# Theoretical calculations and predictions for rare kaon decays


Marc Knecht

Centre de Physique Théorique UMR7332,
CNRS Luminy Case 907, 13288 Marseille cedex 09 - France
knecht@cpt.univ-mrs.fr



Kaons at CERN Workshop, Sept. 11-14, 2023

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- $K \to \gamma^{(*)}\gamma^{(*)}$
- $K \to \pi\gamma^{(*)}$
- $K \to \pi\gamma\gamma^{(*)}$
- $K \to \gamma\gamma\gamma^{(*)}$
- $\cdots$

$$\gamma^* \to \ell^+\ell^- \qquad \ell = e, \mu$$

Decay rates highly suppressed

Large long-distance component

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predictions? $\longrightarrow$ determination of the low-energy constants

Most strategies implemented in the strong sector cannot easily be transposed to the weak sector

Need to take a step back

strong sector: ChPT $\longrightarrow$ three-flavour QCD

weak sector: ChPT $\longrightarrow$ three-flavour QCD "augmented" by four-fermion operators

$$\mathcal{L}_{\mathrm{QCD}}^{\mathrm{B}} + \sum_I C_I^{\mathrm{B}} Q_I^{\mathrm{B}} \to \mathcal{L}_{\mathrm{QCD}} + \sum_I C_I(\nu) Q_I(\nu)$$

SM only required to provide the list of appropriate operators and the values of their "couplings" at some scale $\nu_0 \sim 1$ GeV

problem shifted to the evaluation of the matrix elements of the $Q_I$'s

problem simplifies in the 't Hooft limit $N_c \to \infty$, $N_c \alpha_s = \mathrm{cst}$

$$N_c \to \infty?$$

Large-$N_c$ limit hasn't been very succesful in understanding $\Delta S = 1$ transitions

$-$ fails to provide a quantitative understanding of the $\Delta I = 1/2$ rule

$-$ fails to explain the amount of direct CPV in $K \to \pi\pi$

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Reasons for these failures understood:

— large-$N_c$ limit does not account for large S-wave $\pi\pi$ FS rescattering

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Why then consider large-$N_c$ limit for rare kaon decays?

— in many cases, no $\pi\pi$ in final states

— cancellations are not necessarily at work (case-by-case study)

— may help to understand some features of the amplitude
(QCD SD singularities and how to deal with them, non-VMD features,...)

— may provide some guidance for constructing phenomenological models

# Which QCD SD singularities?

As a template, consider flavour conserving two-photon transitions of neutral spin-0 mesons $\longrightarrow$ described by a transition form factor

$$i\int d^4x e^{iq\cdot x}\langle 0|T\{j^\mu\left(\frac{x}{2}\right)j^\nu\left(-\frac{x}{2}\right)\}|M^0(k)\rangle = \mathcal{F}_M^{\mu\nu}(q_1,q_2),\ q_1 = k/2+q,\ q_2 = k/2-q$$

$$j_\rho = \sum_{q=u,d,s} e_q\bar{q}\gamma_\rho q \qquad e_u = \frac{2}{3},\ e_d = e_s = -\frac{1}{3}$$

satisfying

$$\mathcal{F}_M^{\mu\nu}(q_1,q_2) = \mathcal{F}_M^{\mu\nu}(q_2,q_1) \quad q_{1\mu}\mathcal{F}_M^{\mu\nu}(q_1,q_2) = 0$$

i.e. $[J^{PC} = 0^{++} \to C_M = 0,\, J^{PC} = 0^{-+} \to A_M = B_M = 0]$

$$\mathcal{F}_M^{\mu\nu}(q_1,q_2) = i\frac{A_M(q_1^2,q_2^2)}{M_M^2}Q^{\mu\nu}(q_1,q_2) + iB_M(q_1^2,q_2^2)P^{\mu\nu}(q_1,q_2) + C_M(q_1^2,q_2^2)\epsilon^{\mu\nu\alpha\beta}q_{1\alpha}q_{2\beta}$$

$$P^{\mu\nu}(q_1,q_2) = q_1^\nu q_2^\mu - (q_1\cdot q_2)\eta^{\mu\nu}, \quad Q_{\mu\nu}(q_1,q_2) = q_2^2 q_1^\mu q_1^\nu + q_1^2 q_2^\mu q_2^\nu - (q_1\cdot q_2)q_1^\mu q_2^\nu - q_1^2 q_2^2\eta^{\mu\nu}$$

