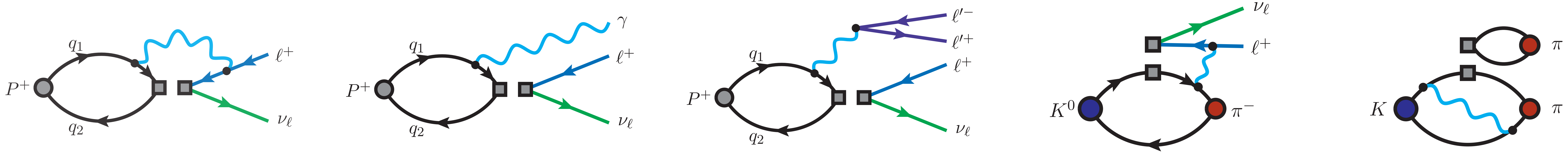


Lattice QCD+QED in relation to kaon decays

Matteo Di Carlo

13th September 2023

THE UNIVERSITY of EDINBURGH

Kaons@CERN
11-14 September 2023

Outline of this talk

1. **Why** are **isospin-breaking** and QED corrections **relevant**?

2. **How** are these effects **included in lattice calculations**?

3. **What** can the **lattice** do for **kaon physics**?

1. Why

Indirect searches of new physics using CKM matrix unitarity constraints

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

in the Standard Model:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

1. Why

Indirect searches of new physics using CKM matrix unitarity constraints

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

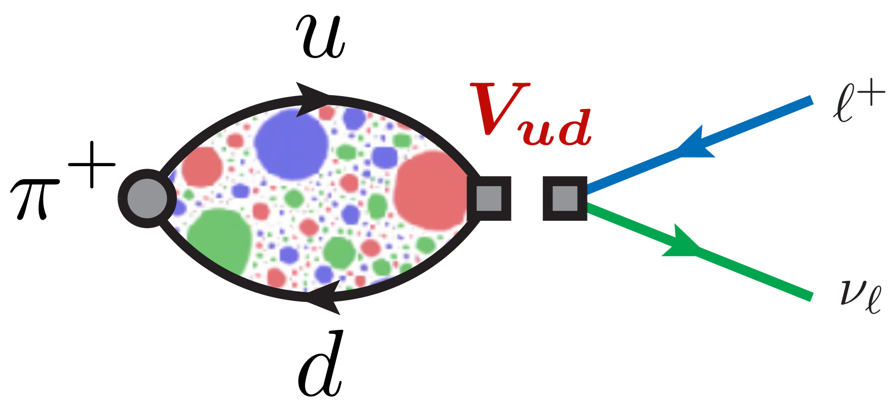
in the Standard Model:

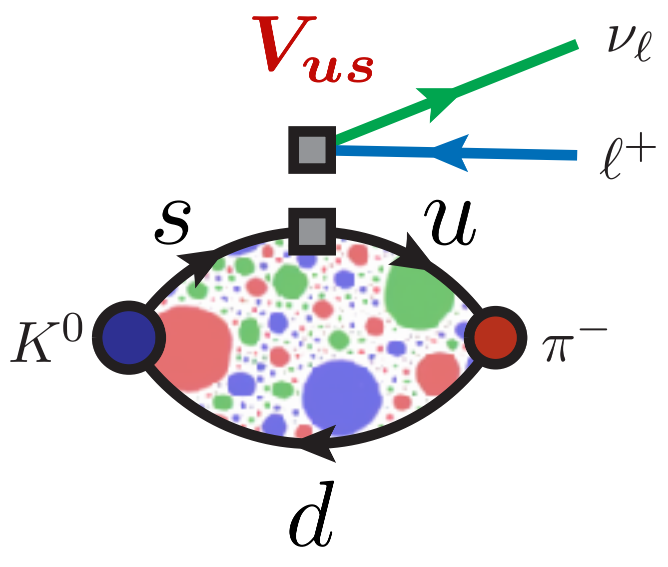
$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

Matrix elements can be extracted e.g. from leptonic and semileptonic decays of mesons

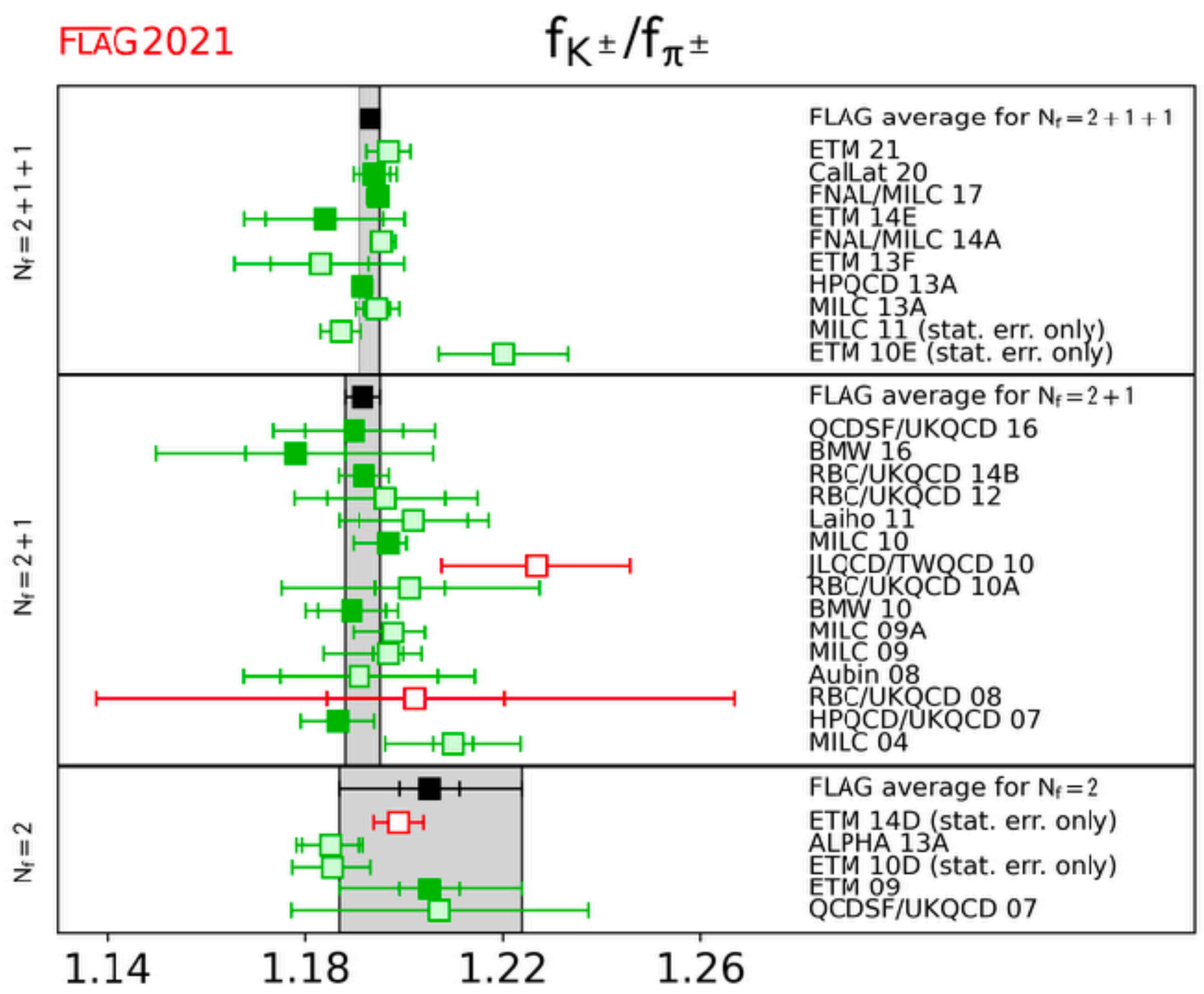
$$\underbrace{\frac{\Gamma\left[K \to \ell\nu_\ell(\gamma)\right]}{\Gamma\left[\pi \to \ell\nu_\ell(\gamma)\right]}}_{\text{experiments}} \propto \left|\frac{V_{us}}{V_{ud}}\right|^2 \underbrace{\left(\frac{f_K}{f_\pi}\right)^2}_{\text{QCD}}$$

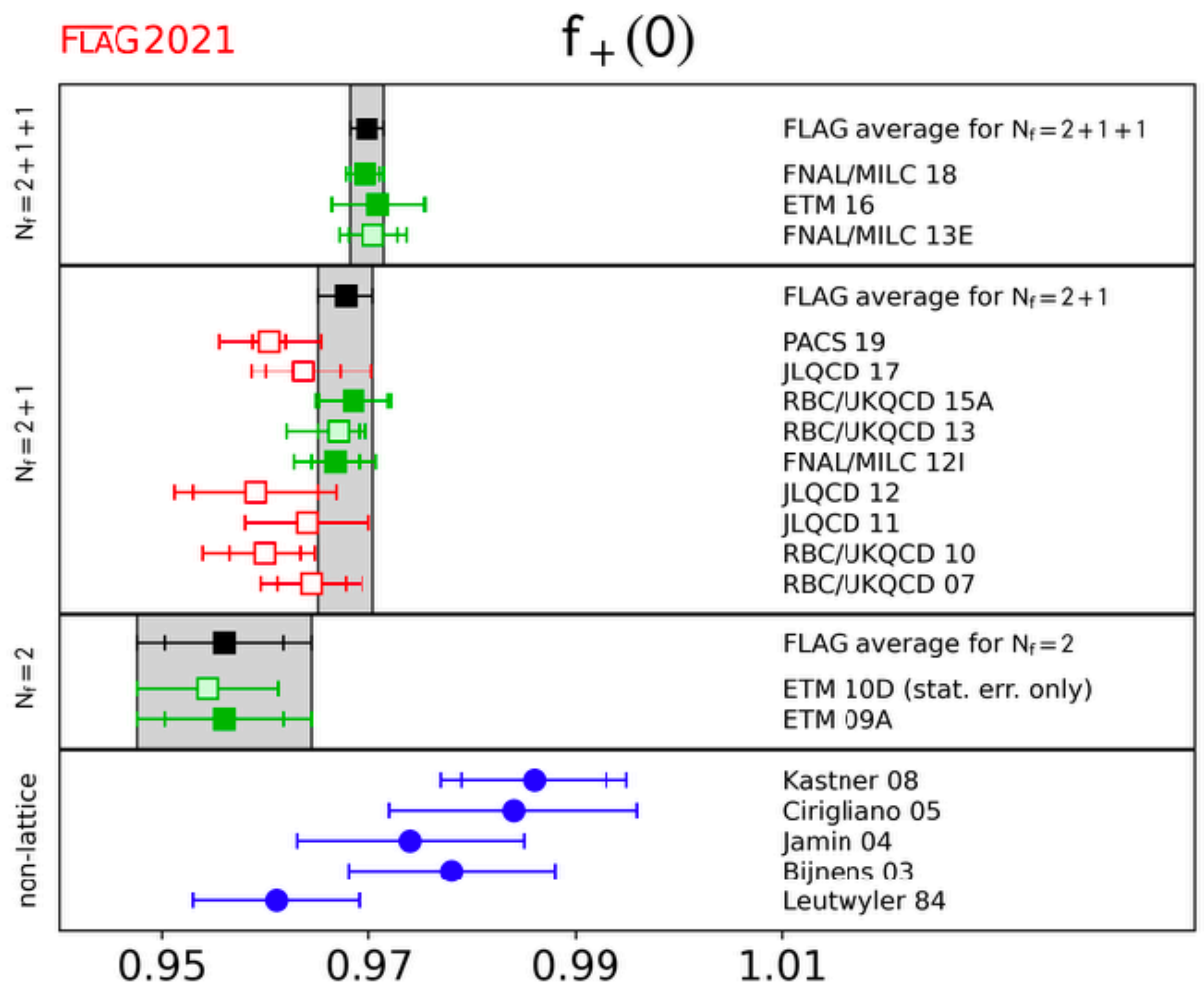
$$\underbrace{\Gamma\left[K \to \pi\ell\nu_\ell(\gamma)\right]}_{\text{experiments}} \propto |V_{us}|^2 \underbrace{|f_+^{K\pi}(0)|^2}_{\text{QCD}}$$





Leptonic and semi-leptonic decays from lattice QCD





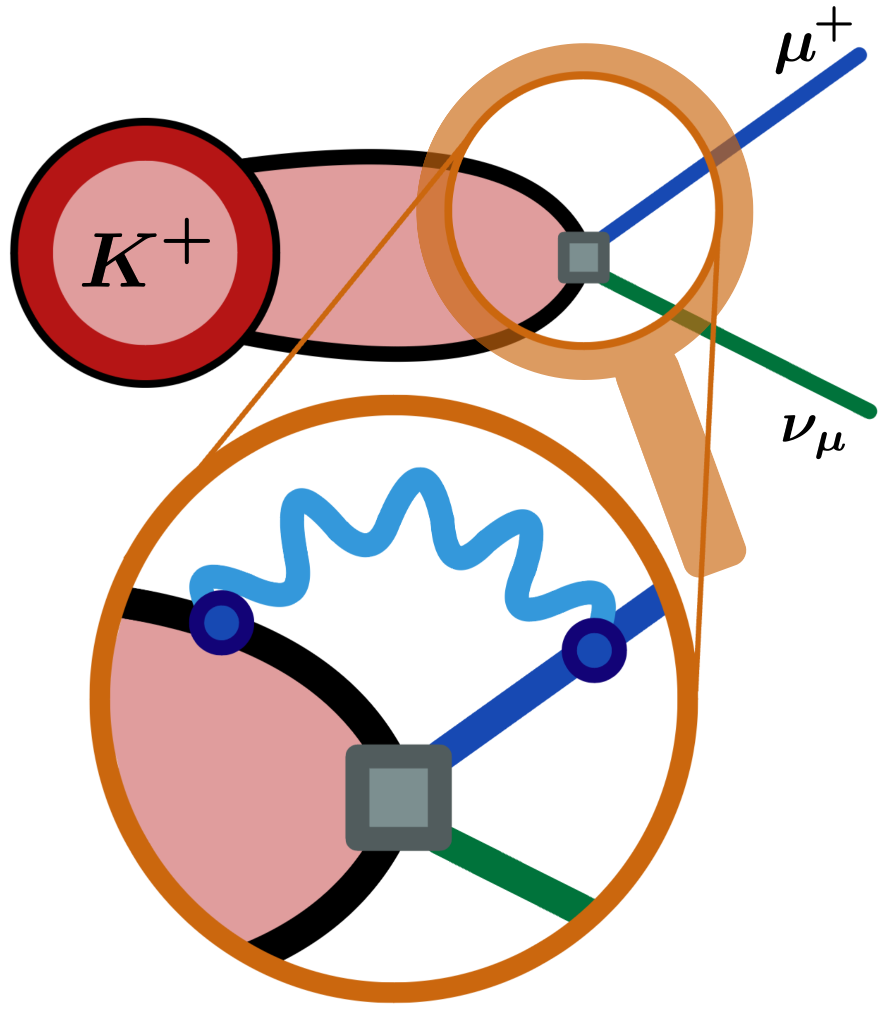
$$f_{K^\pm}/f_{\pi^\pm} = 1.1934\,(19) \qquad f_+^{K\pi}(0) = 0.9698\,(17)$$

FLAG 2021 Flavour Lattice Averaging Group

f_K/f_π and $f_+^{K\pi}(0)$ determined from lattice QCD with **sub percent precision!**

FLAG Review 2021. EPJC **82**, 869 (2022)

QED and isospin-breaking effects



Current level of precision requires the inclusion of **isospin-breaking corrections** due to

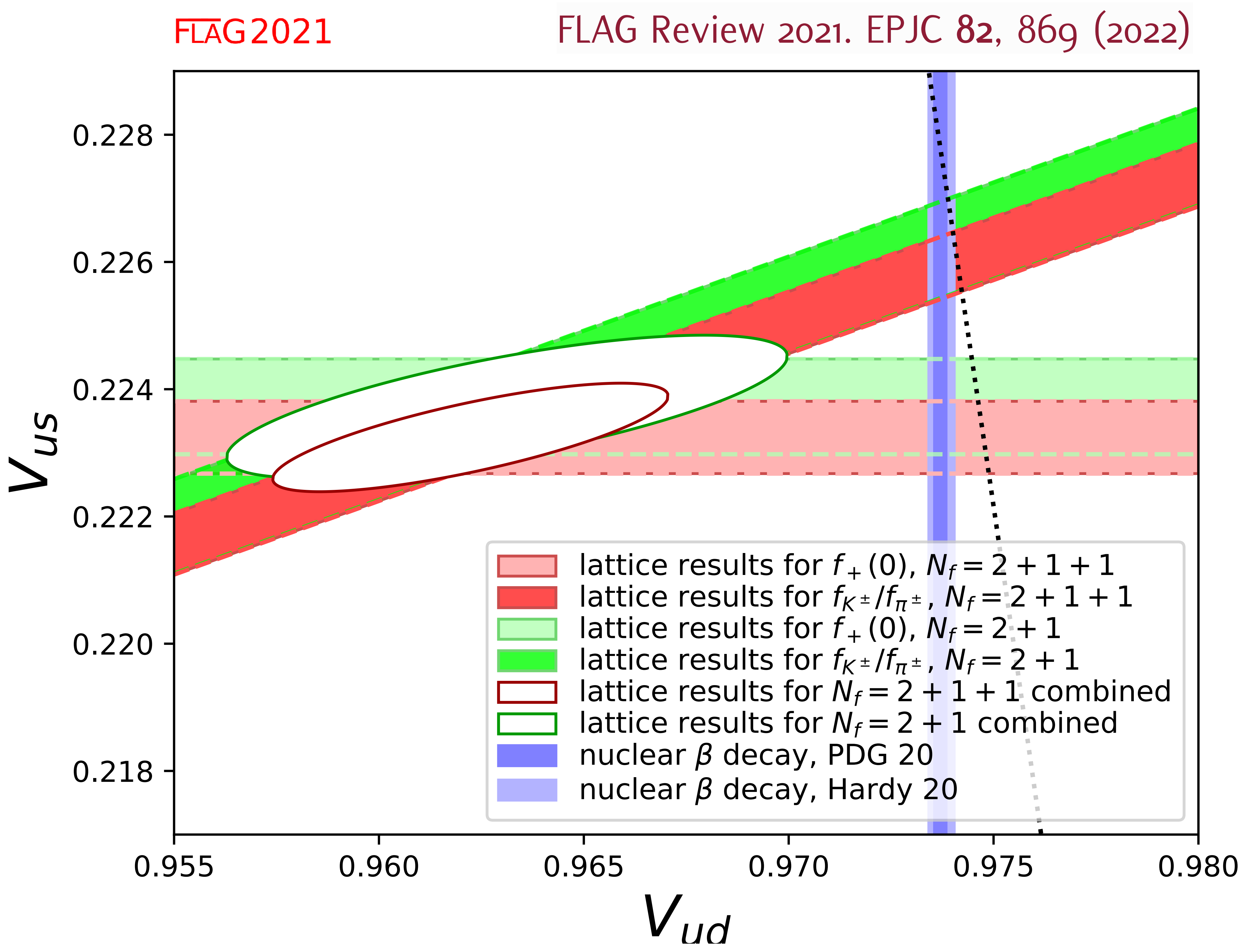
- strong effects $\quad [m_u - m_d]_{\mathrm{QCD}} \neq 0$
- electromagnetic effects $\quad \alpha \neq 0$

$\sim \mathcal{O}(1\%)$

$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi}) \qquad \Gamma(K \to \pi\ell\nu_\ell) \propto |V_{us}|^2 \, |f_+^{K\pi}(0)|^2 \, (1 + \delta R^\ell_{K\pi})$$

- results from χPT **currently quoted in the PDG** — V.Cirigliano & H.Neufeld, PLB **700** (2011)
- these are fully **non-perturbative** (structure dependent)
- first-principle **lattice calculations** are possible!

First-row CKM unitarity tests



Different **tensions** in the V_{us}-V_{ud} plane:

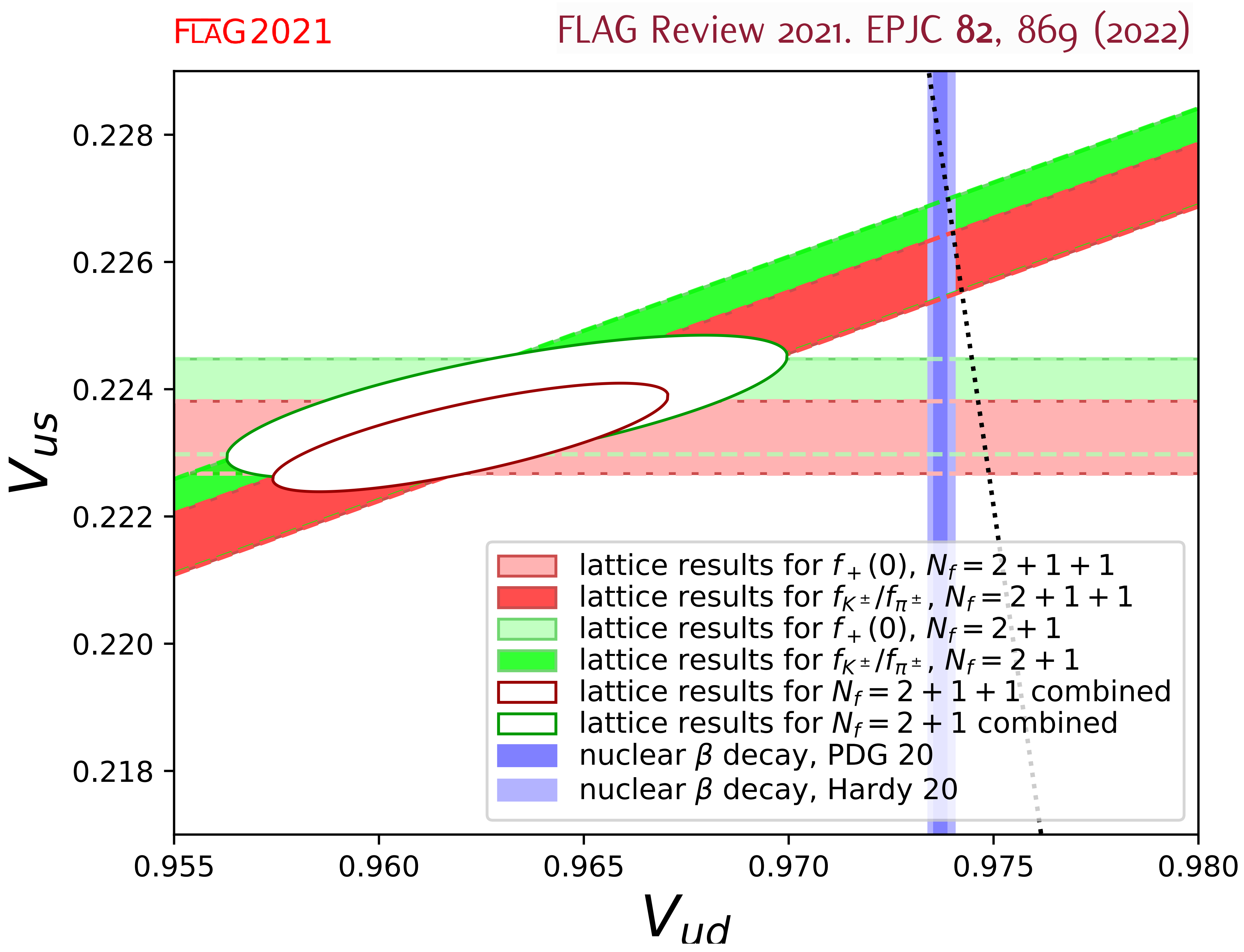
$$|V_u|^2 - 1 = 2.8\sigma$$

$$|V_u|^2 - 1 = 5.6\sigma \qquad |V_u|^2 - 1 = 3.3\sigma$$

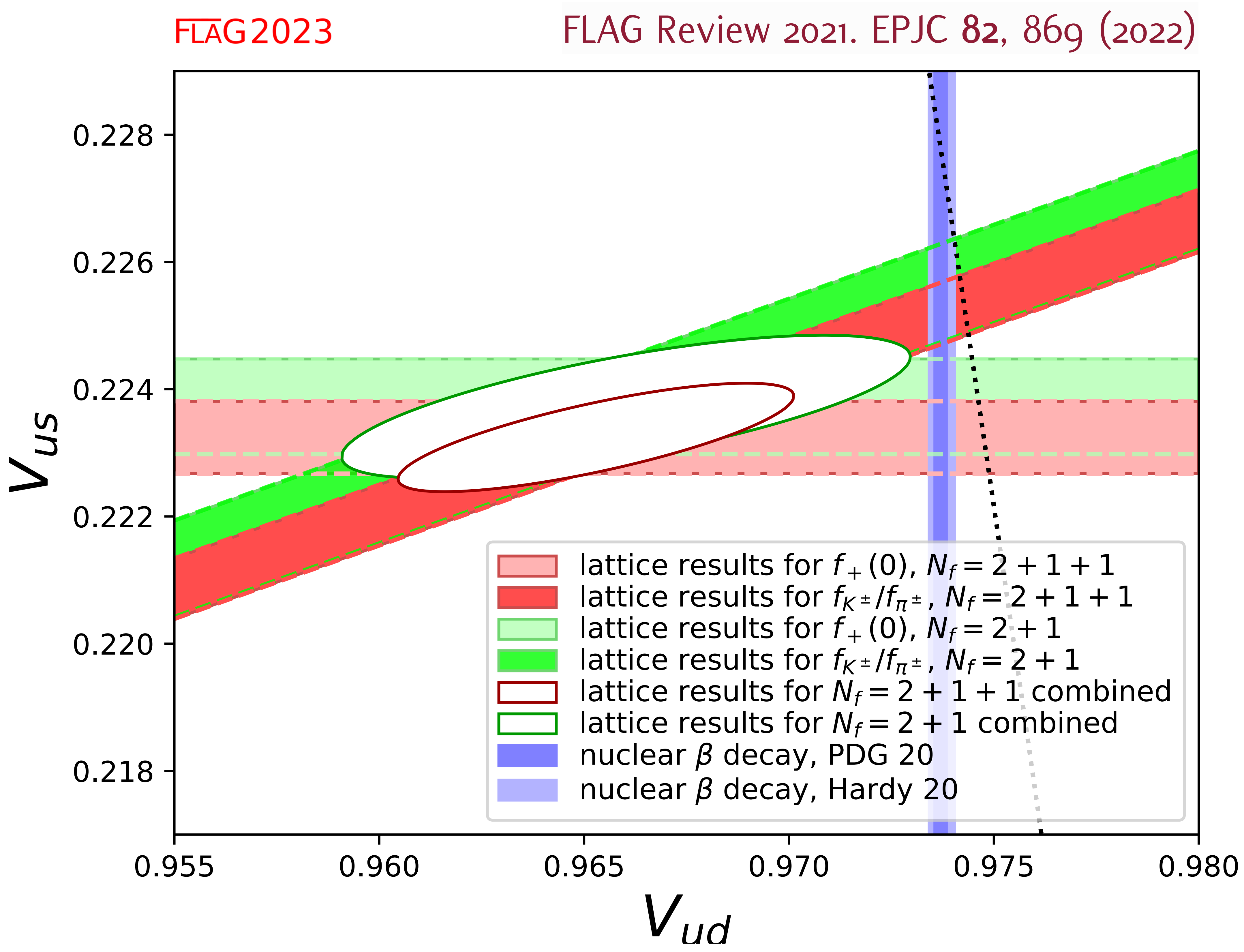
$$|V_u|^2 - 1 = 3.1\sigma \qquad |V_u|^2 - 1 = 1.7\sigma$$

Experimental and **theoretical** control of these quantities is of crucial importance to solve the issue

