Semileptonic kaon decays in the Standard Model

Peter Stoffer

Physik-Institut, University of Zurich
and Paul Scherrer Institut

EPJC **74** (2014) 2749

with G. Colangelo and E. Passemar: EPJC **75** (2015) 172

12th September 2023

Kaons@CERN 2023

University of Zurich[UZH]

PAUL SCHERRER INSTITUT PSI

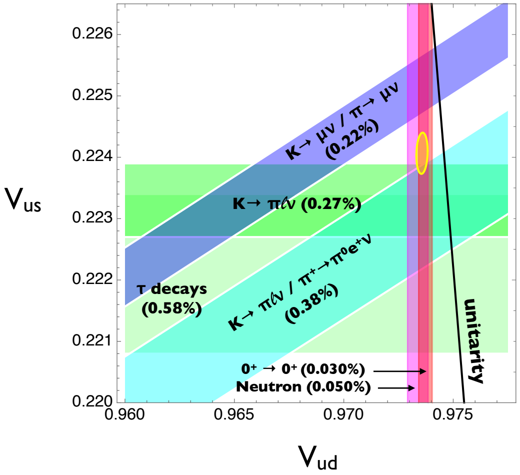
funded by Swiss National Science Foundation

Outline

Leptonic and semileptonic kaon decays

- $K_{\ell 2}$:
  - determination of $|V_{us}/V_{ud}|$ from $\Gamma_{K_{\mu 2(\gamma)}}/\Gamma_{\pi_{\mu 2(\gamma)}}$ together with lattice input
  - lepton universality probed by $R^K_{e/\mu} = \Gamma_{K_{e2(\gamma)}}/\Gamma_{K_{\mu 2(\gamma)}}$
- $K_{\ell 3}$:
  - alternative determination of $|V_{us}/V_{ud}|$ from $\Gamma_{K_{\ell 3(\gamma)}}/\Gamma_{\pi_{e3(\gamma)}}$
  - determination of $|V_{us}|$ with lattice input
  - knowledge of form factor allows to predict $K \to \pi\nu\bar{\nu}$

Leptonic and semileptonic kaon decays



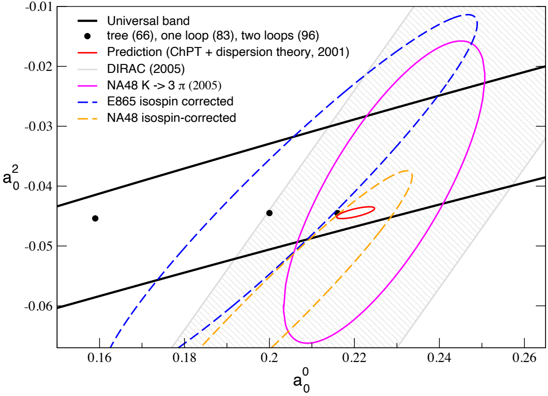
→ Bryman, Cirigliano, Crivellin, Inguglia, ARNPS **72** (2022) 69-91

Leptonic and semileptonic kaon decays

- $K_{\ell 4}$:
  - provides information on $\pi\pi$-scattering lengths a_0^0, a_0^2
  - K_{e4} very precisely measured $\Rightarrow$ **test of χPT**

    $\rightarrow$ Geneva-Saclay, E865, NA48/2
  - **best source** of information on the χPT low-energy constants **L_1^r, L_2^r and L_3^r**
  - happens at **very low energy**, where χPT is expected to converge best
  - background for $K \rightarrow \pi\nu\bar{\nu}$
  - form-factor determination allows to predict $K \rightarrow \pi\pi\nu\bar{\nu}$

Leptonic and semileptonic kaon decays



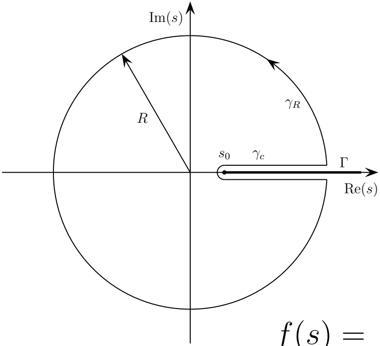
→ Colangelo, PoS KAON (2008) 038

→ Colangelo, Gasser, Rusetsky, EPJC **59** (2009) 777-793

Overview

Dispersion relations

causality implies **analyticity**:



Cauchy integral formula:

