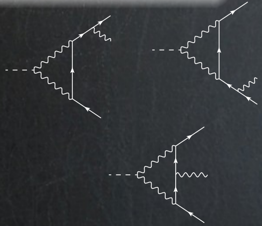
Bergström, Z.Ph.C 20 (1983): $\delta(0.95) = -13.8\,\%$

Dorokhov et al., EPJC 55 (2008): $\delta(0.95) = -13.3\,\%$

↪ original KTeV vs. SM discrepancy reduced to 2σ level

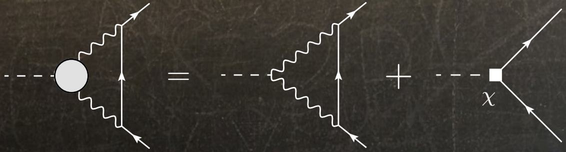
↪ can be thought as model-independent

↪ $\chi^{(\text{r})}_{\text{LMD}}(M_\rho) = 2.2 \pm 0.9$



TH, Kampf and Novotný,
EPJC 74 (2014)

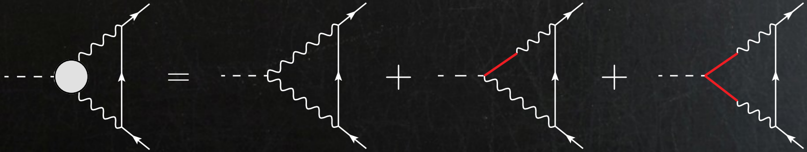
Theoretical descriptions for $\pi^0 \to e^+e^-$



Chiral Perturbation Theory (χPT)

$\Updownarrow$

Resonance Chiral Theory (RχT), large N_C, dispersive techniques, ...



$\pi^0 \to e^+e^-$ branching ratio

Standard Model prediction

At LO in QED and ChPT expansion:

$$\left[\text{using}\quad z = -\frac{1-\beta}{1+\beta}\,,\quad \beta = \sqrt{1-\frac{4m_e^2}{M_\pi^2}}\right]$$

$$\frac{B(\pi^0 \to e^+e^-)}{B(\pi^0 \to \gamma\gamma)}$$

$$= 2\beta\left(\frac{\alpha}{\pi}\frac{m_e}{M_\pi}\right)^2\left\{\left[-\frac{5}{2}+\chi^{(\mathrm{r})}(\mu)+\frac{3}{2}\log\frac{m_e^2}{\mu^2}+\frac{1}{2\beta}\left(\mathrm{Li}_2\, z-\mathrm{Li}_2\,\frac{1}{z}\right)\right]^2+\left[\frac{\pi}{2\beta}\log(-z)\right]^2\right\}$$

Theoretical predictions and models suggest $\chi^{(\mathrm{r})}(770\,\mathrm{MeV}) \sim$ 2–3

Knecht et al., PRL 83 (1999)
Dorokhov and Ivanov, PRD 75 (2007)
TH and Leupold, EPJC 75 (2015)
Hoferichter et al., PRL 128 (2022)

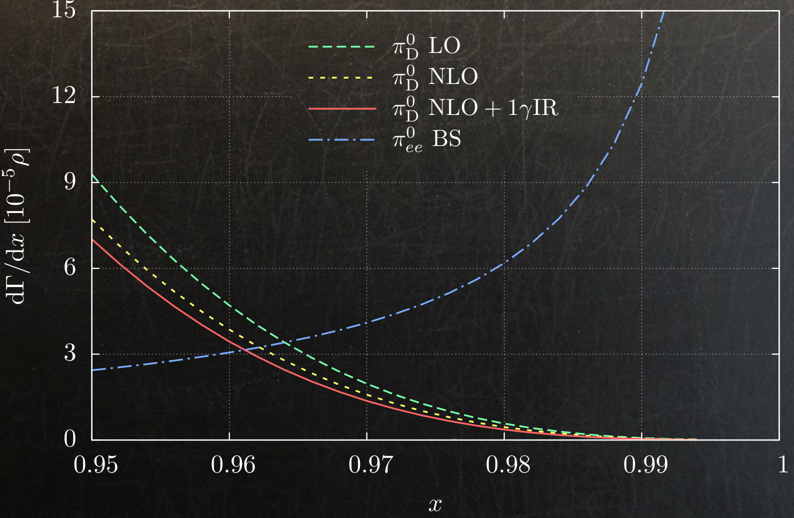
$$B(\pi^0 \to e^+e^-) \approx (6.21 + 0.15\widetilde{\chi}) \times 10^{-8}\,, \qquad \widetilde{\chi} \equiv 2\Big[\chi^{(\mathrm{r})}(770\,\mathrm{MeV}) - \tfrac{5}{2}\Big] \in (-1,1)$$

The experiment-friendly choice is then

$$B(\pi^0 \to e^+e^-(\gamma),\, x > 0.95) = \delta(0.95)B(\pi^0 \to e^+e^-) \approx (5.87 + 0.14\widetilde{\chi}) \times 10^{-8}$$

$\pi^0 \to e^+e^-$ branching ratio

Choice of the cut on x



Dalitz decay of the neutral pion

Introduction

Quantity measured by KTeV

$$\left.\frac{\Gamma(\pi^0 \to e^+e^-(\gamma)\,,\ x > 0.95)}{\Gamma(\pi^0 \to e^+e^-\gamma(\gamma)\,,\ x > 0.2319)}\right|_{\mathrm{KTeV}} = (1.685 \pm 0.064 \pm 0.027) \times 10^{-4}$$

↪ precise theoretical description of Dalitz decay essential

Dalitz decay

- first studied by *Richard H. Dalitz*, PPSA 64 (1951)
- second most important decay channel of the neutral pion
  ↪ branching ratio $(1.174 \pm 0.035)\,\%$
- experimental data provide information on singly-virtual π^0 TFF $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, q^2)$
  ↪ in particular about its slope parameter a_π

$$\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, M_\pi^2 x)}{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, 0)} \simeq 1 + a_\pi x\,, \qquad x = \frac{(p_{e^+} + p_{e^-})^2}{M_\pi^2}$$

Radiative corrections for $\pi^0 \to e^+e^-\gamma$

Introduction

- radiative corrections to the total decay rate of the Dalitz decay
  ↪ first addressed (numerically) by *Joseph*, NC 16 (1960)

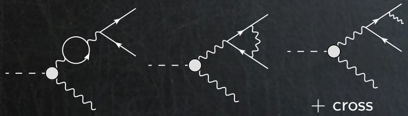
$$\Delta R|_{\text{Jph.}} = 1.05 \times 10^{-4} \quad\longleftrightarrow\quad \delta = 0.88\,\%$$

- pioneering study of corrections to the differential decay rate
  ↪ *Lautrup and Smith*, PRD 3 (1971)
  ↪ soft-photon approximation
  ↪ negative all over the Dalitz plot
  ↪ simple analytical expression for ΔR, $m_e \to 0$

$$\Delta R|_{\text{L\&S}} = \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{8}{9}\log^2\frac{M_\pi}{m_e} - \frac{1}{9}(19-4a_\pi)\log\frac{M_\pi}{m_e} + 2\zeta(3) - \frac{2}{27}\pi^2 + \frac{137}{81} - \frac{63}{108}a_\pi + \mathcal{O}\left(\frac{m_e}{M_\pi}\right)\right]$$

$$= (1.038 + 0.102a_\pi)\times 10^{-4} \quad\longleftrightarrow\quad \delta = (0.874 + 0.086a_\pi)\,\%$$

- extended by *Mikaelian and Smith*, PRD 5 (1972)
  ↪ hard-photon corrections
  ↪ whole range of bremsstrahlung photon energy
  ↪ table of values



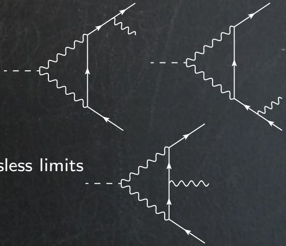
Radiative corrections for $\pi^0 \to e^+e^-\gamma$

Discussion around 1γIR contribution

1γIR (two-photon exchange) contribution at one loop

- First neglected completely by *Lautrup and Smith* or *Mikaelian and Smith*
  ↪ suspected suppression due to (m_e/M_π) dependence of $\pi^0 \to e^+e^-$ amplitude

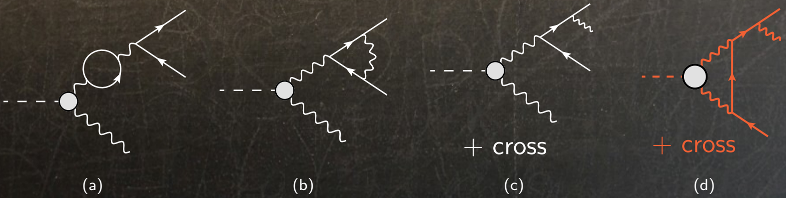
  ↪ supportive arguments appeared based on Low's theorem
    ↪ *Lambin and Pestieau*, PRD 31 (1985)



- Explicit calculation within $m_e \to 0$ and $f(x) \to 1$ approximation
  ↪ *Tupper, Grose, Samuel*, PRD 28 (1983)
  ↪ *Tupper*, PRD 35 (1987)
    ↪ pointed out non-interchangeability of soft-photon and massless limits

Radiative corrections for $\pi^0 \to e^+e^-\gamma$

Exact result for MC



New calculations motivated by needs of NA62 experiment at CERN

$\hookrightarrow$ *TH, Kampf and Novotný*, PRD 92 (2015)

$\hookrightarrow$ unlike before no approximation was used + 1γIR correction

  $\hookrightarrow$ can be partially used also for related decays $\eta \to \ell^+\ell^-\gamma$ etc.