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$$j_\rho = \sum_{q=u,d,s} e_q\bar{q}\gamma_\rho q \qquad e_u = \frac{2}{3},\ e_d = e_s = -\frac{1}{3}$$

$\longrightarrow$ provides the amplitude for $M^0 \to \gamma\gamma$...

$$\langle\gamma(q_1,\lambda_1)\gamma(q_2,\lambda);\mathrm{out}|M_0(k)\rangle = \varepsilon_\mu^{(\lambda_1)}(q_1)\varepsilon_\nu^{(\lambda_2)}(q_2)\mathcal{F}_K^{\mu\nu}(q_1,q_2)|_{q_1^2=q_2^2=0}$$

$$= \frac{M_M^2}{2}\varepsilon^{(\lambda_1)}(q_1)\cdot\varepsilon^{(\lambda_2)}(q_2)B_M(0,0) + i\epsilon^{\mu\nu\alpha\beta}\varepsilon_\mu^{(\lambda_1)}(q_1)\varepsilon_\nu^{(\lambda_2)}(q_2)q_{1\alpha}q_{2\beta}C_M(0,0)$$

... but also for $M^0 \to \gamma\ell^+\ell^-$ or $M^0 \to \ell_1^+\ell_1^-\ell_2^+\ell_2^-$

$$\langle\ell^+(p_+)\ell^-(p_-)\gamma(q_2,\lambda);\mathrm{out}|M_0(k)\rangle \propto \varepsilon_\nu^{(\lambda_2)}(q_2)\mathcal{F}_K^{\mu\nu}(q_1,q_2)|_{q_1^2=m_{\ell\ell}^2,q_2^2=0}$$

and even for $M^0 \to \ell^+\ell^-$ (involves a convergent loop integral)

$$\lim_{x\to 0} T\{j^\mu\left(\frac{x}{2}\right)j^\nu\left(\frac{-x}{2}\right) \sim -i\frac{1}{2\pi^2}A_\tau(0)\,\epsilon^{\mu\nu\rho\tau}\partial_\rho\frac{1}{x^2} \qquad A_\rho = \sum_{q=u,d,s} e_q^2\bar{q}\gamma_\rho q$$

$$\lim_{q\to\infty}\frac{q^2}{M_M^2}A_M((k/2+q)^2,(k/2-q)^2) = \lim_{q\to\infty} B_M((k/2+q)^2,(k/2-q)^2) \propto \frac{1}{q^2}$$

$$\lim_{q\to\infty} C_M((k/2+q)^2,(k/2-q)^2) \propto \frac{1}{q^2}$$

# Anatomy of $K \to \gamma^* \gamma^*$ at $N_c \to \infty$

What about $K \to \gamma^* \gamma^*$ ?

One expects a similar description, with the additional zero-momentum insertion of the weak $|\Delta S| = 1$ lagrangian of order $\mathcal{O}(G_F)$

$$\mathcal{F}_K^{\mu\nu}(q_1, q_2) \; = \; i \int d^4x \int d^4y \, e^{iq\cdot x} \langle 0 | T\{ j^\mu\left(\frac{x}{2}\right) j^\nu \left(-\frac{x}{2}\right) i\mathcal{L}_{\rm non-lept}^{|\Delta S|=1}(y) \} | K(k) \rangle$$

$$\mathcal{L}_{\rm non-lept}^{|\Delta S|=1}(x) = -\frac{G_{\rm F}}{\sqrt{2}} V_{us}^* V_{ud} \sum_{I=1}^{6} C_I(\nu) Q_I(x;\nu) \; + \; {\rm h.c.}$$

However natural, this definition brings with it some difficulties

$$\lim_{x\to 0} T\{ j^\mu(x) [(\bar{s}^i u_j)_{V-A} (\bar{u}^k d_l)_{V-A}](y)\} \sim -\frac{1}{18\pi^4} \delta_j^k [\bar{s}^i \gamma^\rho (1-\gamma_5) d_l](0) \left( \delta_\rho^\mu \Box - \partial_\rho \partial^\mu \right) \frac{1}{(x^2)^2}$$

This short-distance singularity is no longer integrable

$\mathcal{F}_K^{\mu\nu}(q_1, q_2)$ is not well defined in (three-flavour) QCD!

