Lattice calculations and implications for rare kaon decays

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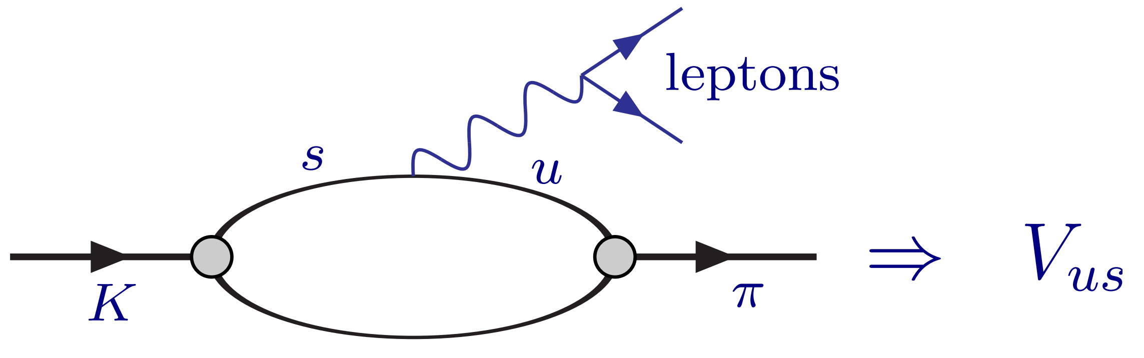
Well-studied quantities in lattice kaon physics

1. Leptonic decay constant f_K



$$\langle 0|A_\mu|K(p)\rangle = f_K p_\mu\,,$$

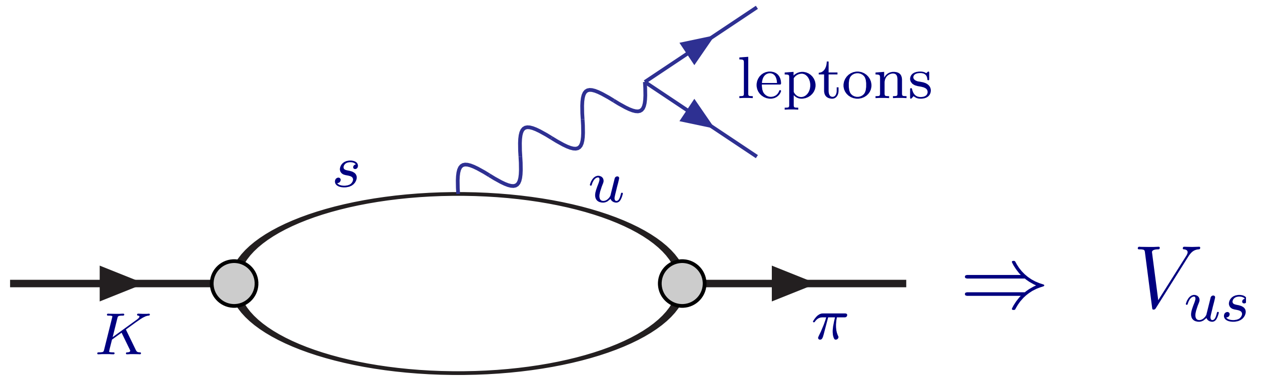
$$\Gamma^{(0)} = \frac{G_F^2|V_{us}|^2 f_K^2}{8\pi} m_K^3 r_\ell^2 \left(1 - r_\ell^2\right)^2$$

$$r_\ell = \frac{m_\ell}{m_K}$$

$f_K = 155.7(3)$ MeV

FLAG Review 2021, Y.Aoki et al., arXiv:2111.09849

2. $K_{\ell 3}$ decays



$$\langle \pi(p_\pi)|\bar{s}\gamma_\mu u|K(p_K)\rangle = f_0(q^2)\frac{m_K^2 - m_\pi^2}{q^2} q_\mu$$

$$+ f_+(q^2)\left[(p_\pi + p_K)_\mu - \frac{m_K^2 - m_\pi^2}{q^2} q_\mu\right]$$

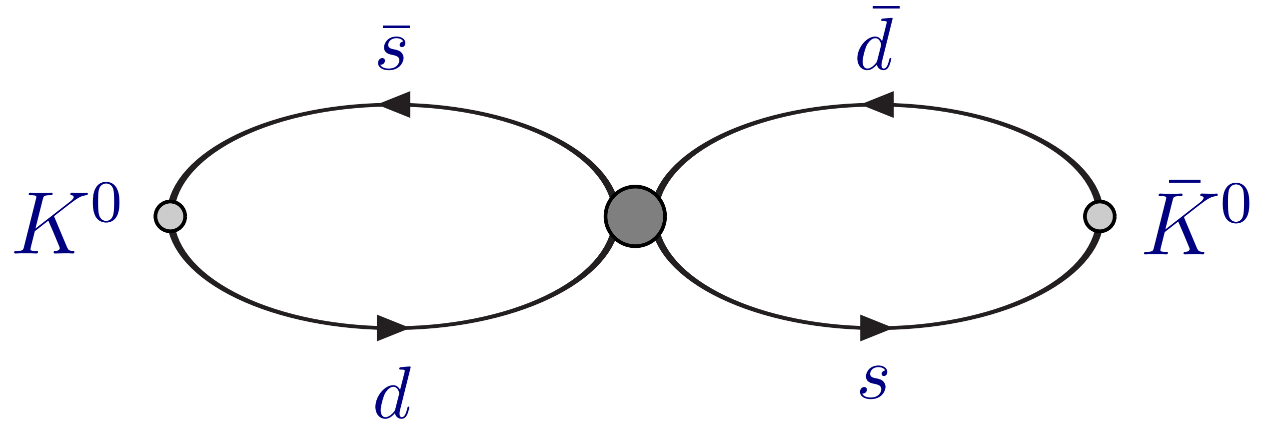
where $q = p_K - p_\pi$.