$\hookrightarrow$ C++ code returns correction for any given point of Dalitz plot (x and y)

  $\hookrightarrow$ MC generator of NA62 experiment

Latest measurement of the $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,q^2)$ slope

$\hookrightarrow$ *Lazzeroni et al.*, PLB 768 (2017)

$$a_\pi^{\text{NA62}} = 3.68(57)\,\%$$

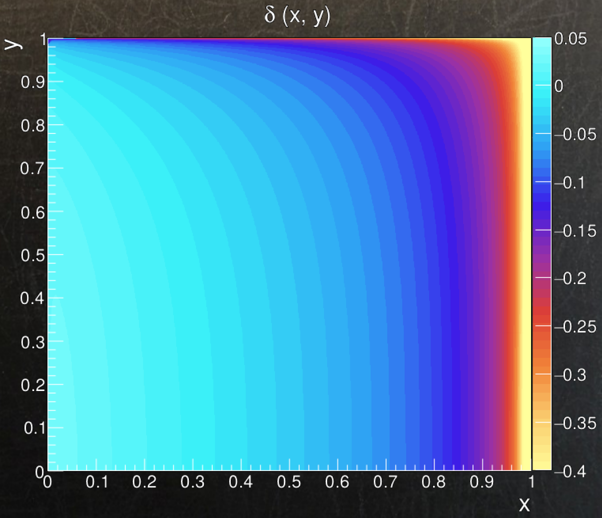
Radiative corrections for $\pi^0 \to e^+e^-\gamma$

The overall NLO correction $\delta(x,y)$ given in percent (Dalitz-plot corrections)

y x	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.99
0.01	2.761	2.714	2.599	2.449	2.273	2.061	1.786	1.402	0.803	-0.357	-5.657
0.02	2.756	2.720	2.622	2.480	2.300	2.073	1.774	1.355	0.703	-0.546	-5.859
0.03	2.669	2.639	2.552	2.419	2.242	2.012	1.704	1.267	0.586	-0.716	-6.125
0.04	2.558	2.531	2.452	2.327	2.155	1.925	1.611	1.164	0.464	-0.874	-6.372
0.05	2.437	2.412	2.340	2.221	2.053	1.824	1.509	1.054	0.341	-1.025	-6.601
0.06	2.311	2.288	2.221	2.108	1.944	1.717	1.400	0.940	0.216	-1.172	-6.815
0.07	2.184	2.163	2.099	1.990	1.830	1.605	1.288	0.824	0.092	-1.315	-7.017
0.08	2.056	2.036	1.975	1.870	1.714	1.491	1.173	0.707	-0.033	-1.455	-7.211
0.09	1.928	1.909	1.851	1.749	1.596	1.374	1.057	0.588	-0.157	-1.593	-7.397
0.10	1.801	1.783	1.726	1.628	1.477	1.257	0.940	0.469	-0.281	-1.729	-7.578
0.15	1.170	1.154	1.105	1.016	0.874	0.661	0.345	-0.131	-0.900	-2.394	-8.424
0.20	0.546	0.532	0.486	0.402	0.266	0.057	-0.258	-0.738	-1.520	-3.048	-9.219
0.25	-0.079	-0.092	-0.135	-0.217	-0.350	-0.556	-0.871	-1.355	-2.148	-3.704	-9.995
0.30	-0.713	-0.726	-0.768	-0.847	-0.978	-1.184	-1.499	-1.988	-2.790	-4.372	-10.77
0.35	-1.366	-1.378	-1.419	-1.497	-1.627	-1.833	-2.149	-2.641	-3.454	-5.058	-11.56
0.40	-2.044	-2.056	-2.097	-2.174	-2.304	-2.509	-2.827	-3.324	-4.146	-5.773	-12.37
0.45	-2.759	-2.771	-2.811	-2.887	-3.017	-3.222	-3.543	-4.044	-4.875	-6.525	-13.22
0.50	-3.521	-3.533	-3.572	-3.648	-3.777	-3.983	-4.306	-4.811	-5.653	-7.324	-14.12
0.55	-4.344	-4.356	-4.395	-4.470	-4.599	-4.806	-5.130	-5.640	-6.492	-8.186	-15.08
0.60	-5.249	-5.261	-5.299	-5.373	-5.501	-5.708	-6.034	-6.549	-7.410	-9.128	-16.12
0.65	-6.262	-6.273	-6.310	-6.383	-6.510	-6.717	-7.044	-7.563	-8.435	-10.18	-17.28
0.70	-7.425	-7.435	-7.470	-7.541	-7.666	-7.871	-8.198	-8.721	-9.603	-11.37	-18.60
0.75	-8.802	-8.811	-8.844	-8.910	-9.031	-9.232	-9.558	-10.08	-10.98	-12.77	-20.14
0.80	-10.51	-10.52	-10.54	-10.60	-10.72	-10.91	-11.23	-11.76	-12.66	-14.49	-22.02
0.85	-12.78	-12.78	-12.80	-12.85	-12.95	-13.13	-13.44	-13.96	-14.86	-16.72	-24.47
0.90	-16.21	-16.21	-16.21	-16.23	-16.29	-16.43	-16.71	-17.21	-18.11	-20.00	-28.00
0.95	-23.17	-23.14	-23.08	-23.01	-22.96	-22.98	-23.14	-23.53	-24.36	-26.26	-34.45
0.99	-54.29	-54.07	-53.44	-52.50	-51.35	-50.15	-49.03	-48.16	-47.76	-48.47	-55.83

Radiative corrections for $\pi^0 \to e^+e^-\gamma$

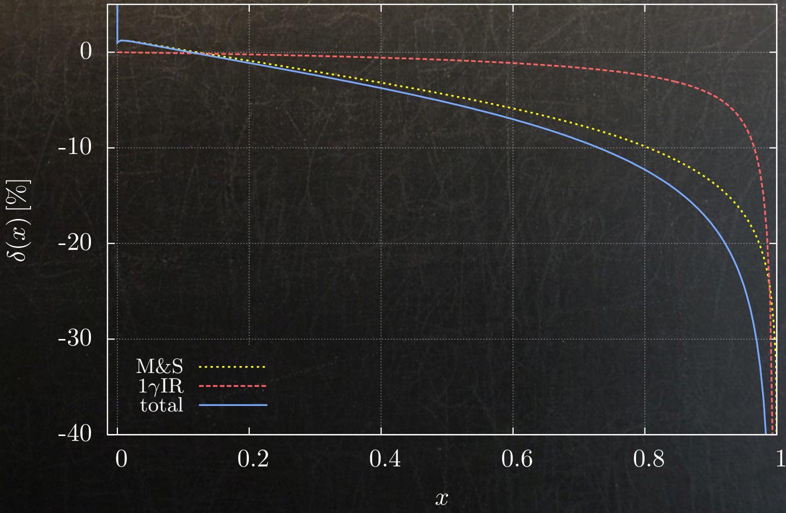
The overall NLO correction $\delta(x, y)$ given in percent (Dalitz-plot corrections)



(web of *M. Koval' (NA62)*)

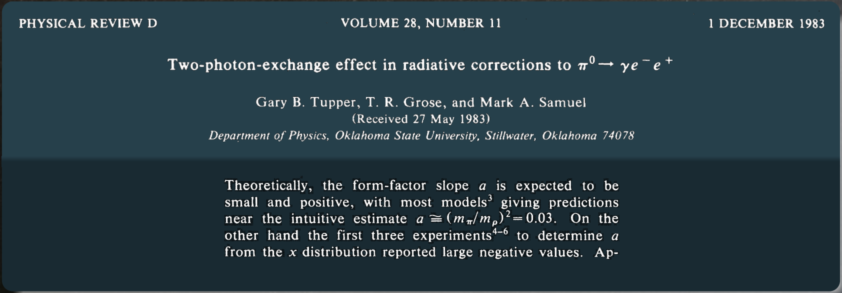
Radiative corrections for $\pi^0 \to e^+e^-\gamma$