Does not affect $\varepsilon^{(\lambda_1)\mu}(q_1) \varepsilon^{(\lambda_2)\nu}(q_2) \mathcal{F}_K^{\mu\nu}(q_1, q_2)|_{q_1^2 = q_2^2 = 0}$

Suitable for $K \to \gamma\gamma$ but not, as it stands, for e.g. $K \to \gamma \ell^+ \ell^-$, $K \to \ell^+ \ell^-$,...

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Suitable for $K \to \gamma\gamma$ but not, as it stands, for e.g. $K \to \gamma \ell^+ \ell^-$, $K \to \ell^+ \ell^-$,...

Is this singularity really there? If yes, how is it dealt with?

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$$\mathcal{F}_K^{\mu\nu}(q_1, q_2) = -\frac{G_{\text{F}}}{\sqrt{2}} V_{us} V_{ud} \left[ i \frac{a(q_1,q_2)}{M_K^2} Q^{\mu\nu}(q_1,q_2) + ib(q_1,q_2) P^{\mu\nu}(q_1,q_2) + c(q_1,q_2) \epsilon^{\mu\nu\alpha\beta} q_{1\alpha} q_{2\beta} \right]$$

In the absence of CPV $[C_I(\nu) = C_I^*(\nu)]$

$$K_L \to \gamma^* \gamma^* \; \longleftrightarrow \; c(q_1,q_2) \qquad K_S \to \gamma^* \gamma^* \; \longleftrightarrow \; a(q_1,q_2), b(q_1,q_2)$$

In the large-$N_c$ limit, the four-quark operators factorize into products of quark bilinears

$$
\begin{aligned}
c(q_1^2, q_2^2) \;=\; & +\frac{2}{9}\sqrt{2}\,[C_1 - C_4]\, q_2^2 \left[\mathcal{H}^K(q_1^2, q_2^2) + \tilde{\mathcal{H}}^K(q_1^2, q_2^2)\right] \times \Pi(q_2^2) \\
& +\frac{2}{9}\sqrt{2}\,[C_1 - C_4]\, q_1^2 \left[\mathcal{H}^K(q_2^2, q_1^2) + \tilde{\mathcal{H}}^K(q_2^2, q_1^2)\right] \times \Pi(q_1^2) \\
& + \text{finite}
\end{aligned}
$$

## Divergence contained in the VP-type function

$$
i\int d^4x\, e^{iq\cdot x}\langle 0|T\{[\bar{q}\gamma_\mu q](x)[\bar{q}\gamma_\nu q](0)\}|0\rangle = (q_\mu q_\nu - q^2\eta_{\mu\nu})\Pi(q^2) \quad q = u, d.
$$

$$
\Pi(s) = -N_c \frac{1}{16\pi^2}\frac{8}{3}\frac{1}{d-4}\left[1 + \frac{3}{16}\frac{\alpha_s}{\pi}N_c + \cdots\right] + \text{finite}
$$

$$
(\Gamma^K_{VV})_{\rho\mu}(q,k) \equiv \int d^4x\, e^{iq\cdot x}\langle 0|T\{[\bar{d}\gamma_\rho d](x)[\bar{s}\gamma_\mu d](0)\}|K^0(k)\rangle = i\epsilon_{\rho\mu\alpha\beta}q_1^\alpha q_2^\beta \mathcal{H}^K(q_1^2, q_2^2)
$$

$$
(\tilde{\Gamma}^K_{VV})_{\rho\mu}(q,k) \equiv \int d^4x\, e^{iq\cdot x}\langle 0|T\{[\bar{s}\gamma_\rho s](x)[\bar{s}\gamma_\mu d](0)\}|K^0(k)\rangle = i\epsilon_{\rho\mu\alpha\beta}q_1^\alpha q_2^\beta \tilde{\mathcal{H}}^K(q_1^2, q_2^2)
$$