First-row CKM unitarity tests



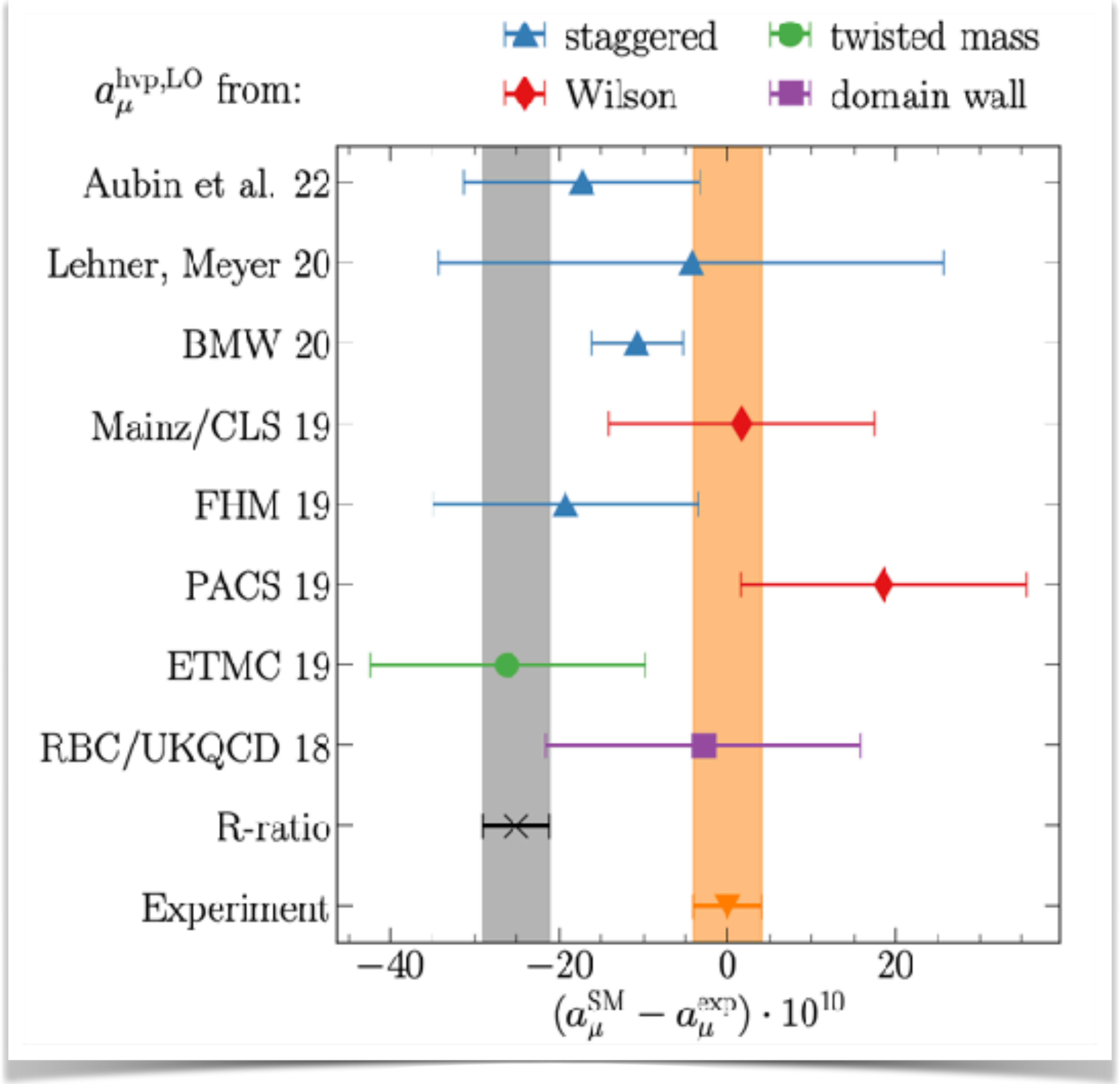
with QED corrections
from lattice calculation



without QED corrections
from lattice calculations

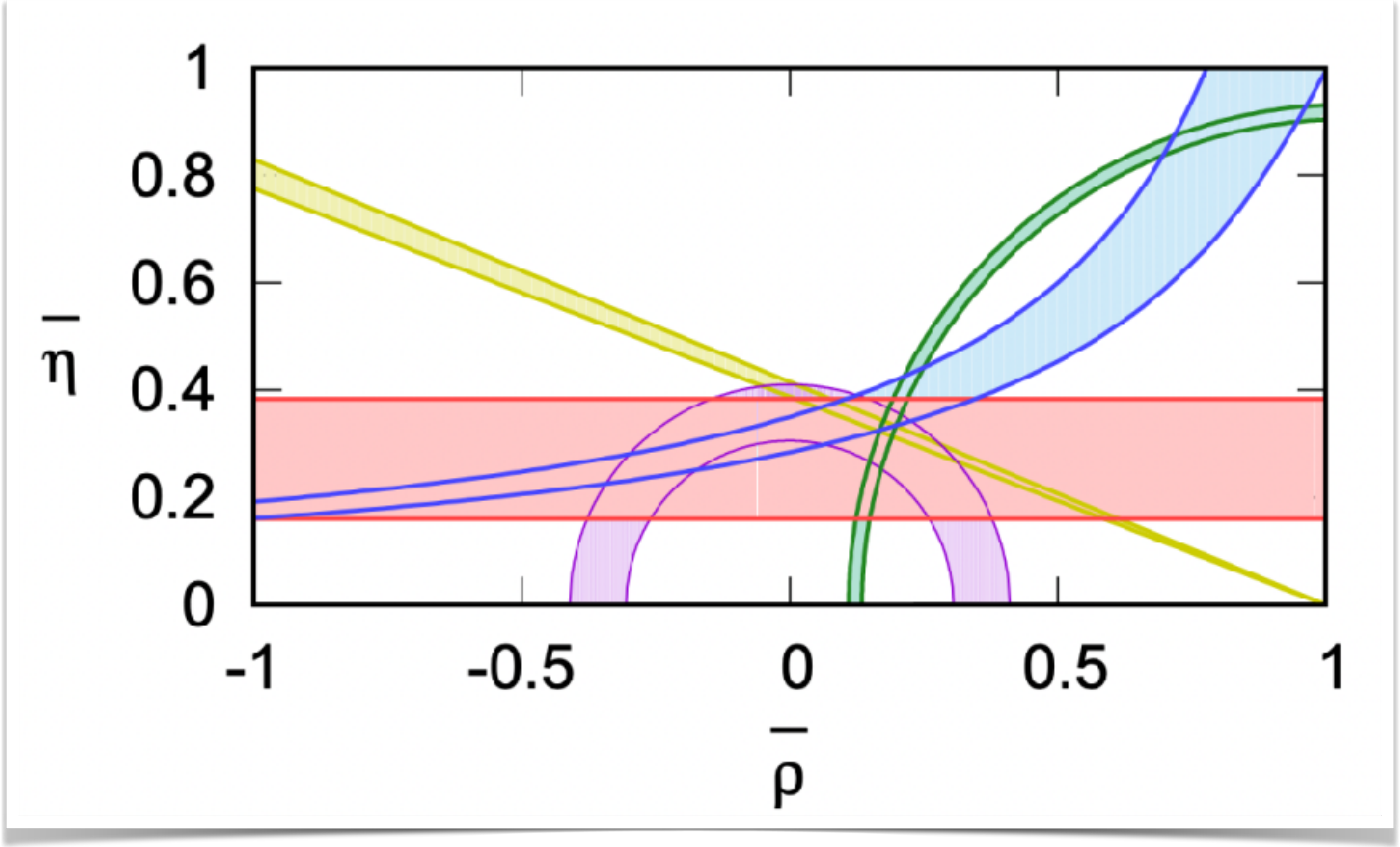
Some other motivations…

from S.Kuberski @Lattice2023



HVP contribution to muon g-2
target precision of O(0.1%)

from R.Abbott et al., PRD **102** (2020)



Study of CP violation in the SM
target precision of O(10%)

Lattice QCD in a [small] nutshell

- **QCD** on a **discrete and finite Euclidean** space-time

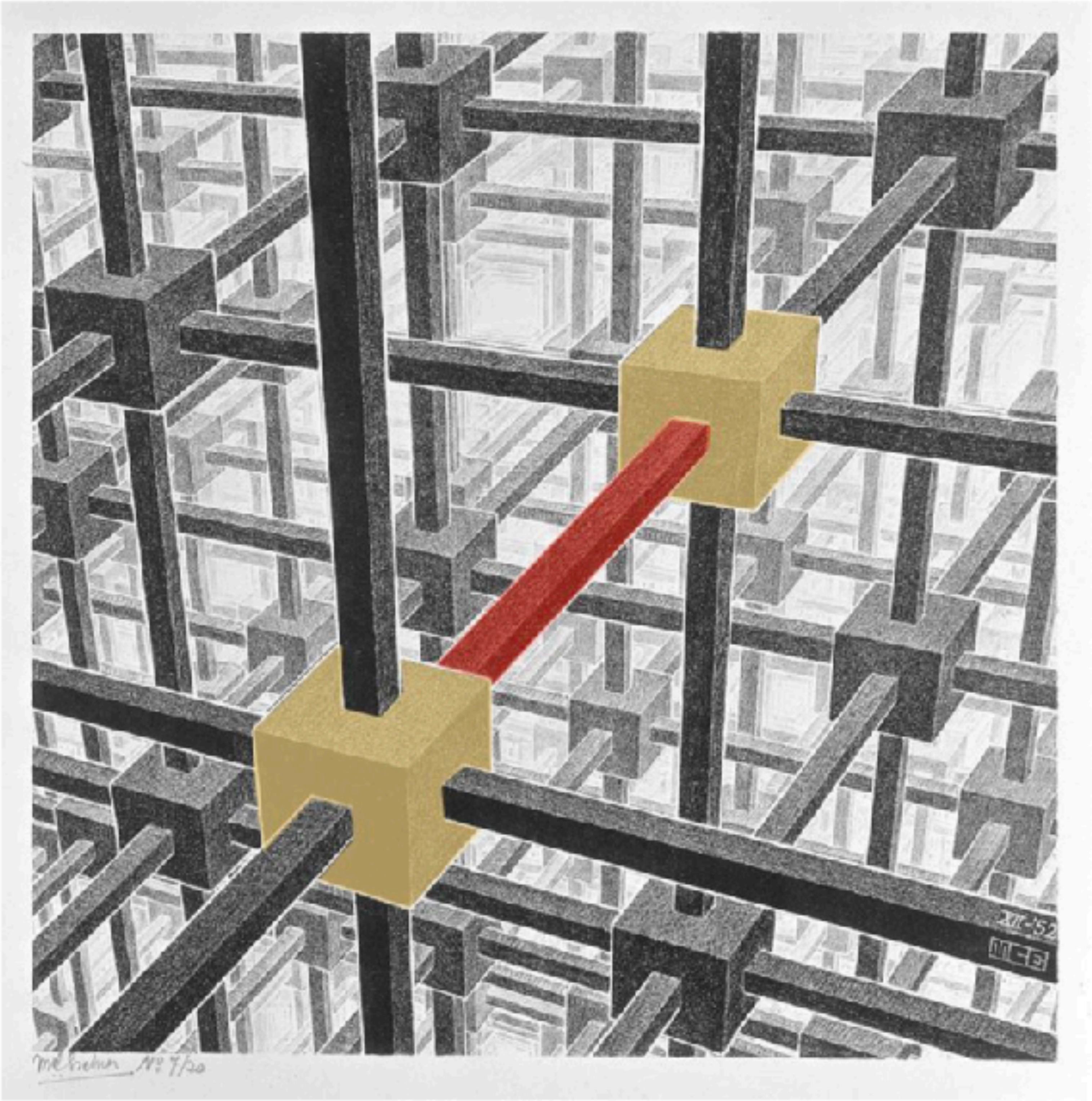
- Based on **Feynman path integrals**

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}[U]\mathcal{D}[\psi, \bar{\psi}]\, \mathcal{O}[\psi, \bar{\psi}, U]\, \mathrm{e}^{-S[\psi, \bar{\psi}, U]}$$

- Path integral solved using **Monte Carlo methods**

- Physical **QCD** results obtained, after **renormalization**, by taking the **continuum & infinite-volume limit**

- Usual setup for lattice simulations: exact **isospin symmetry**, i.e. $m_u = m_d \equiv m_{ud}$ and $\alpha = 0$



M.C. Escher, "Cubic space division" (1953)

2. How

Computing **QED corrections on a finite-sized lattice** is challenging:

- **long-range interactions** don't like **finite volumes** with **periodic boundary conditions**

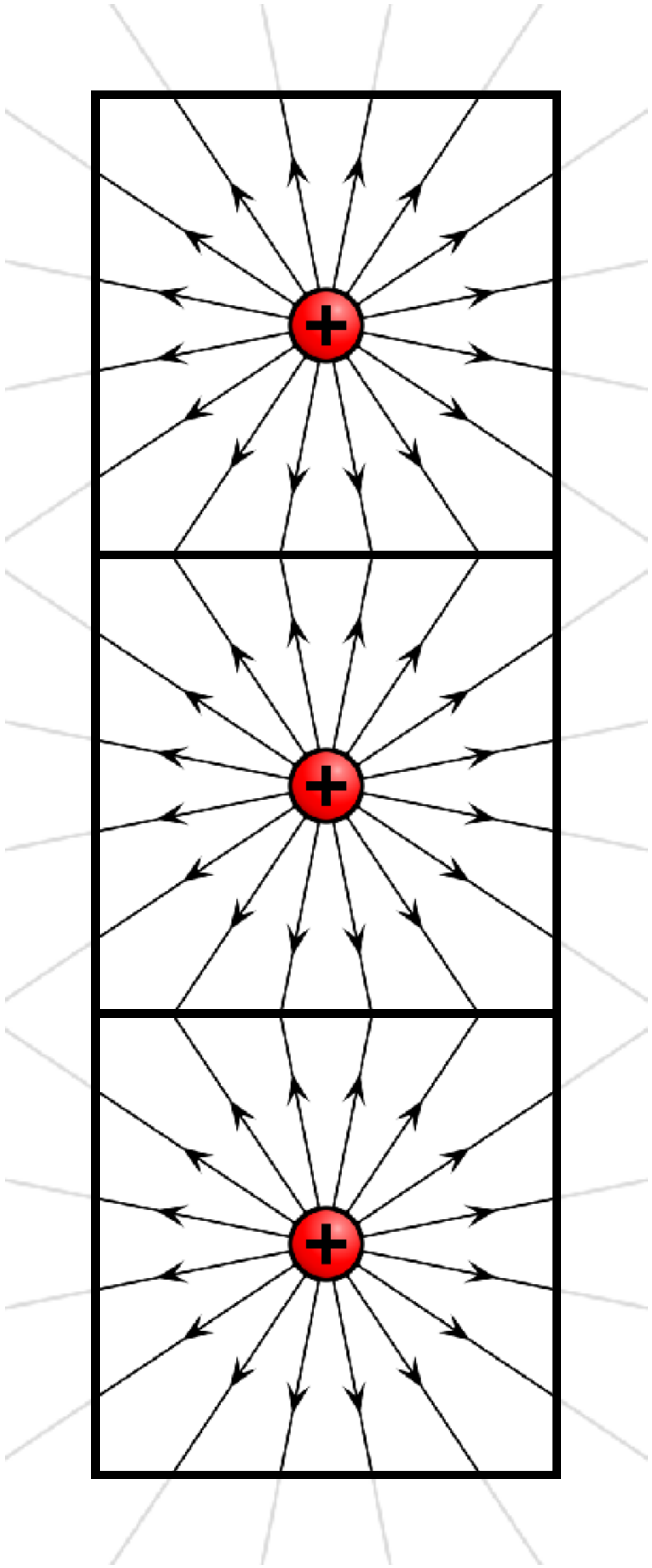
- **finite-volume effects** can be sizeable and power-like

  M.Hayakawa & S.Uno, PTP **120** (2008) / Z.Davoudi & M.Savage, PRD **90** (2014) / S.Borsanyi et al., Science **347** (2015)

- logarithmic **infrared divergences** arise when studying decays

  V.Lubicz et al., PRD **95** (2017)

- QCD and QCD+QED are **different theories** which require separate renormalisation and scale-setting



Charged states in a finite box

Gauss law: only zero net charge is allowed in a finite volume with periodic boundary conditions

$$Q = \int_{\text{p.b.c.}} \mathrm{d}^3\mathbf{x}\; j_0(t,\mathbf{x}) = \int_{\text{p.b.c.}} \mathrm{d}^3\mathbf{x}\; \boldsymbol{\nabla}\cdot\boldsymbol{E}(t,\mathbf{x}) = 0$$

Charged states in a finite box

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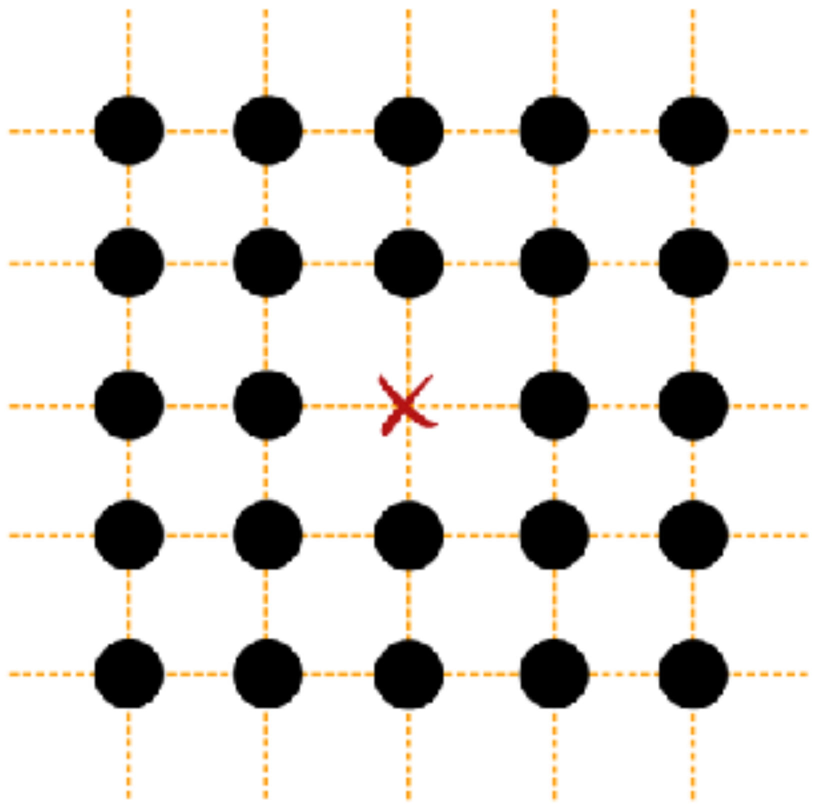
$$Q = \int_{\text{p.b.c.}} \mathrm{d}^3\mathbf{x}\; j_0(t,\mathbf{x}) = \int_{\text{p.b.c.}} \mathrm{d}^3\mathbf{x}\; \boldsymbol{\nabla}\cdot\boldsymbol{E}(t,\mathbf{x}) = 0$$

Possible solutions:

QED$_L$

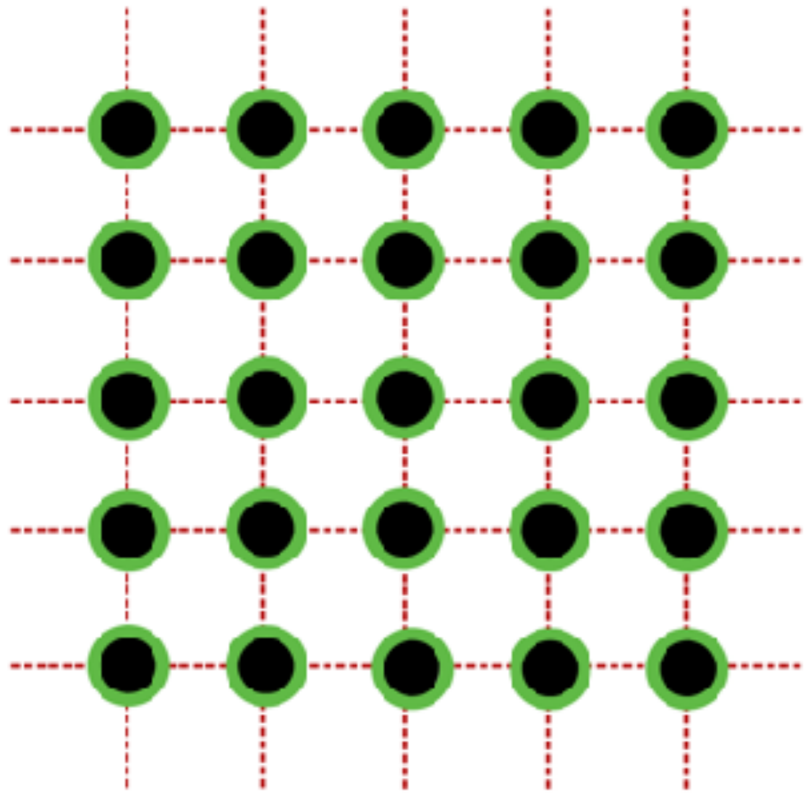


$\Omega_3 = 2\pi\mathbb{Z}^3/L$

remove spatial zero-mode of the photon field

M.Hayakawa & S.Uno, PTP **120** (2008)

QED$_m$

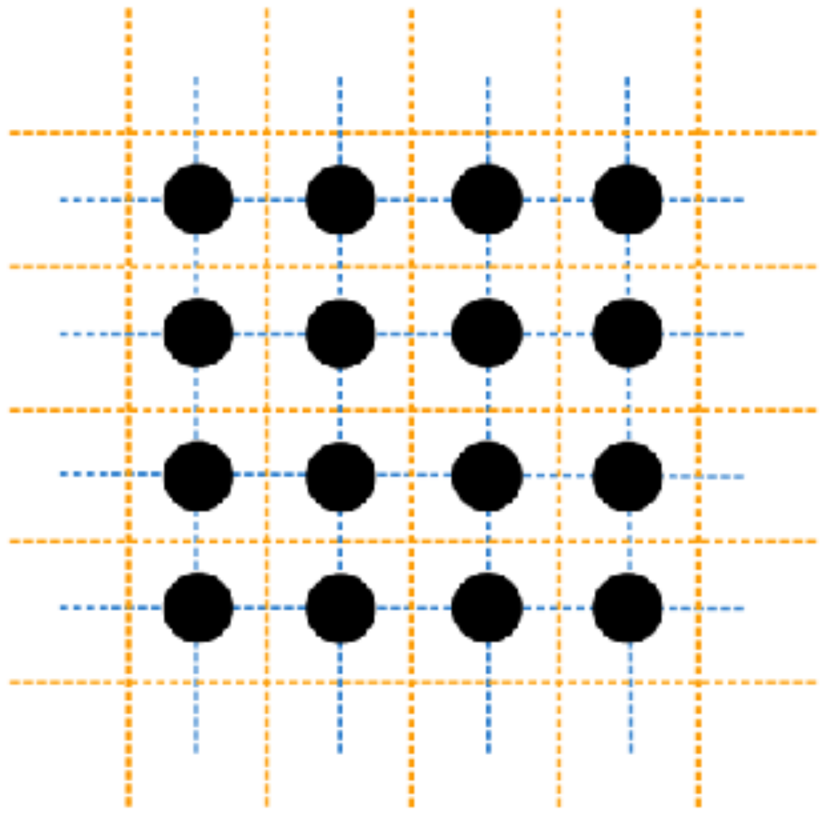


$\Omega_4 = 2\pi\{\mathbb{Z}^3/L, \mathbb{Z}/T\}$

use massive photon m_γ

M.G.Endres et al., [1507.08916]

QED$_{C^*}$



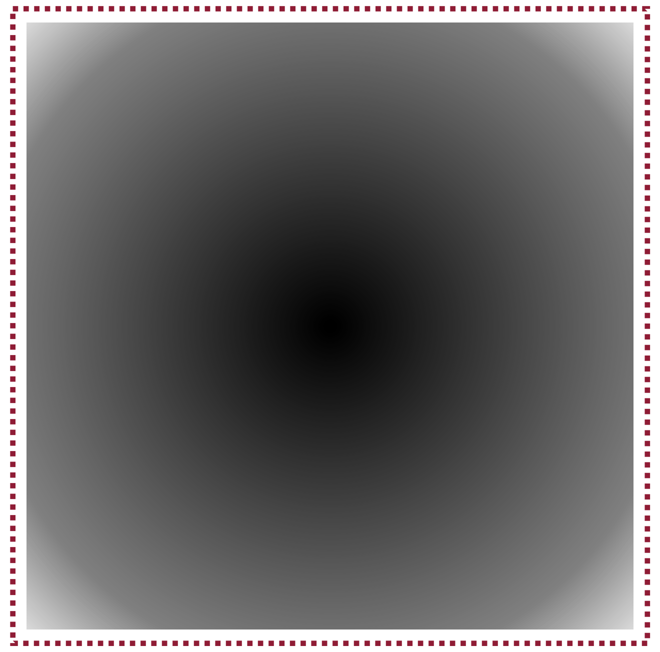
$\Omega_3 = 2\pi\mathbb{Z}^3/L$ $\quad \Omega_3' = (2\,\mathbb{Z}^3 + \bar{\mathbf{n}})\pi/L$

employ C* boundary conditions

A.S.Kronfeld & U.-J.Wiese, NPB **357** (1991)

B.Lucini et al., JHEP **02** (2016)

QED$_\infty$



$\Omega_4 = \mathbb{R}^4$

infinite-volume reconstruction

X.Feng & L.Jin, PRD **100** (2019)

N.Christ et al., [2304.08026]

Implementing QCD+QED on the lattice

- RM123 perturbative approach

G.M.de Divitiis et al. (RM123), PRD 87 (2013)

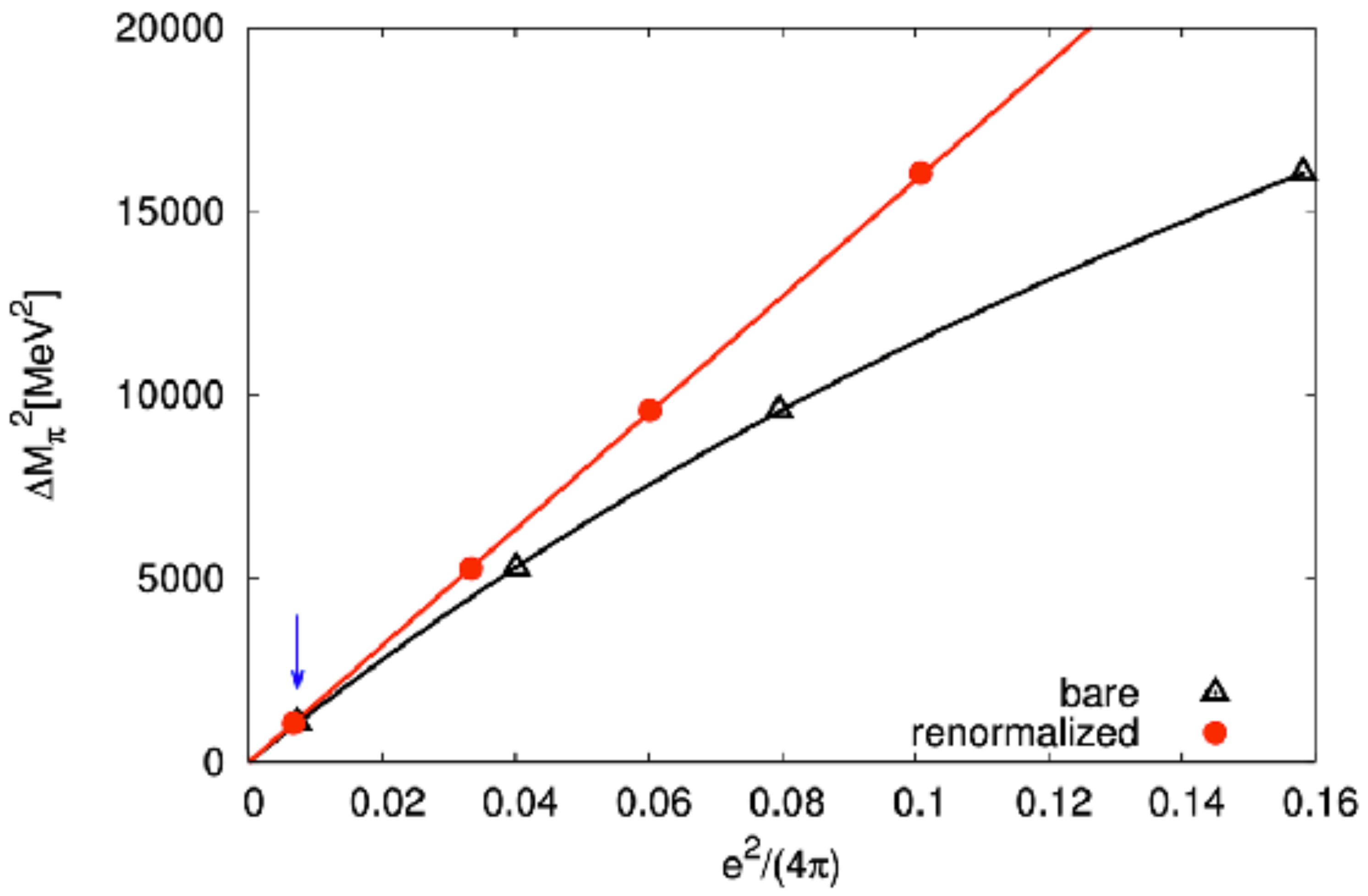
$$\langle \mathcal{O} \rangle = \int \mathcal{D}\Phi \, \mathcal{O} \, e^{-S_{\text{iso}} - \Delta S} = \langle \mathcal{O} \rangle_{\text{iso}} + \langle \Delta S \, \mathcal{O} \rangle_{\text{iso}} + \ldots$$

Pros: only evaluate QCD observables

Cons: need to compute many diagrams:

- Full QCD+QED lattice simulations

S.Borsanyi et al., Science 347 (2015)



Pros: simpler observables:

Cons: need of dedicated gauge configurations

3. What

Lattice QCD+QED calculations can provide IB corrections for several hadronic observables:

- hadron masses & quark masses
- HVP contribution to muon g-2
- leptonic & semileptonic weak decay rates
- CP violation parameters
- ...

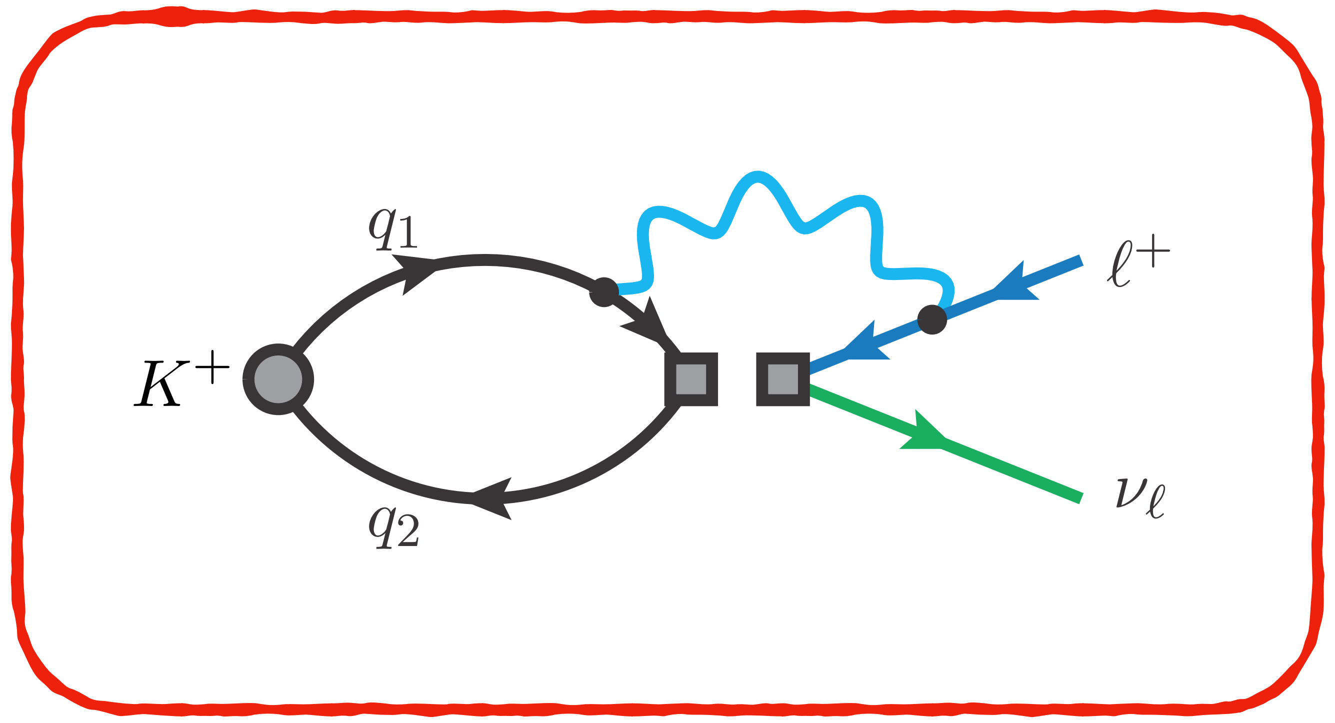
As hadronic uncertainties **decrease**, such corrections become more and more **relevant**!

This is a growing research field: improvements expected in the foreseeable future...