Results



Radiative corrections for $\pi^0 \to e^+e^-\gamma$

Size of the correction



PHYSICAL REVIEW D VOLUME 28, NUMBER 11 1 DECEMBER 1983

Two-photon-exchange effect in radiative corrections to $\pi^0 \to \gamma e^- e^+$

Gary B. Tupper, T. R. Grose, and Mark A. Samuel

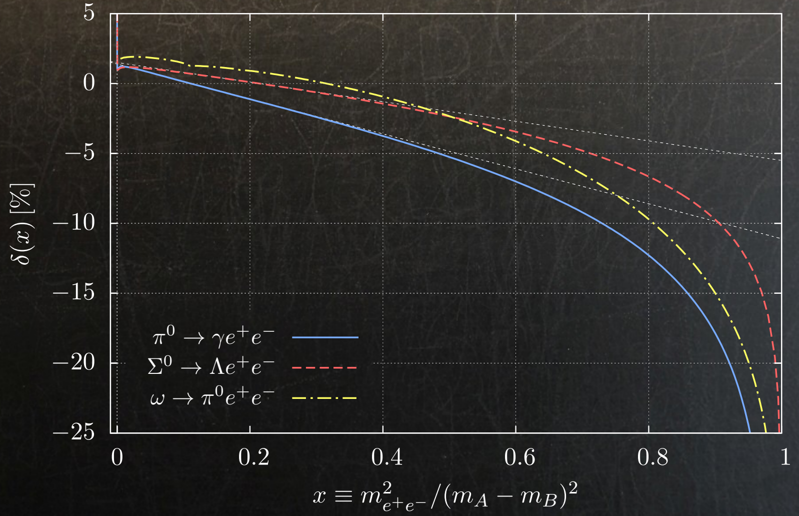
(Received 27 May 1983)

Department of Physics, Oklahoma State University, Stillwater, Oklahoma 74078

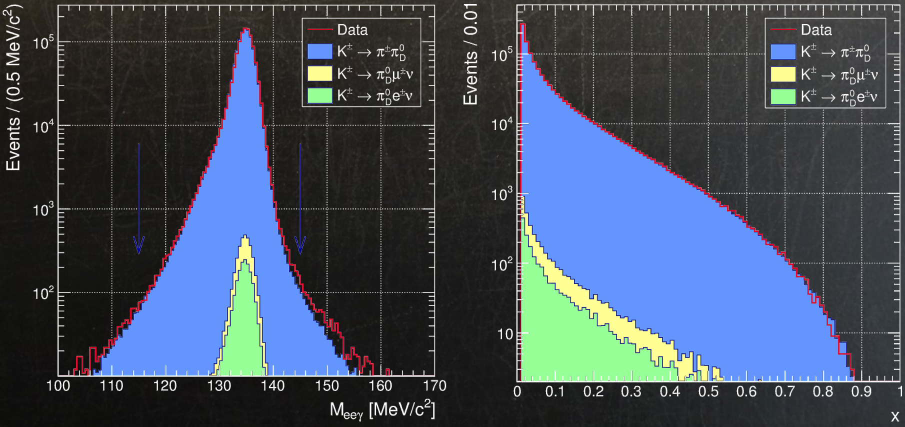
Theoretically, the form-factor slope a is expected to be small and positive, with most models[3] giving predictions near the intuitive estimate $a \cong (m_\pi/m_\rho)^2 = 0.03$. On the other hand the first three experiments[4–6] to determine a from the x distribution reported large negative values. Ap-

Inclusive NLO QED radiative corrections for Dalitz decays $A \to Be^+e^-$

Comparison plot



Result of π^0 form-factor-slope measurements



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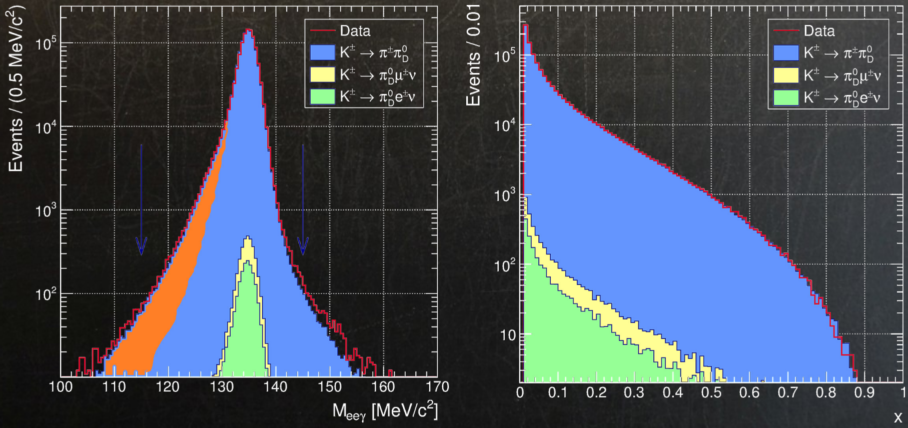
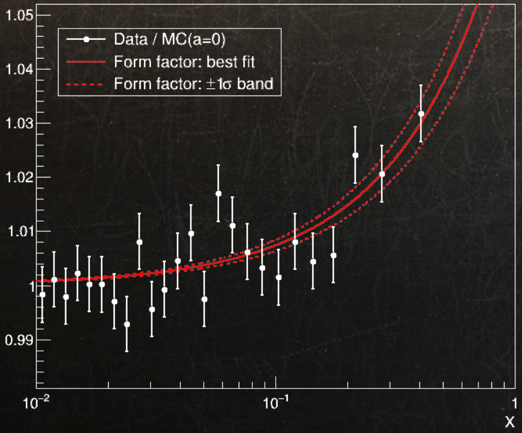
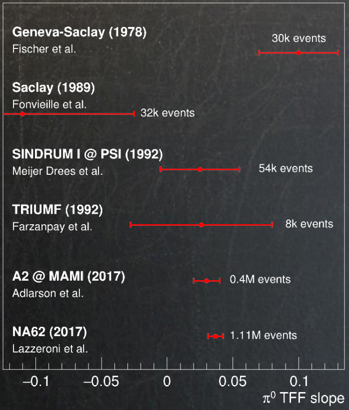


Result of π^0 form-factor-slope measurements





Radiative corrections for $\pi^0 \to e^+e^-\gamma$

Determination of ratio R

Precise and reliable determination of $R \equiv \dfrac{\Gamma(\pi^0 \to e^+e^-\gamma)}{\Gamma(\pi^0 \to \gamma\gamma)}$

↪ for small slope and up to NLO radiative corrections

$$R \simeq \frac{\alpha}{\pi} \iint (1 + a_\pi x)^2 (1 + \delta(x,y)) \frac{(1-x)^3}{4x} \left[1 + y^2 + \frac{4m_e^2}{M_\pi^2 x}\right] \mathrm{d}x\mathrm{d}y$$

Choose $a_\pi^{\text{univ}} = 3.55(70)\,\%$, covers

source	VMD	LMD	THS	dispers.	Padé aps.	NA62	A2	PDG
a_π [%]	3.00	2.45	2.92(4)	3.15(9)	3.21(19)	3.68(57)	3.0(1.0)	3.35(31)
b_π [10^{-3}]	0.90	0.74	0.87(2)	1.14(4)	1.04(22)	×	×	×

$$R = 1.1978(5)_{a_\pi}(3)_{\text{NNLO}}\,\%$$

Constraint: $1 \simeq \mathcal{B}(\pi^0 \to \gamma\gamma) + \mathcal{B}(\pi^0 \to e^+e^-\gamma(\gamma)) + \mathcal{B}(\pi^0 \to e^+e^-e^+e^-)$

	R	$\mathcal{B}(\pi^0 \to \gamma\gamma)$	$\mathcal{B}(\pi^0 \to e^+e^-\gamma(\gamma))$
TH et al., PRL 122 (2018)	$1.1978(6)\,\%$	$98.8131(6)\,\%$	$1.1836(6)\,\%$
PDG	$1.188(35)\,\%$	$98.823(34)\,\%$	$1.174(35)\,\%$

‘Recent’ KTeV measurement

Abouzaid et al., PRD 100 (2019)
$\hookrightarrow$ based on 1999 data and *E. Abouzaid*, Ph.D. thesis (2007)