$$\begin{aligned} c(q_1^2, q_2^2) &= +\frac{2}{9}\sqrt{2}\,[C_1 - C_4]\,q_2^2\left[\mathcal{H}^K(q_1^2, q_2^2) + \tilde{\mathcal{H}}^K(q_1^2, q_2^2)\right] \times \Pi(q_2^2) \\ &\quad +\frac{2}{9}\sqrt{2}\,[C_1 - C_4]\,q_1^2\left[\mathcal{H}^K(q_2^2, q_1^2) + \tilde{\mathcal{H}}^K(q_2^2, q_1^2)\right] \times \Pi(q_1^2) \\ &\quad + \text{finite} \end{aligned}$$

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no divergence in $c(0,0)$!

$$c(q_1^2, q_2^2) = +\frac{2}{9}\sqrt{2}\,[C_1 - C_4]\, q_2^2 \left[\mathcal{H}^K(q_1^2, q_2^2) + \tilde{\mathcal{H}}^K(q_1^2, q_2^2)\right] \times \Pi(q_2^2)$$
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$$+\,\mathrm{finite}$$

Finite part essentially given by

$$\frac{8}{9} C_1 F_K \left[\mathcal{A} + \mathcal{H}(q_1^2, q_2^2, M_K^2)\right]$$

where

$$\mathcal{W}_{\mu\nu}(q_1, q_2) \equiv \int d^4x_1 \int d^4x_2\, e^{i(q_1\cdot x_1 + q_2\cdot x_2)} \langle 0 | T\{[\bar{u}\gamma_\mu u](x_1)[\bar{u}\gamma_\nu u](x_2)\partial^\rho[\bar{u}\gamma_\rho\gamma_5 u](0)\}|0\rangle$$
$$= \epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta \mathcal{H}(q_1^2, q_2^2, (q_1+q_2)^2)$$

and

$$\mathcal{A} = -\frac{N_c}{2\pi^2}$$

is the ABJ anomaly

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and

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link with the usual ChPT expression obtained from the WZW lagrangian for $c(0,0)$, but with $\eta'$ pole included (realistic description requires to go beyond the large-$N_c$ limit)

J. F. Donoghue, B. R. Holstein, Y. C. R. Lin, Nucl. Phys. B 277, 651 (1986)

D. Gomez Dumm, A. Pich, Phys. Rev. Lett. 80, 4633 (1998)

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$$

and

$$
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is the ABJ anomaly

What about $q_1^2 \neq 0$ or even $q_1^2 \neq 0$ and $q_2^2 \neq 0$?

When the photons are virtual, one needs to deal with the SD singularity

Solution has to be provided by the SM

For $K \to \gamma\ell^+\ell^-$ this solution comes in the form of

$$\mathcal{L}_{\text{lept}}^{\Delta S=1}(x) = -\frac{G_{\text{F}}}{\sqrt{2}} V_{us} V_{ud}\, C_{7V}(\nu) Q_{7V}(x)$$

Provides the local counterterm that absorbs the divergence in $\Pi(s)$

$$\Pi(s), C_{7V}^{\text{B}} \longrightarrow \Pi_{\overline{\text{MS}}}(s;\nu), C_{7V}(\nu)$$

The amplitude for $K \to \gamma\ell^+\ell^-$ $\mathcal{M}_\lambda^{K\to\gamma\ell^+\ell^-}(k,p_+,p_-) = \varepsilon_\mu^{(\lambda)}(q)\mathcal{M}_L^\mu(k,p_+,p_-)$

receives, in addition to the non-local component

$$\mathcal{M}_\mu^{\rm nloc}(k,p_+,p_-;\nu) = -ie^2\times\bar{\rm u}(p_-)\gamma^\rho{\rm v}(p_+)\times\frac{1}{s}\times\mathcal{F}_{\mu\rho}(q,k-q;\nu) \qquad s=(k-q)^2=(p_++p_-)^2$$

an additional, "local" component

$$\mathcal{M}_\mu^{\rm loc}(k,p_+,p_-;\nu) = \frac{1}{i}\int d^4x\langle\ell^+(p_+)\ell^-(p_-)|T\{j_\mu(0)i\mathcal{L}_{\rm lept}^{|\Delta S|=1}(x)\}|K(k)\rangle$$