3. What

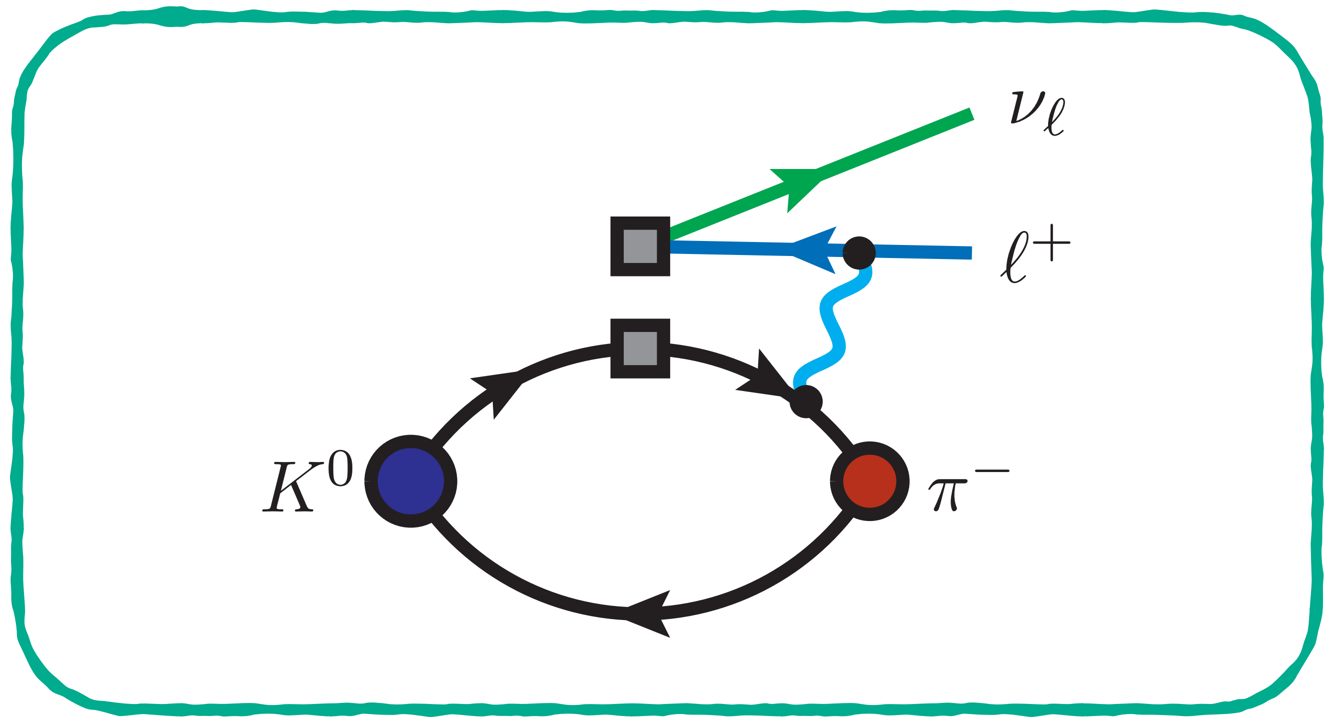
The focus of this talk will be on isospin-breaking corrections to kaon weak decays

- many conceptual and computational **challenges**
- stimulating topic: **different groups** are working on these calculations



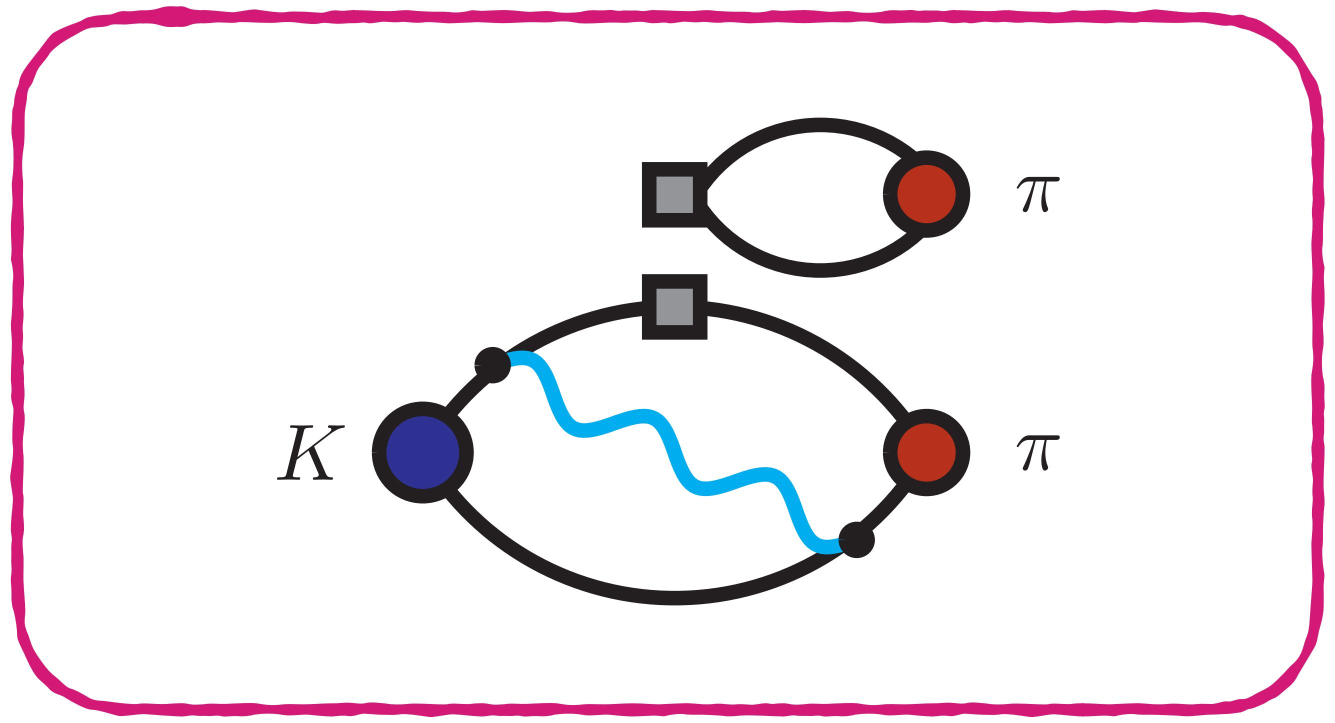
leptonic decays

$K \to \ell\nu_\ell$



semi-leptonic decays

$K \to \pi\ell\nu_\ell$

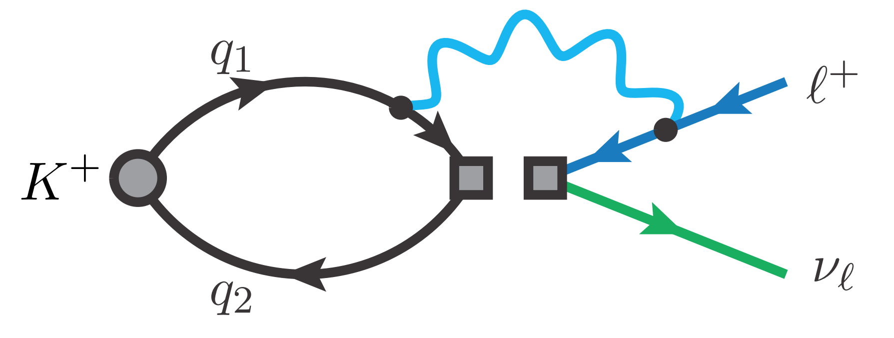


hadronic decays

$K \to \pi\pi$

today — 1–5 years — 5+ years

Leptonic kaon decays



Goal: precision determination of $|V_{us}|$ & test of first-row unitarity

Relevance: sub-percent precision on f_K requires inclusion of IB effects

Status:

- solid theoretical framework
- two lattice QCD+QED calculations for $\Gamma(K_{\mu 2})/\Gamma(\pi_{\mu 2})$:
  1. RM123 + Southampton collaboration (QED_L) D.Giusti et al., PRL **120** (2018) / MDC et al., PRD **100** (2019)
  2. RBC-UKQCD collaboration (QED_L) P.Boyle, MDC et al., JHEP **02** (2023)
- ongoing calculation based on recent alternative QED_∞ method N.Christ et al., [2304.08026]

RM123S

1904.08731

- $\Gamma(K_{\mu 2})$ and $\Gamma(\pi_{\mu 2})$ separately
- Twisted Mass fermions
- multiple volumes and 3 lattice spacings
- unphysical pion masses ($\gtrsim$ 230 MeV)
- RM123 method + QED$_L$

PHYSICAL REVIEW D **100**, 034514 (2019)

Editors' Suggestion

Light-meson leptonic decay rates in lattice QCD + QED

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Isospin-breaking corrections to light-meson leptonic decays from lattice simulations at physical quark masses

Peter Boyle,[a,b] Matteo Di Carlo,[b] Felix Erben,[b] Vera Gülpers,[b] Maxwell T. Hansen,[b] Tim Harris,[b] Nils Hermansson-Truedsson,[c,d] Raoul Hodgson,[b] Andreas Jüttner,[e,f] Fionn Ó hÓgáin,[b] Antonin Portelli,[b] James Richings[b,e,g] and Andrew Zhen Ning Yong[b]

RBC-UKQCD

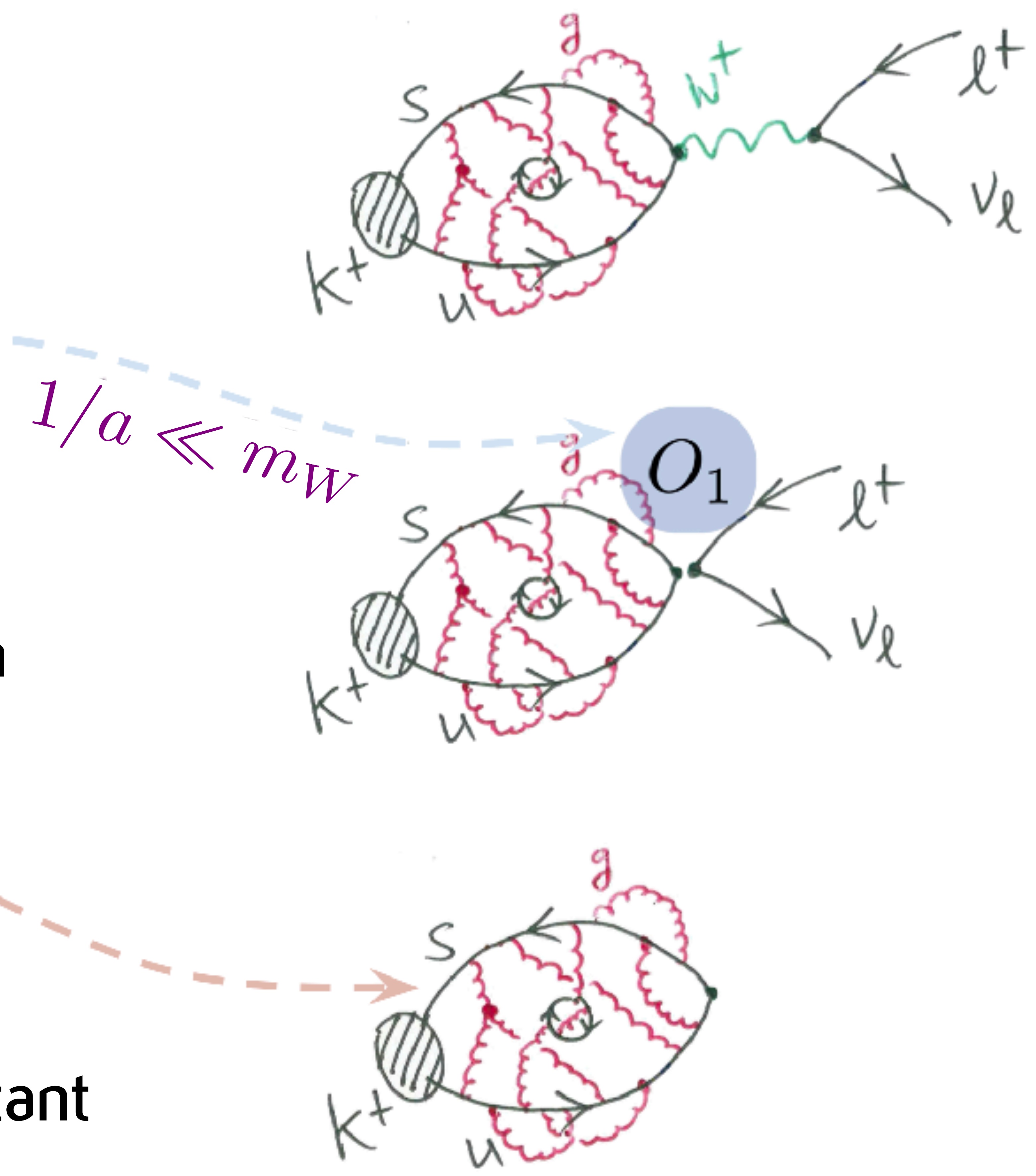
2211.12865

- ratio $\Gamma(K_{\mu 2})$ / $\Gamma(\pi_{\mu 2})$
- Domain Wall fermions
- single volume and lattice spacing
- physical quark masses
- RM123 method + QED$_L$

Leptonic decay rate

Can be studied in an **effective Fermi theory** with the W-boson integrated out and the local interaction described by

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left[\bar{q}_2\, \gamma_\mu (1-\gamma_5)\, q_1\right] \left[\bar{\nu}_\ell\, \gamma^\mu (1-\gamma_5)\, \ell\right]$$



In the **PDG convention**, the tree-level decay rate takes the form

$$\Gamma_P^{\text{tree}} = \frac{G_F^2}{8\pi} m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2 m_P \left[f_{P,0}\right]^2$$

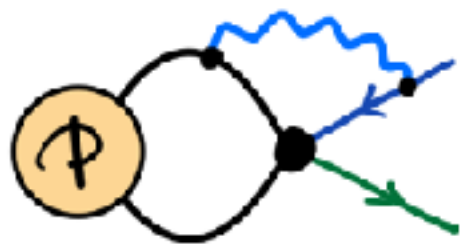
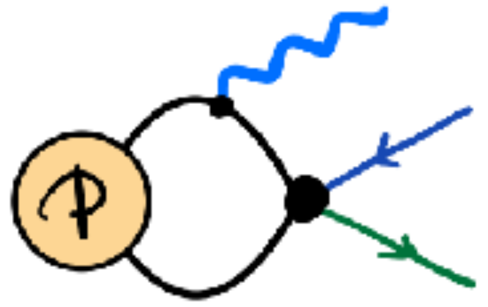
with the non-perturbative dynamic encoded in the **decay constant**

$$\mathcal{Z}_0 \langle 0 | \bar{q}_2\, \gamma_0 \gamma_5\, q_1 | P, \mathbf{0} \rangle^{(0)} = i\, m_{P,0}\, f_{P,0}$$

Leptonic decay rate at $\mathcal{O}(\alpha)$

When including radiative corrections many subtleties arise, for example:

- **IR divergences** appear in intermediate steps of the calculation   F. Bloch & A. Nordsieck, PR 52 (1937) 54

$$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\mathrm{IR}} \to 0} \Big\{$$

$$+$$

$$\Big\}$$

IR finite   IR divergent   IR divergent

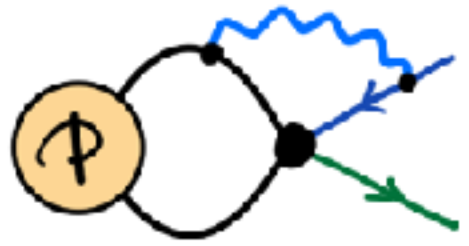
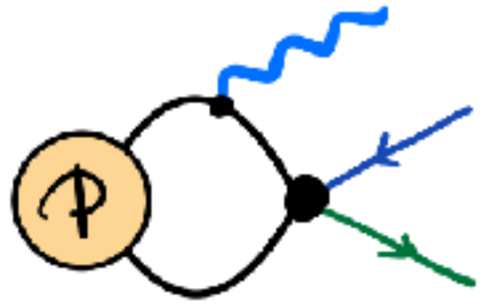
- new **UV divergences**: include QED corrections to the renormalization of the weak Hamiltonian

- the **decay constant** $f_{P,0}$ becomes an ambiguous and unphysical quantity: one needs to introduce a **scheme** to give a meaning to "QCD" or "iso-QCD"

Leptonic decay rate at $\mathcal{O}(\alpha)$

F. Bloch & A. Nordsieck, PR **52** (1937)

The RM123+Soton recipe

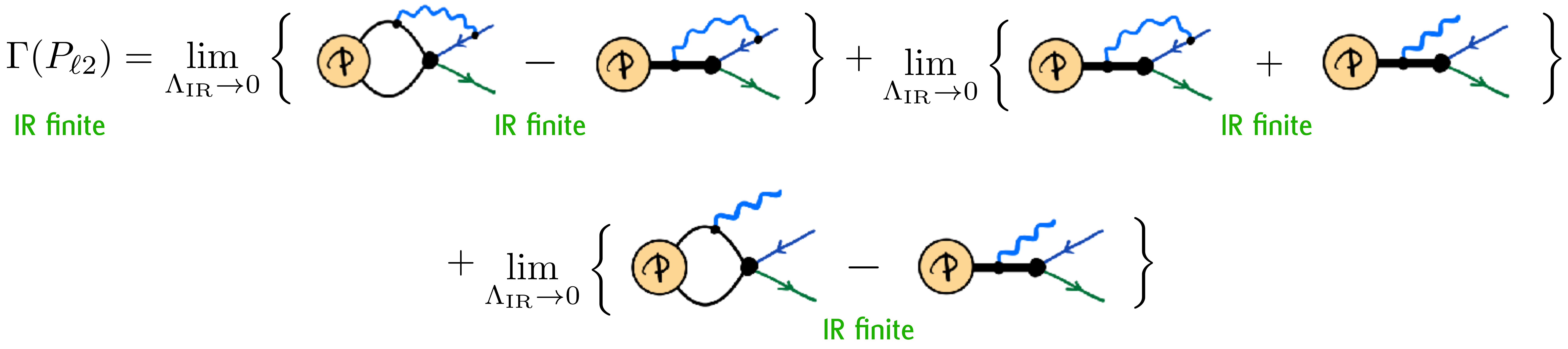
$$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\mathrm{IR}} \to 0} \Big\{$$  $$+$$  $$\Big\}$$

IR finite — IR divergent — IR divergent

Leptonic decay rate at $\mathcal{O}(\alpha)$

F. Bloch & A. Nordsieck, PR **52** (1937)
N. Carrasco et al., PRD **91** (2015)
V. Lubicz et al., PRD **95** (2017)
D. Giusti et al., PRL **120** (2018)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

The RM123+Soton recipe



Leptonic decay rate at $\mathcal{O}(\alpha)$

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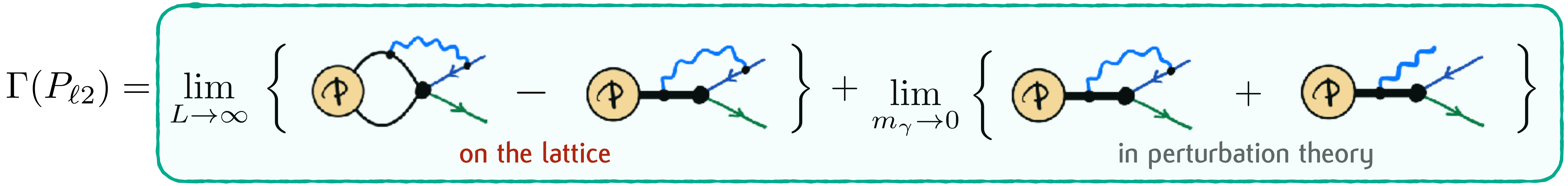
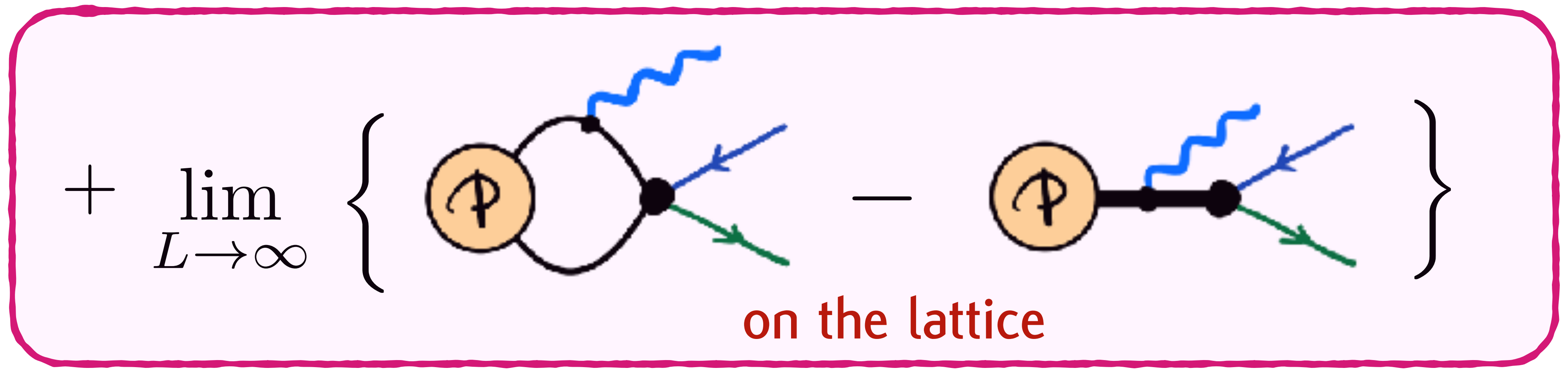
The RM123+Soton recipe





enough for $K_{\mu 2}$ and $\pi_{\mu 2}$

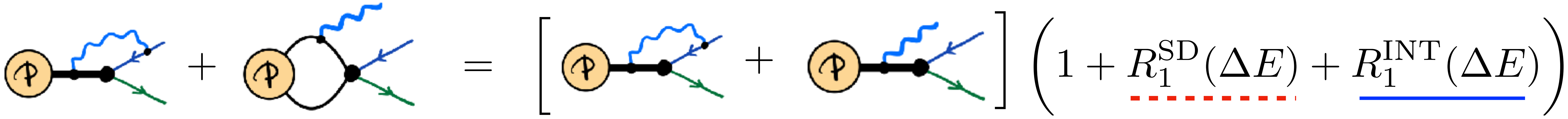
finite-volume scaling well studied

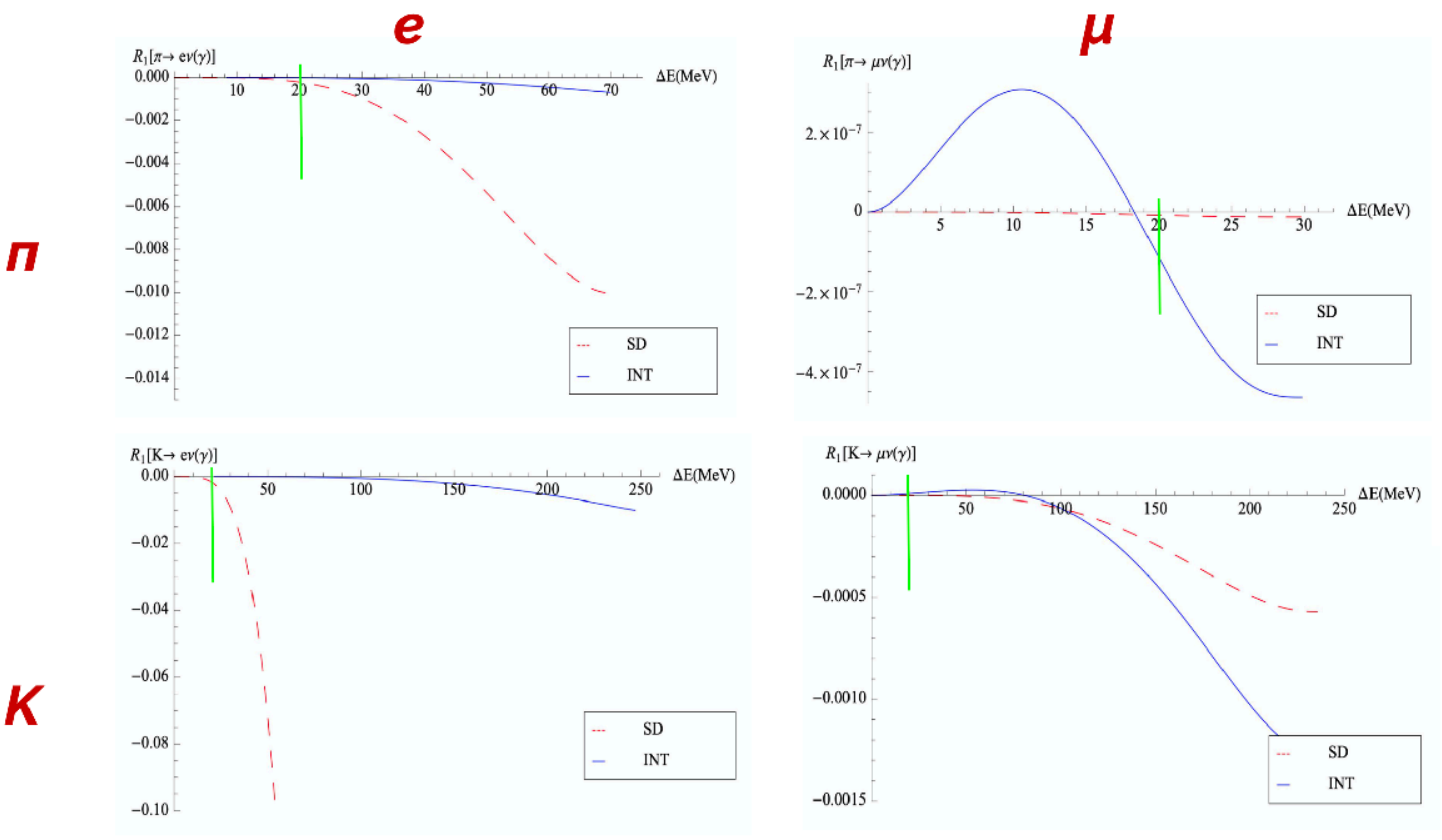
V.Lubicz et al., PRD **95** (2017) N.Tantalo et al., [1612.00199v2]
MDC et al., PRD **105** (2022)

relevant for K_{e2} and π_{e2}
& decays of heavier mesons

G.M. de Divitiis et al., [1908.10160]
R. Frezzotti et al., PRD **103** (2021)
A. Desiderio et al., PRD **102** (2021)
C. Kane et al., [1907.00279 & 2110.13196]
D. Giusti et al., [2302.01298]
R.Frezzotti et al., [2306.05904]

Real photon emission and structure dependence

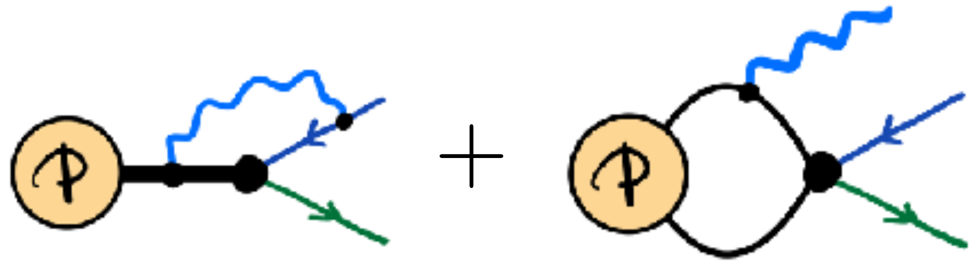




Calculation at O(p4) in χPT

N. Carrasco et al., PRD **91** (2015)

Real photon emission and structure dependence



$$= \delta R_{\rm pt}(\Delta E) + \delta R_1^{\rm SD}(\Delta E) + \delta R_1^{\rm INT}(\Delta E)$$

	$\pi_{e2[\gamma]}$	$\pi_{\mu2[\gamma]}$	$K_{e2[\gamma]}$	$K_{\mu2[\gamma]}$
δR_0	$^{(*)}$	0.0411 (19)	$^{(*)}$	0.0341 (10)
$\delta R_{\rm pt}(\Delta E_\gamma^{max})$	-0.0651	-0.0258	-0.0695	-0.0317
$\delta R_1^{\rm SD}(\Delta E_\gamma^{max})$	$5.4\ (1.0) \times 10^{-4}$	$2.6\ (5) \times 10^{-10}$	1.19 (14)	$2.2\ (3) \times 10^{-5}$
$\delta R_1^{\rm INT}(\Delta E_\gamma^{max})$	$-4.1\ (1.0) \times 10^{-5}$	$-1.3\ (1.5) \times 10^{-8}$	$-9.2\ (1.3) \times 10^{-4}$	$-6.1\ (1.1) \times 10^{-5}$
ΔE_γ^{max} (MeV)	69.8	29.8	246.8	235.5

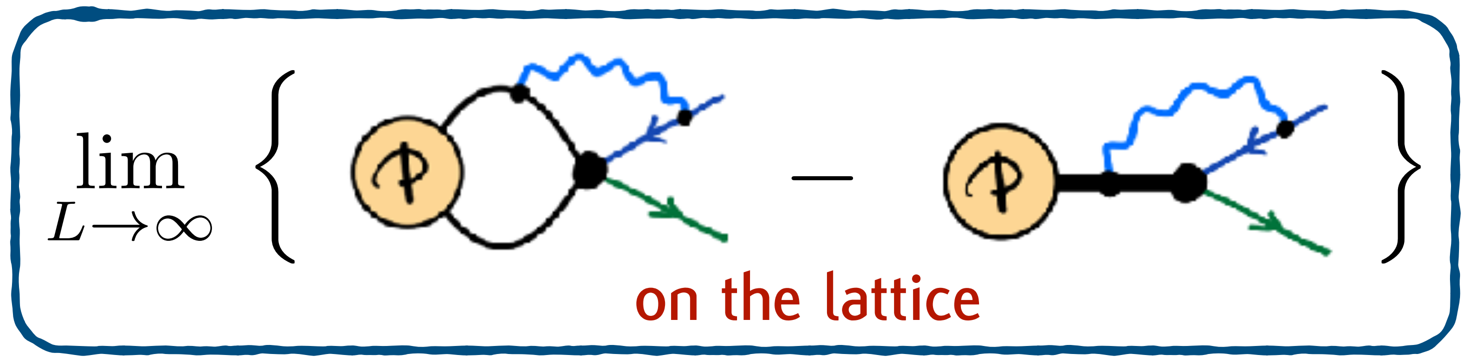
$^{(*)}$ *Not yet evaluated by numerical lattice QCD+QED simulations.*