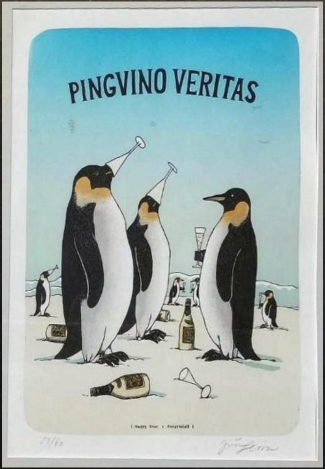
$$\frac{\Gamma(\pi^0 \to e^+e^-\gamma)}{\Gamma(\pi^0 \to \gamma\gamma)} = 1.1559(46)(106)\,\%$$

PDG average: $R = 1.188(35)\,\%$
$\hookrightarrow$ most recent (archived ALEPH data) *Beddall and Beddall*, EPJC 54 (2008)

TH, Goudzovski and Kampf, PRL 122 (2018): $R = 1.1978(6)\,\%$
$\hookrightarrow$ chosen $a_\pi^{\text{univ}} = 3.55(70)\,\%$
$\hookrightarrow$ conservative estimate for uncertainty

$\Longrightarrow$ $3.6\,\sigma$ discrepancy between theory and experiment

Need for new measurements
$\hookrightarrow$ R, improvement on a_π (maybe b_π) welcome



(Auditorium Gong gallery, Ostrava, Czechia)

Radiative corrections for $K^+ \to \pi^+ \ell^+ \ell^-$ decays

Radiative decays $K^+ \to \pi^+ \ell^+ \ell^-$

Motivation

Flavor-changing (strangeness-changing) neutral-current weak transitions
$\hookrightarrow$ absent at tree level in Standard Model
$\hookrightarrow$ manifest in radiative non-leptonic kaon decays like $K^+ \to \pi^+ \ell^+ \ell^- (\gamma)$, $\ell = e, \mu$
  $\hookrightarrow$ interesting probe of SM quantum corrections and beyond

Underlying long-distance-dominated radiative modes (transitions) $K^+ \to \pi^+ \gamma^* (\gamma)$ studied before
$\hookrightarrow$ calculated in Chiral Perturbation Theory (ChPT) enriched with electroweak perturbations
$\hookrightarrow$ *Ecker, Pich, de Rafael*, NPB 291 (1987), 303 (1988), at leading order (LO) (at one-loop level)

beyond LO: including the dominant unitarity corrections from $K \to 3\pi$
$\hookrightarrow$ *D'Ambrosio, Ecker, Isidori, Portolés*, JHEP 08 (1998)
$\hookrightarrow$ *Gabbiani*, PRD 59 (1999)

Radiative decays $K^+ \to \pi^+\ell^+\ell^-$

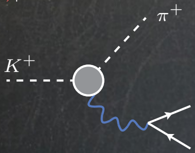
Form-factor parametrization

LO appears at $\mathcal{O}(p^4)$ + unitarity loop correction from $\pi\pi$ rescattering ↓

↪ universally used parametrization for the fit: $V_+(z) = a_+ + b_+ z + V_+^{\pi\pi}(z)$

↪ *Ecker et al.*, NPB 291 (1987), *D'Ambrosio et al.*, JHEP 08 (1998)

$$\frac{\mathrm{d}\Gamma_+}{\mathrm{d}z} = \frac{G_\mathsf{F}^2\alpha^2 M_K^5}{3(4\pi)^5}\lambda^{3/2}(z)\sqrt{1-\frac{4r_\ell^2}{z}}\left(1+\frac{2r_\ell^2}{z}\right)|V_+(z)|^2$$



LFU

↪ a_+s and b_+s should be the same for both (e and μ) channels

↪ discrepancy due to NP via SD effects

Moreover, the ratio deviates significantly from the VMD ansatz

$$\text{VMD:}\quad \frac{b_+}{a_+} = \frac{M_K^2}{M_\rho^2} \approx 0.4\,, \qquad \text{exp.:}\quad \frac{b_+}{a_+} \approx 1.25$$

Measurement of quadratic term $c_+ z^2$ may further test the VMD hypothesis

ℓ	a_+	b_+	exp.
e	$-0.587(10)$	$-0.655(44)$	E865
e	$-0.578(16)$	$-0.779(66)$	NA48/2
μ	$-0.575(39)$	$-0.813(145)$	NA48/2
μ	$-0.575(13)$	$-0.722(43)$	NA62(2022)

← JHEP 11 (2022) 011

improve precision ⟶ radiative corrections, studied earlier: *Kubis and Schmidt*, EPJC 70 (2010)

Radiative decays $K^+ \to \pi^+\ell^+\ell^-$

Recent NA62 analysis

Measure (one-photon-)inclusive process $K^+ \to \pi^+\mu^+\mu^-(\gamma)$
$\hookrightarrow$ EM effects subtracted in terms of NLO radiative corrections

Separate $K^+ \to \pi^+\mu^+\mu^-(\gamma)$ final-state phase space into two parts
$\hookrightarrow$ soft-photon 3-body process $K^+ \to \pi^+\mu^+\mu^-(\gamma)$
$\hookrightarrow$ hard-photon 4-body process $K^+ \to \pi^+\mu^+\mu^-\gamma$
$\hookrightarrow$ based on the Lorentz-invariant kinematical conditions $2p_\pi \cdot p_\gamma \gtrless 100\ \text{MeV}^2$
  $\hookrightarrow$ cutoff value is optimized with respect to the resolution of the NA62 detector system

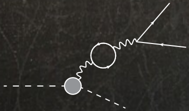
$\longrightarrow$ Ratio of the 4-body to 3-body integrated decay widths found to be $(1.64 \pm 0.02)\,\%$

3-body part
$\hookrightarrow$ radiative corrections studied also earlier $\longrightarrow$ starting point *Kubis et al.*, EPJC 70 (2010)
$\hookrightarrow$ NLO virtual and bremsstrahlung corrs. (integrated over photon energies and emission angles)
  $\hookrightarrow$ implementation going beyond the soft-photon approximation

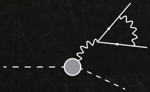
Radiative decays $K^+ \to \pi^+\ell^+\ell^-$

Radiative corrections: Lepton part

Vacuum-polarization contribution



QED vertex correction



lepton bremsstrahlung

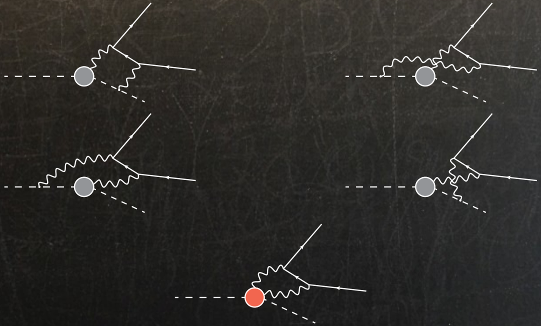
Radiative decays $K^+ \to \pi^+\ell^+\ell^-$

Radiative corrections: Meson part

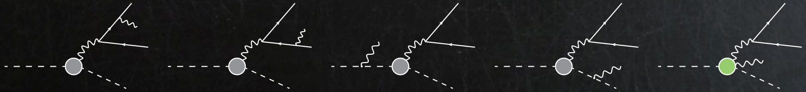
Radiative decays $K^+ \to \pi^+\ell^+\ell^-$

Radiative corrections: Lepton–meson interplay

Contributions linking lepton and meson currents



Interference of lepton and meson bremsstrahlung



$\hookrightarrow$ antisymmetric under $\ell^+ \leftrightarrow \ell^-$ $\longrightarrow$ cancel in $\mathrm{d}\Gamma/\mathrm{d}s$ $\Leftrightarrow$ does not contribute to FF extraction

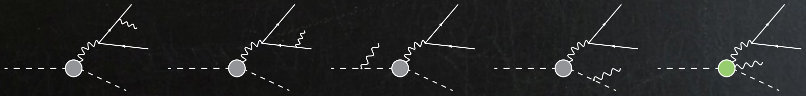
Radiative decays $K^+ \to \pi^+\ell^+\ell^-$

4-body part

- $\hookrightarrow$ LO (scalar) QED contributions where the real photon is radiated from lepton (meson) legs
- $\hookrightarrow$ radiation from effective $K^+ \to \pi^+\gamma^*$ vertex $\longrightarrow$ gauge invariant
  - $\hookrightarrow$ represented in terms of $F(s)$:

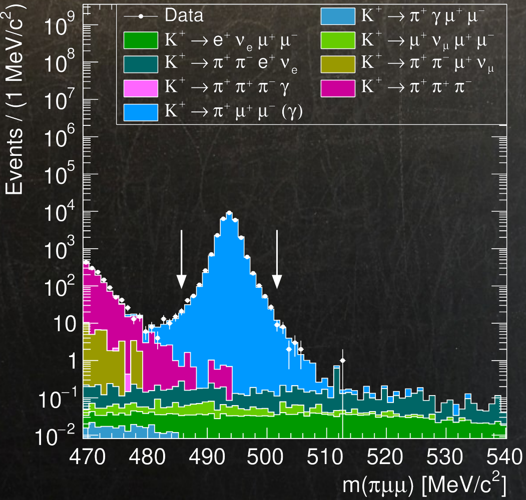
$$\mathcal{M}_{\rho\sigma}\big(K^+(P) \to \pi^+(r)\gamma^*_\rho(k_1)\gamma_\sigma(k_2)\big) = e^2F(k_1^2)\left\{k_1^2\left(r_\rho\frac{P_\sigma}{P\cdot k_2} - P_\rho\frac{r_\sigma}{r\cdot k_2} + g_{\rho\sigma}\right)\right\} + e^2\tilde{\kappa}F(k_1^2)\big[(k_1\cdot k_2)g_{\rho\sigma} - k_{1\sigma}k_{2\rho}\big]$$

- $\hookrightarrow$ term proportional to $\tilde{\kappa}$ represents estimate of associated uncertainty
  - $\hookrightarrow$ small in given set-up



Radiative decays $K^+ \to \pi^+\ell^+\ell^-$

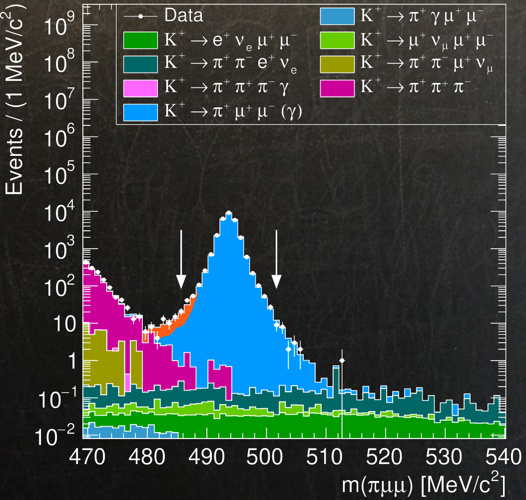
Spectra



(Thanks to *Ľ. Bičian (NA62)*)

Radiative decays $K^+ \to \pi^+\ell^+\ell^-$

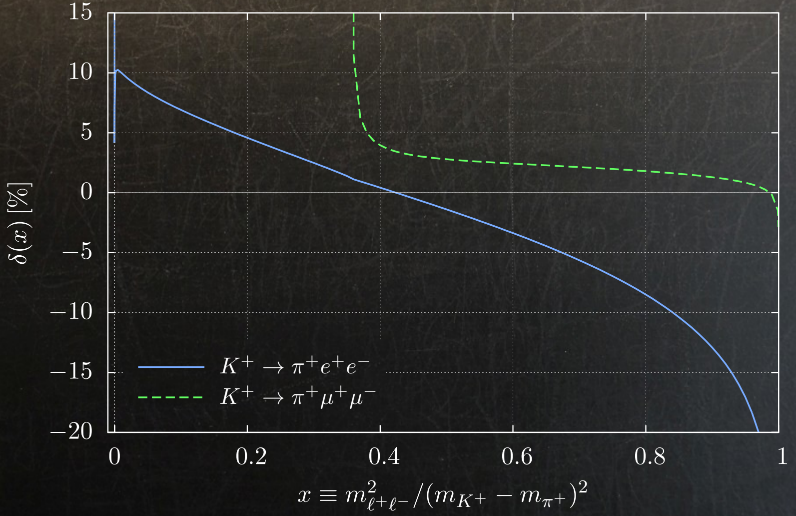
Spectra



(Thanks to *Ľ. Bičian (NA62)*)

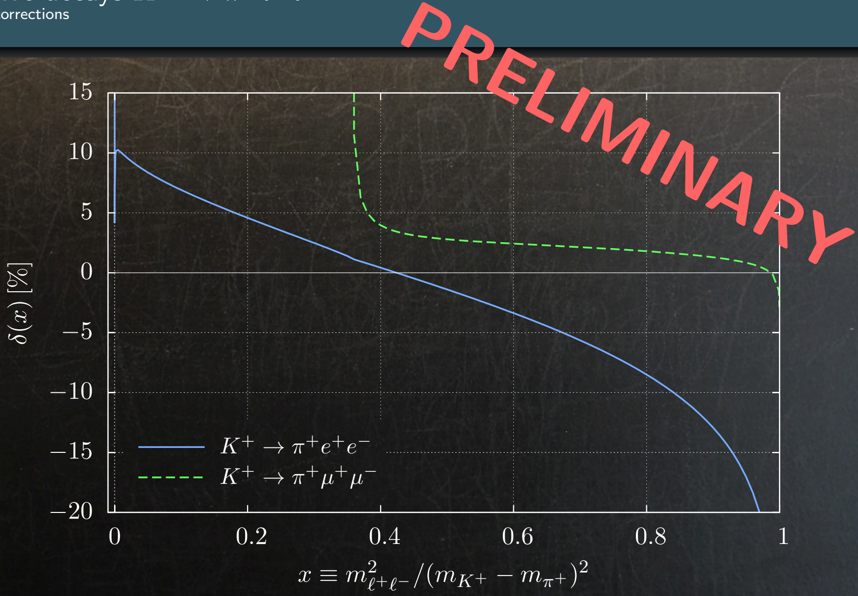
Radiative decays $K^+ \to \pi^+\ell^+\ell^-$

Radiative corrections



Radiative decays $K^+ \to \pi^+ \ell^+ \ell^-$

Radiative corrections



$K^+ \to \pi^+ 4e$ decay

$K^+ \to \pi^+ e^+ e^- e^+ e^-$

Introduction

The long-distance-dominated $K^+ \to \pi^+ \gamma^*$ transition essential also for $K^+ \to \pi^+ 4e$
$\hookrightarrow$ one also needs to consider $K^+ \to \pi^+ \gamma^* \gamma^*$ transition

Neutral-pion exchange ($K^+ \to \pi^+ \pi^{0*}$, $\pi^{0*} \to 4e$) clearly dominates when $m_{4e} \simeq M_{\pi^0}$
$\hookrightarrow$ overall branching ratio saturated by contribution of associated narrow π^0 peak
  $\hookrightarrow$ directly determined as $B(K^+ \to \pi^+ 4e) = B(K^+ \to \pi^+ \pi^0) B(\pi^0 \to 4e)$

Challenging to observe $K^+ \to \pi^+ 4e$ away from $m_{4e} \simeq M_{\pi^0}$
$\hookrightarrow$ suppressed decay rate $\longrightarrow$ attractive to study possible effects of BSM physics
$\hookrightarrow$ to identify new-physics-scenario contribution $\longrightarrow$ need for (rough) estimate of SM rate
  $\hookrightarrow$ new-physics effects spotted as deviations from such SM predictions

$K^+ \to \pi^+ e^+ e^- e^+ e^-$

Introduction

No number for BR in literature better than order-of-magnitude estimate
$\hookrightarrow$ naturally believed that is unlikely to exceed $\mathcal{O}(10^{-10})$ (*Hostert, Pospelov*, PRD 105 (2022))
$\hookrightarrow$ suppressed with respect to

$$B(K^+ \to \pi^+ e^+ e^-) \approx 3 \times 10^{-7}$$

$$B(K^+ \to \pi^+ \gamma\gamma, \text{ non-res.}) \approx 1 \times 10^{-6}$$

simply due to phase-space factors and additional QED vertices by $\mathcal{O}(\alpha^2)$
$\hookrightarrow$ indeed, non-resonant topologies give rise to

$$B(K^+ \to \pi^+ 4e, \text{ non-res.}) = 7.2(7) \times 10^{-11}$$

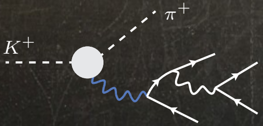
$\longrightarrow$ possible BSM scenarios are being explored
Hostert, Pospelov, PRD 105 (2022)
$\hookrightarrow$ $K \to \pi 4e$ decays proceed via $K \to \pi(X' \to XX)$ intermediate states
  $\hookrightarrow$ cascade of dark-sector particles $X^{(\prime)}$
  $\hookrightarrow$ underlying dynamics potentially significantly enhanced compared to the SM case

$\longrightarrow$ searches in suitable experiments
$\hookrightarrow$ more precise knowledge of SM background essential
  $\hookrightarrow$ ideally at level suited for Monte Carlo (MC) implementation