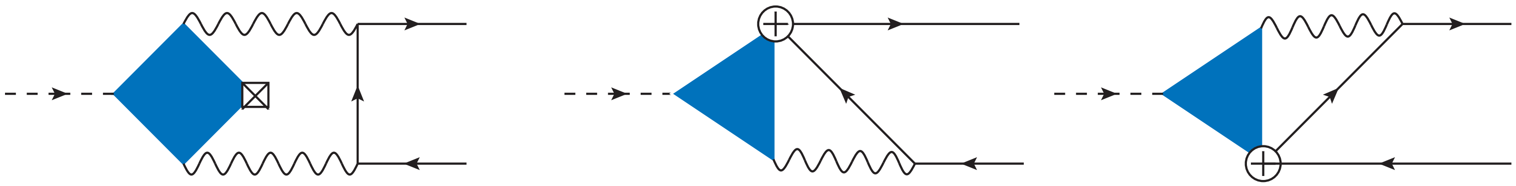
For $K_L$ this means

$$\begin{aligned} c(0,s) \;=\; & +\frac{2}{9}\sqrt{2}\,[C_1-C_4]\,s\left[\mathcal{H}^K(0,s)+\tilde{\mathcal{H}}^K(0,s)\right]\times\Pi_{\overline{\rm MS}}(s;\nu) \\ & +\frac{1}{3}\sqrt{2}\,s\left[\mathcal{H}^K(0,s)+\tilde{\mathcal{H}}^K(0,s)\right]\times\frac{C_{7V}(\nu)}{4\pi\alpha} \\ & +\cdots \end{aligned}$$

$$\nu\frac{d}{d\nu}\left[\frac{2}{3}(C_1-C_4)\Pi_{\overline{\rm MS}}(s;\nu)+\frac{C_{7V}(\nu)}{4\pi\alpha}\right]=0$$

What about $K \to \ell^+\ell^-$?

General structure rather clear



Problem: loop integrals do not converge

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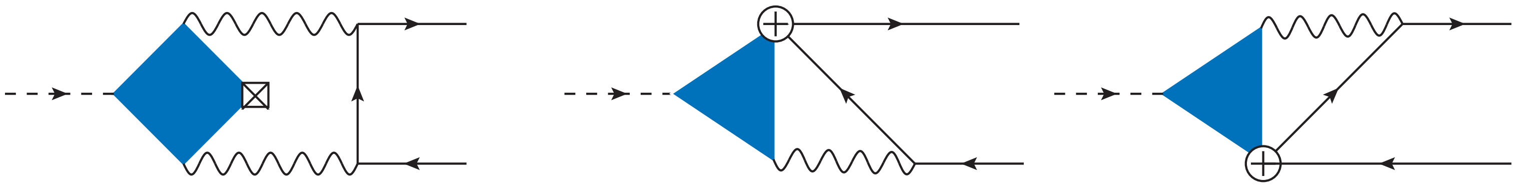
General structure rather clear



Problem: loop integrals do not converge

Requires a counterterm of order $\mathcal{O}(\alpha^2 G_F)$

Cannot be

$$\mathcal{L}_{\rm SD}^{|\Delta S|=1}(x) = \frac{G_F}{\sqrt{2}}\,\frac{\alpha(M_Z)}{2\pi\sin^2\theta_{\rm w}}\left[V_{ts}^* V_{td} Y(x_t) + V_{cs}^* V_{cd} Y_{\rm NL}\right]\left[Q_{7V}(x) - Q_{7A}(x)\right]$$

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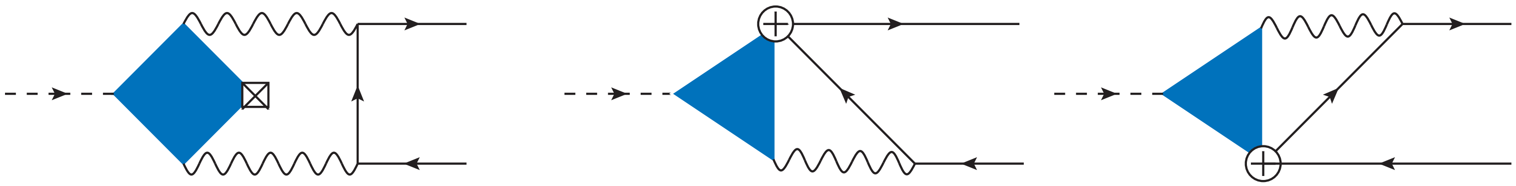