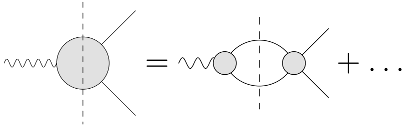
$$f(s) = \frac{1}{2\pi i} \oint_\gamma \frac{f(s')}{s' - s} ds'$$

deform integration path:

$$f(s) = f(0) + \frac{s}{\pi} \int_{s_0}^{\infty} \frac{\mathrm{Im} f(s')}{(s' - s - i\epsilon) s'} ds'$$

Unitarity

unitarity of S-matrix determines discontinuities



$$f(s) = |f(s)|e^{i\delta(s)+\dots}$$

Watson's final-state theorem for elastic region:

$\delta =$ scattering phase shift

Example: $K_{\ell 3}$ scalar form factor

twice-subtracted **Omnès representation**:

$$\bar{f}_0(t) = \exp\left\{ \frac{t}{\Delta_{K\pi}} \ln \bar{f}_0(\Delta_{K\pi}) + \frac{t(t-\Delta_{K\pi})}{\pi} \int_{t_{K\pi}}^{\infty} \frac{\delta(t')}{t'(t'-\Delta_{K\pi})(t'-t-i\epsilon)} dt' \right\}$$

free parameter: $\bar{f}_0(\Delta_{K\pi})$ at Callan–Treiman point

$\Delta_{K\pi} = M_K^2 - M_\pi^2 \Rightarrow$ use low-energy theorem

→ Bernard, Oertel, Passemar, Stern, PLB **638** (2006) 480, PRD **80** (2009) 034034

Advantages of dispersion relations

- based on analyticity and unitarity $\Rightarrow$ model independence
- summation of rescattering
- connect different energy regions

Applications to $K_{\ell 4}$

- as a means to compute isospin breaking at two loops in χPT:

  $\rightarrow$ Bernard, Descotes-Genon, Knecht, EPJC **73** (2013) 2478

- numerical solution, Omnès methods:

  $\rightarrow$ Colangelo, Passemar, Stoffer, EPJC **75** (2015) 172

Definition of the $K_{\ell 4}$ decay

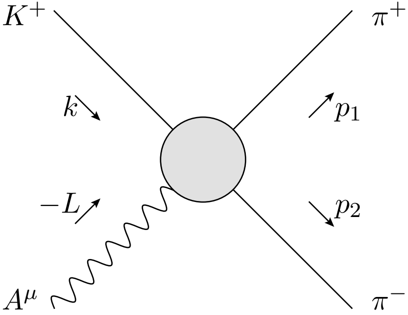
decay of a kaon into two pions and a lepton pair

$$K^+(p) \to \pi^+(p_1)\pi^-(p_2)\ell^+(p_\ell)\nu_\ell(p_\nu)$$

with $\ell \in \{e, \mu\}$

(other modes involving neutral pions related by isospin symmetry)

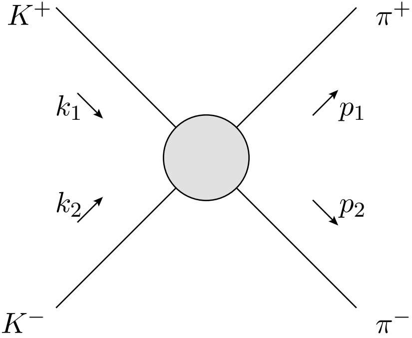
Hadronic part of $K_{\ell 4}$ as $2 \to 2$ scattering



Mandelstam variables:

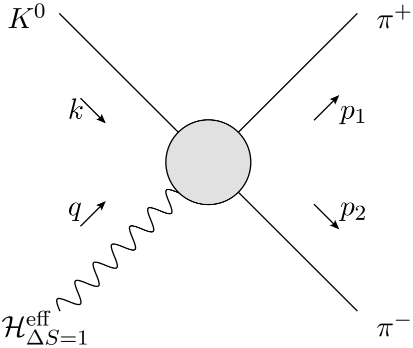
$$s = (p_1 + p_2)^2, \quad t = (k - p_1)^2, \quad u = (k - p_2)^2$$