Confirmed by numerical lattice calculation

A. Desiderio et al., PRD **102** (2021)
R. Frezzotti et al., PRD **103** (2021)

Decay rate at $\mathcal{O}(\alpha)$

Virtual decay rate



$$\Gamma(P_{\ell 2}) = \Gamma_P^{\text{tree}}(1 + \delta R_P) \quad \blacktriangleright \quad \Gamma_P^{\text{tree}} = \frac{G_F^2}{8\pi} m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2 m_P \left[f_{P,0}\right]^2 \quad \blacktriangleright \quad \delta R_P = 2\left(\frac{\delta \mathcal{A}_P}{\mathcal{A}_{P,0}} - \frac{\delta m_P}{m_{P,0}} + \frac{\delta \mathcal{Z}}{\mathcal{Z}_0}\right)$$

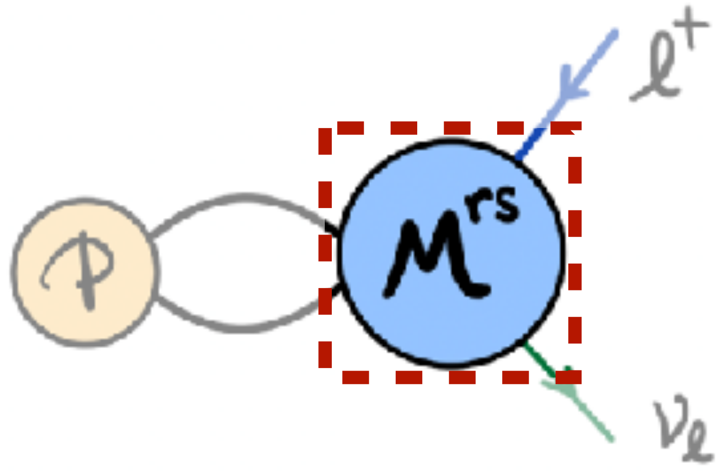
PDG convention

- $\boldsymbol{\delta \mathcal{A}_P}$ correction to the bare matrix element $\mathcal{M}_P^{rs}(\mathbf{p}_\ell) = \langle \ell^+, r, \mathbf{p}_\ell; \nu_\ell, s, \mathbf{p}_\nu | O_W | P^+, \mathbf{0} \rangle$
- $\boldsymbol{\delta m_P}$ correction to the meson mass
- $\boldsymbol{\delta \mathcal{Z}}$ correction to the renormalization of the weak operator
  1. cancels in the ratio $\Gamma(K_{\ell 2})/\Gamma(\pi_{\ell 2})$ (if mass-indep. renormalization scheme is used)
  2. same correction contributes also to semileptonic decays

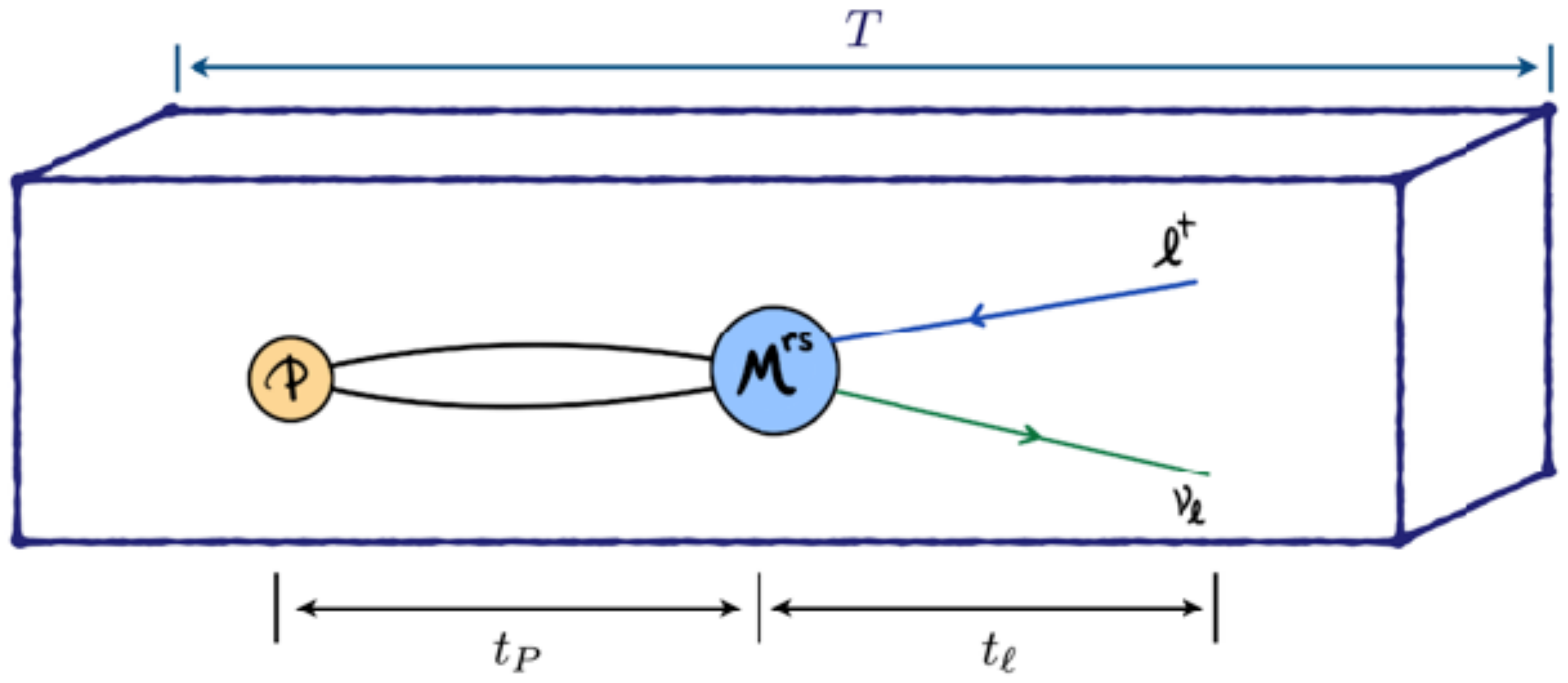
MDC et al., PRD **100** (2019)
MDC et al., [1911.00938]

From Euclidean lattice correlators to matrix elements

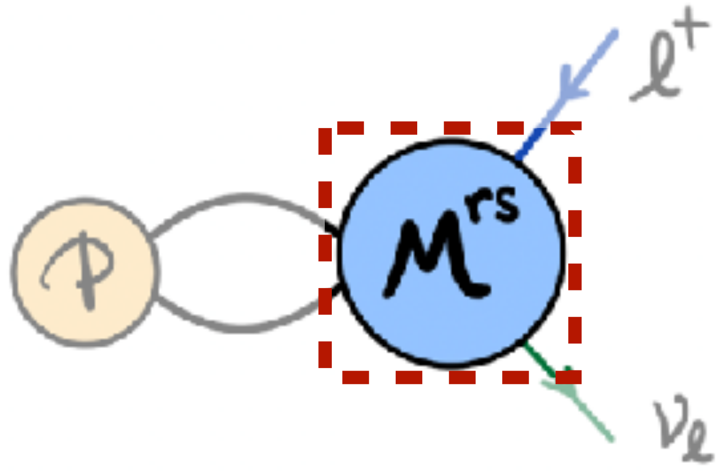
Our goal:

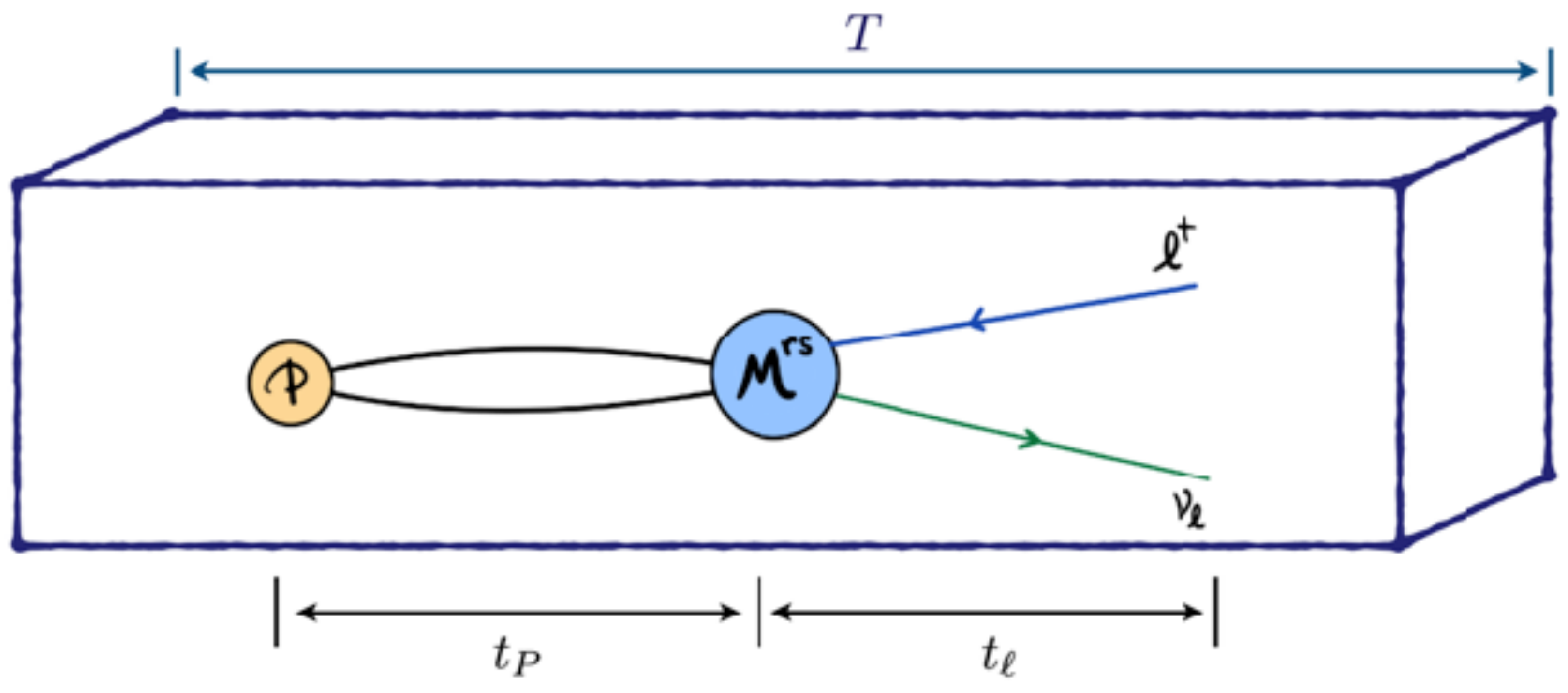


How we realise it:



From Euclidean lattice correlators to matrix elements

Our goal: 

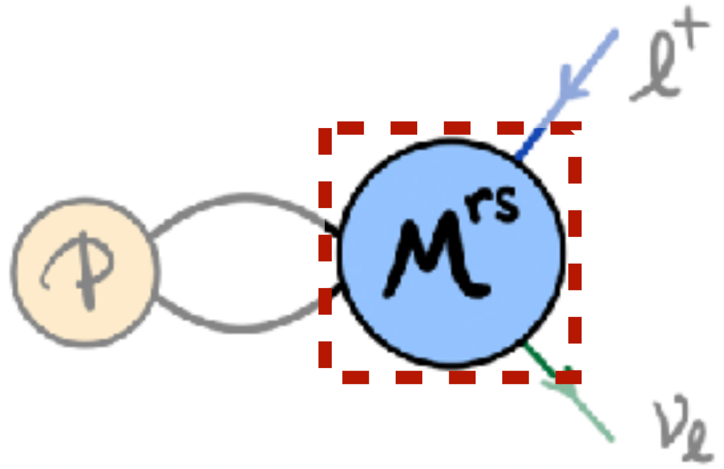
How we realise it: 

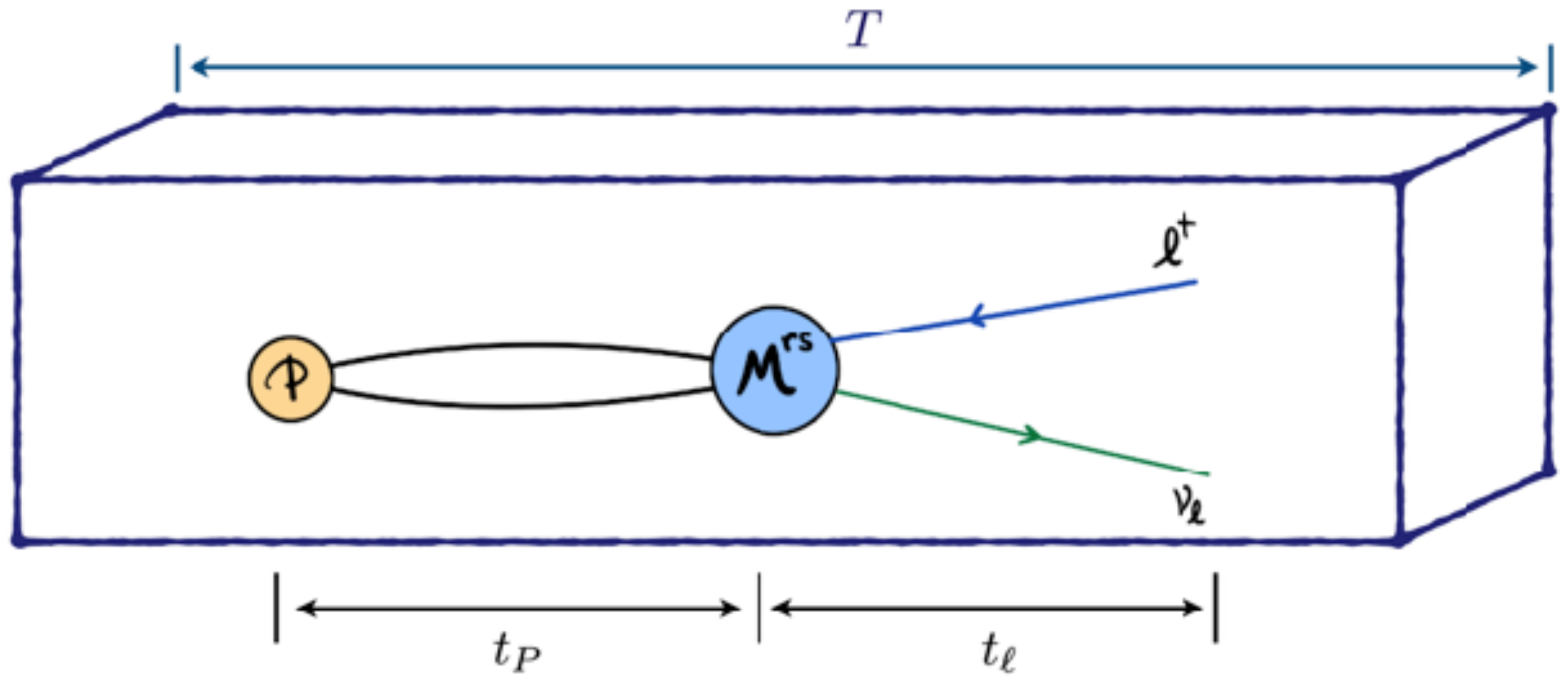
Tree-level decay amplitude: $|\mathcal{M}_0(\mathbf{p}_\ell)|^2 = |\mathcal{A}_{P,0}|^2 |\mathcal{L}_0(\mathbf{p}_\ell)|^2$

$$= \langle 0|A^0(t)\,\phi^\dagger(0)|0\rangle \longrightarrow \frac{Z_{P,0}\mathcal{A}_{P,0}}{2m_{P,0}}\; \mathrm{e}^{-m_{P,0}t}$$

$$= \langle 0|\phi(t)\,\phi^\dagger(0)|0\rangle \longrightarrow \frac{Z_{P,0}^2}{2m_{P,0}}\; \mathrm{e}^{-m_{P,0}t}$$

From Euclidean lattice correlators to matrix elements

Our goal: 

How we realise it: 

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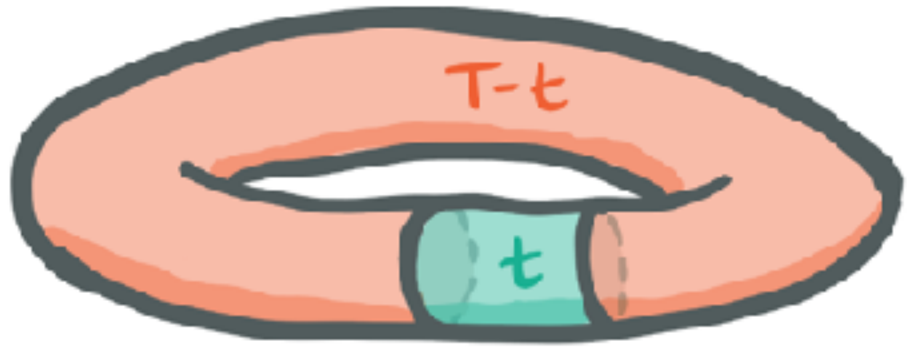
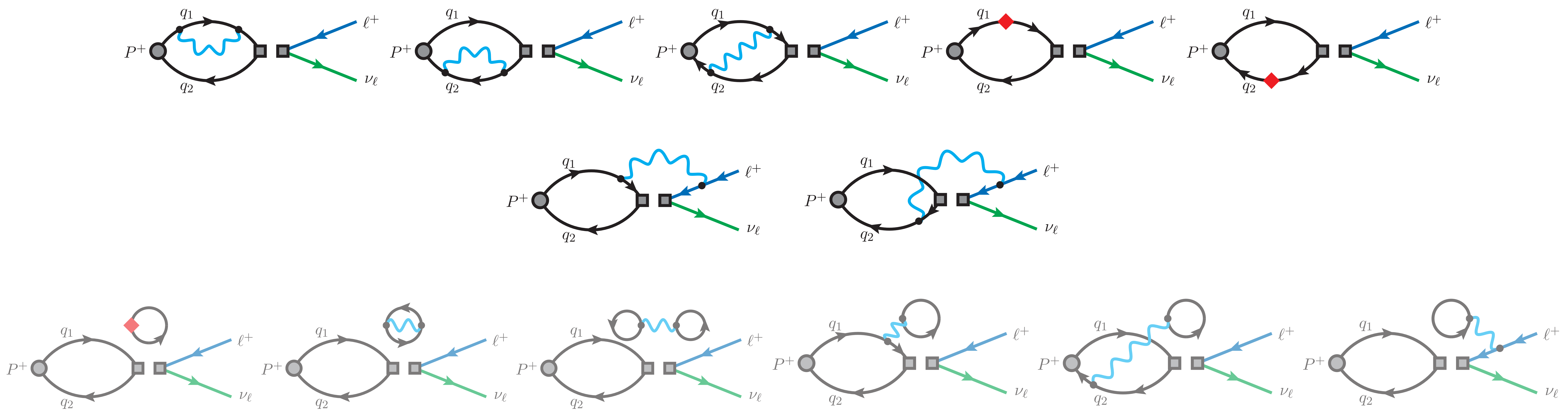


IB corrections to the decay amplitude

G.M.de Divitiis et al. [RM123], PRD 87 (2013)

RM123 perturbative method: expand lattice path-integral around isosymmetric point $\alpha = m_u - m_d = 0$



Both RM123S and RBC-UKQCD calculations are performed in the **electro-quenched approximation: sea quarks electrically neutral**

IB corrections to the decay amplitude

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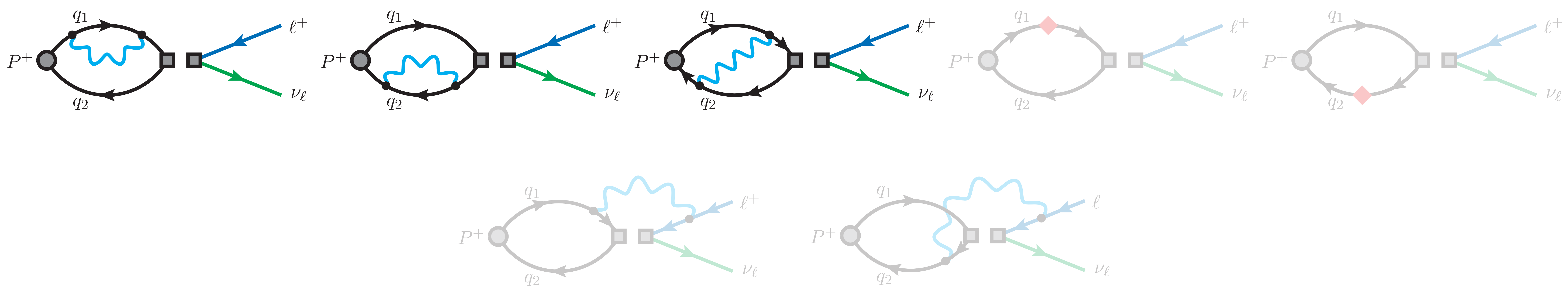
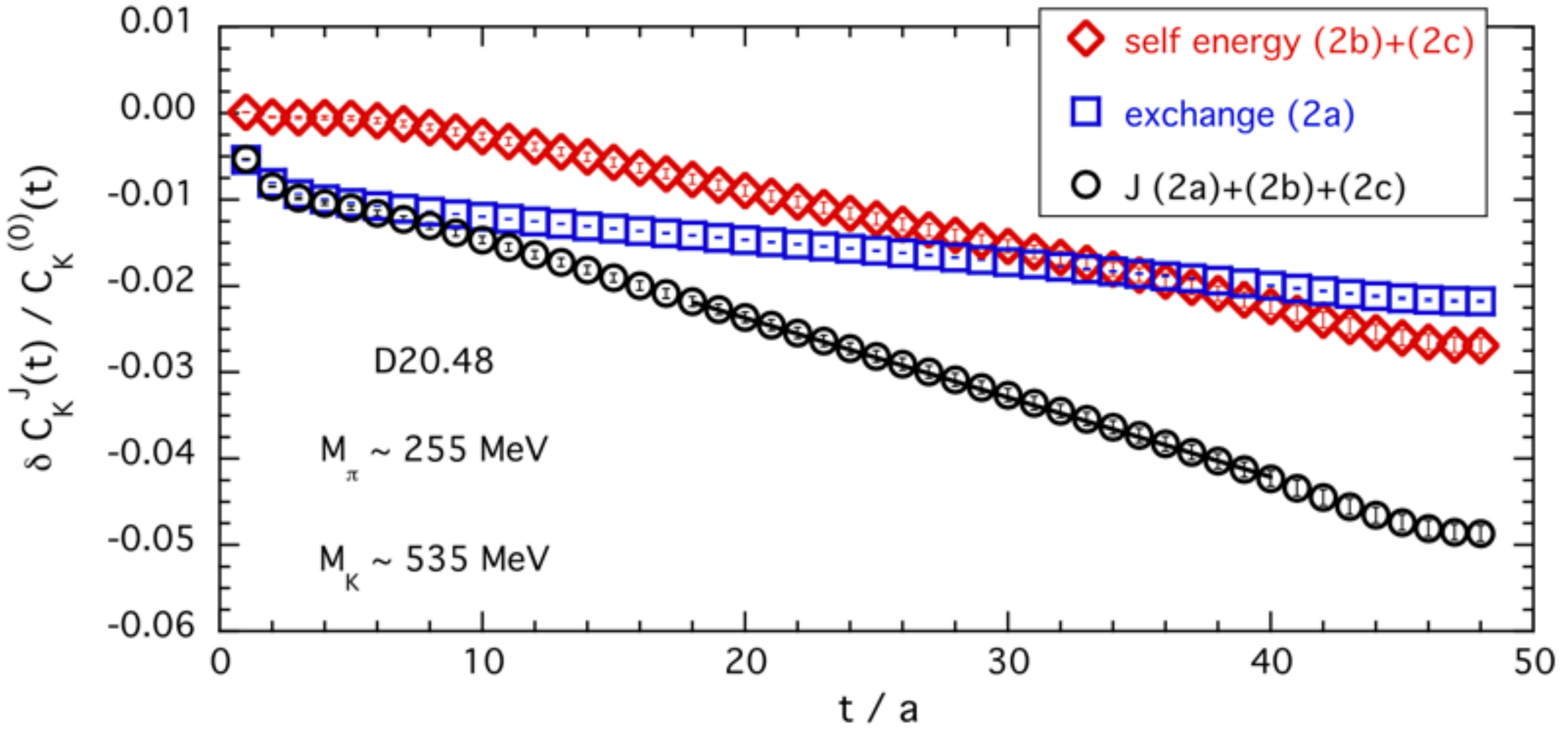
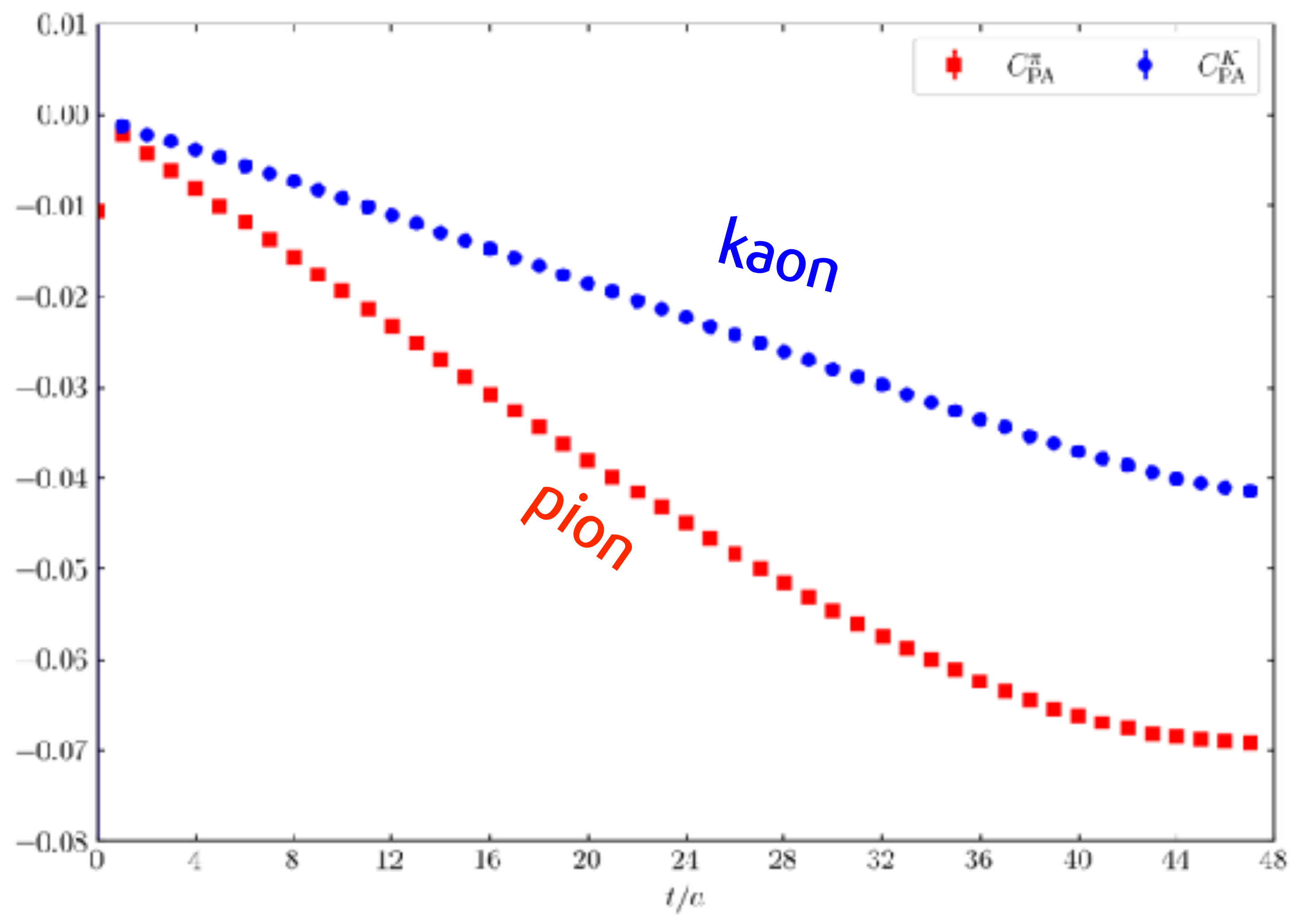




MDC et al., PRD 100 (2019)



P.Boyle, MDC et al., JHEP 02 (2023)