$f_0(0) = 0.9698(17)$

- Shape of form factor also computed.

FLAG Review 2021, Y.Aoki et al., arXiv:2111.09849 from ETM (arXiv:1602.04113) and FNAL/MILC (arXiv:1809.02827) collaborations.

3. K^0-$\bar{K}^0$ mixing



$$\langle \bar{K}^0|\bar{s}\gamma^\mu(1-\gamma^5)d\;\bar{s}\gamma_\mu(1-\gamma^5)d|K^0\rangle = \frac{8}{3} f_K^2 m_K^2 B_K(\mu)$$

$$\hat{B}_K \equiv \alpha_s(\mu)^{-\gamma_0/2\beta_0}\left(1 + O(\alpha_s(\mu)\right) B_K(\mu)$$

$\hat{B}_K = 0.717(18)(16)$

FLAG Review 2021, Y.Aoki et al., arXiv:2111.09849 from ETM (arXiv:1505.06639) collaboration.

1. Introductory remarks

- In this talk I will focus on topics in kaon physics which we have only relatively recently learned how to handle using Lattice QCD.

- Outline of Talk:

  1. Introductory remarks
  2. $\Delta m_K = m_{K_L} - m_{K_S}$
  3. Long-distance contributions to ϵ_K
  4. The rare decays $K \to \pi \ell^+ \ell^-$
  5. The rare decays $K^+ \to \pi^+ \nu\bar{\nu}$
  6. Radiative decays $K^+ \to \ell^+ \nu_\ell \gamma$
  7. $K \to \pi\pi$ decays
  8. Summary and Conclusions

- The rare FCNC processes and small quantities are excellent places to search for the effects of new physics *Beyond the Standard Model*. I will only be able to sketch the main issues.

- Items 2-5 and 7 undertaken with the RBC-UKQCD collaborations. Item 6 with RM123 (ETMC).

Generic Issues: continuation to Euclidean space - (an illustrative example)

- Imagine that we wish to compute a matrix element of the form $M = \langle f(\mathbf{p}_f)\,|\,O(0)\,|\,i(\mathbf{p}_i)\rangle$.
- (For illustration let $\mathbf{p}_i = \mathbf{0}$ and $|\,i\rangle$ be the lightest single-particle state with its quantum numbers and mass m_i .)
- To prepare for a lattice computation we proceed as follows:

$$M = \int d^3\mathbf{x}_i \int d^3\mathbf{x}_f\, e^{-i\mathbf{p}_f\cdot\mathbf{x}_f} \langle 0|\,\phi_f(\mathbf{x}_f,t_f)\,O(0)\,\phi_i^\dagger(\mathbf{x}_i,t_i)\,|0\rangle \quad \text{with } t_i < 0 < t_f$$

$$= \sum_{n_i,n_f} \langle 0|\,\phi_f(0)|n_f(\mathbf{p}_f)\rangle\,\langle n_f(\mathbf{p}_f)\,|\,O(0)\,|n_i(\mathbf{0})\rangle\,\langle n_i(\mathbf{0})\,|\phi_i^\dagger(0)\,|0\rangle\; e^{-iE_{n_i}|t_i|}e^{-iE_{n_f}t_f}$$

$$\underset{\text{Eucl}}{\rightarrow}\; e^{-m_i|t_i|}\sum_{n_f} \langle 0\,|\,\phi_f(0)\,|\,n_f(\mathbf{p}_f)\rangle\langle n_f(\mathbf{p}_f)\,|\,O(0)\,|\,i(\mathbf{0})\rangle\langle i(\mathbf{0})\,|\,\phi_i^\dagger(0)\,|\,0\rangle\; e^{-E_{n_f}t_f}$$

- If $|\,f\rangle$ is not the ground state then its contribution to the correlation function at large t_f is subleading and methods have to be developed to extract the required matrix element, e.g. $K \to \pi\pi$ decays.

Generic Issues: Matrix Elements of bilocal operators

- For some interesting processes, e.g. $K \to \pi\nu\bar{\nu}$ decays, we need to evaluate matrix elements of bilocal operators of the form