Similar to $K\pi \to K\pi$ or $KK \to \pi\pi$



but: physical region for $K\pi$ scattering: $E > M_K + M_\pi$

Similar to $K \to 2\pi$



$\to$ Büchler, Colangelo, Kambor, Orellana (2001)

with application in rare K_S decays

$\to$ Colangelo, Stucki, Tunstall, EPJC **76** (2016) 11, 604

Form factors

- Lorentz structure allows four form factors in the hadronic matrix element ($P = p_1 + p_2$, $Q = p_1 - p_2$):

$$\left\langle \pi^+(p_1)\pi^-(p_2) \middle| A_\mu(0) \middle| K^+(k) \right\rangle = -i \frac{1}{M_K} \left(P_\mu \boldsymbol{F} + Q_\mu \boldsymbol{G} + L_\mu \boldsymbol{R} \right)$$

$$\left\langle \pi^+(p_1)\pi^-(p_2) \middle| V_\mu(0) \middle| K^+(k) \right\rangle = -\frac{\boldsymbol{H}}{M_K^3} \epsilon_{\mu\nu\rho\sigma} L^\nu P^\rho Q^\sigma$$

- contribution of R **helicity suppressed**: invisible in K_{e4}
- H related to **chiral anomaly**: chirally suppressed
- concentrate here on F and G
- form factors are functions of the Mandelstam variables s, t and u

Analytic properties

- $F(s,t,u)$ and $G(s,t,u)$ have a right-hand branch cut in the complex s-plane, starting at the $\pi\pi$-threshold
- left-hand cut present due to crossing
- analogous situation in t- and u-channel

Reconstruction theorem

→ Stern, Sazdjian, Fuchs (1993), Ananthanarayan, Buettiker (2001), . . .

- define function with **only right-hand cut** of f_0, the first partial wave of F:

$$M_0(s) := P(s) + \frac{s^2}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\operatorname{Im} f_0(s')}{(s' - s - i\epsilon){s'}^2} ds'$$

- similar functions take care of the right-hand cuts of all other S- and P-waves (also crossed channels)
- all the **discontinuities** are **split up** into functions of a single variable
- neglect imaginary parts of D- and higher waves

Reconstruction theorem

form factors decomposed into functions of one Mandelstam variable only:

$$F(s,t,u) = M_0(s) + \frac{u-t}{M_K^2} M_1(s) + (\text{functions of } t \text{ or } u),$$

$$G(s,t,u) = \tilde{M}_1(s) + (\text{functions of } t \text{ or } u).$$

- generalization of Khuri–Treiman equations
- violated only at $\mathcal{O}(p^8)$ in χPT

Omnès representation

function M_0 contains only right-hand cut of the partial wave f_0: difference is the 'inhomogeneity' $\hat{M}_0$:

$$f_0(s) = M_0(s) + \hat{M}_0(s)$$

inhomogeneous **Omnès problem**:

$$\mathrm{Im} M_0(s) = (M_0(s) + \hat{M}_0(s)) e^{-i\delta_0^0(s)} \sin \delta_0^0(s)$$

Watson's theorem: δ_0^0 is elastic $\pi\pi$ phase shift

Omnès representation

Omnès function takes care of rescattering:

$$\Omega_l^I(s) := \exp\left\{\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\delta_l^I(s')}{(s'-s-i\epsilon)s'}ds'\right\}$$

δ_l^I: elastic $\pi\pi$ or $K\pi$ phase shifts

write dispersion relation for $\dfrac{M_0(s)}{\Omega_0^0(s)}$

Omnès representation

Omnès solution for the functions $M_0(s)$, $M_1(s)$, $\tilde{M}_1(s)$, etc.:

$$M_0(s) = \Omega_0^0(s)\left\{P(s) + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\Lambda^2} \frac{\hat{M}_0(s')\sin\delta_0^0(s')}{|\Omega_0^0(s')|(s'-s-i\epsilon)s'^3}ds'\right\}$$

P: subtraction polynomial
$\hat{M}_i$: inhomogeneities, angular averages of all the functions M_i

Obtained dispersive representation

- problem parametrized by 9 **subtraction constants**
- input: elastic $\pi\pi$- and $K\pi$-scattering **phase shifts**
- energy dependence fully determined by the dispersion relation

Obtained dispersive representation

- set of coupled integral equations:
  $\Rightarrow M_0(s), M_1(s), \ldots$: DR involving $\hat{M}_0(s), \hat{M}_1(s), \ldots$
  $\Rightarrow \hat{M}_0(s), \hat{M}_1(s), \ldots$: angular integrals over $M_0(s), M_1(s), \ldots$
- system solved by iteration
  (alternatively: direct matrix inversion
  $\rightarrow$ Gasser, Rusetsky, EPJC **78** (2018) 11, 906)
- problem linear in the subtraction constants
  $\Rightarrow$ construct 9 basis solutions

Determination of the subtraction constants

- fit to data of the high-statistics experiments E865 and NA48/2
- soft-pion theorems as additional constraints
- chiral input for the subtraction constants that are not well determined by data