IB corrections to the decay amplitude

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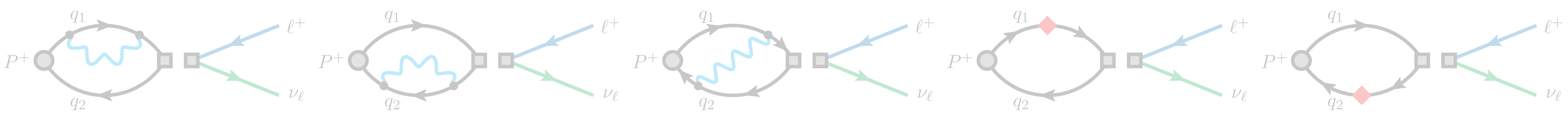
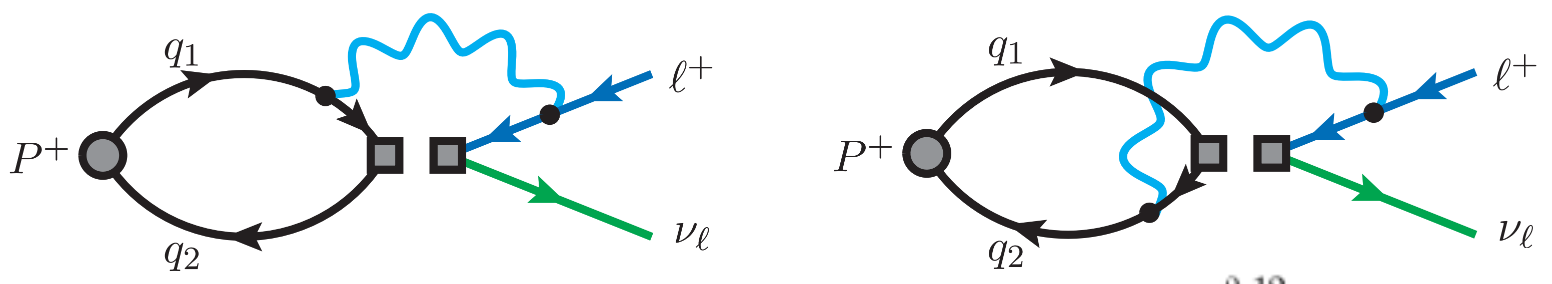
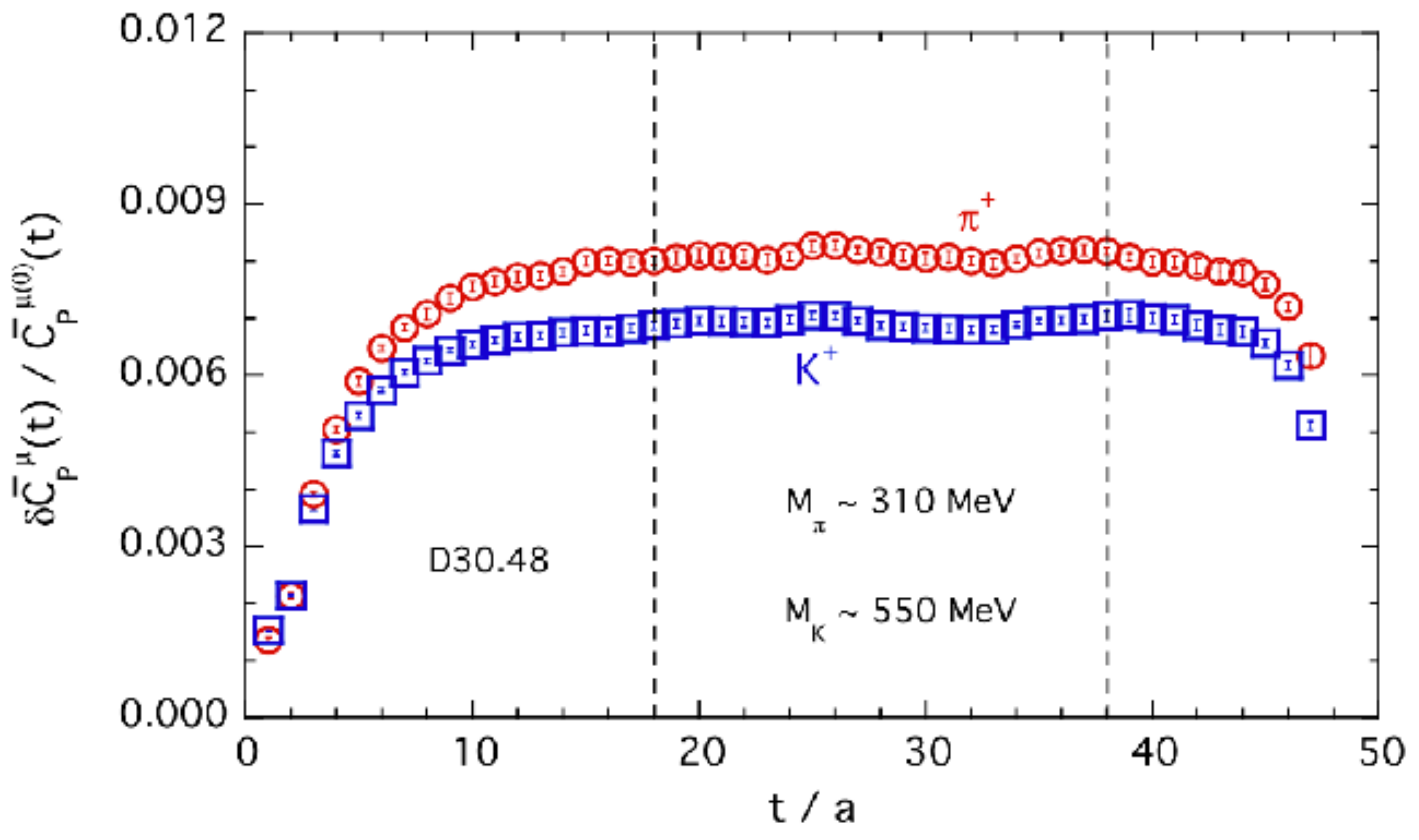
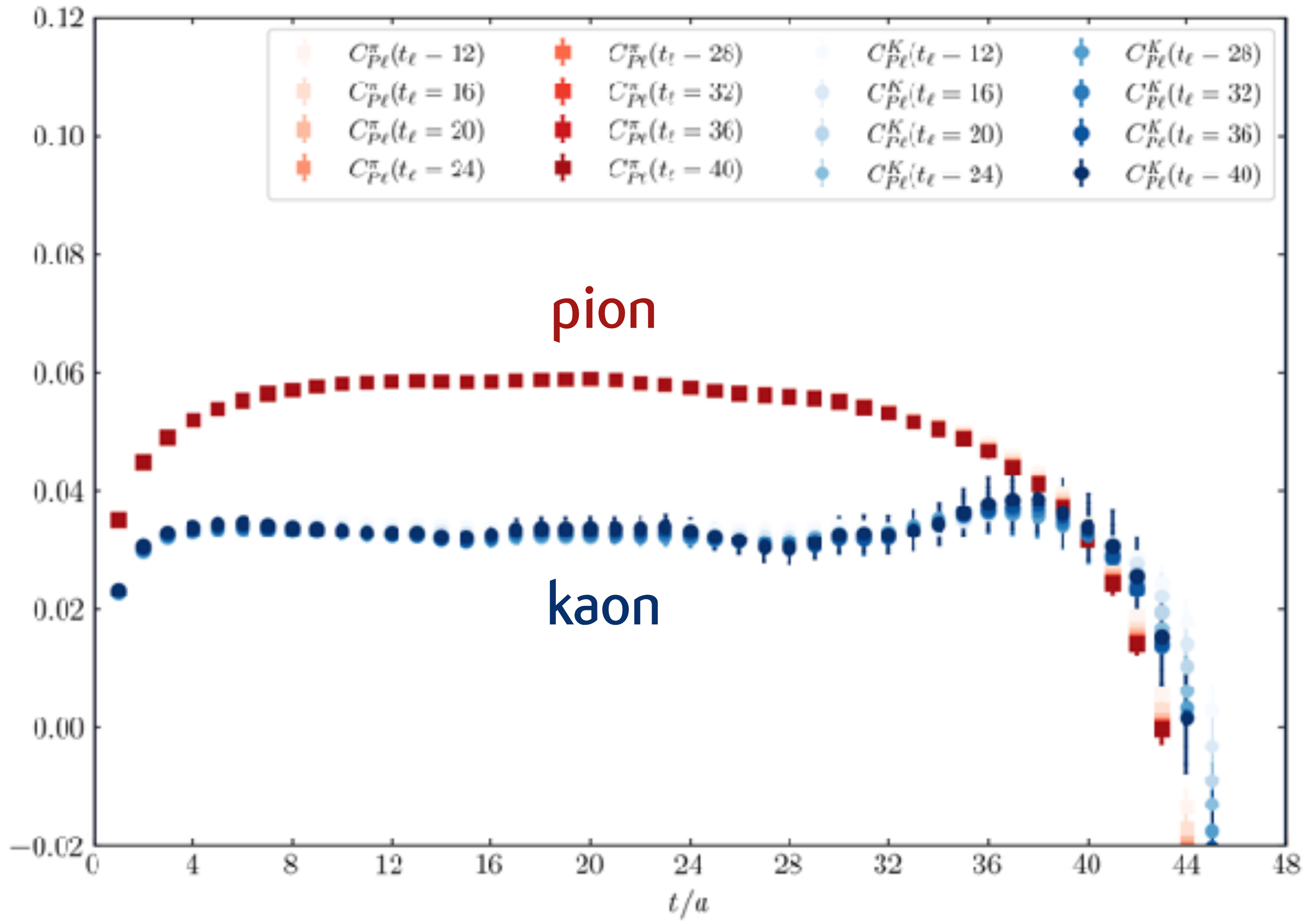






MDC et al., PRD 100 (2019)



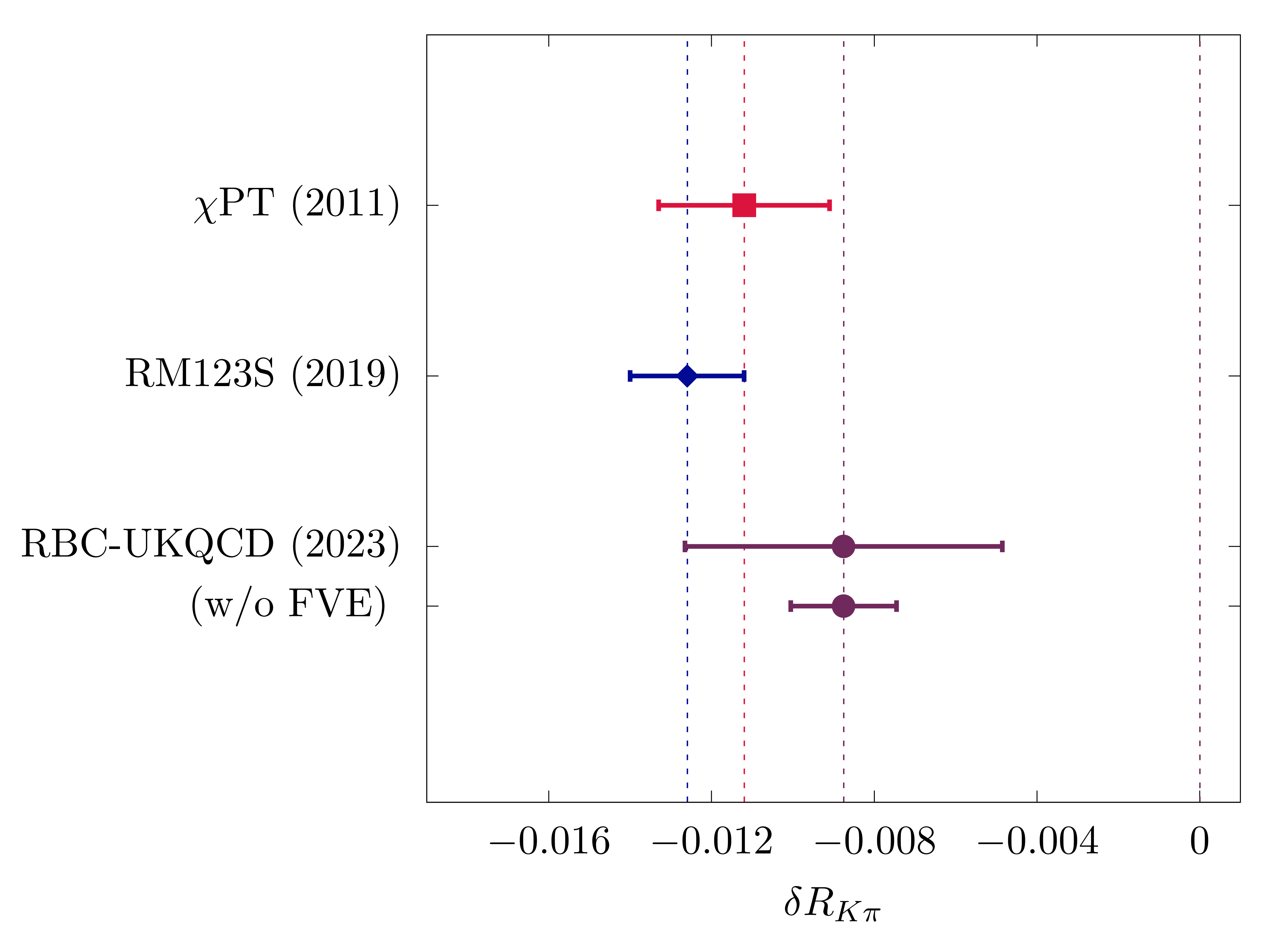
P.Boyle, MDC et al., JHEP 02 (2023)

Results for $\delta R_{K\pi}$

V. Cirigliano et al., PLB **700** (2011)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

- ■ $\delta R_{K\pi} = -0.0112\,(21)$
- ◆ $\delta R_{K\pi} = -0.0126\,(14)$
- ● $\delta R_{K\pi} = -0.0086\,(13)(39)_{\text{vol.}}$

$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi})$$



- Strong evidence that $\delta R_{K\pi}$ **can be computed from first principles non-perturbatively on the lattice!**
- Results highlight crucial role of **finite-volume effects**: ongoing effort to tame such systematic uncertainty
- **Errors on** $|V_{us}|/|V_{ud}|$ from **theoretical** inputs can become comparable with those from **experiments**

Finite-volume effects in QED$_L$

Leptonic decay rate

V. Lubicz et al., PRD **95** (2017)
N. Tantalo et al., [1612.00199v2]
MDC et al., PRD **105** (2022)

$$\Gamma_0(L) = \Gamma_0^{\text{tree}} \left\{ 1 + 2\,\frac{\alpha}{4\pi}\, Y(L) \right\}$$

$$Y(L) - Y(\infty) = Y_{\log}(L) + Y_0 + \frac{1}{m_P L}\, Y_1 + \frac{1}{(m_P L)^2}\, Y_2 + \frac{1}{(m_P L)^3}\, Y_3 + \mathrm{O}(1/L^4) + \mathrm{O}(\mathrm{e}^{-\alpha L})$$

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$m_\pi L \approx 3.9$ | ≈ -3.96 | ≈ -2.24 | ≈ 3.37 | ?

- structure independent ("**universal**") terms ✔
- **structure dependent** contribution at $O(1/L^2)$ ✔
- **sizeable pointlike** contribution at $O(1/L^3)$ ✔
- higher order effects ✘

Prospects for $|V_{us}/V_{ud}|$

An estimate of the error budget

$$\frac{|V_{\rm us}|}{|V_{\rm ud}|} = \left[\frac{\Gamma(K^+ \to \mu^+\nu_\mu[\gamma])}{\Gamma(\pi^+ \to \mu^+\nu_\mu[\gamma])}\frac{m_K}{m_\pi}\frac{(m_\pi^2 - m_\mu^2)}{(m_K^2 - m_\mu^2)}\right]^{1/2}_{\rm exp} \frac{f_{\pi,0}}{f_{K,0}}\left(1 - \frac{1}{2}\,\delta R_{K\pi}\right)$$

- From RM123+Soton calculation: $\delta R_{K\pi} = -0.0126\,(14)$

$[f_{K,0}/f_{\pi,0}]$		$\|V_{us}/V_{ud}\|$
FLAG19 2+1+1 average	1.1966 (18)	$0.23131\ (28)_{\rm exp}\ (17)_{\delta R}\ (35)_{f_P}$

- From RBC-UKQCD calculation: $\delta R_{K\pi} = -0.0086\,(13)(39)_{\rm vol.}$

$[f_{K,0}/f_{\pi,0}]$		$\|V_{us}/V_{ud}\|$
FLAG21 2+1 average	1.1930 (33)	$0.23154\ (28)_{\rm exp}\ (15)_{\delta R}\ (45)_{\delta R,{\rm vol.}}\ (65)_{f_P}$

- the uncertainty on $[f_{K,0}/f_{\pi,0}]$ dominates in the error budget
- if improved, precision from **lattice** becomes competitive with **experiments**

Future steps for leptonic decays

There's room for **improvements,** achievable in the next few years.

Most importantly:

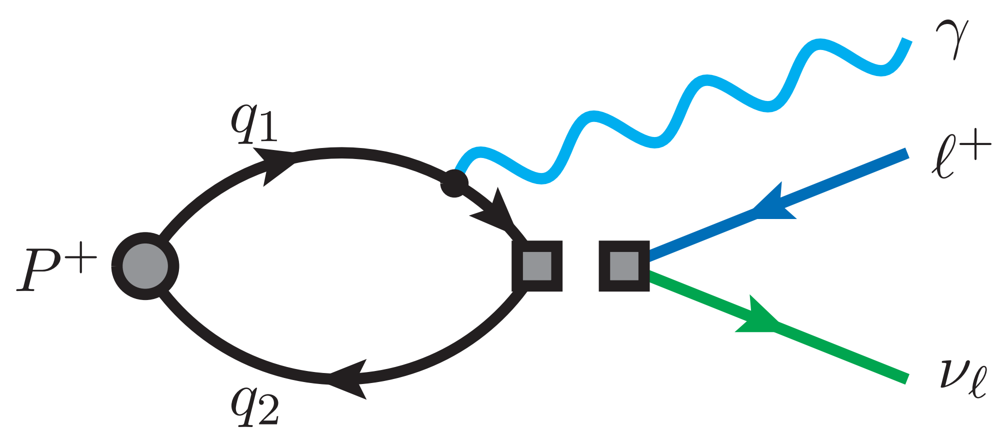
1. improve **control of QED_L finite-volume effects**
   - new **QED_r formulation** proposed: unknown effects pushed to $O(1/L^4)$ MDC @Lattice2023
   - ongoing **calculation with QED_∞** method: exponentially suppressed effects N.Christ et al., [2304.08026]

2. improve **iso-QCD calculations of decay constants**

3. include electromagnetic and strong **IB effects from sea quarks** ("electro-unquenching")

Other leptonic decays under study

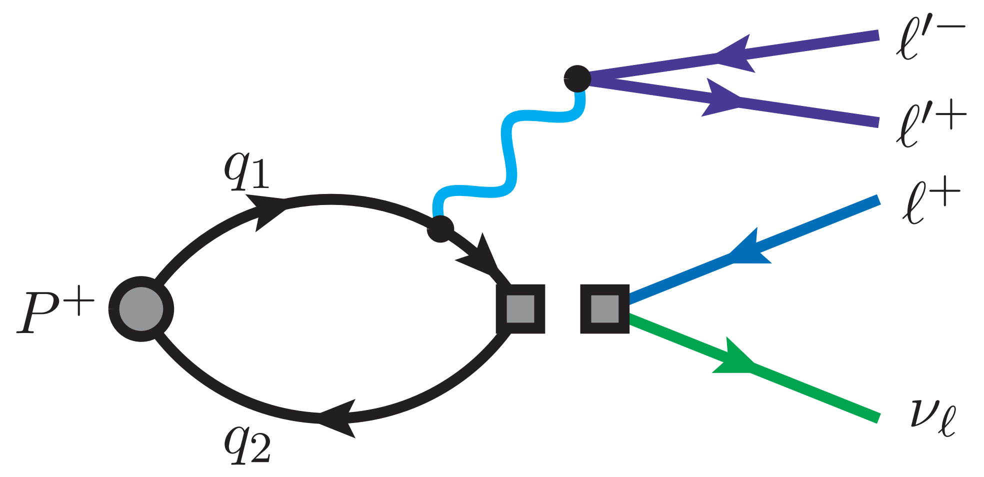
real photon emission



- Lattice evaluations of form factors $F_{V,A}$
- Interesting comparison with experimental results (KLOE, PIBETA, E787, ISTRA+ & OKA) highlights 3-4σ tensions on $K \to \mu\nu_\mu\gamma$

G.M.de Divitiis et al., [1908.10160]
A.Desiderio et al., PRD **102** (2021)
R.Frezzotti et al., PRD **103** (2021)
C.Kane et al., [1907.00279 & 2110.13196]
D.Giusti et al., [2302.01298]
R.Frezzotti et al., [2306.05904]

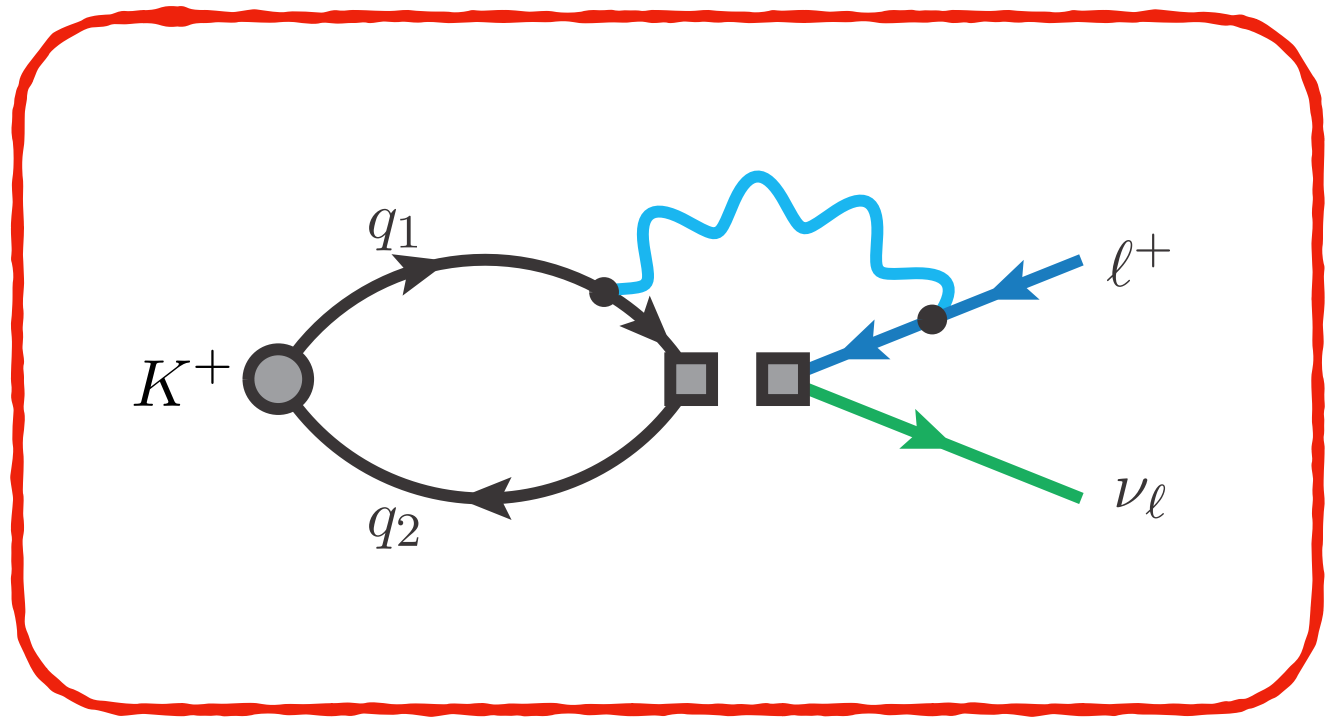
virtual photon emission



- Exploratory calculation of relevant form factors using smeared spectral function reconstruction
- Applied to D_s decay, but extension to K decay is planned

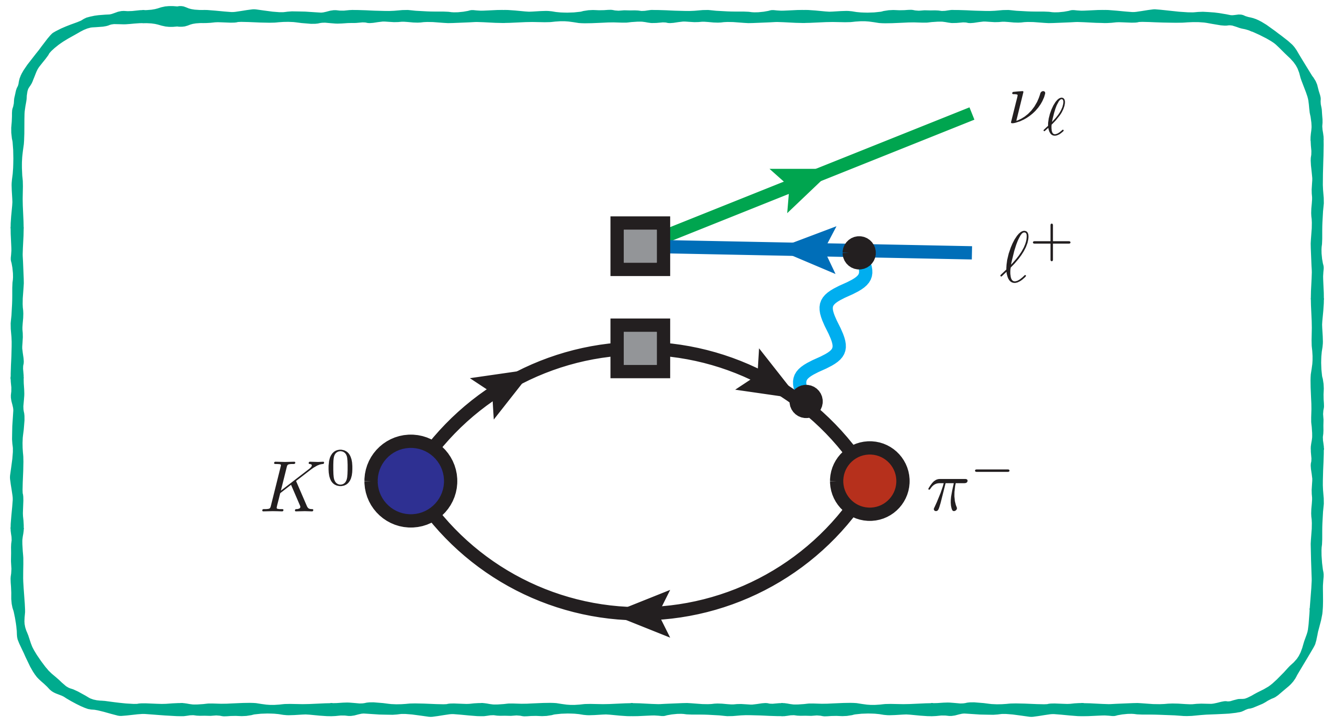
R.Frezzotti et al., [2306.07228]

A look into the future...