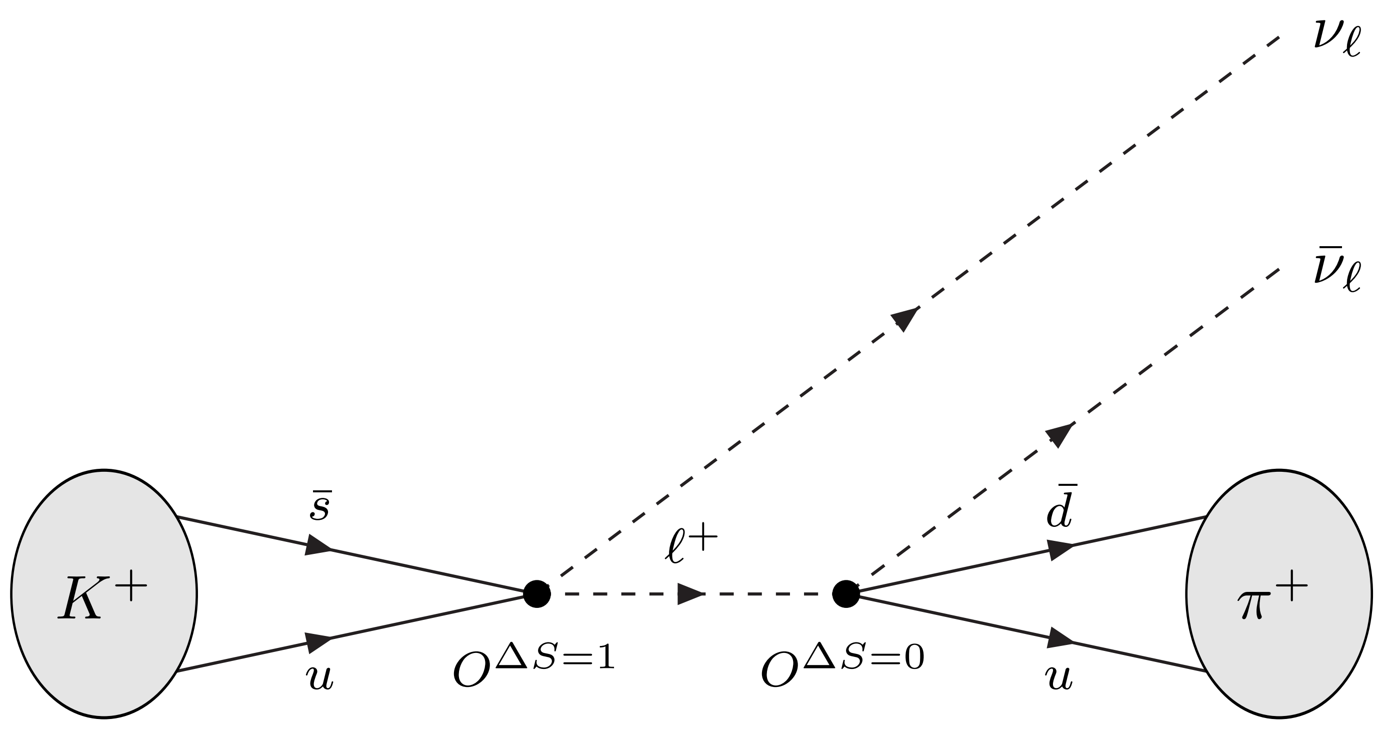
$$\int d^4x \, \langle f \,|\, O_2(x)\, O_1(0) \,|\, i \rangle .$$

- Typically the two operators may be four-fermion weak operators as in the illustrated example or combinations of weak and electromagnetic currents.

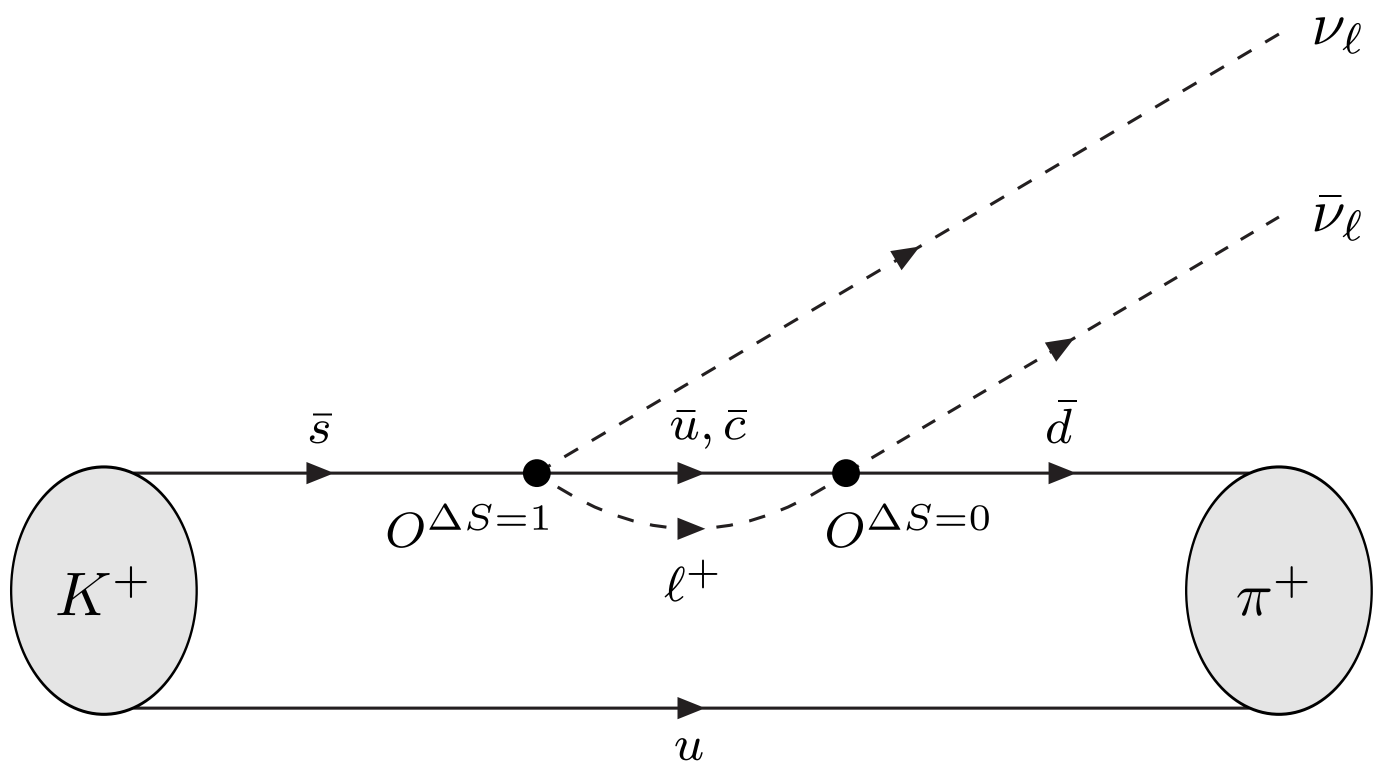
- The non-perturbative renormalisation of local operators $O_{1,2}$ is now standard. However there may be additional ultraviolet divergences as $x \to 0$.