$K^+ \to \pi^+ e^+ e^- e^+ e^-$

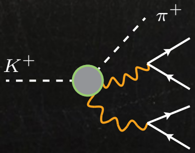
Standard Model prediction: Topologies

One-photon-exchange topology

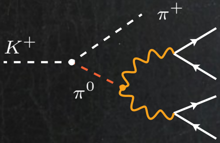


(1)

Two-photon-exchange topology



(2a)



(2b)

TH, PRD 106 (2022)

$K^+ \to \pi^+ e^+ e^- e^+ e^-$

One-photon-exchange topology: Matrix element

$K^+ \to \pi^+ \gamma^*$ transition

$$\begin{aligned}\mathcal{M}_\rho\big(K^+(P) \to \pi^+(r)\gamma^*_\rho(k)\big) &\equiv i \int \mathrm{d}^4 x\, e^{ikx} \langle \pi(r) |\, T\big[J^{\mathrm{EM}}_\rho(x)\mathcal{L}^{\Delta S=1}(0)\big] \,| K(P)\rangle \\ &= \frac{e}{2} F(k^2)\big[(P-r)^2(P+r)_\rho - (P^2 - r^2)(P-r)_\rho\big]\end{aligned}$$

↪ based on gauge and Lorentz symmetries

↪ simplified when coupled to a conserved current:

$$\mathcal{M}_\rho\big(K^+(P) \to \pi^+(r)\gamma^*_\rho(k)\big) \overset{\text{eff.}}{=} eF(k^2)k^2 r_\rho$$

The lepton part of the amplitude then amounts to

$$\begin{aligned}&\mathcal{M}^\rho_{\gamma^* \to 4e} \equiv \mathcal{M}^\rho\big(\gamma^{*\rho} \to e^-(p_1)e^+(p_2)e^-(p_3)e^+(p_4)\big) \\ &= \mathcal{M}^\rho_\gamma(p_1,p_2;p_3,p_4) + \mathcal{M}^\rho_\gamma(p_3,p_4;p_1,p_2) - \mathcal{M}^\rho_\gamma(p_1,p_4;p_3,p_2) - \mathcal{M}^\rho_\gamma(p_3,p_2;p_1,p_4)\end{aligned}$$

↪ overall amplitude for the topology (1) reads

$$\begin{aligned}\mathcal{M}^{(1)}_{K \to \pi 4e} &= \mathcal{M}_\rho\big(K^+(P) \to \pi^+(r)\gamma^*_\rho(k)\big)\, \frac{1}{k^2}\, \mathcal{M}^\rho_{\gamma^* \to 4e} \\ &= e^4 F\big((P-r)^2\big)\, r_\rho\, \widetilde{\mathcal{M}}^\rho_{\gamma^* \to 4e}\end{aligned}$$

$K^+ \to \pi^+ e^+ e^- e^+ e^-$

Two-photon-exchange topology: Matrix element

The two-photon transition of topology (2a) can be written, approximately, as follows:

$$\begin{aligned}
&\mathcal{M}^{(a)}_{\rho\sigma}(K(P) \to \pi(r)\gamma^*_\rho(k_1)\gamma^*_\sigma(k_2))\\
&\simeq e^2 F(k_1^2)\Big\{(k_1^2 r_\rho - r\cdot k_1 k_{1\rho})\,\frac{(2P-k_2)_\sigma}{2P\cdot k_2 - k_2^2} - (k_1^2 P_\rho - P\cdot k_1 k_{1\rho})\,\frac{(2r+k_2)_\sigma}{2r\cdot k_2 + k_2^2}\\
&\qquad\qquad + (k_1^2 g_{\rho\sigma} - k_{1\rho}k_{1\sigma})\\
&\qquad\qquad + \kappa\big[(k_1\cdot k_2)g_{\rho\sigma} - k_{1\sigma}k_{2\rho}\big]\Big\}\\
&+ \{k_1 \leftrightarrow k_2, \rho \leftrightarrow \sigma\}
\end{aligned}$$

$\hookrightarrow$ in this model depends on a single form factor (the same $F(s)$)

$\hookrightarrow$ useful when measuring $F(s) \longrightarrow$ radiative corrections for the $K^+ \to \pi^+\ell^+\ell^-$ decay

$\hookrightarrow$ one of the photons on-shell

Soft-photon regime $\longrightarrow$ approximation justified

Hard photons $\longrightarrow$ free parameter $|\kappa| \lesssim 1$ introduced to cover model uncertainty

$\hookrightarrow$ physical results do not seem to be sensitive to this parameter

For $K^+ \to \pi^+ 4e$, we assume it is good enough (at least) as an order-of-magnitude guess

$\hookrightarrow$ numerically negligible (one order of magnitude) compared to the topology (1)

$K^+ \to \pi^+ e^+ e^- e^+ e^-$

Neutral-pion pole contribution: Matrix element

pion-pole enhancement + simplicity
↪ proceed with the on-shell pion form factor and only consider the LO formula

↪ combining with a π^0-width-regulated propagator
↪ matrix element for the two-photon transition of the topology (2b):

$$\mathcal{M}^{(b)}_{\rho\sigma}(K(P) \to \pi(r)\gamma^*_\rho(k_1)\gamma^*_\sigma(k_2)) = -\frac{ie^2 G_{27}}{12\pi^2}\frac{2(P-r)^2 + 5M_K^2 - 7M_\pi^2}{(P-r)^2 - M_{\pi^0}^2 + iM_{\pi^0}\Gamma_{\pi^0}}\,\epsilon_{\rho\sigma(k_1)(k_2)}$$

[shorthand notation $\epsilon_{\rho\sigma\alpha\beta}k_1^\alpha k_2^\beta = \epsilon_{\rho\sigma(k_1)(k_2)}$]

$K^+ \to \pi^+ e^+ e^- e^+ e^-$

Phase space

The differential decay width for the $K \to \pi 4e$ process

$$\mathrm{d}\Gamma = \frac{1}{4}\,\frac{1}{2M_K}\,|\mathcal{M}_{K\to\pi 4e}|^2 \mathrm{d}\Phi_5(P;r,p_1,\dots,p_4)$$

with differential phase space

$$\mathrm{d}\Phi_5(P;r,p_1,\dots,p_4) = (2\pi)^4\delta^{(4)}(P-r-\sum_i p_i)\,\frac{\mathrm{d}^3 r}{(2\pi)^3\,2E_r}\,\frac{\mathrm{d}^3 p_1}{(2\pi)^3\,2E_{p_1}}\cdots\frac{\mathrm{d}^3 p_4}{(2\pi)^3 2\,E_{p_4}}$$

$|\mathcal{M}_{K\to\pi 4e}|^2$ depends on the particles' momenta

↪ subsequent integral largely nontrivial ⟶ use MC ⟶ normalization?

Turns out, in general, the branching ratio B can be obtained as

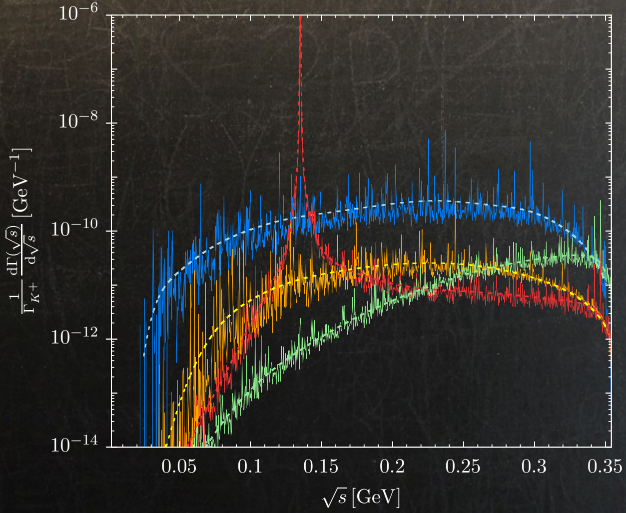
$$B = S\,\frac{1}{2M}\,\frac{1}{\Gamma_0}\,\Phi\,\frac{1}{N}\sum_{N\text{ events}}\overline{|\mathcal{M}|^2}$$

↪ rescaled average of the matrix element squared over the phase space × phase-space volume Φ

- ↪ N events randomly and evenly distributed in the momentum space

$K^+ \to \pi^+ e^+ e^- e^+ e^-$

Contributions to the branching ratio



[large MC samples generated by A. Shaikhiev, E. Goudzovski]

$K^+ \to \pi^+ e^+ e^- e^+ e^-$

Contributions to the branching ratio

Branching ratio calculated using Monte Carlo event generator technique:

$$B = \frac{1}{\Gamma_0} \frac{1}{4} \frac{1}{2M_K} \Phi_5 \frac{1}{N} \sum_{N \text{ events}} \overline{|\mathcal{M}|^2}$$