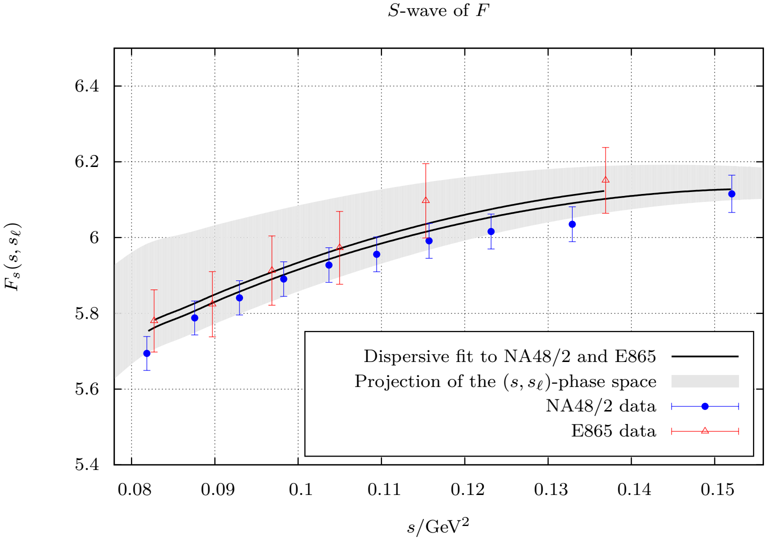
Isospin-breaking corrections

$\rightarrow$ EPJC **74** (2014) 2749

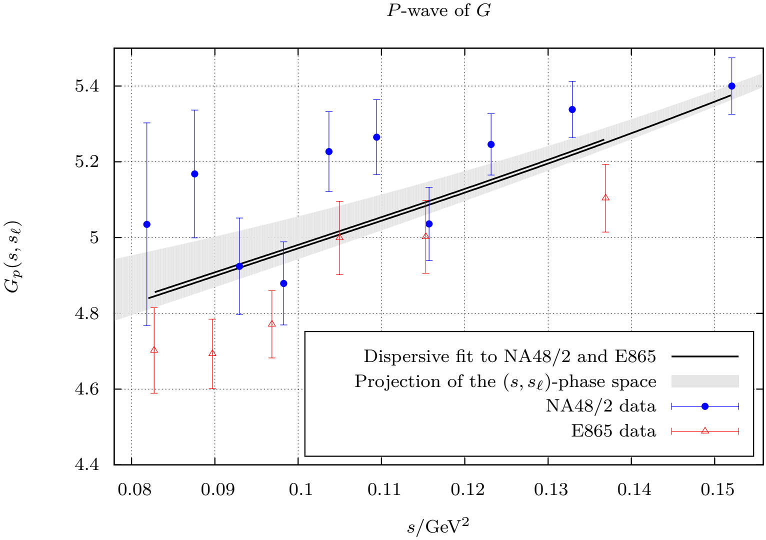
isospin-breaking effects beyond the ones taken into account in NA48/2 analysis:

- based on one-loop χPT calculation
- strong-isospin breaking effects $\propto (m_u - m_d)$
- meson-mass difference effects $\propto Ze^2$
- purely photonic effects $\propto e^2$
- compared to Coulomb factor and PHOTOS MC

Fit results for partial waves



Fit results for partial waves



Matching to χPT

- matching to χPT at the level of subtraction constants in Omnès form: separate rescattering effects
- fit to 2-dimensional data set of NA48/2
- L_9^r can be determined from dependence on s_ℓ

Matching at NNLO

- many poorly known LECs C_i^r at NNLO
- include additional constraints in the fit: require good chiral convergence
- input: C_i^r contribution to subtraction constants with $\pm 50\%$ uncertainty
- fit the C_i^r contribution
- not all sets of C_i^r input lead to a good chiral convergence: prefer BE14

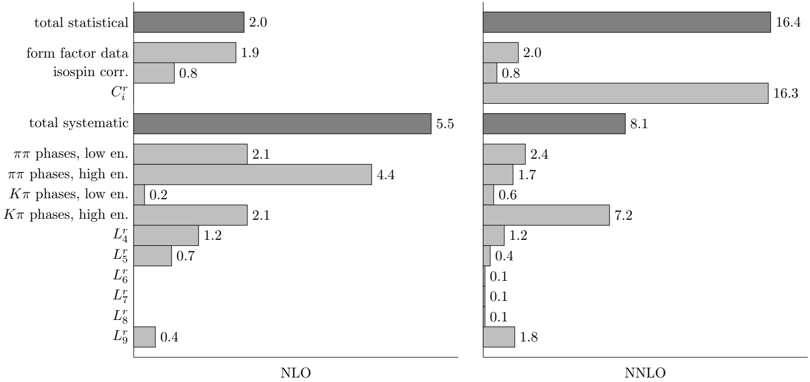
  → Bijnens, Ecker, ARNPS **64** (2014) 149