- In the evaluation of Δm_K and $K \to \pi\ell^+\ell^-$ decay amplitudes GIM, gauge and chiral symmetries protect the appearance of additional divergences.

- In the evaluation of ϵ_K and $K \to \pi\nu\bar{\nu}$ decay amplitudes additional divergences do occur and must be renormalised.



Type 1



Type 2

2. $\Delta m_K = m_{K_L} - m_{K_S}$

N.H.Christ, T.Izubuchi, CTS, A.Soni, J.Yu, arXiv:1212.5931

Z.Bai, N.H.Christ, T.Izubuchi, CTS, A.Soni, J.Yu, arXiv1406.0916; Z.Bai, N.H.Christ, CTS, EPJ Web Conf. 175 (2018) 13017,

- Δm_K is given by:

$$\Delta m_K = m_{K_L} - m_{K_S} = 2M_{\bar{0}0} = 2\mathcal{P}\sum_n \frac{\langle \bar{K}^0|H_{\text{eff}}^{\Delta S=1}|n\rangle \langle n|H_{\text{eff}}^{\Delta S=1}|K^0\rangle}{m_K - E_n} = 3.483(6)\times 10^{-12}\,\text{MeV}$$

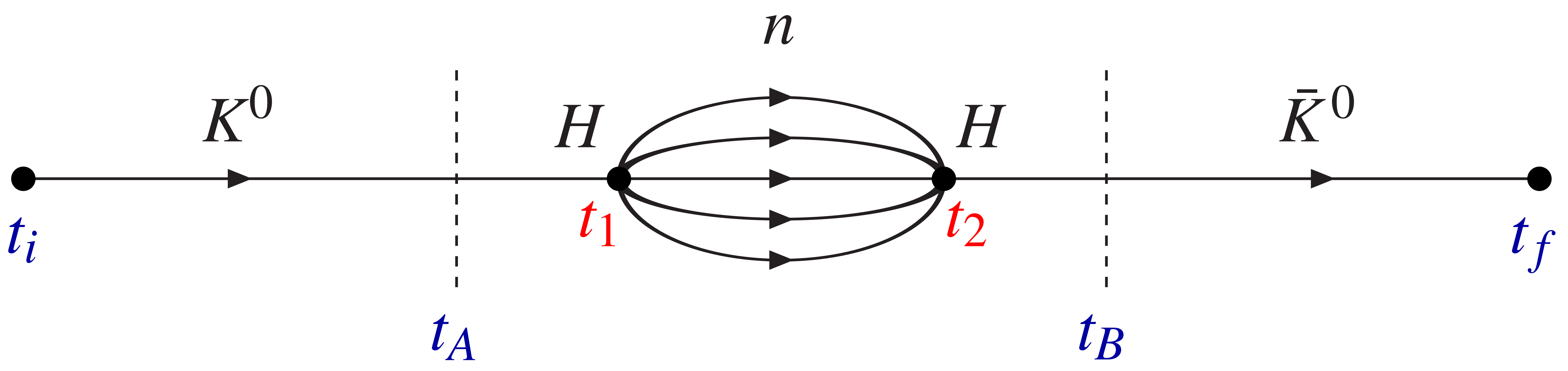
where

$$H_{\text{eff}}^{\Delta S=1} = \sum_{q,q'=u,c} V^*_{q's} V_{qd}\,(C_1 Q_1^{q'q} + C_2 Q_2^{q'q}) + \text{h.c.}$$

and

$$Q_1^{q'q} = (\bar{s}_i q'_j)_{V-A}\,(\bar{q}_j d_i)_{V-A} \quad \text{and} \quad Q_2^{q'q} = (\bar{s}_i q'_i)_{V-A}\,(\bar{q}_j d_j)_{V-A}\,.$$