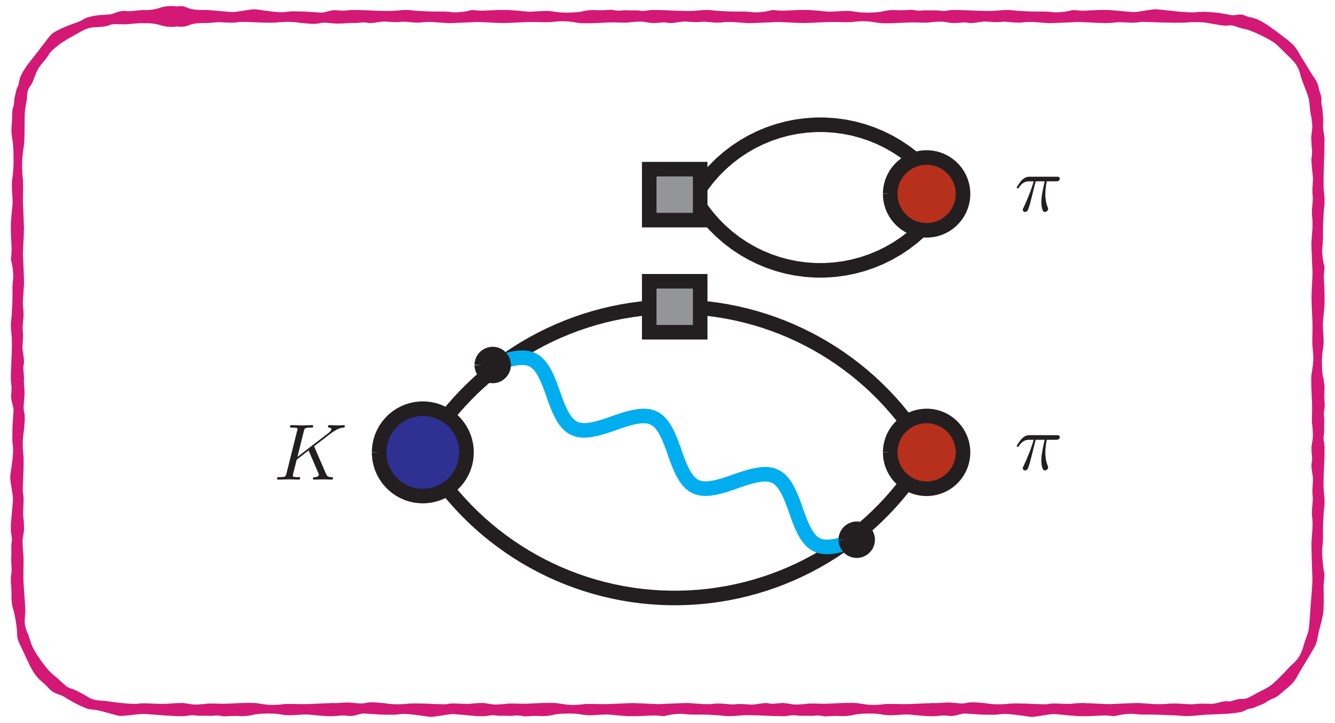
leptonic decays

$K \to \ell \nu_\ell$



semi-leptonic decays

$K \to \pi \ell \nu_\ell$



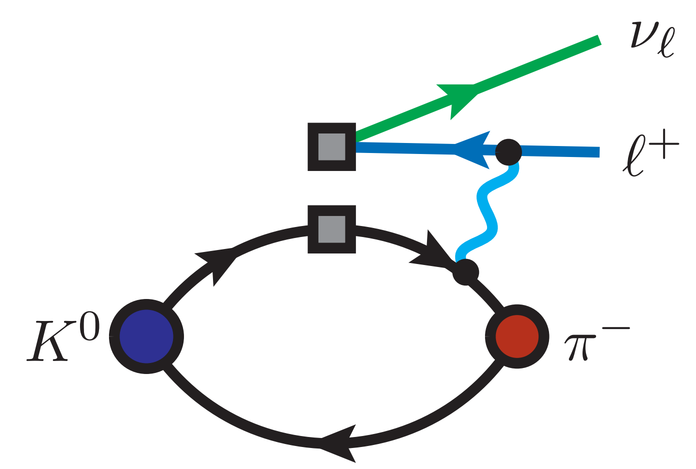
hadronic decays

$K \to \pi\pi$

today — 1–5 years — 5+ years

Semileptonic kaon decays

Goal: precision determination of $|V_{us}|$ & test of first-row unitarity

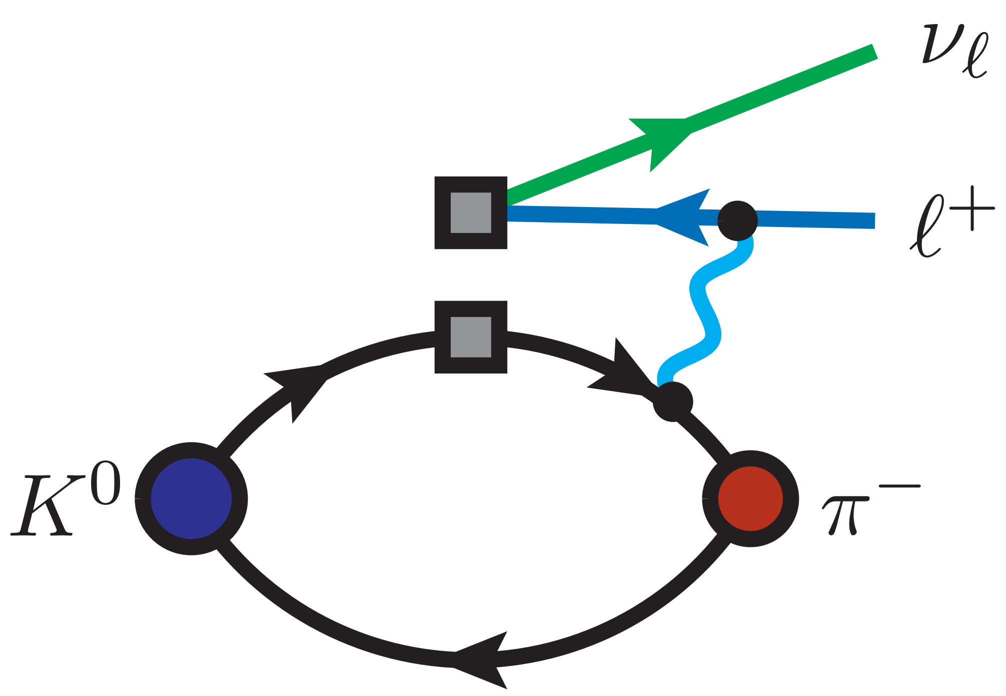


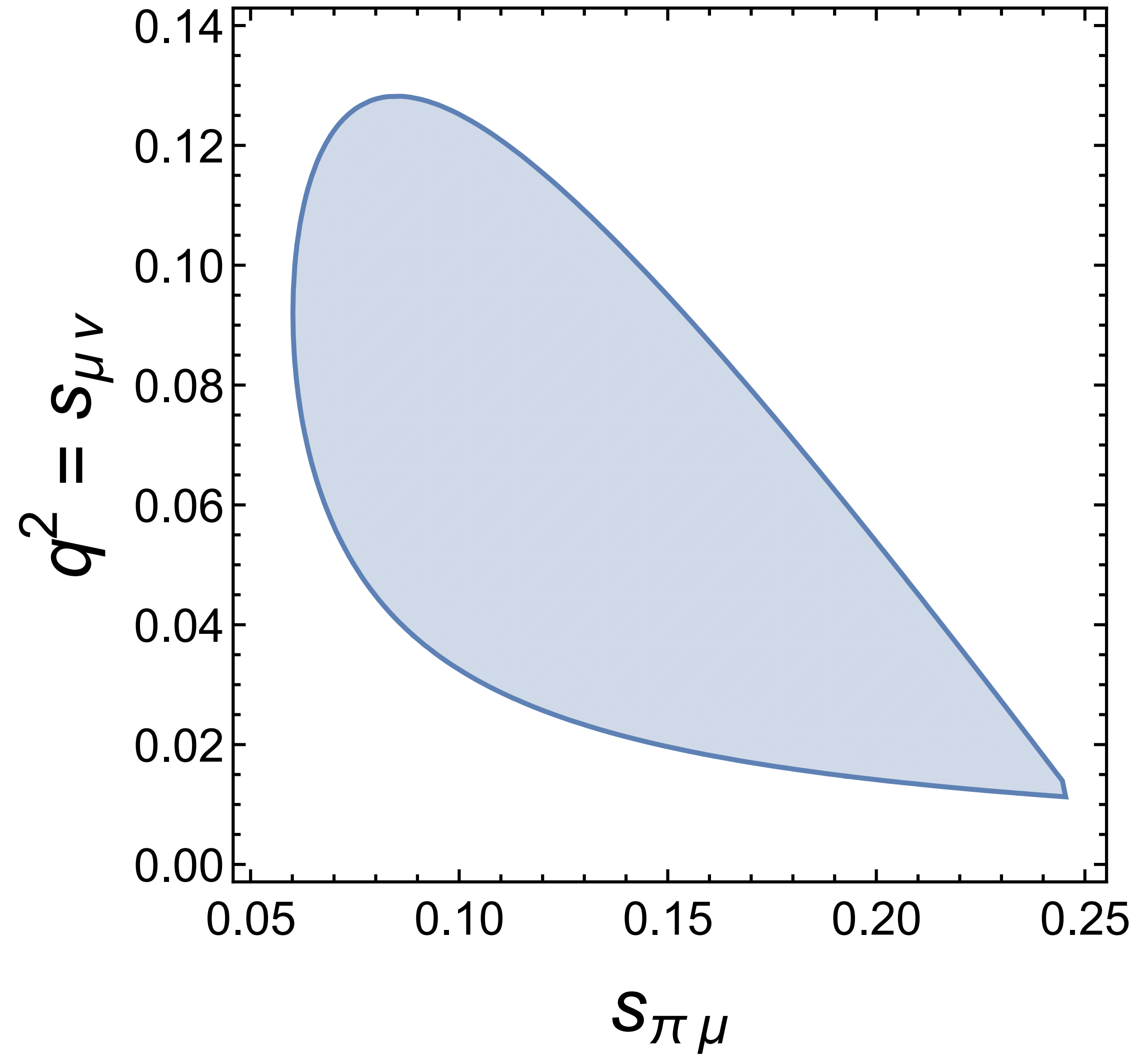
Relevance: sub-percent precision on $f^+(0)$ requires inclusion of IB effects

Status:

- no complete lattice QCD+QED calculations
- difficulties of finite-volume QED calculations identified — C.Sachrajda et al., [1910.07342]
- recent proposal using QED$_\infty$ method — N.Christ et al., [2304.08026] / N.Christ @Lattice2023

QED corrections to semileptonic decays





Although the RM123+Soton method could in principle be applied, additional **difficulties** arise compared to **leptonic decays**:

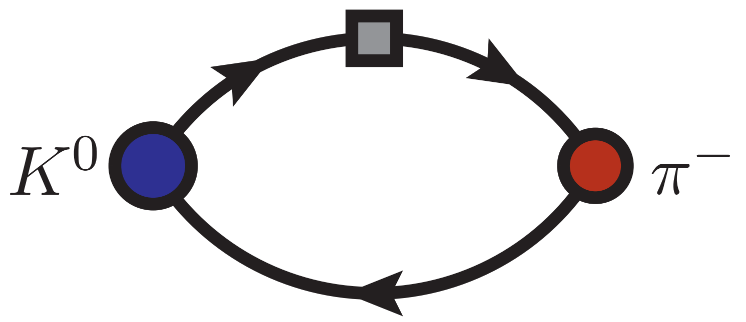
- integration over **three-body phase-space**
- problems of **analytical continuation** when intermediate on shell states are lighter than external ones
- evaluating **finite-volume corrections** potentially more complicated

Solutions to these issues are under study by different groups.

Hopefully we'll see progress in the next few years...

Extension of RM123S approach

- Without QED corrections:



$$\langle \pi(p_\pi)|\bar{s}\gamma^\mu u|K(p_K)\rangle = \boldsymbol{f_+(q^2)}\left[(p_\pi + p_K)^\mu - \frac{m_K^2 - m_\pi^2}{q^2}\, q^\mu\right] + \boldsymbol{f_0(q^2)}\,\frac{m_K^2 - m_\pi^2}{q^2}\, q^\mu$$

An appropriate observable to study is the **differential decay rate:** $s_{\pi\ell} = (p_\pi + p_\ell)^2,\ q^2 = (p_K - p_\pi)^2$

$$\frac{\mathrm{d}^2\Gamma^{(0)}}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} = G_F^2|V_{us}|^2\left[a_1(q^2, s_{\pi\ell})\,|\boldsymbol{f_+(q^2)}|^2 + a_2(q^2, s_{\pi\ell})\,\boldsymbol{f_+(q^2)f_0(q^2)} + a_3(q^2, s_{\pi\ell})\,|\boldsymbol{f_0(q^2)}|^2\right]$$

Extension of RM123S approach

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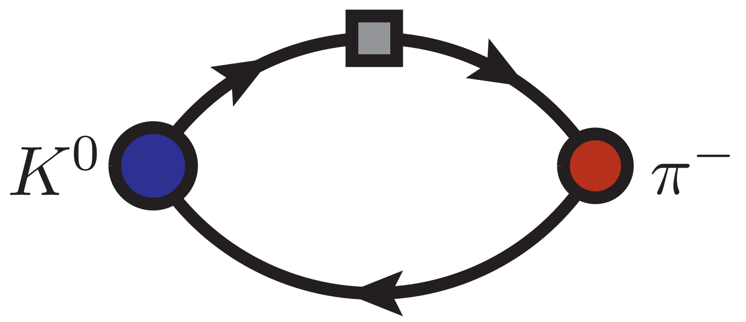

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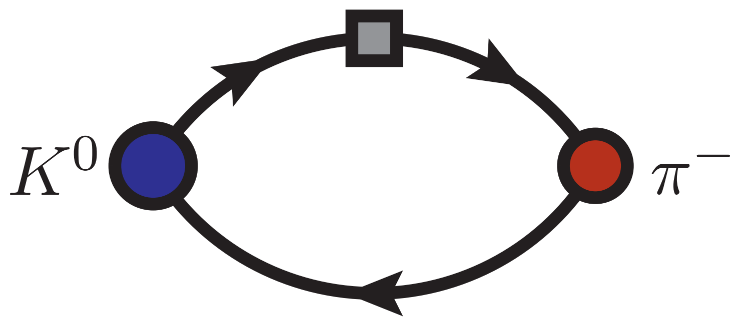
- Including QED, we can treat IR divergences using the **RM123S method**: C.Sachrajda et al., [1910.07342]

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} = \lim_{\Lambda_{\mathrm{IR}}\to 0}\left[\frac{\mathrm{d}^2\Gamma_0}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} - \frac{\mathrm{d}^2\Gamma_0^{\mathrm{pt}}}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}}\right] + \lim_{\Lambda_{\mathrm{IR}}\to 0}\left[\frac{\mathrm{d}^2\Gamma_0^{\mathrm{pt}}}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} + \frac{\mathrm{d}^2\Gamma_1}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}}\right]$$

Extension of RM123S approach

Q: Can experiments provide results for differential decay rates?

- Without QED corrections:



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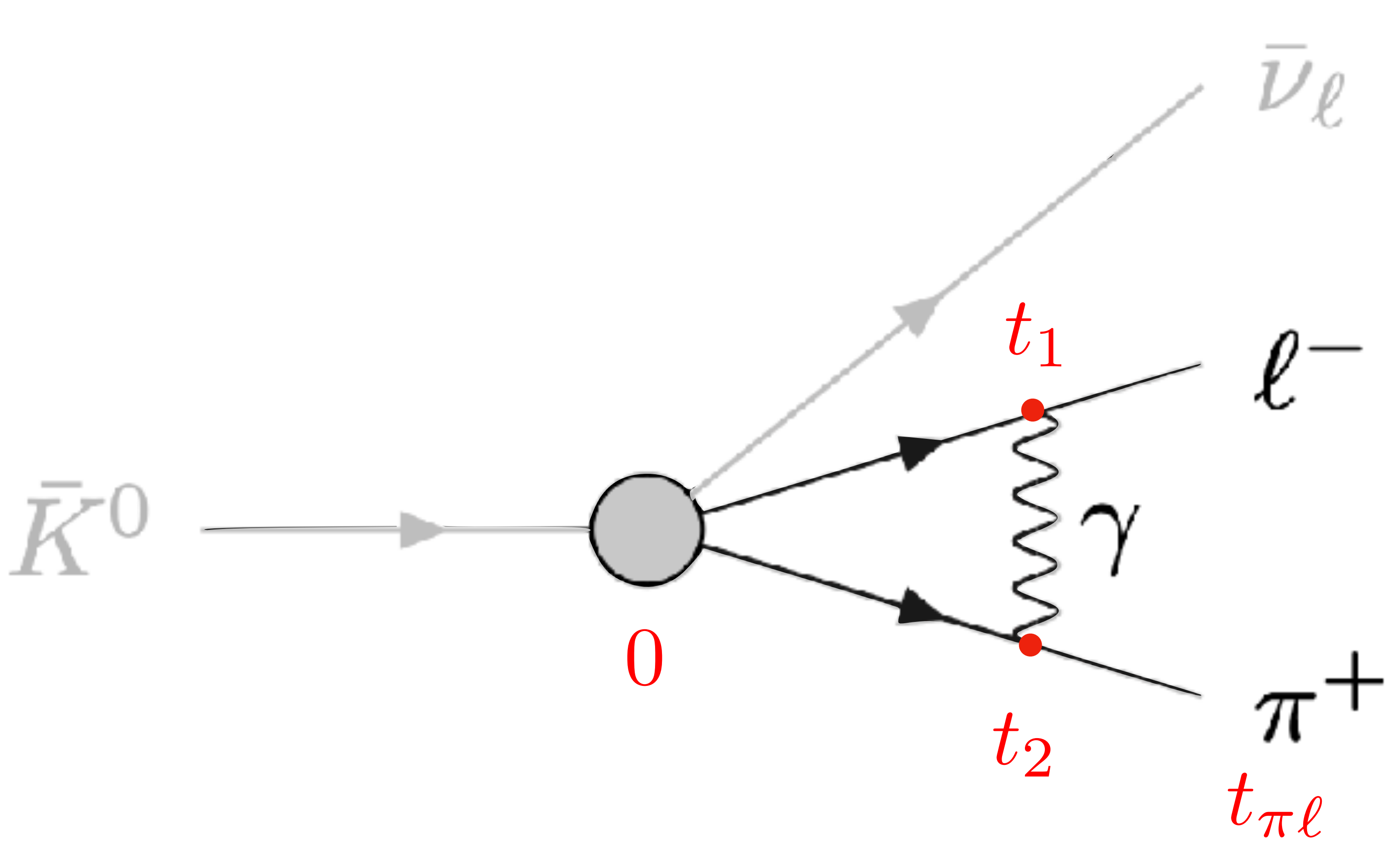
Appearance of growing exponentials

For <u>generic kinematics</u>, the **physical observable cannot be obtained from leading exponentials** of Euclidean lattice correlators



$$\int_0^{t_{\pi\ell}} \mathrm{d}t_1\,\mathrm{d}t_2 \longrightarrow \frac{1}{\Delta E}\left(\frac{1}{\Delta\omega_\ell + \omega_k} + \frac{1}{\Delta\omega_\pi + \omega_k}\right) + \frac{\mathbf{e}^{-t_{\pi\ell}\,\boldsymbol{\Delta E}}}{\Delta E}\left(\frac{1}{\Delta\omega_\ell - \omega_k} + \frac{1}{\Delta\omega_\pi - \omega_k}\right) + \ldots$$

with $\boldsymbol{\Delta E} = E_{\text{int}} - E_{\text{ext}}$ and $\omega_k = |\mathbf{k}|$

- generates growing exponentials if $\boldsymbol{\Delta E} = E_{\text{int}} - E_{\text{ext}} < 0$
- they need to be **identified and subtracted** to get physical amplitude
- however, **there might be a corner of the phase space** where this issue does not arise

A recent proposal with QED$_\infty$

X.Feng & L.Jin, PRD **100** (2019)
N.Christ et al., [2304.08026]

These are **long-distance issues** that could be treated using a recently proposed method called "infinite-volume reconstruction" (IVR)

Radiative corrections = **convolution** of hadronic correlators with infinite-volume QED kernels

$$\Delta\mathcal{O} = \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\, \mathcal{H}(t,\mathbf{x})\, f_{\mathrm{QED}}(t,\mathbf{x}) = \Delta\mathcal{O}^{(s)} + \Delta\mathcal{O}^{(l)}$$

Separate correlator into **short** and **long** distance parts:

$$\Delta\mathcal{O}^{(s)} \approx \frac{1}{2}\int_{-t_s}^{t_s} \mathrm{d}t \int_{L^3} \mathrm{d}^3\mathbf{x}\, \mathcal{H}^L(t,\mathbf{x}) f_{\mathrm{QED}}(t,\mathbf{x})$$

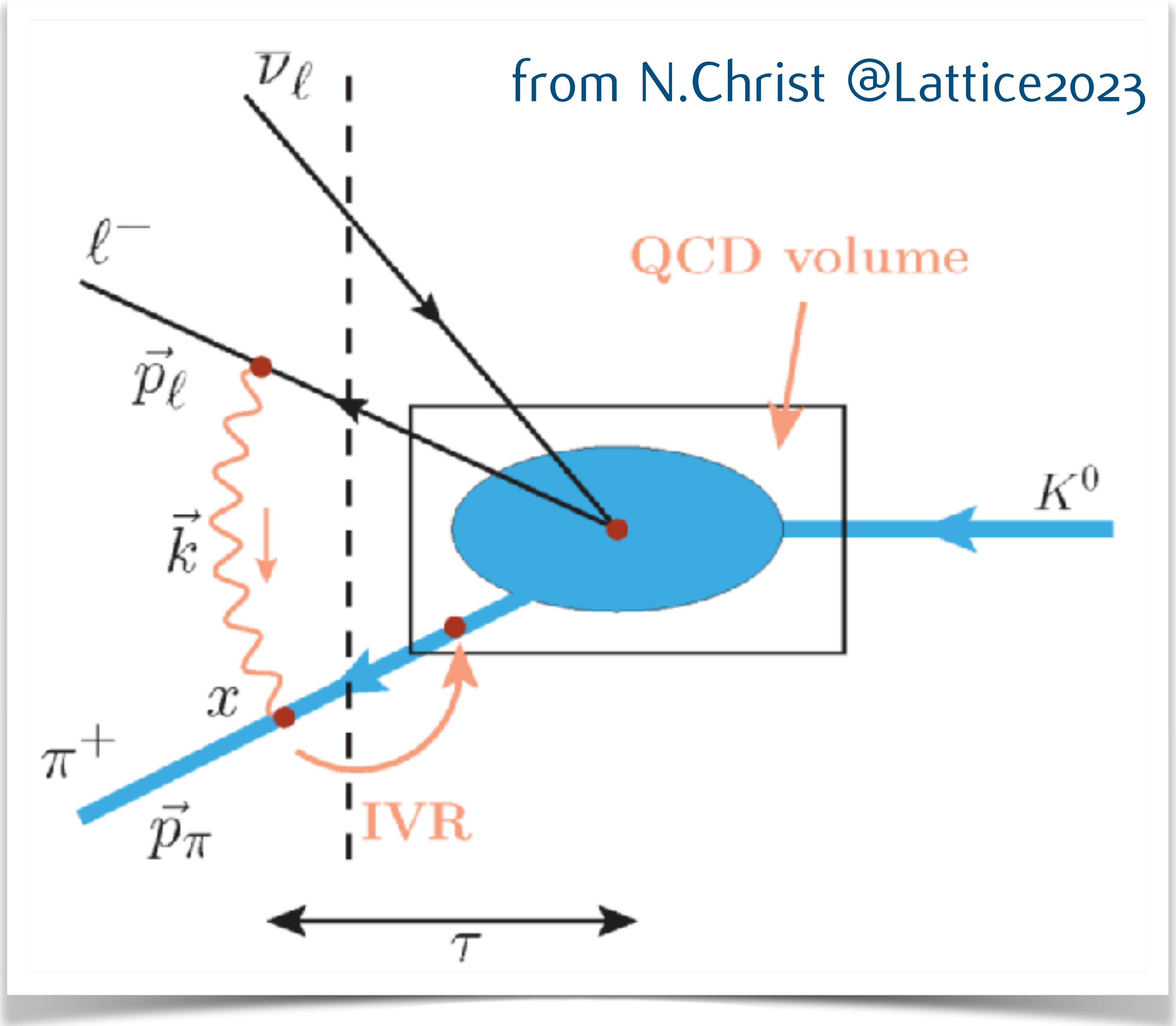
$$\Delta\mathcal{O}^{(l)} \approx \int_{L^3} \mathrm{d}^3\mathbf{x}\, \mathcal{H}^L(t_s,\mathbf{x})\, \mathcal{F}_{\mathrm{QED}}(t_s,\mathbf{x})$$

A recent proposal with QED$_\infty$

X.Feng & L.Jin, PRD **100** (2019)
N.Christ et al., [2304.08026]

Recipe for semileptonic (and leptonic) decays:

- work in Minkowski space-time + analytic QED
- compute short distance in finite Euclidean volume
- use IVR for long-distance single pion propagation

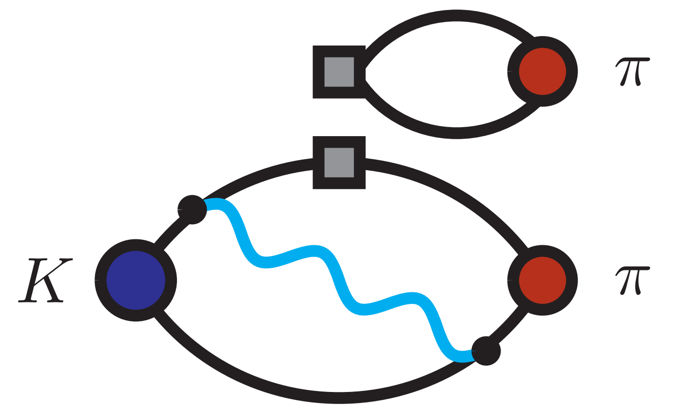


Difficult but potentially promising...

- no growing exponentials
- exponentially suppressed finite-volume effects
- simple cancellation of IR divergences
- separation of scales
- modelling of long-distance QCD

Hadronic kaon decays

Goal: precision determination of $\mathrm{Re}(\epsilon'/\epsilon)$ & study of CP violation



Relevance: IB effects will be dominant source of systematic error, once continuum limit will be performed (work in progress)

Status:

- no complete lattice QCD+QED calculation
- lattice QCD calculations by RBC-UKQCD collaboration
- strategy for calculation of IB effects proposed
- first step: Coulomb corrections to $\pi^+\pi^+$ scattering

R.Abbott et al., PRD **102** (2020)
Z.Bai et al., PRL **115** (2015)

N.Christ et al., PRD **106** (2022)
N.Christ & X.Feng, EPJ Web Conf. **175** (2018)
Y.Cai & Z.Davoudi, [1812.11015]

Current status of ϵ'/ϵ

If **isospin-symmetry** is conserved, then the CP violation parameters can be expressed as

$$\frac{\epsilon'}{\epsilon} = \frac{i e^{i(\delta_2-\delta_0)}}{\sqrt{2}\epsilon} \frac{\mathrm{Re}(A_2)}{\mathrm{Re}(A_0)} \left[\frac{\mathrm{Im}(A_2)}{\mathrm{Re}(A_2)} - \frac{\mathrm{Im}(A_0)}{\mathrm{Re}(A_0)} \right]$$

$$A_I = \langle (\pi\pi)_I | H_W^{\Delta S=1} | K \rangle$$

$\delta_I = \pi\pi$ scattering phase shifts

($I =$ isospin)

1. **RBC-UKQCD** performed **first calculation** of ϵ' in 2015 — Z.Bai et al., PRL **115** (2015)
2. **Improved result** in 2020: 3.5x more statistics + improved systematics — R.Abbott et al., PRD **102** (2020)

lattice: $\mathrm{Re}(\epsilon'/\epsilon) = 21.7\,(2.6)_{\mathrm{stat.}}(8.0)_{\mathrm{sys.}} \times 10^{-4}$

experiments: $\mathrm{Re}(\epsilon'/\epsilon) = 16.6\,(2.3) \times 10^{-4}$

Systematic error budget (from C.Kelly @Lattice2023)

- (~12%) Perturbation theory in Wilson coeffs to match 3f – 4f weak EFT at m_c
  - Improve with 4f calculation (active charm) : computationally infeasible?
  - Non-perturbative calculation of matching matrix : investigation underway

    [M.Tomii, *PoS* LATTICE2018 (2019) 216]

- (~23%) Lack of EM+isospin-breaking contributions in lattice calculation
  - Lattice measurement of these effects extremely challenging but approach is being formulated. [*Phys.Rev.D* 106 (2022) 1, 014508] [Christ, *PoS* LATTICE2021 (2022) 312]

    → estimated using χPT results
    V.Cirigliano et al., JHEP **02** (2020)

- (~12%) Use of single lattice spacing to compute I=0 amplitude
  - Repeat calculation with multiple, finer lattice spacings: **my current focus**

Intense work by RBC-UKQCD to reduce ~12% error due to use of single lattice spacing

—› IB correction will soon become relevant!

Isospin-breaking corrections

IB corrections are usually O(1%), but the "$\Delta I = 1/2$ rule" can give a ~20x enhancement in ϵ'/ϵ

A calculation of these effects is **very challenging!**

- **Lüscher & Lellouch-Lüscher formalisms** that relate finite-volume quantities (energy levels & correlation functions) to infinite-volume observables (scattering phase shifts & decay amplitudes) **need to be corrected for long range QED interactions**
- $\pi\pi$ final states with $I = 0$ and $I = 2$ are not independent anymore and **can mix**: it's a **coupled two-channel problem**

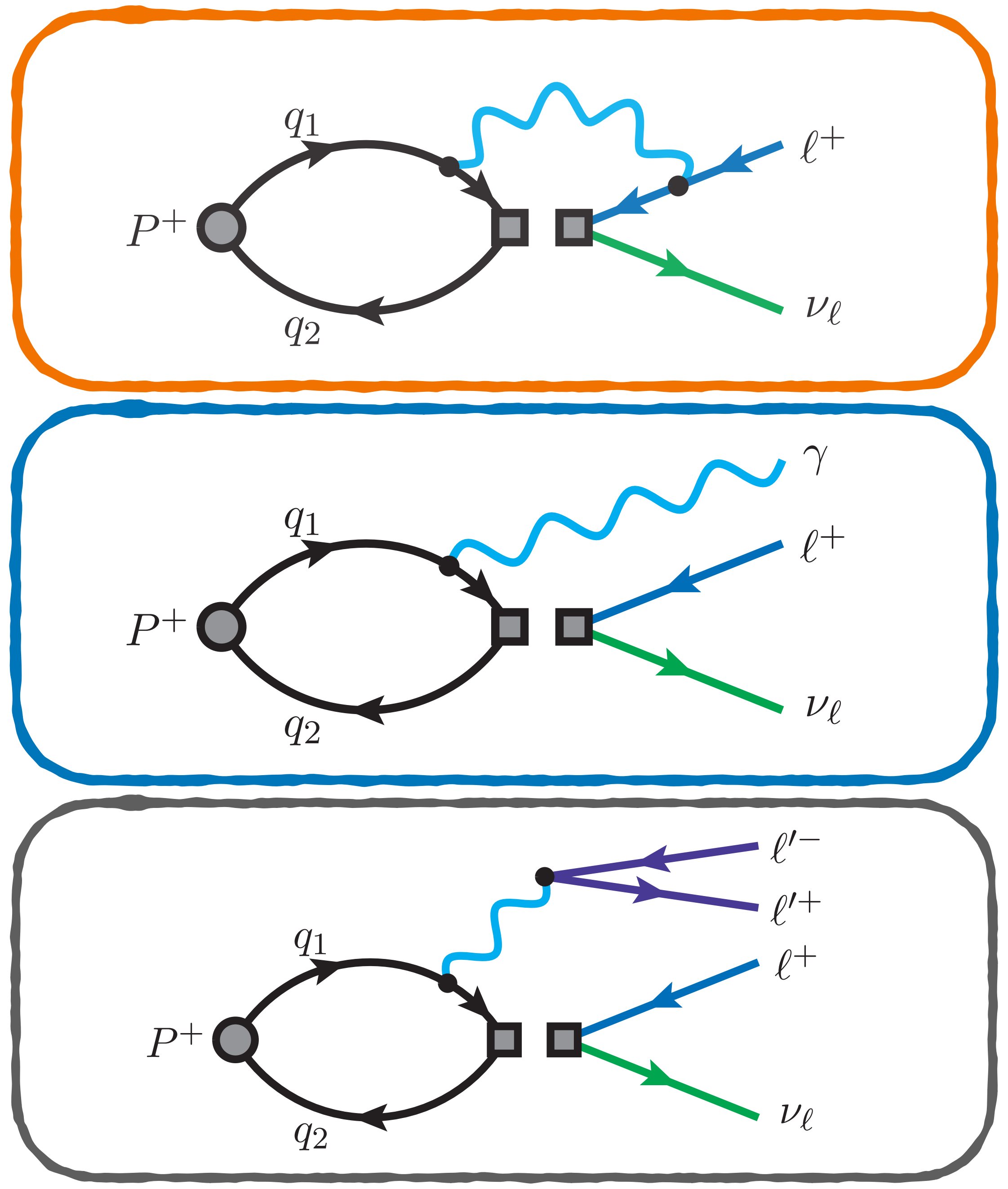
First step done: include QED corrections from Coulomb interaction to $\pi^+\pi^+$ scattering phase shift

Y.Cai & Z.Davoudi, [1812.11015] / N.Christ & X.Feng, EPJ Web Conf. **175** (2018) / N.Christ et al., PRD **106** (2022)

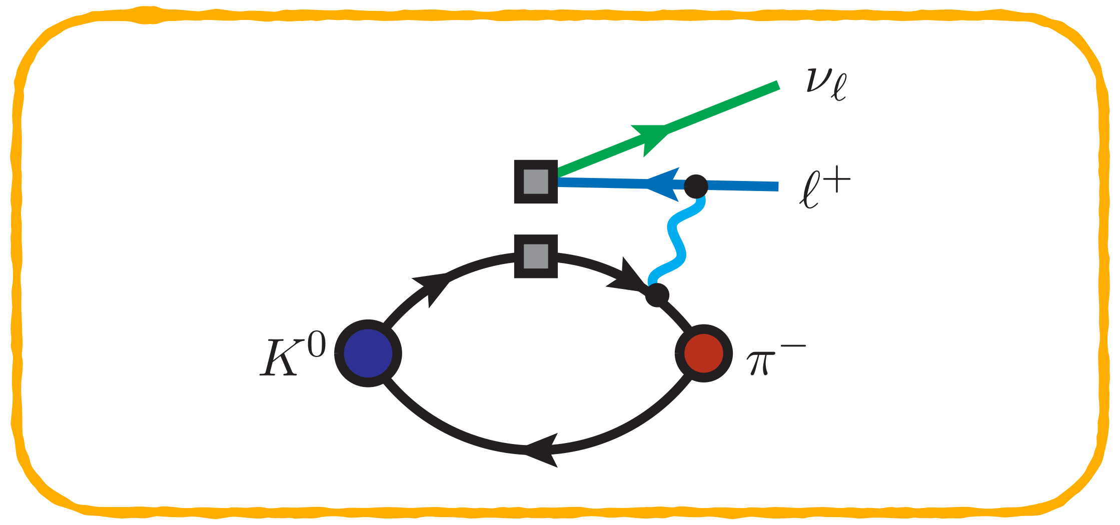
Conclusions

(all references in backup slides)