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$$\Pi_{\overline{\mathrm{MS}}}(s;\nu) \sim \frac{1}{4\pi^2}\frac{N_c}{3}\left[-\ln(-s/\nu^2)\left(1+\frac{3}{8}N_c\frac{\alpha_s(\nu)}{\pi}+\cdots\right)+\cdots\right]$$

Features a logarithmic behaviour

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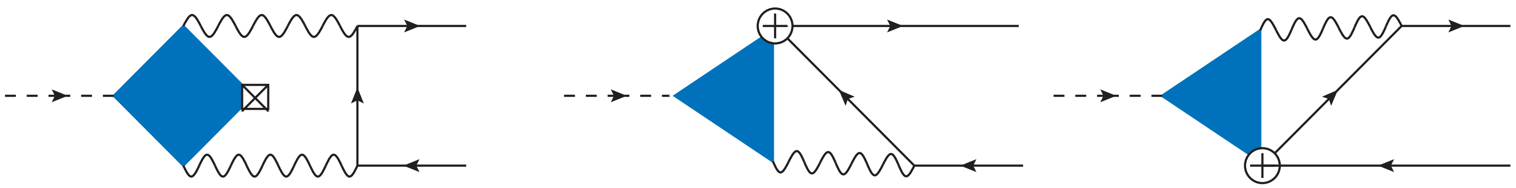

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Features a logarithmic behaviour

Cannot be achieved with a *finite* number of resonances

$K \to \pi\gamma^*$ at $N_c \to \infty$

$$\mathcal{A}(K^+ \to \pi^+ \ell^+ \ell^-) = e^2 \bar{\mathrm{u}}(p_{\ell^-})\gamma^\rho \mathrm{v}(p_{\ell^+}) \times \frac{1}{s}\left[(k-p)_\rho (M_K^2 - M_\pi^2) - s(k+p)_\rho\right] \times \frac{W_+(s)}{16\pi^2 M_K^2}$$

$$\mathcal{A}(K_S \to \pi^0 \ell^+ \ell^-) = e^2 \bar{\mathrm{u}}(p_{\ell^-})\gamma^\rho \mathrm{v}(p_{\ell^+}) \times \frac{1}{s}\left[(k-p)_\rho (M_K^2 - M_\pi^2) - s(k+p)_\rho\right] \times \frac{\mathcal{W}_S(s)}{16\pi^2 M_K^2}$$

$$\mathcal{W}_{+,S} = G_F M_K^2 \left[a_{+,S} + b_{+,S}\frac{s}{M_K^2} + W^{\pi\pi}_{+,S}(s; \alpha_{+,S}, \beta_{+,S})\right]$$

$$
\begin{aligned}
\frac{\mathcal{W}_+(s)}{16\pi^2 M_K^2} \;=\; & +\frac{G_F}{\sqrt{2}} V_{us} V_{ud} \times \Bigg\{ f_+(s) \times \left[ \frac{2}{3} \Pi_{\overline{\rm MS}}(s;\nu)(C_1 - C_4) + \frac{C_{7V}(\nu)}{4\pi\alpha} \right] \\
& + (C_2 + C_4) \times \Bigg[ - F_K \mathcal{P}^\pi(s, M_K^2) - \frac{2}{3} F_\pi \mathcal{P}^K(s, M_\pi^2) + \frac{1}{3} F_\pi \tilde{\mathcal{P}}(s, M_\pi^2) \\
& \quad - \frac{2}{3} \frac{F_K F_\pi}{M_K^2 - M_\pi^2} \left( 3 M_\pi^2 \frac{F_V^\pi(s) - 1}{s} - 2 M_K^2 \frac{F_u^K(s) - 1}{s} + M_K^2 \frac{F_s^K(s) + 1}{s} \right) \Bigg] \Bigg\} \\
& + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathcal{W}_S(s)}{16\pi^2 M_K^2} \;=\; & -\frac{G_F}{\sqrt{2}} V_{us} V_{ud} \times \Bigg\{ f_+(s) \times \left[ \frac{2}{3} \Pi_{\overline{\rm MS}}(s;\nu)(C_1 - C_4) + \frac{C_{7V}(\nu)}{4\pi\alpha} \right] \\
& + (C_1 - C_4) \times \Bigg[ \frac{2}{3} \frac{F_\pi F_K M_K^2}{M_K^2 - M_\pi^2} \, \frac{F_u^K(s) + F_s^K(s)}{s} \\
& \qquad - \frac{F_\pi}{3} \mathcal{P}^K(s, M_\pi^2) - \frac{F_\pi}{3} \mathcal{P}^K(s, M_\pi^2) \Bigg] \Bigg\} \\
& + \cdots
\end{aligned}
$$