Δm_K — Correlation Function



- The above correlation function gives ($T = t_B - t_A + 1$)

$$C_4(t_A, t_B; t_i, t_f) = Z_K^2\, e^{-m_K(t_f - t_i)} \sum_n \frac{\langle \bar{K}^0 | H_{\text{eff}}^{\Delta S=1} | n \rangle \langle n | H_{\text{eff}}^{\Delta S=1} | K^0 \rangle}{(m_K - E_n)^2} \left\{ e^{(m_K - E_n)T} - (m_K - E_n)T - 1 \right\}$$

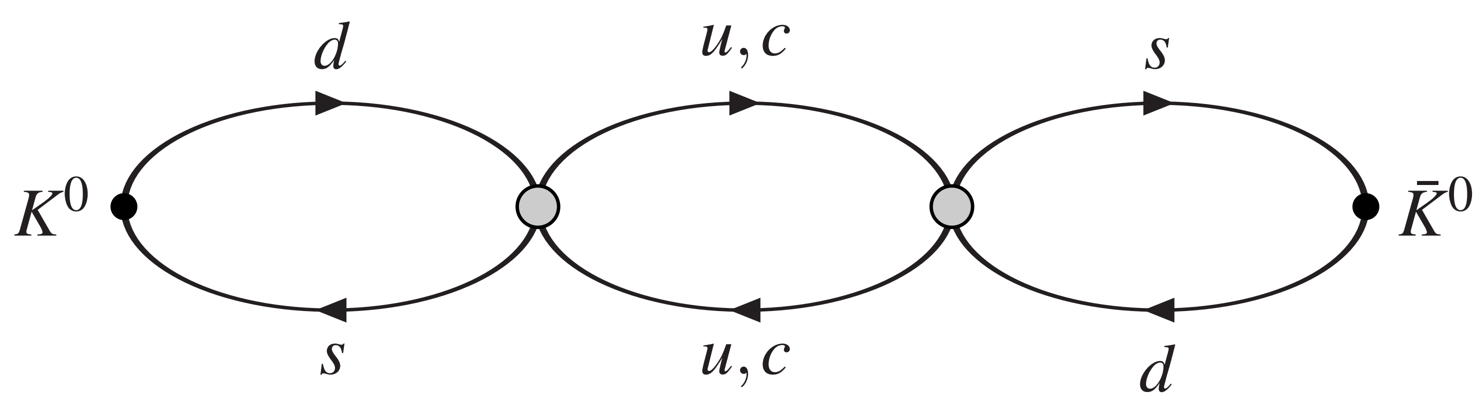
- From the coefficient of T we can deduce Δm_K.

- Generic Issues

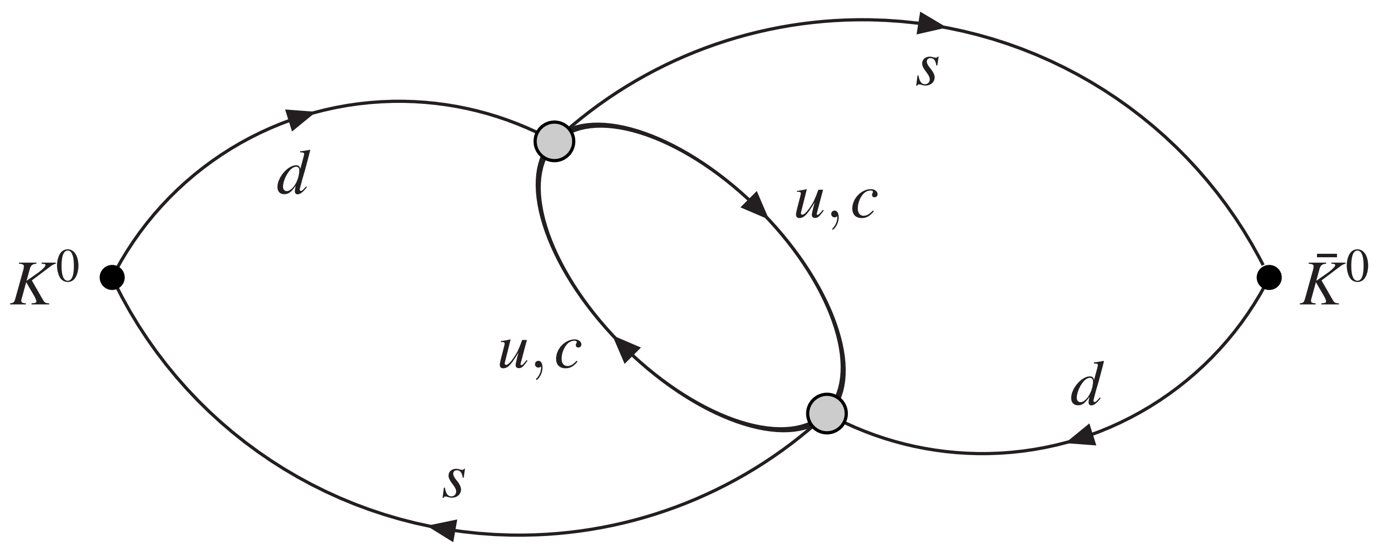
1. The presence of intermediate states with $E_n < m_K$ (e.g. $\pi\pi$ states) leads to terms in the correlation function which grow exponentially with T.
2. In addition there are finite-volume effects which are not exponentially small.