	$B(\sqrt{s} < 120\,\text{MeV})$	$B(\sqrt{s} > 150\,\text{MeV})$	B
(1)	5.60×10^{-12}	5.44×10^{-11}	6.70×10^{-11}
(2a)	3.11×10^{-13}	3.85×10^{-12}	4.60×10^{-12}
(2b)	1.40×10^{-13}	1.97×10^{-12}	$7.0(3) \times 10^{-6}$
κ	7.08×10^{-15}	3.69×10^{-12}	3.72×10^{-12}
$\sum$	$6.1(4) \times 10^{-12}$	$6.0(6) \times 10^{-11}$	$7.2(7) \times 10^{-11}$

$B(K^+ \to \pi^+ 4e) \simeq B(K^+ \to \pi^+ \pi^0) B(\pi^0 \to 4e)$

$\hookrightarrow B(K^+ \to \pi^+ \pi^0) = 20.67(8)\,\%$ and $B(\pi^0 \to 4e) = 3.38(16) \times 10^{-5}$

TH, PRD 106 (2022)

$K^+ \to \pi^0 e^+ \nu_e \gamma$ (aka $K^+_{e3\gamma}$) decay

$K^+_{e3\gamma}$ decay

Historical development

Fruitful era of publications peaked in late 60s
$\hookrightarrow$ *Fischbach, Smith*, PR 184 (1969); *Fearing, Fischbach, Smith*, PRL 24 (1970)

Holstein, PRD 41 (1990)
$\hookrightarrow$ ChPT calculation, tree level at NLO

Bijnens, Ecker, Gasser, NPB 396 (1993)
$\hookrightarrow$ NLO ChPT calculation at one-loop level

Kubis, Müller, Gasser, Schmid, EPJC 50 (2007)
$\hookrightarrow$ phenomenological approach for IB
$\hookrightarrow$ existing $\mathcal{O}(p^4)$ ChPT results for SD
$\hookrightarrow$ extended by some $\mathcal{O}(p^6)$ ChPT pieces

Khriplovich, Rudenko, PAN 74 (2011)
$\hookrightarrow$ only consider LO (IB with constant form factors)

$K^+_{e3\gamma}$ decay

What is the problem?

For $R = \dfrac{\text{“}\Gamma(K_{e3\gamma})\text{”}}{\Gamma(K_{e3})}$ disagreement between (most recent/precise results)

$\hookrightarrow$ theory: *Kubis et al.*, EPJC 50 (2007)

$\hookrightarrow$ experiment: NA62, JHEP 09 (2023)

NA62	preliminary (2021) 0.599(6) %	final (2023) 0.609(6) %
theory	0.640(8) %	

discrepancy $\approx 3\,\sigma \to 2\,\sigma$

$\hookrightarrow$ improved photon veto relevant for the $K_{e3\gamma(\gamma)}$ part

uncertainties at the level $\approx 1\,\%$, ‘missing effect’ $\approx 5\,\%$

$K^+_{e3\gamma}$ decay

Theoretical description

In terms of hadronic tensors

$$V(A)_{\mu\nu} = i\int \mathrm{d}^4x\, e^{iqx}\langle\pi^0|TV^{\mathsf{em}}_\mu(x)[\overline{s}\gamma_\nu(\gamma_5)u](0)|K^+\rangle\,, \quad F_\nu = \langle\pi^0|\overline{s}\gamma_\nu u|K^+\rangle\,,$$

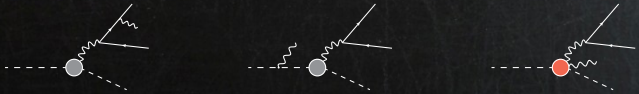
satisfying Ward identities $q^\mu V_{\mu\nu} = q^\mu(V^{\mathsf{IB}}_{\mu\nu} + V^{\mathsf{SD}}_{\mu\nu}) = q^\mu V^{\mathsf{IB}}_{\mu\nu} = F_\nu$ and $q^\mu A_{\mu\nu} = 0$

$$\mathcal{M}(K_{e3\gamma})$$

$$= \frac{G_{\mathsf{F}}}{\sqrt{2}}\, eV^*_{us}\, \overline{u}(p_\nu)\, \epsilon^{\mu *}(q)\gamma^\nu(1-\gamma_5)\Bigg[\underbrace{V^{\mathsf{IB}}_{\mu\nu} - \frac{F_\nu(t)}{2p_e\cdot q}(2p_{e\mu} + \not{q}\gamma_\mu)}_{\text{Internal bremsstrahlung}} \;+\; \underbrace{V^{\mathsf{SD}}_{\mu\nu} - A_{\mu\nu}}_{\text{Structure-dependent}}\Bigg]v(p_e)$$

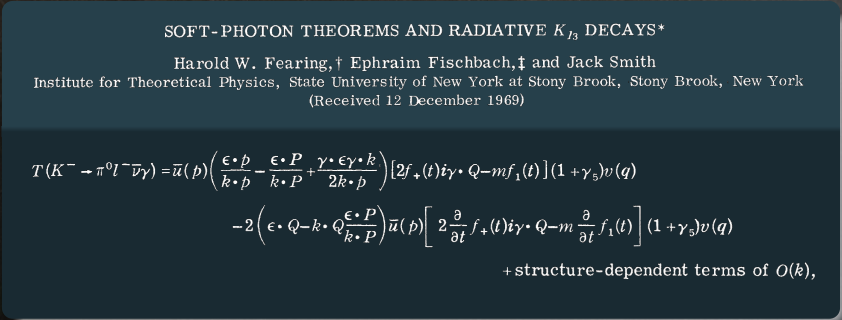
The K_{e3} form factor: $F_\nu(t) = \frac{1}{\sqrt{2}}\big[(p_K+p_\pi)_\nu f_+(t) + (p_K-p_\pi)_\nu f_-(t)\big] \longrightarrow \sqrt{2}p_{\pi\nu}f_+(t)$

$$V^{\mathsf{IB}}_{\mu\nu} \simeq \sqrt{2}p_{\pi\nu}\left\{\frac{p_{K\mu}}{p_K\cdot q}f_+(t) + \mathcal{O}(f'_+)\right\}, \quad t = (p_K-p_\pi)^2$$



$K^+_{e3\gamma}$ decay

Theoretical description



SOFT-PHOTON THEOREMS AND RADIATIVE K_{l3} DECAYS*

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(Received 12 December 1969)

$$T(K^- \to \pi^0 l^- \bar{\nu}\gamma) = \bar{u}(p)\left(\frac{\epsilon\cdot p}{k\cdot p} - \frac{\epsilon\cdot P}{k\cdot P} + \frac{\gamma\cdot\epsilon\gamma\cdot k}{2k\cdot p}\right)[2f_+(t)i\gamma\cdot Q - mf_1(t)](1+\gamma_5)v(q)$$

$$-2\left(\epsilon\cdot Q - k\cdot Q\frac{\epsilon\cdot P}{k\cdot P}\right)\bar{u}(p)\left[2\frac{\partial}{\partial t}f_+(t)i\gamma\cdot Q - m\frac{\partial}{\partial t}f_1(t)\right](1+\gamma_5)v(q)$$

$$+\text{structure-dependent terms of } O(k),$$

$K^+_{e3\gamma}$ decay

Quantity in question

$$R(E_{\rm min},\theta_{\rm min}) = \frac{\Gamma\big(K^+\to\pi^0e^+\nu\gamma(\gamma), E^* > E_{\rm min}, \theta^*_{e\gamma} > \theta_{\rm min}\big)}{\Gamma\big(K^+\to\pi^0e^+\nu(\gamma)\big)}$$

Numerator: $K^+_{e3\gamma}$ with cuts + any number of other (soft/hard) photons
↪ frequent choice is $E_{\rm min} = 30\,\mathrm{MeV}$
↪ since *Bijnens et al.*, also cut on emission angle, e.g. $\theta_{\rm min} = 20^\circ$
  ↪ motivated from K^0 sector

Denominator: inclusive K^+_{e3}
↪ arbitrarily many photons allowed in the final state, energy/emission angle unconstrained

$K^+_{e3\gamma}$ decay

Theoretical predictions and experimental results ($E_{\text{min}} = 30$ MeV)

	year	E_{min}	θ_{min}	$R\ [10^{-2}]$	
		Theory			
FFS	1969	30 MeV	0°	2.04	
Holstein	1990	30 MeV	0°	2.098(4)	
*Bijnens et al.**	1993	30 MeV	20°	0.611(18)	
Kubis et al.	2007	30 MeV	20°	0.640(8)	
Khriplovich et al.†	2011	30 MeV	20°	0.54(7)	
		Experiment			
ISTRA+	2007	30 MeV	20°	0.64(4)	4.5k
OKA	2021	30 MeV	20°	0.587(18)	32k
NA62	2023	30 MeV	20°	0.609(6)	130k