$$(\Gamma^{\pi}_{VP})_\rho(q,p) = i\int d^4x\, e^{iq\cdot x}\langle\pi^+(p)|T\{\frac{1}{2}[\bar{u}\gamma_\rho u - \bar{d}\gamma_\rho d](x)[\bar{u}i\gamma_5 d](0)\}|0\rangle$$

$$(\Gamma^{K}_{VP})_\rho(q,k) \equiv i\int d^4x\, e^{iq\cdot x}\langle 0|T\{[\bar{u}\gamma_\rho u](x)[\bar{s}i\gamma_5 u](0)\}|K^+(k)\rangle$$

$$(\tilde{\Gamma}^{K}_{VP})_\rho(q,k) \equiv i\int d^4x\, e^{iq\cdot x}\langle 0|T\{[\bar{s}\gamma_\rho s](x)[\bar{s}i\gamma_5 u](0)\}|K^+(k)\rangle$$

$$q^\rho(\Gamma^{K}_{VP})_\rho(q,k) = -\sqrt{2}F_K\frac{M_K^2}{m_s+\hat{m}}, \qquad q^\rho(\tilde{\Gamma}^{K}_{VP})_\rho(q,k) = +\sqrt{2}F_K\frac{M_K^2}{m_s+\hat{m}}.$$

$$\begin{aligned}(m_s+\hat{m})(\Gamma^{K}_{VP})_\rho(q,k) &= \sqrt{2}F_K M_K^2\frac{(2k-q)_\rho}{(q-k)^2-M_K^2}F_u^K(q^2) + \sqrt{2}F_K M_K^2\frac{F_u^K(q^2)-1}{q^2}q_\rho \\ &\quad + \sqrt{2}[q^2k_\rho-(q\cdot k)q_\rho]\mathcal{P}^K(q^2,(q-k)^2)\end{aligned}$$

$$\begin{aligned}(m_s+\hat{m})(\tilde{\Gamma}^{K}_{VP})_\rho(q,k) &= \sqrt{2}F_K M_K^2\frac{(2k-q)_\rho}{(q-k)^2-M_K^2}F_s^K(q^2) + \sqrt{2}F_K M_K^2\frac{F_s^K(q^2)+1}{q^2}q_\rho \\ &\quad + \sqrt{2}[q^2k_\rho-(q\cdot k)q_\rho]\tilde{\mathcal{P}}^K(q^2,(q-k)^2)\end{aligned}$$

$$\langle K^+(k')|[\bar{u}\gamma_\rho u](0)|K^+(k)\rangle = (k'+k)_\rho F_u^K(q^2), \quad \langle K^+(k')|[\bar{s}\gamma_\rho s](0)|K^+(k)\rangle = (k'+k)_\rho F_s^K(q^2)$$

Lowest-meson-dominance approximation to the large-$N_c$ limit:

one resonance per channel (pseudo-scalar channel is saturated by the kaon state)

$$\mathcal{P}^{\pi}_{\text{LMD}}(q^2,(q+p)^2)=0, \quad \mathcal{P}^{K}_{\text{LMD}}(q^2,(q-k)^2)=0, \quad \tilde{\mathcal{P}}^{K}_{\text{LMD}}(q^2,(q-k)^2)=0.$$

$$F^{\pi}_{V;\,\text{VMD}}(q^2)=\frac{M_\rho^2}{M_\rho^2-q^2}, \quad F^{K}_{u;\,\text{VMD}}(q^2)=\frac{M_\rho^2}{M_\rho^2-q^2}, \quad F^{K}_{s;\,\text{VMD}}(q^2)=\frac{M_\phi^2}{q^2-M_\phi^2}.$$