Generalisation of Luscher's quantisation condition , N.H.Christ, X.Feng, G.Martinelli and C.T.S., arXiv:1504.01170

Δm_K — The diagrams



Type 1



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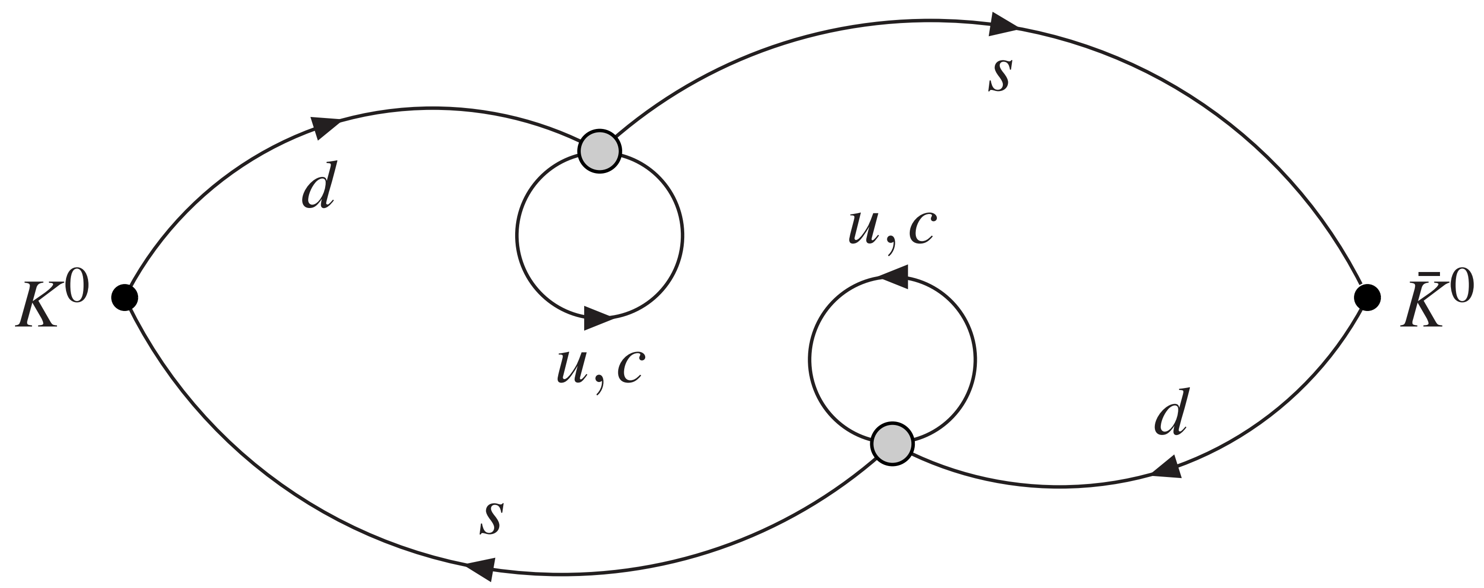

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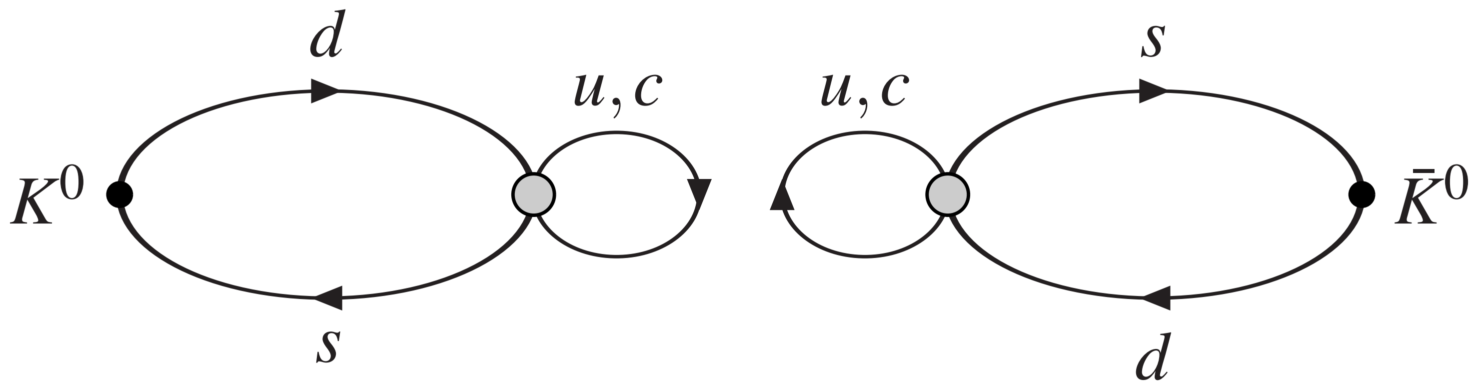
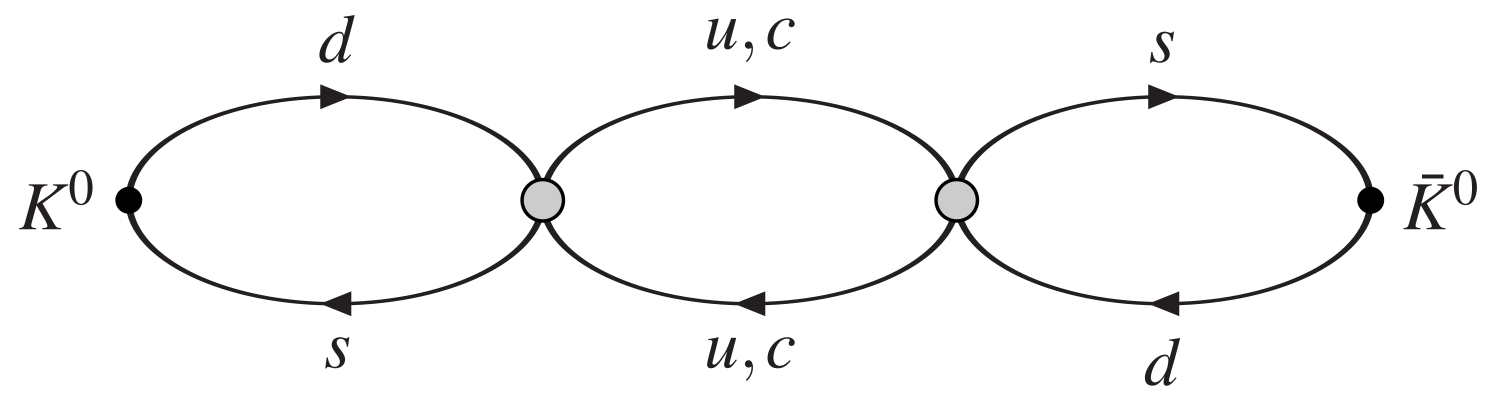