Low-energy constants

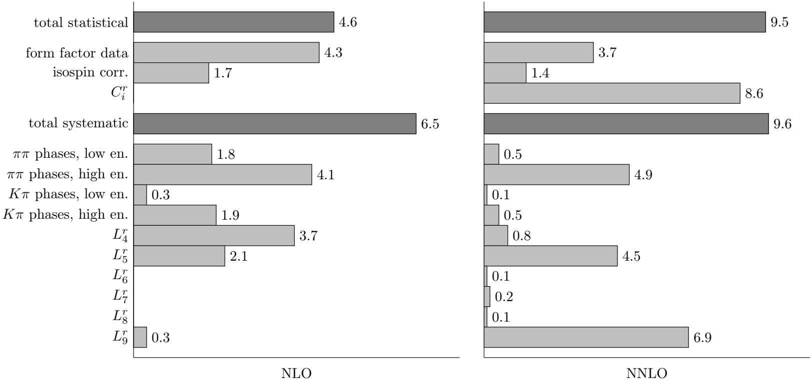
Results for the LECs using χPT at NLO and NNLO.

	NLO	NNLO	Bijnens, Ecker (2014)
$10^3 \cdot L_1^r$	$0.51(2)(6)$	$0.69(16)(8)$	$0.53(6)$
$10^3 \cdot L_2^r$	$0.89(5)(7)$	$0.63(9)(10)$	$0.81(4)$
$10^3 \cdot L_3^r$	$-2.82(10)(7)$	$-2.63(39)(24)$	$-3.07(20)$
χ^2/dof	$141/116 = 1.2$	$124/122 = 1.0$	

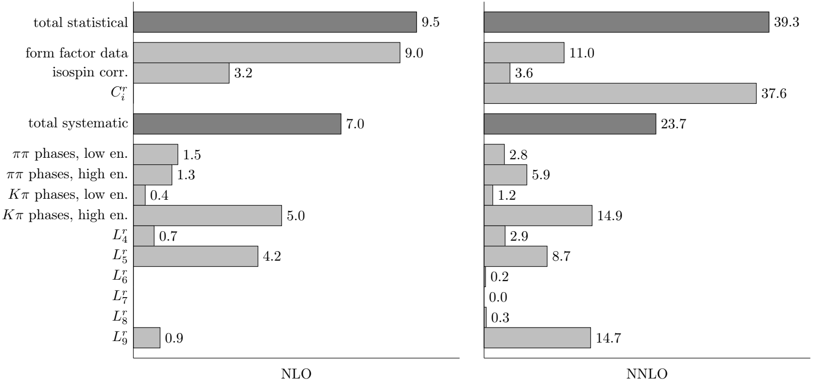
Error budget: L_1^r



Error budget: L_2^r



Error budget: L_3^r



Electron mode K_{e4}

What could be done with higher statistics?

- s_ℓ-dependence of F and G can be used to **extract L_9^r**
  $\Rightarrow$ relation to pion charge radius
- determination of $\boldsymbol{L_1^r}$, $\boldsymbol{L_2^r}$, $\boldsymbol{L_3^r}$ with even higher precision
- (better) determination of linear combinations of C_i^r
- include 1-loop radiative corrections for $K_{e4}(\gamma)$ in PHOTOS Monte Carlo $\rightarrow$ EPJC **74** (2014) 2749

Muon mode $K_{\mu 4}$

- larger values of s_ℓ
- form factor R is accessible
- s-dependence of R contains $\boldsymbol{L_4^r}$, $\boldsymbol{L_5^r}$ and L_9^r
- information on $\boldsymbol{K\pi}$ **scattering**

Summary

- parametrization valid up to and including $\mathcal{O}(p^6)$
- model independence
- resummation of rescattering effects
- very precise data available
- determination of LECs from matching to χPT

Summary

- even higher statistics could be useful for better determination of L_i^r and combinations of C_i^r
- better data on s_ℓ-dependence would enable independent determination of L_9^r
- radiative corrections should be included in Monte Carlo
- new form factor and further LECs accessible in $K_{\mu 4}$