This simple picture with a single resonance does not work in the case of $\Pi_{\overline{\rm MS}}(s;\nu)$

$$\Pi_{\overline{\rm MS}}(s;\nu)=\frac{2f_\rho^2 M_\rho^2}{M_\rho^2-s}+\cdots$$

The asymptotic logarithmic behaviour must involve an *infinite* number of resonances

$$\Pi_{\overline{\rm MS}}(s;\nu)=\frac{2f_\rho^2 M_\rho^2}{M_\rho^2-s}+\frac{1}{4\pi^2}\frac{N_c}{3}\left[\frac{5}{3}-\ln(M^2/\nu^2)-\Psi\left(1-\frac{s}{M^2}\right)\right]$$

$$\Psi\left(1-\frac{s}{M^2}\right)=-\gamma_E+1+\sum_{n\geq 0}\frac{1}{n+2}\,\frac{s}{s-2M^2-nM^2}$$

$$\Psi\left(1-\frac{s}{M^2}\right)\sim\ln(-s/M^2)-\frac{3}{2}\frac{M^2}{s}+\cdots\qquad[s\to-\infty]$$

In QCD, the Adler function $s\partial\Pi_{\overline{\rm MS}}(s;\nu)/\partial s$ does not have a term $\propto 1/s$ in its asymtotic expansion (in the chiral limit)

S. Peris, M. Perrottet, E. de Rafael, JHEP 05, 011 (1998)

$$M^2=\frac{16\pi^2}{3}\frac{3}{N_c}f_\rho^2 M_\rho^2\sim 1.1\ \mathrm{GeV}$$

## A few (preliminary) results

$$a_+ + a_S = -\frac{1}{\sqrt{2}} V_{ud} V_{us} \frac{16}{3} \pi^2 \frac{F_K F_\pi}{M_\rho^2} (C_1 - 5C_2) > 0$$

$$b_+ + b_S = -\frac{1}{\sqrt{2}} V_{ud} V_{us} 8\pi^2 \frac{F_K F_\pi}{M_\rho^2} \frac{M_K^2}{M_\rho^2} (C_1 - 3C_2) > 0$$

$$a_S = \begin{cases} +1.1 & \text{[NLO, HV]} \\ +0.7 & \text{[NLO, NDR]} \end{cases} \qquad b_S = \begin{cases} +0.8 & \text{[NLO, HV]} \\ +0.5 & \text{[NLO, NDR]} \end{cases}$$

$$\frac{a_S}{b_S} \frac{M_K^2}{M_\rho^2} \sim 0.6$$

$$a_+ < 0 \qquad b_+ < 0 \qquad a_S > 0 \qquad b_S > 0$$

with input from A. J. Buras et al., Nucl. Phys. B 423, 349 (1994)

Work in progress with G. D'Ambrosio and S. Neshatpour

# Summary

ChPT provides the natural framework for the theoretical study of kaon physics

Difficult to make predictions for radiative decay modes of kaons without at least some knowledge of counterterms

Implement large-$N_c$ approach directly in the underlying theory: QCD with three active flavours extended by an appropriate set of four-fermion operators (SM LEEFT)

Form factors for FCNC-induced transitions exhibit a more complex structure than in the strong sector

Necessary to address the issue of QCD short-distance singularities

- either work in QCD with four active flavours and in the limit $V_{td} = 0$ (GIM)

G. Isidori, G. Martinelli, P. Turchetti, Phys. Lett. B 633, 75 (2006)

- or need to consider an infinite tower of zero-width resonances in order to reproduce the logarithmic high-energy behaviour of some QCD correlators

Study of $K \to \pi\gamma^*$ is at in an advanced stage, interesting results ($a_S > 0$)

Other radiative decay modes will be studied within this framework: $K \to \gamma^*\gamma^*$, ...