Type 4

Δm_K — Renormalisation



- The renormalisation of each local $\Delta S = 1$ operator is "standard".

- GIM cancellation and the chiral structure of $H_{\text{eff}}^{\Delta S=1}$

  $\Rightarrow$ no new UV divergences.

$$H_{\text{eff}}^{\Delta S=1} = \sum_{q,q'=u,c} V_{q's}^* V_{qd} \, (C_1 Q_1^{q'q} + C_2 Q_2^{q'q}) + \text{h.c.}$$

where $Q_1^{q'q} = (\bar{s}_i q'_j)_{V-A} \, (\bar{q}_j d_i)_{V-A}$ and $Q_2^{q'q} = (\bar{s}_i q'_i)_{V-A} \, (\bar{q}_j d_j)_{V-A}$.

- For example, in the type 1 diagram above, the two $\Delta S = 1$ operators are joined by u and c-quark propagators.

- GIM $\Rightarrow$ the propagators come in the combination

$$S_u - S_c = i \left(\frac{\not{p} + m_u}{p^2 - m_u^2} - \frac{\not{p} + m_c}{p^2 - m_c^2} \right)$$
$$= -i \frac{\not{p}(m_c^2 - m_u^2)}{(p^2 - m_u^2)(p^2 - m_c^2)} + O(\frac{m_u, m_c}{p^2})$$

- Chiral structure of operators $\Rightarrow$ cannot have odd powers of m $\Rightarrow$ no additional UV divergences.

We know how to evaluate Δm_K.

Δm_K — Numerical Results

- The numerical implementation of the theoretical framework has been in progress for some time, first as a proof of principle at unphysical quark masses and more recently at physical quark masses. N.H.Christ, T.Izubuchi, CTS, A.Soni, J.Yu, arXiv:1212.5931

  Z.Bai, N.H.Christ, T.Izubuchi, CTS, A.Soni, J.Yu, arXiv11406.0916; Z.Bai, N.H.Christ, CTS, EPJ Web Conf. 175 (2018) 13017,

  - The emphasis now is on the control and reduction of systematic uncertainties.

- The most recent preliminary result, presented at Lattice 2022, to be compared to the physical value $\Delta m_K = 3.483(6)\,\text{MeV}$, is

$$\Delta m_K = 5.8\,(0.6)_{\text{stat}}\,(2.3)_{\text{syst}} \times 10^{-12}\,\text{MeV}$$

B. Wang, arXiv:2301.01387, journal paper in preparation.

- The result was obtained from a computation using 152 configurations on a $64^3 \times 128$ lattice, with $a^{-1} = 2.36\,\text{GeV}$.
  - Finite-volume correction is estimated to be $\Delta m_K^{\text{FV}} = -\,0.22(7)\,\text{MeV}$.
  - The largest systematic uncertainty, is due to discretisation effects resulting from the large value of m_c.
  - This uncertainty is estimated from extensive scaling studies of quantities at different values of a.

- Ultimately, in order to have the discretisation errors under control, computations will be performed on finer lattices and the results extrapolated to the continuum limit.

Δm_K - Prospects

- To reduce the finite-lattice spacing errors particularly, but not exclusively those resulting from the large value of m_c, requires computations at several values of a, e.g.

$$64^3 \times 256\,,\; a^{-1} = 2.76\,\text{GeV}, \quad \text{cost} = 2\ \text{Exaflop} - \text{hours}$$
$$96^3 \times 384\,,\; a^{-1} = 4.14\,\text{GeV}, \quad \text{cost} = 10\ \text{Exaflop} - \text{hours}$$
$$128^3 \times 512\,,\; a^{-1} = 5.51\,\text{GeV}, \quad \text{cost} = 32\ \text{Exaflop} - \text{hours}$$

$\Rightarrow$ 5 % result in 2026

- Other sources of systematic error (statistical, finite-volume, V_{td}, $V_{ts} \neq 0$ etc.) can be controlled within this precision.