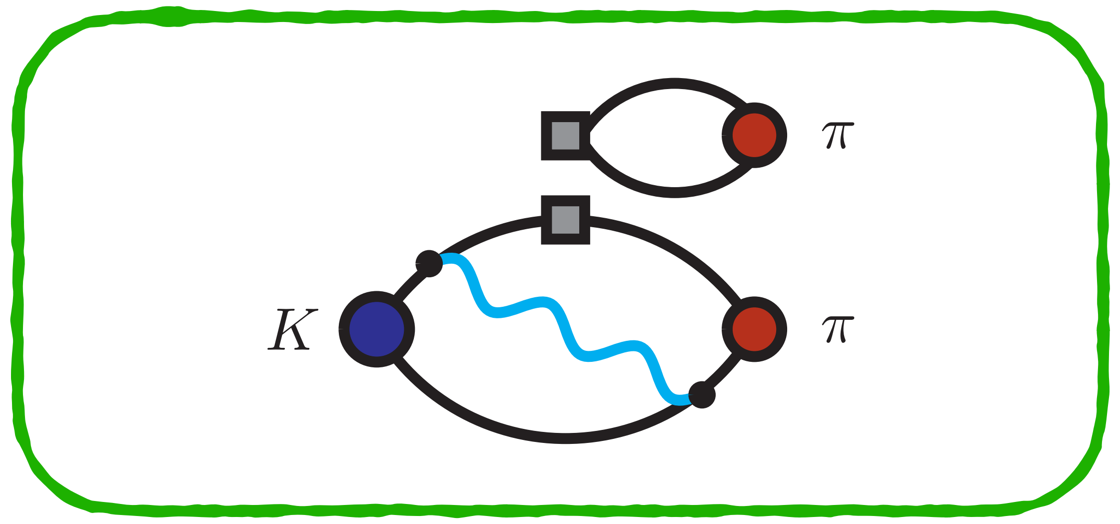
Overview



leptonic decays



semileptonic decays



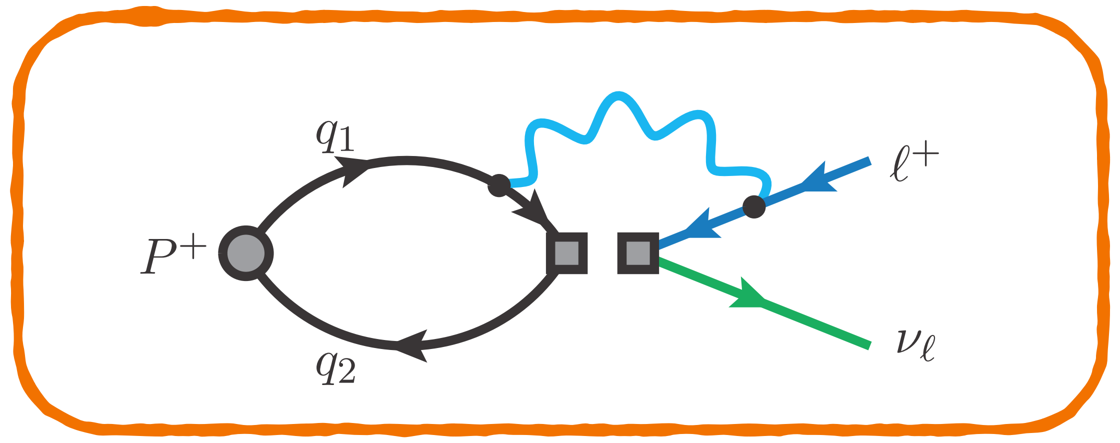
hadronic decays

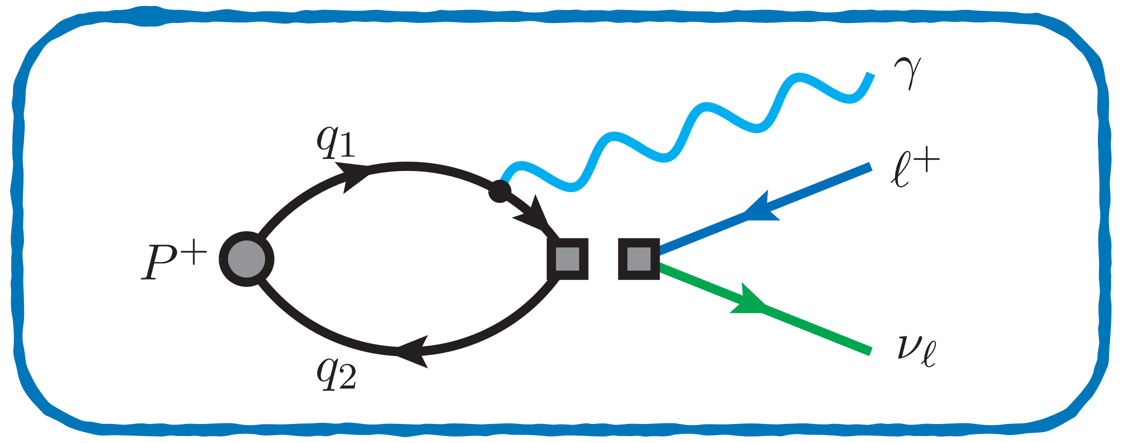
today — 1–5 years — 5+ years

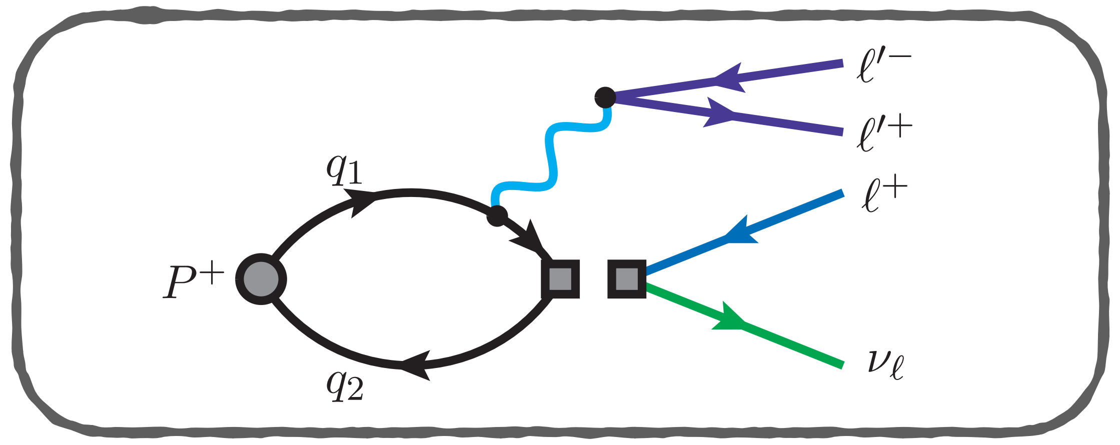
Conclusions

(all references in backup slides)

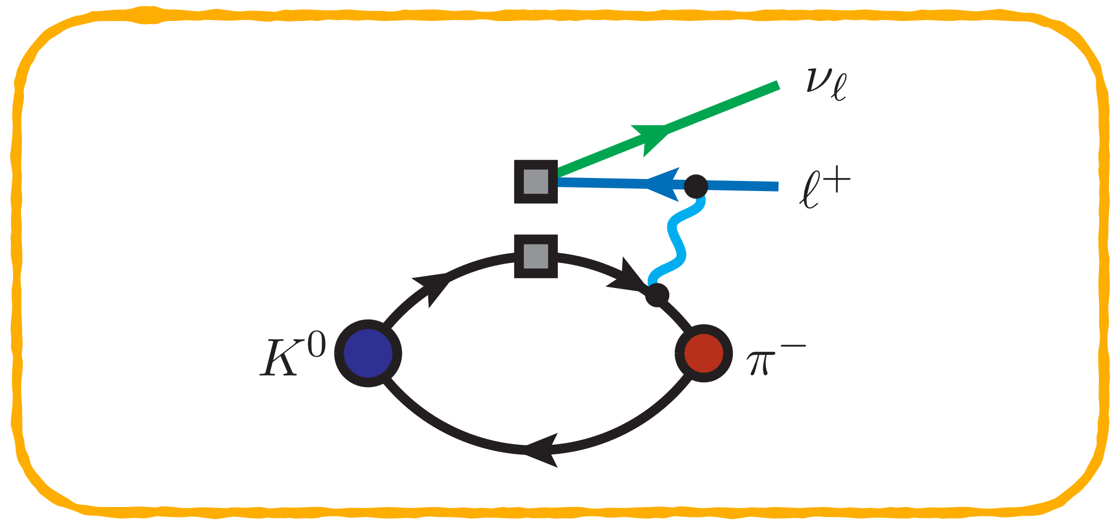
Overview



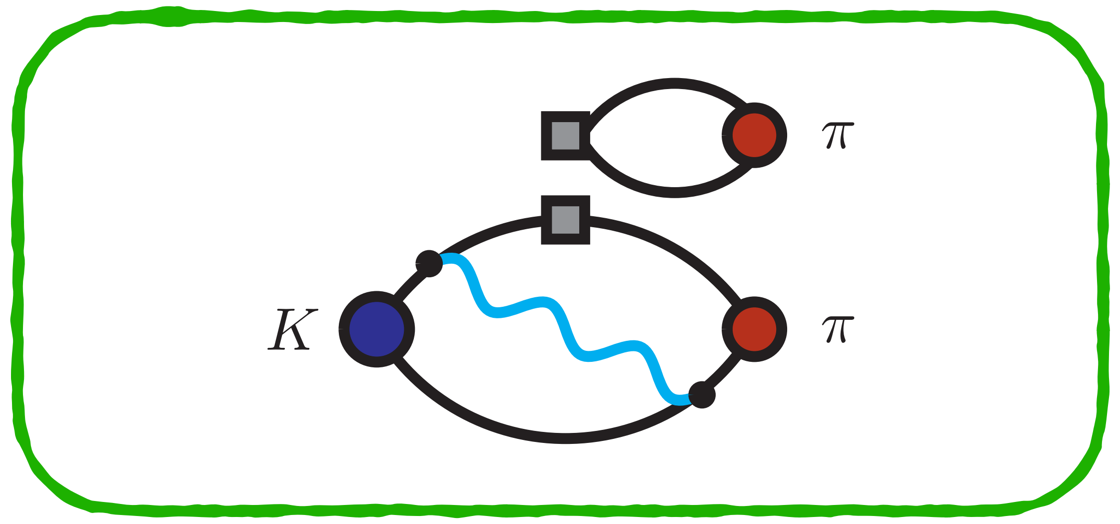




leptonic decays



semileptonic decays



hadronic decays

... an interesting future ahead!

today — 1–5 years — 5+ years

Thank you

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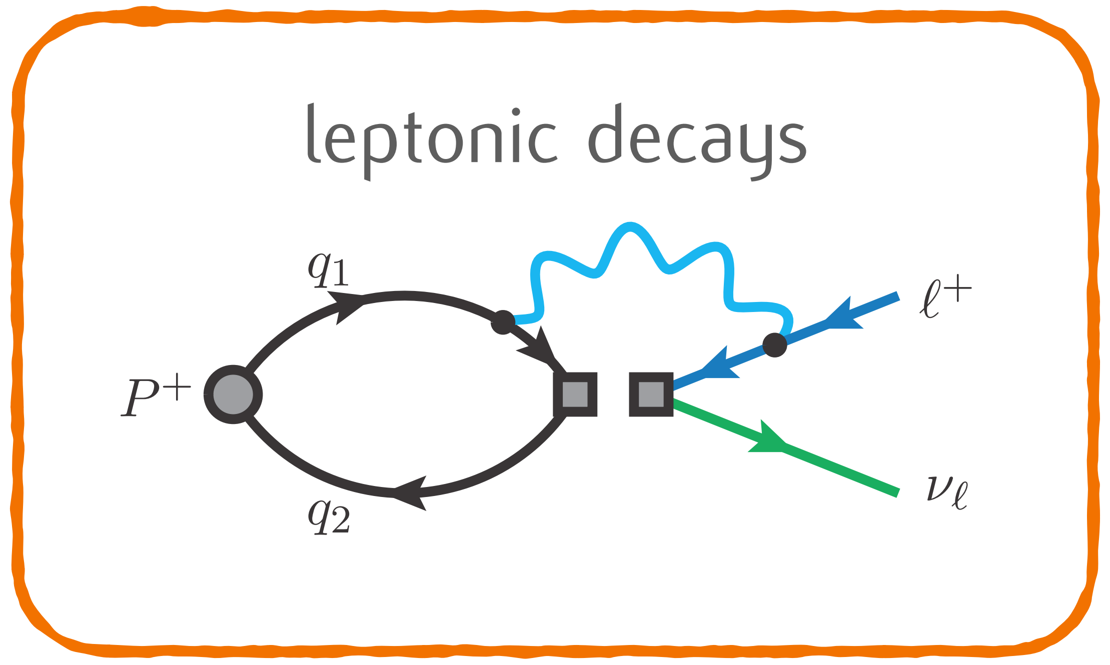
Backup slides

References



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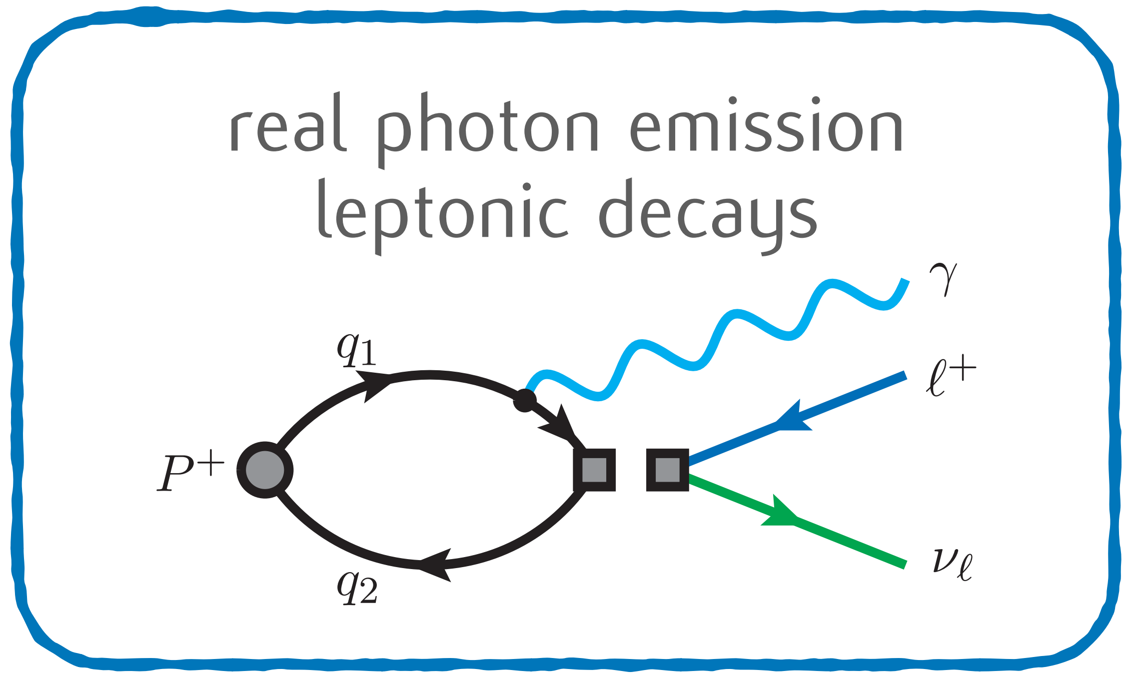
V. Lubicz et al., PRD 95 (2017)
N.Tantalo et al., [1612.00199v2]
D. Giusti et al., PRL 120 (2018)
MDC et al., PRD 100 (2019)
MDC et al., PRD 105 (2022)
P.Boyle, MDC et al., JHEP 02 (2023)
N.Christ et al., [2304.08026]



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C. Kane et al., [1907.00279 & 2110.13196]
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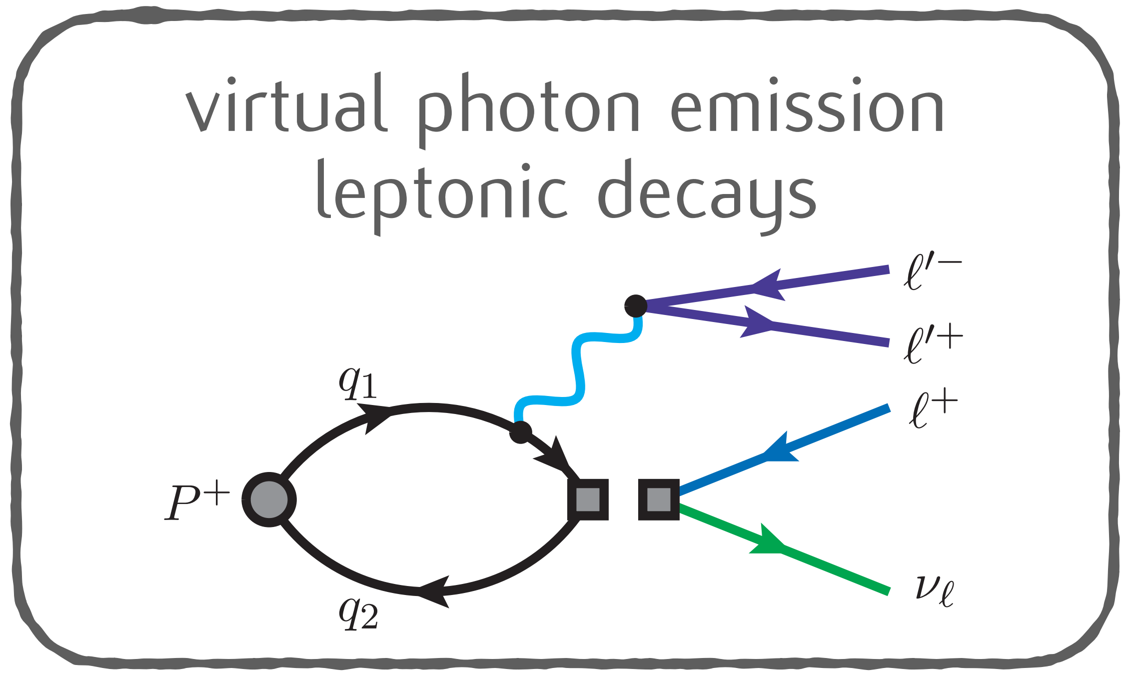
A.Desiderio et al., PRD 102 (2021)
D. Giusti et al., [2302.01298]
R.Frezzotti et al., [2306.05904]



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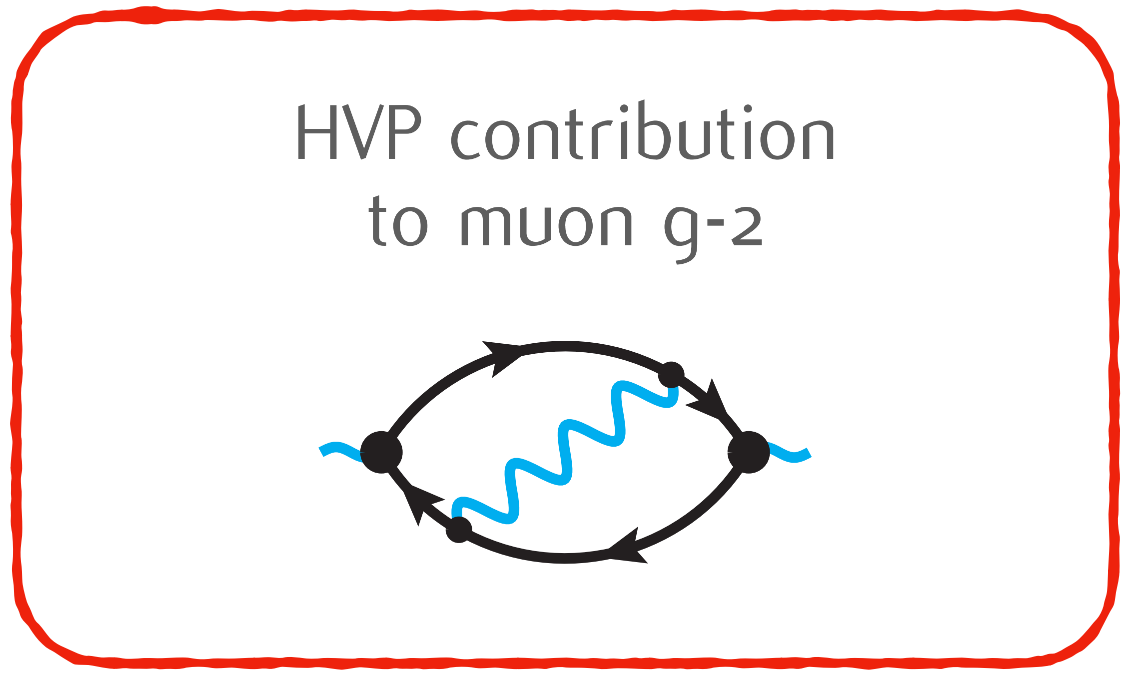
R.Frezzotti et al., [2306.07228]

White Paper: Phys. Rept. 887 (2020)



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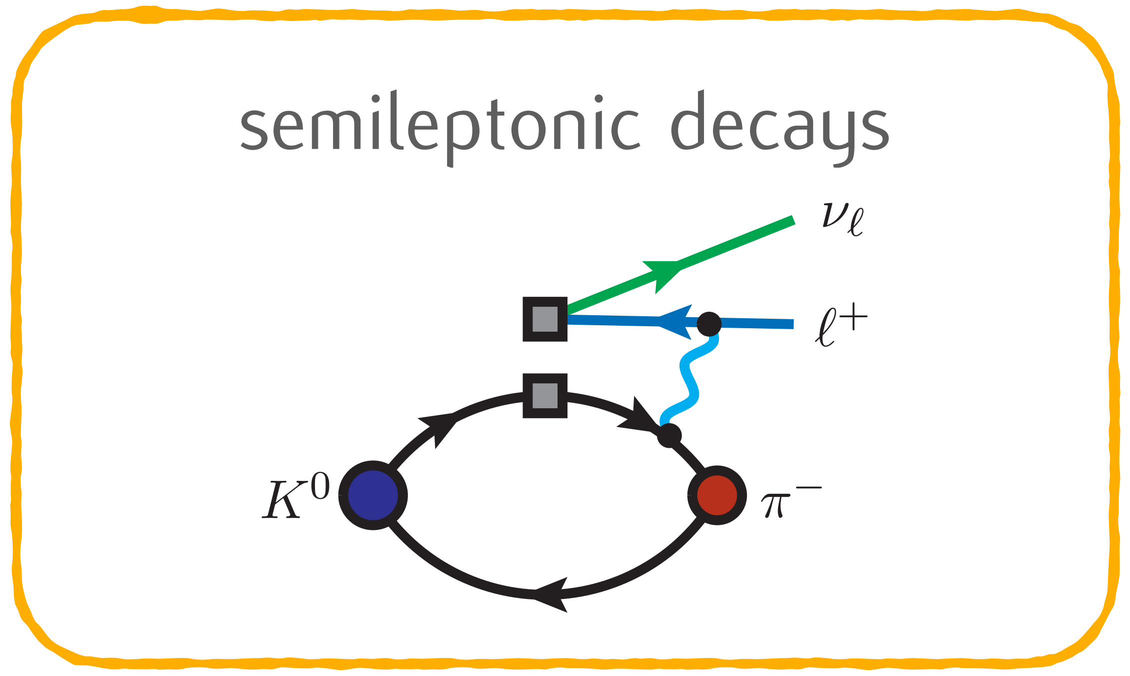
N.Christ et al., [2304.08026]



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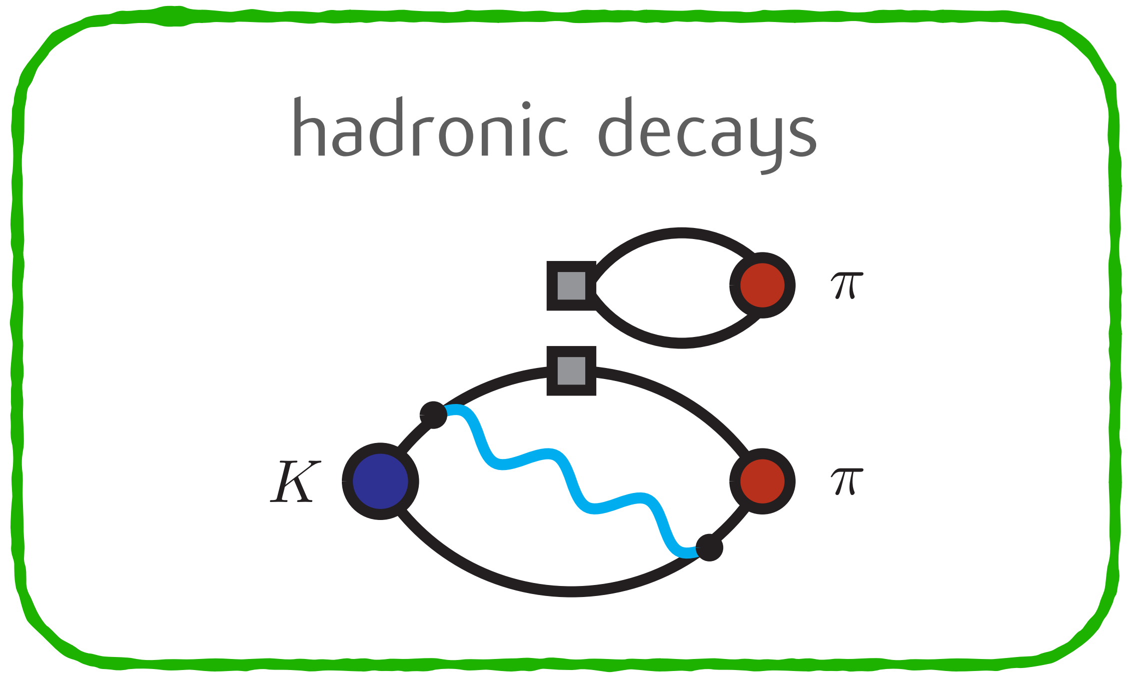


Experimental input from PDG

Pion

$$\Gamma_{\pi\to\mu\nu}/\Gamma_\pi = (99.98770 \pm 0.00004)\%$$

$$\Gamma_\pi = 1/\tau_\pi = 1/[(2.6033 \pm 0.0005) \cdot 10^{-8} s]$$

$$\Gamma_{\pi\to\mu\nu} = 3.8408\,(7) \cdot 10^7\, s^{-1}$$

Kaon

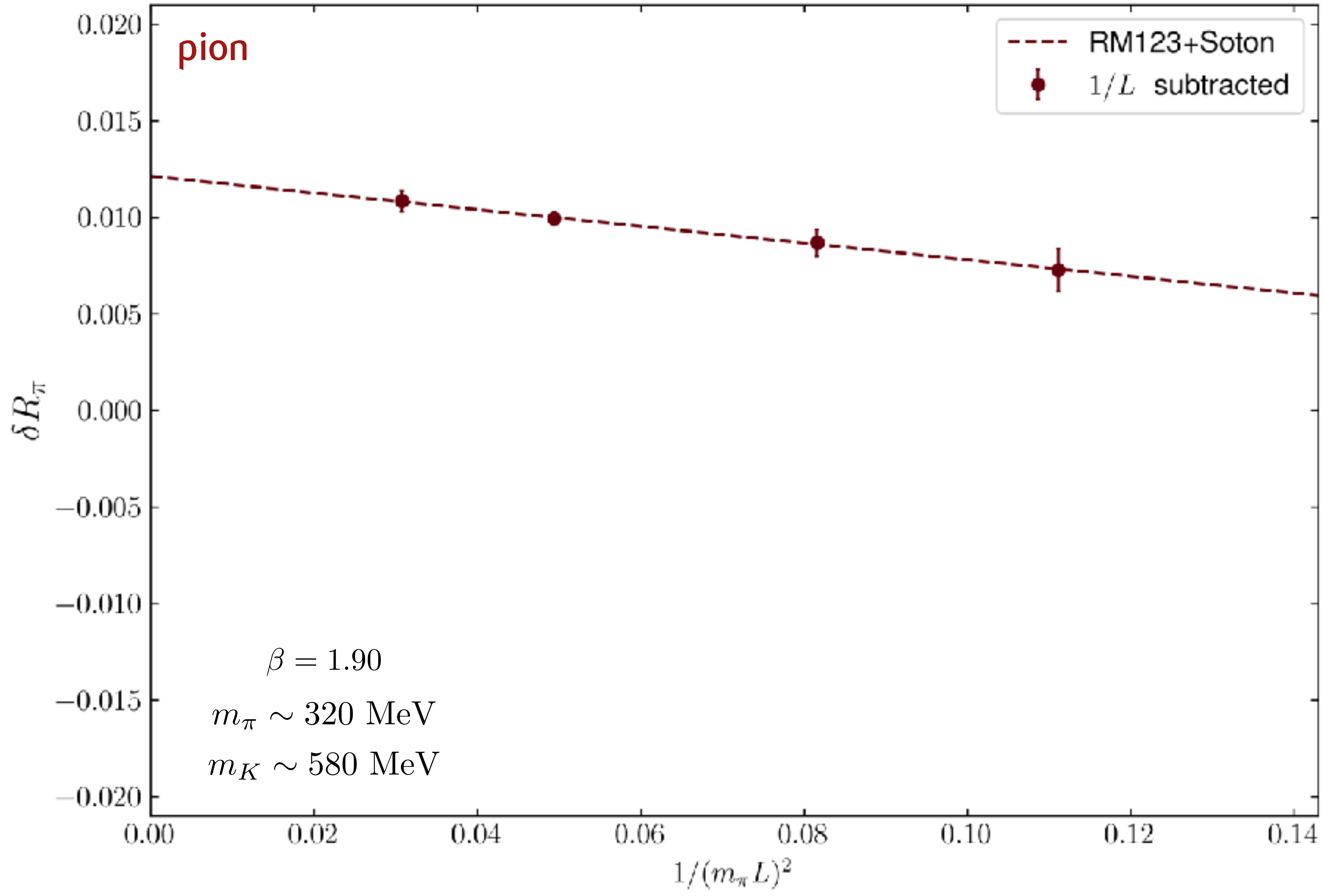
$$\Gamma_{K\to\mu\nu}/\Gamma_K = 63.56 \pm 0.11)\%$$

$$\Gamma_K = 1/\tau_K = 1/[(1.2380 \pm 0.0020) \cdot 10^{-8} s]$$

$$\Gamma_{K\to\mu\nu} = 5.134\,(12) \cdot 10^7\, s^{-1}$$

$$\frac{\Gamma_{K\to\mu\nu}}{\Gamma_{\pi\to\mu\nu}} = 1.3367\,(32)$$

Comparing with RM123+Soton result



from MDC et al., PRD 100 (2019) [RM123+Soton]

Subtracting:

1. universal FVEs up to $1/L$

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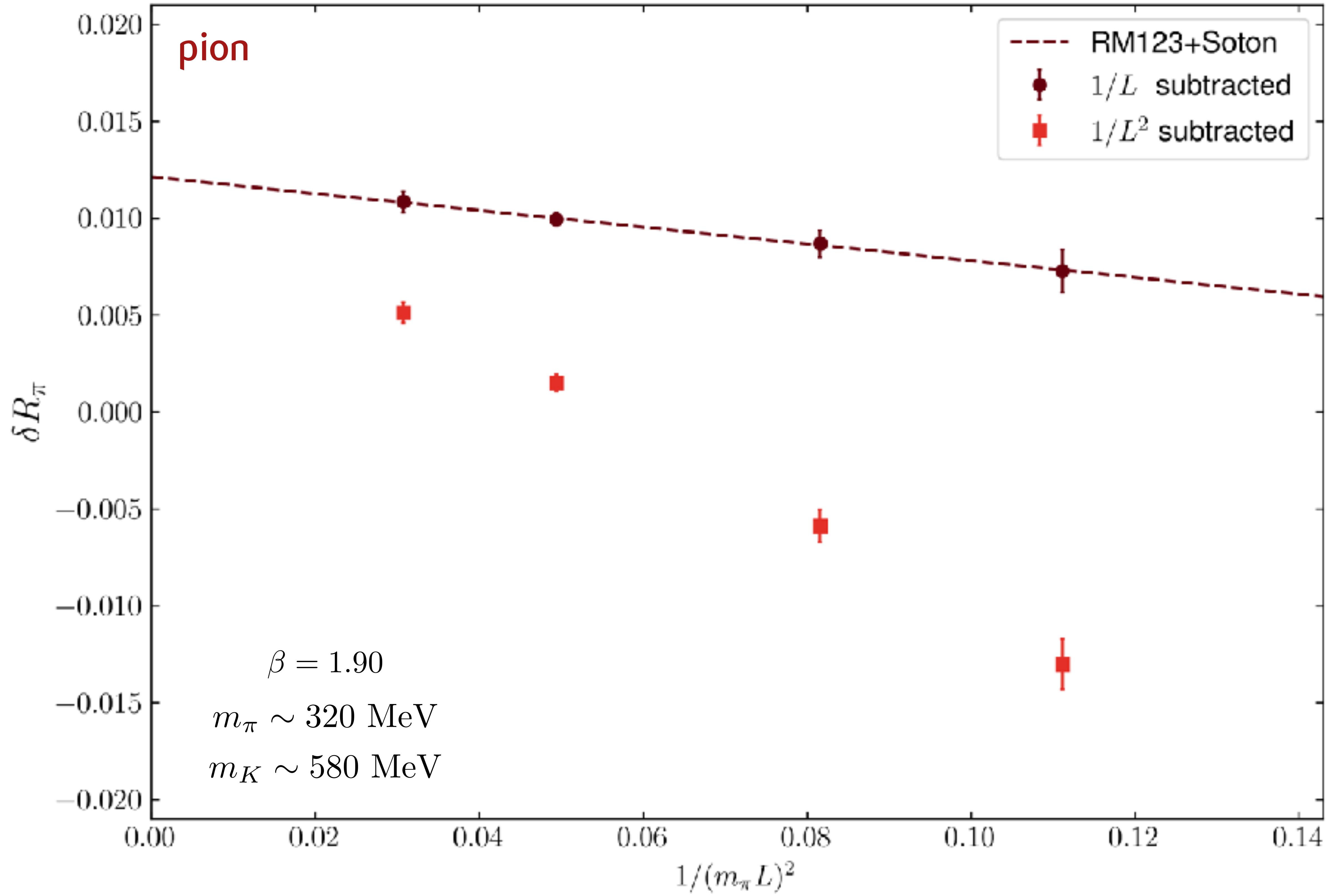

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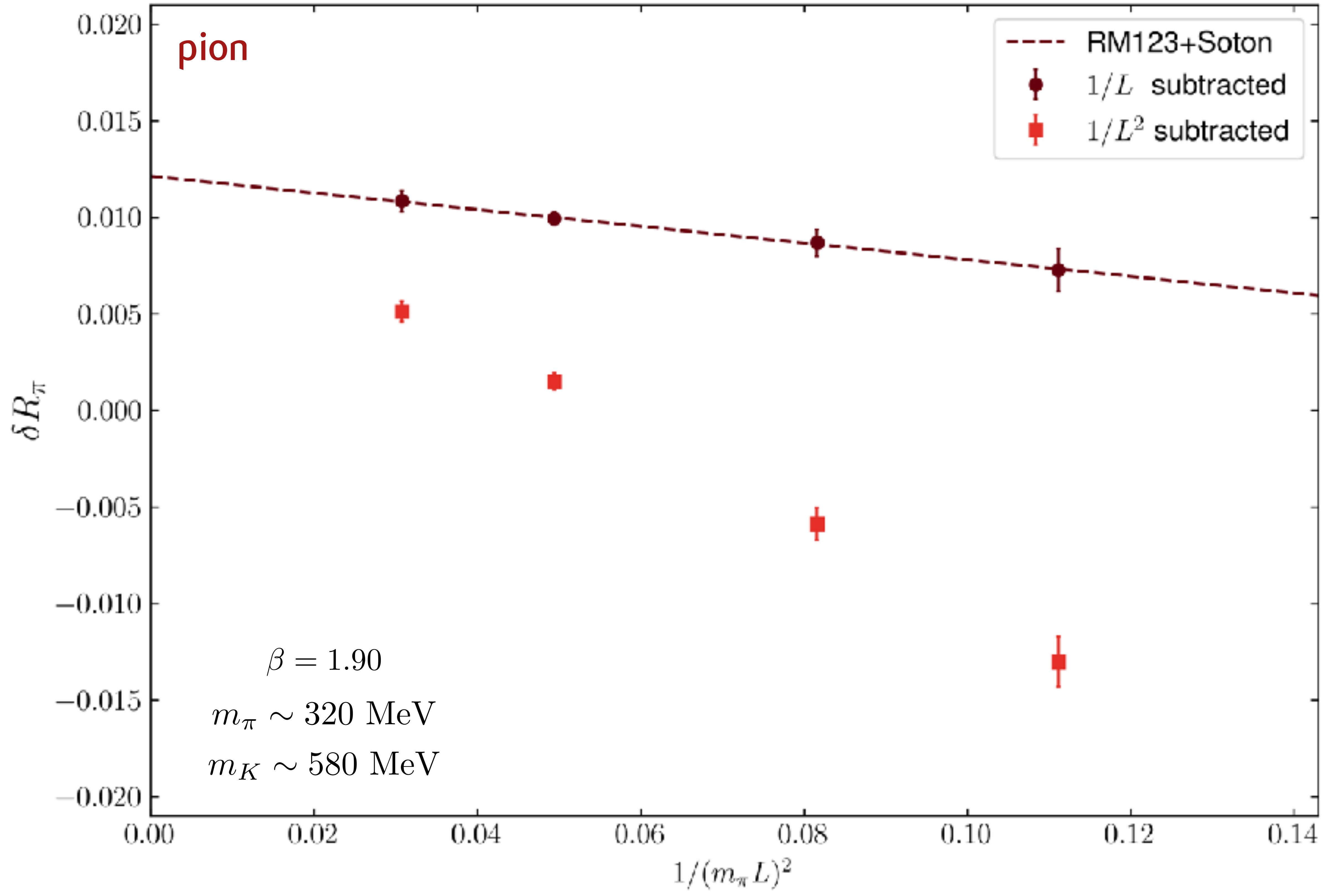

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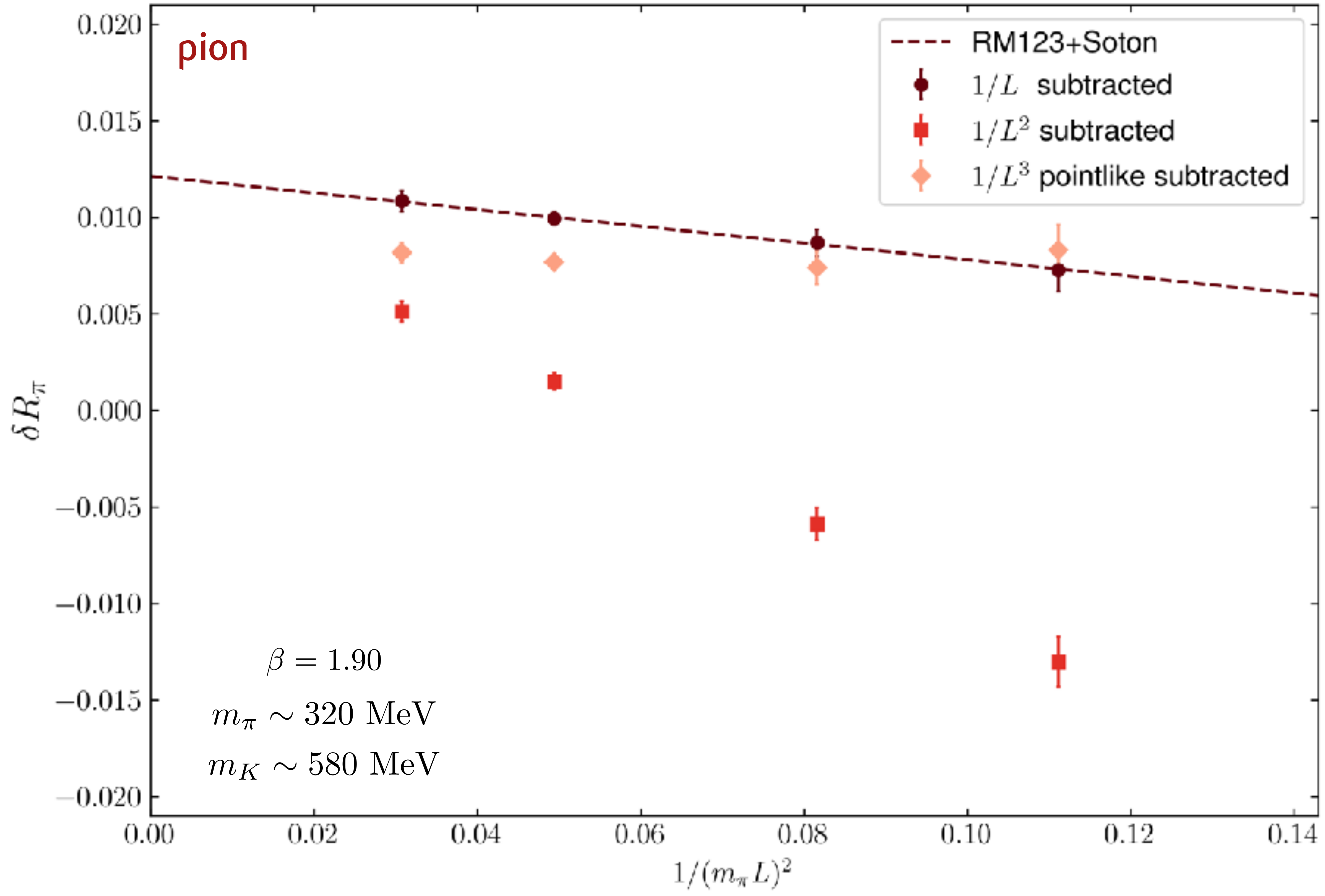
Subtracting:

1. universal FVEs up to $1/L$
2. pointlike $1/L^2$
3. structure-dependent $1/L^2$

> include the pointlike limit $Y_{\rm pt}^{(2)}(L)$ setting $F_A^\pi = 0$, and notice that the structure-dependent contribution at $\mathcal{O}(1/L^2)$ is negligible with respect to the pointlike one. In total, there

MDC et al., PRD 105 (2022)

Comparing with RM123+Soton result



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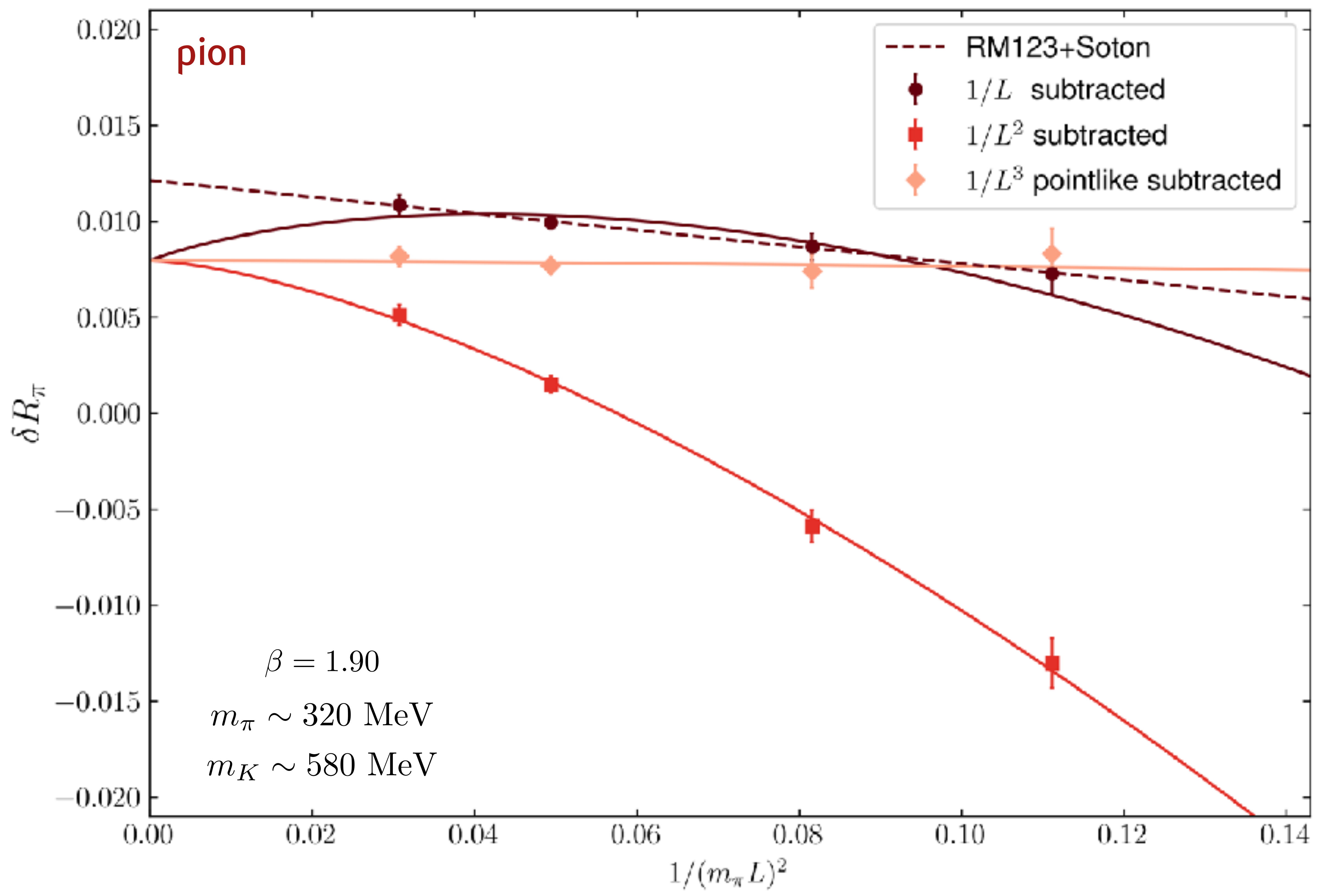
   MDC et al., PRD 105 (2022)

4. pointlike $1/L^3$

$$\Delta^{(3,\rm pt)}(\delta R_P) = \left(\frac{2\alpha}{4\pi}\right) \frac{32\pi^2 c_0 (2 + r_\ell^2)}{(m_P L)^3 (1 + r_\ell^2)^3}$$

MDC et al., PRD 105 (2022)
N. Tantalo et al., [1612.00199v2]

Comparing with RM123+Soton result



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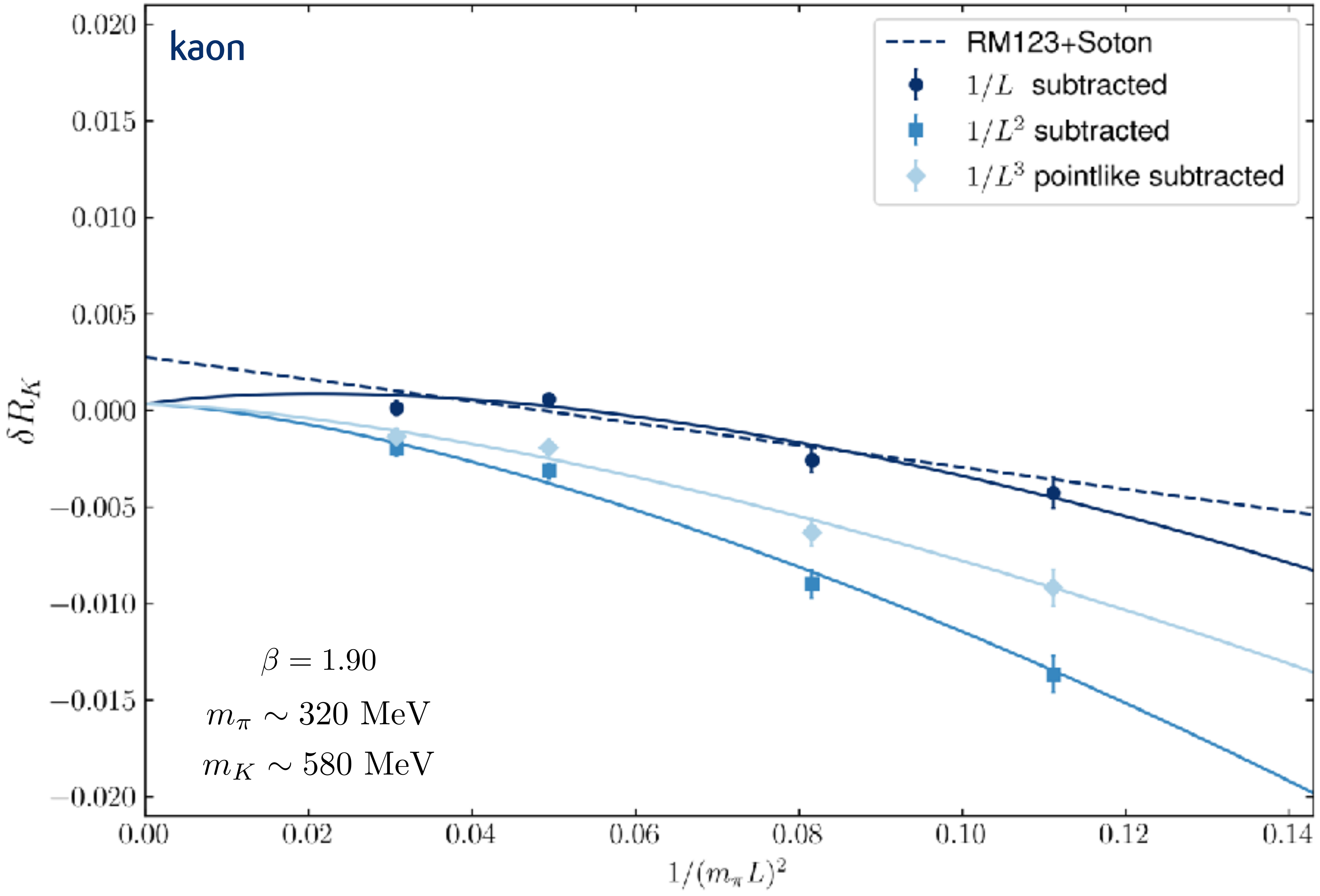

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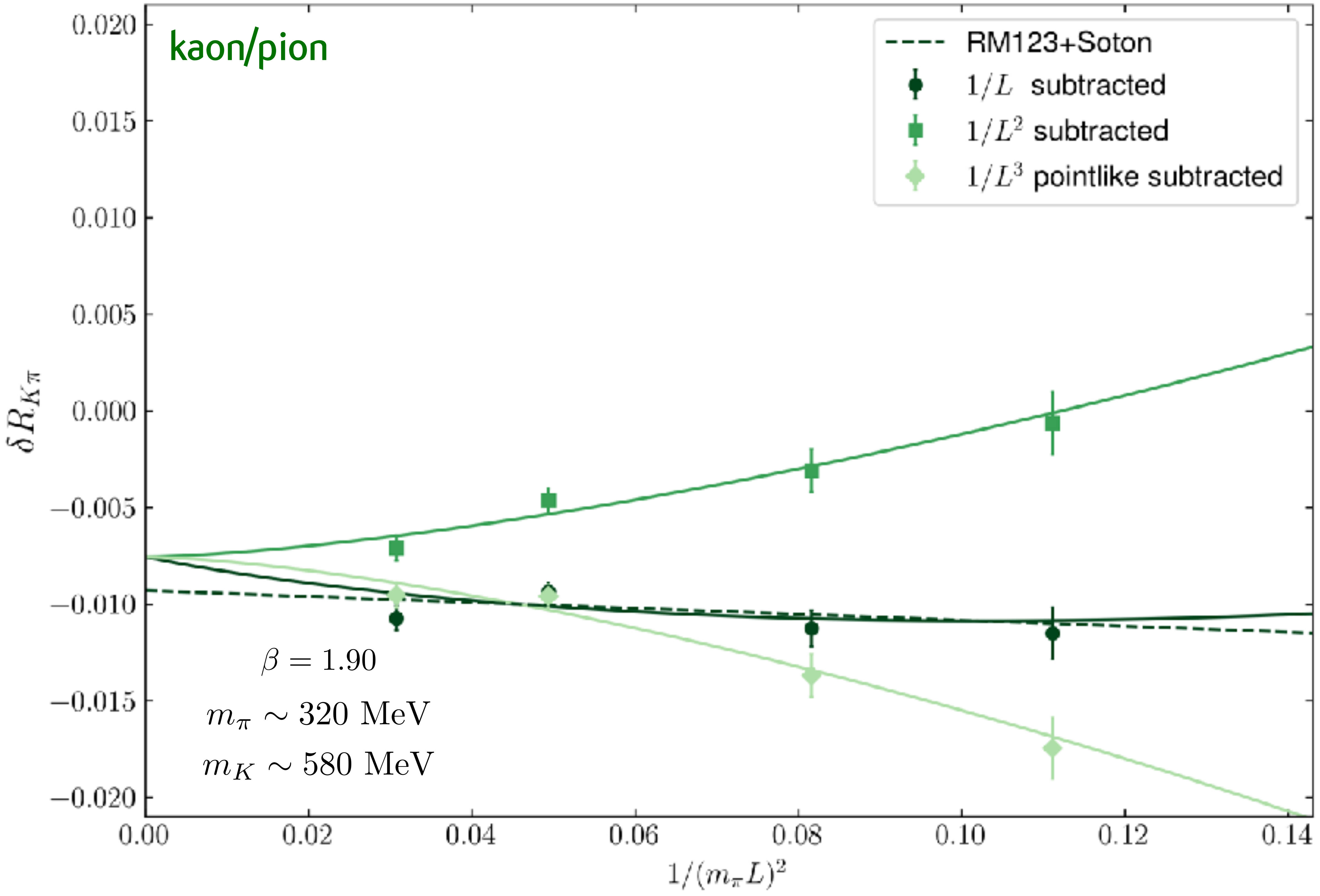

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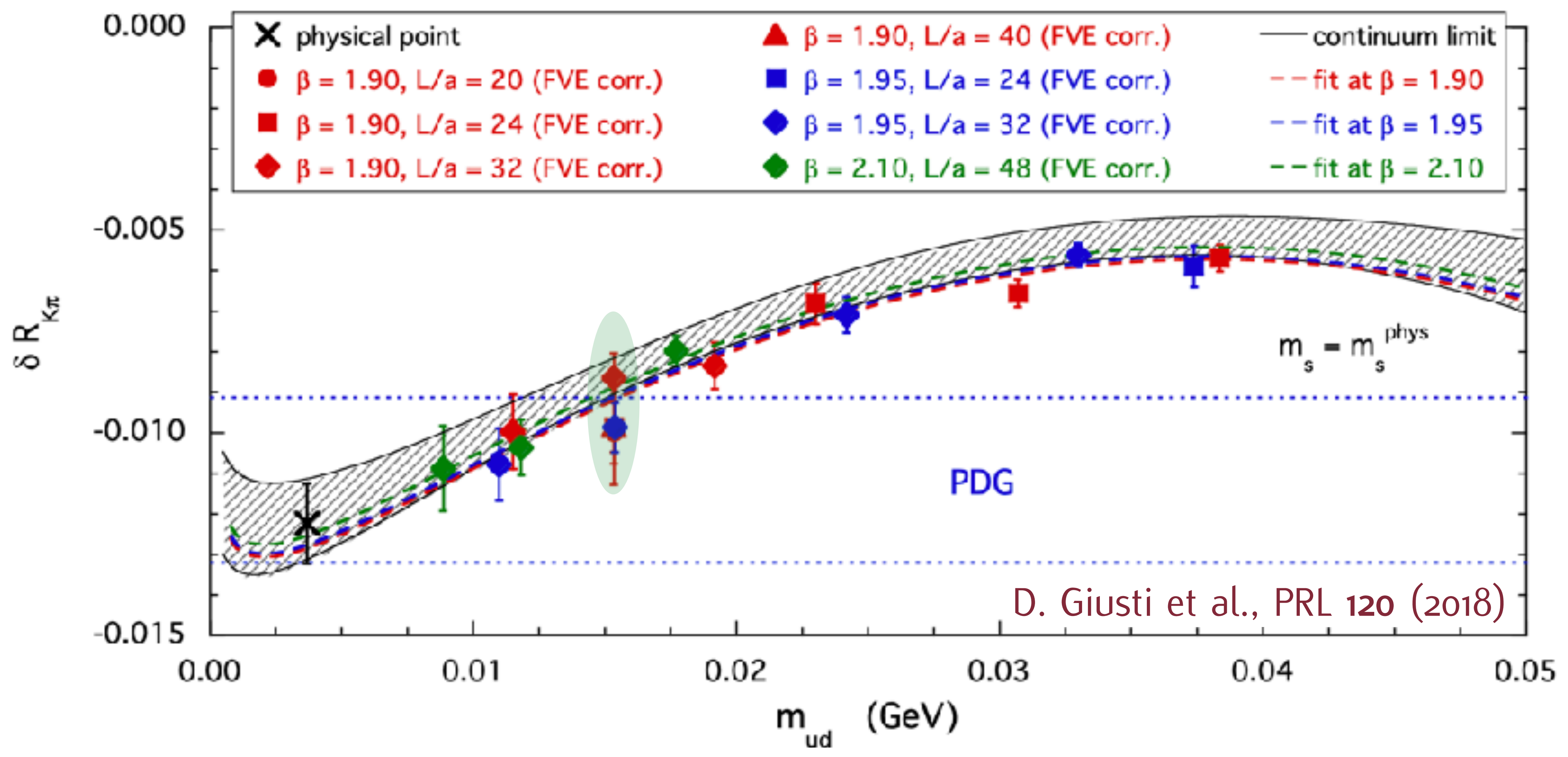
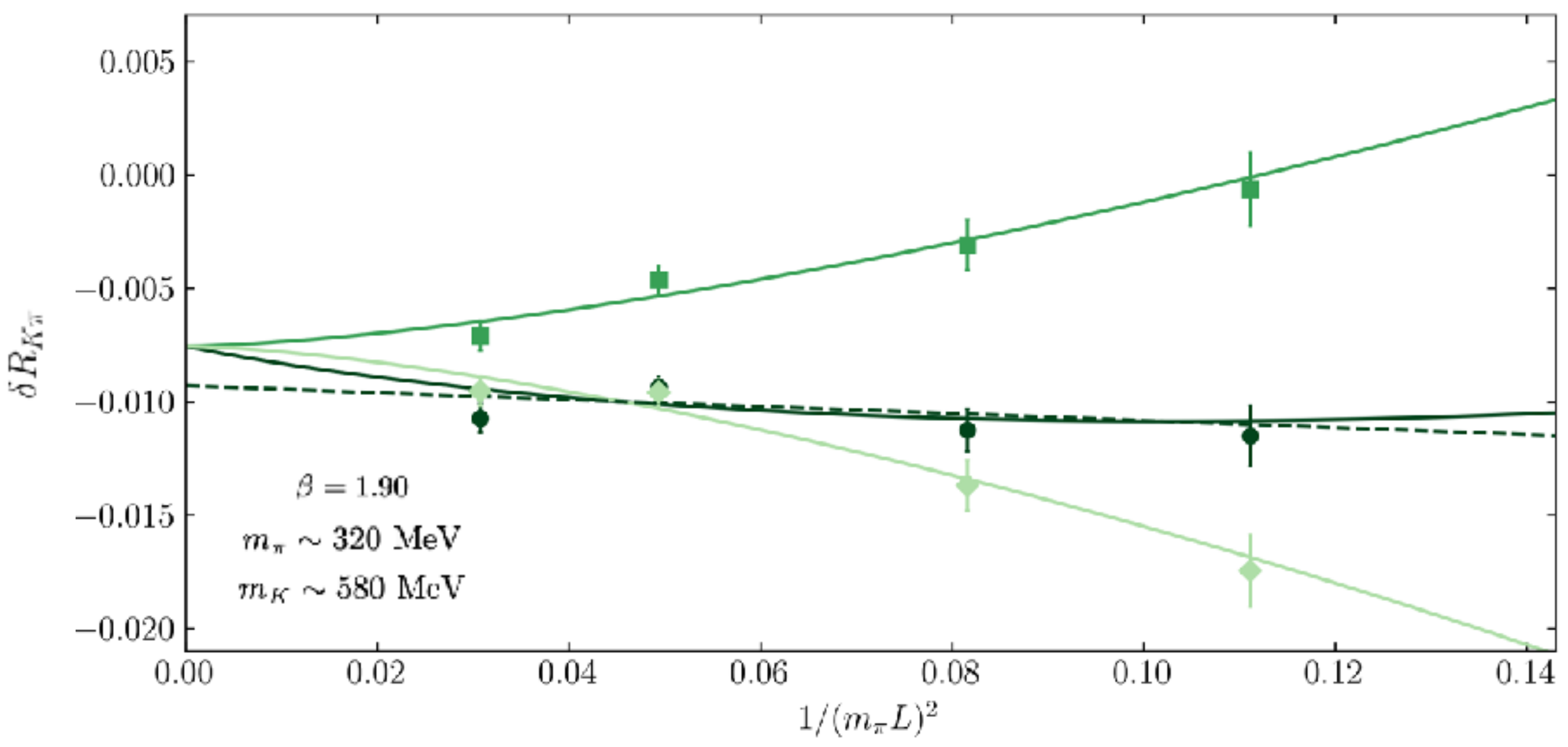
Comparing with RM123+Soton result

Combining all pieces of information





pure speculation!

possible small shift of RM123S result?