* *Bijnens et al.* obtained $B(K^+_{e3\gamma}, 30\,\text{MeV}, 20^\circ) = 3.0 \times 10^{-4}$

$\hookrightarrow$ up-to-date values for physical constants with uncertainties, double precision, higher statistics

- $\hookrightarrow$ $B(K^+_{e3\gamma}, 30\,\text{MeV}, 20^\circ) = 3.10(6) \times 10^{-4}$
- $\hookrightarrow$ $B(K^+_{e3}) = 5.07(4)\ \%$ (PDG)

† *Khriplovich et al.* obtained $B(K^+_{e3\gamma}, 30\,\text{MeV}, 20^\circ) = 2.72((40)) \times 10^{-4}$

$K^+_{e3\gamma}$ decay

Theoretical predictions and experimental results (3 regions)

$[10^{-2}]$	$R(10\ \text{MeV}, 10^\circ)$	$R(30\ \text{MeV}, 20^\circ)$	$R(10\ \text{MeV}, \text{acos}(0.6\text{–}0.9))$
	Experiment		
ISTRA+	1.81(8)	0.64(4)	0.47(4)
OKA	1.990(27)	0.587(18)	0.532(16)
PDG	–	–	0.525(17)
NA62	1.715(11)	0.609(6)	0.533(4)
	Theory		
*Bijnens et al.**	1.725(52)	0.611(18)	0.534(16)
Kubis et al.	1.804(21)	0.640(8)	0.559(6)

* obtained using the original code of *Bijnens et al.*, with PDG value for $B(K^+_{e3})$

$\hookrightarrow$ up-to-date values for physical constants with uncertainties, double precision, higher statistics

$K^+_{e3\gamma}$ decay

Bijnens et al. vs. *Kubis et al.*

Bijnens et al.

$\hookrightarrow$ NLO ChPT result (at one loop) for $B(K^{+,0}_{\ell 3\gamma}, 30\,\text{MeV}, 20°)$

  $\hookrightarrow$ both for IB and SD parts

$\hookrightarrow$ to obtain R, simply normalize on experimental value for $B(K^+_{e3})$

  $\hookrightarrow$ LO QED in numerator

  $\hookrightarrow$ inclusive decay in denominator

Kubis et al.

$\hookrightarrow$ more general phenomenological approach for IB: $f_+(t) \simeq f_+(0)\left(1 + \lambda_+ \dfrac{t}{M_\pi^2}\right)$

  $\hookrightarrow$ depends on extracted/calculated K^+_{e3} FF parameters from elsewhere

  $\hookrightarrow$ ChPT for SD part + partly $\mathcal{O}(p^6)$

$\hookrightarrow$ makes use of cancellations in R

$\hookrightarrow$ considers LO/LO (regarding QED)

  $\hookrightarrow$ extends to estimate on equivalent inclusive quantities in the ratio (RCs discussed)

Major differences

$\hookrightarrow$ (in)dependence on $f_+(0)$; result not very sensitive to λ_+

  $\hookrightarrow$ *Bijnens et al.* does not contain isospin-breaking corrections

$\hookrightarrow$ inclusive/exclusive quantities

$K^+_{e3\gamma}$ decay

Bijnens et al. vs. *Kubis et al.*

Underlying NLO ChPT expressions in SD part equivalent in both papers

R in *Kubis et al.* defined using phenomenological expression for IB
$\hookrightarrow$ in terms of K^+_{e3} FF parameters
$\hookrightarrow$ normalizations (including $f_+(0)$) in theoretical expressions cancel in the ratio R

$$\mathcal{R} \equiv \frac{\Gamma\big(K^+ \to \pi^0 e^+ \nu\gamma, 30\ \text{MeV}, 20^\circ\big)}{\Gamma\big(K^+ \to \pi^0 e^+ \nu\big)} \simeq \frac{8\alpha}{\pi^4} \frac{b_0 + b_1\lambda_+ + b_2\lambda_+^2}{a_0 + a_1\lambda_+ + a_2\lambda_+^2} = 0.633(2)\ \%$$

Reproduce result from *Bijnens et al.**
$\hookrightarrow$ remove isospin-breaking corrections ($\mathcal{R}/1.022^2$)

OR

$\hookrightarrow$ fit the NLO ChPT result on $f_+(t) \simeq f_+(0)\left(1 + \lambda_+ \frac{t}{M_\pi^2}\right)$
  $\hookrightarrow$ $f_+(0) = 0.973$, $\lambda_+ = 0.033(3)$

$\hookrightarrow$ reintroduce normalization into the numerator

$$R \simeq \underbrace{\frac{4\alpha M_K^5 G_\mathsf{F}^2 |V_{us}|^2}{(2\pi)^7 \Gamma_{K^+}}}_{2.739 \to 2.850(25) \times 10^{-4}} \underbrace{f_+^2(0)\big(b_0 + b_1\lambda_+ + b_2\lambda_+^2\big)}_{1.084 \to 1.087(13)} \underbrace{\frac{1}{B_{\text{exp}}(K^+_{e3})}}_{\frac{1}{5.07(4)\ \%}} = 0.611(18)\ \%$$

$K^+_{e3\gamma}$ decay

Included corrections in R in *Kubis et al.*

How to obtain (inclusive) R from (exclusive) $\mathcal{R}$?

- Some corrections appear both in numerator and denominator and cancel in R
- Isospin-breaking and radiative corrections
  $\hookrightarrow$ for K_{e3} up to $\mathcal{O}\big(p^2e^2, p^2(m_u - m_d)\big)$ calculated in *Cirigliano et al.*, EPJC 23 (2002)
  $\hookrightarrow$ partly cancel in R $\big[$those contributing to $f_+(0)\big]$
  $\hookrightarrow$ denominator (K_{e3})
    $\hookrightarrow$ rest considered by modifying the a_i coefficients of slope expansion
    $\hookrightarrow$ overall effect $\approx -1\,\%$
- numerator
  $\hookrightarrow$ radiative corrections expected small $(\mathcal{O}(1\,\%))$
    $\hookrightarrow$ taken as uncertainty
  $\hookrightarrow$ remaining residual $\mathcal{O}\big(p^2(m_u - m_d)\big)$ corrections considered negligible

Considered final enhancement by 1 % leads to

$$R = \big[1.01(1)\big] \times \mathcal{R} = 0.640(8)\,\%$$

$K^+_{e3\gamma}$ decay

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$K^+_{e3\gamma}$ decay

Current status and possible improvements

1) Improve MC generator

Numerator: $K_{e3\gamma}(E_{\text{min}}, \theta_{\text{min}})$
↪ originally only 1 photon
    ↪ extra photons using PHOTOS
↪ introduce a complete generator for $K^+_{e3\gamma\gamma}$ (not available nor trivial)

2) 'Exclusive' measurement

Numerator: only 1 hard photon above the sensitivity threshold
Denominator: no detectable photons

$$R(E_{\min}, \theta_{\min}) = \frac{\Gamma\big(K^+ \to \pi^0 e^+ \nu\gamma, E^* > E_{\min}, \theta^*_{e\gamma} > \theta_{\min}\big)}{\Gamma\big(K^+ \to \pi^0 e^+ \nu\big)}$$

Summary

Pion decays
↪ NLO QED radiative corrections available

- $\pi^0 \to e^+e^-$
  Vaško and Novotný, JHEP 1110 (2011)
  TH, Kampf and Novotný, EPJC 74 (2014)
  ↪ measure $B(\pi^0 \to e^+e^-)$, extract $\chi^{(\mathrm{r})}(M_\rho)$
- $\pi^0 \to e^+e^-\gamma$
  TH, Kampf and Novotný, PRD 92 (2015)
  ↪ precise determination of R: *TH, Goudzovski and Kampf*, PRL 122 (2018)
  ↪ could be used in future exp. analysis of $K^+ \to \pi^+e^+e^-$

NLO QED radiative corrections for $K^+ \to \pi^+\ell^+\ell^-$
↪ used in recent NA62 analysis ⟶ JHEP 11 (2022) 011

SM estimate of $B(K^+ \to \pi^+ 4e)$
↪ $B(K^+ \to \pi^+ 4e$, non-resonant$) = 7.2(7) \times 10^{-11}$, *TH*, PRD 106 (2022) 7

Tension between theory and experiment in $K_{e3\gamma}$
↪ investigate the origin of ‘missing’ 5 % effect
↪ room for improvement on both (experiment and theory) sides