- *"... an ab initio lattice QCD calculation of Δm_K in the standard model which reaches the experimental accuracy is likely not possible within the next decade."*

"Discovering new physics in rare kaon decays"
RBC & UKQCD Collaborations
T.Blum et al., arXiv:2203.10998

- The sizes of the lattices have been modified since this document was posted to $96^3 \times 192\,@\,a^{-1} = 3.0\,\text{GeV}$, $128^3 \times 566\,@\,a^{-1} = 4.0\,\text{GeV}$ and $160^3 \times 640\,@\,a^{-1} = 5.0\,\text{GeV}$.
  - We have started generating the smaller two lattices.

3. Long distance contribution to ϵ_K

Z.Bai, N.H.Christ, J.M.Karpie, CTS, A.Soni and B.Wang, arXiv:2309.01193
Z.Bai and N.H.Christ, PoS(Lattice2015) (2016) 342

- Indirect CP-violation is conventionally parametrised by

$$\epsilon_K = e^{i\phi_\epsilon} \sin\phi_\epsilon \left(\frac{-\mathrm{Im} M_{\bar{0}0}}{\Delta m_K} + \frac{\mathrm{Im} A_0}{\mathrm{Re} A_0} \right)$$

where $\phi_\epsilon = 43.51^\circ$. $\epsilon_K^{\text{exp}} = 2.228(11) \times 10^{-3}$.

- The challenge now is to compute Im $M_{\bar{0}0}$, and in particular the long-distance contribution.

- The $\Delta S = 2$ quantity Im $M_{\bar{0}0}$ is a quadratic expression in $\lambda_i = V_{id}V_{is}^*$ where $i = u, c, t$.

- Using the unitarity relation $\lambda_u + \lambda_c + \lambda_t = 0$, one of the λ_i can be eliminated and traditionally it is λ_u and the effective $\Delta S = 2$ Hamiltonian is conventionally written in the form:

$$H_{\text{eff}}^{\Delta S=2} = \frac{G_F^2}{16\pi^2} m_W^2 \left[\lambda_c^2 \eta_1 S_0(x_c) + \lambda_t^2 \eta_2 S_0(x_t) + 2\lambda_c\lambda_t\eta_3 S_0(x_c, x_t) \right] O_{LL} + \text{h.c.}$$

where $O_{LL} = (\bar{s}d)_{V-A}\,(\bar{s}d)_{V-A}$, $x_i = m_i^2/m_W^2$, the S_0 are Inami-Lin functions and the η_i are QCD perturbative corrections.

Long distance contribution to ϵ_K (cont.)

- For the calculation of the long distance contribution to ϵ_K , i.e. from scales $> O(m_c^{-1})$, it is convenient to use the unitarity to eliminate λ_c , and to rewrite the $\Delta S = 2$ effective Hamiltonian as (setting $x_u = 0$)

$$H_{\text{eff}}^{\Delta S=2} = \frac{G_F^2}{16\pi^2} m_W^2 \left[\lambda_u^2 \eta_1' S_0(0,0,x_c) + \lambda_t^2 \eta_2' S_0(x_t,x_t,x_c) + 2\lambda_u\lambda_t\eta_3' S_0(x_t,0,x_c)\right] O_{LL} + \text{h.c.}$$

- The propagators in each box diagram come in the combination:

$$\sum_{i=u,c,t} \lambda_i S_i = \lambda_u (S_u - S_c) + \lambda_t (S_t - S_c)$$

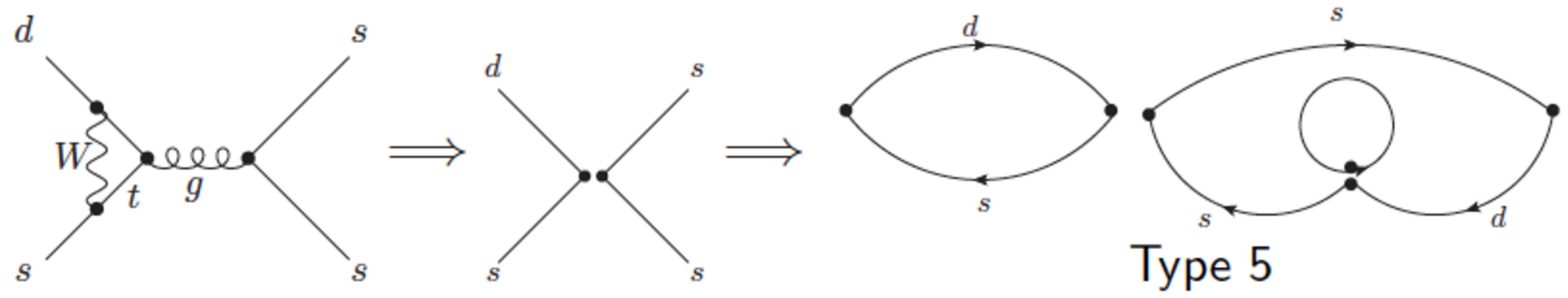
and the notation in $S_0(x_1, x_2, x_3)$ is that in one line we have $S_1 - S_3$ and in the other we have $S_2 - S_3$.

- The reason for this choice is that:

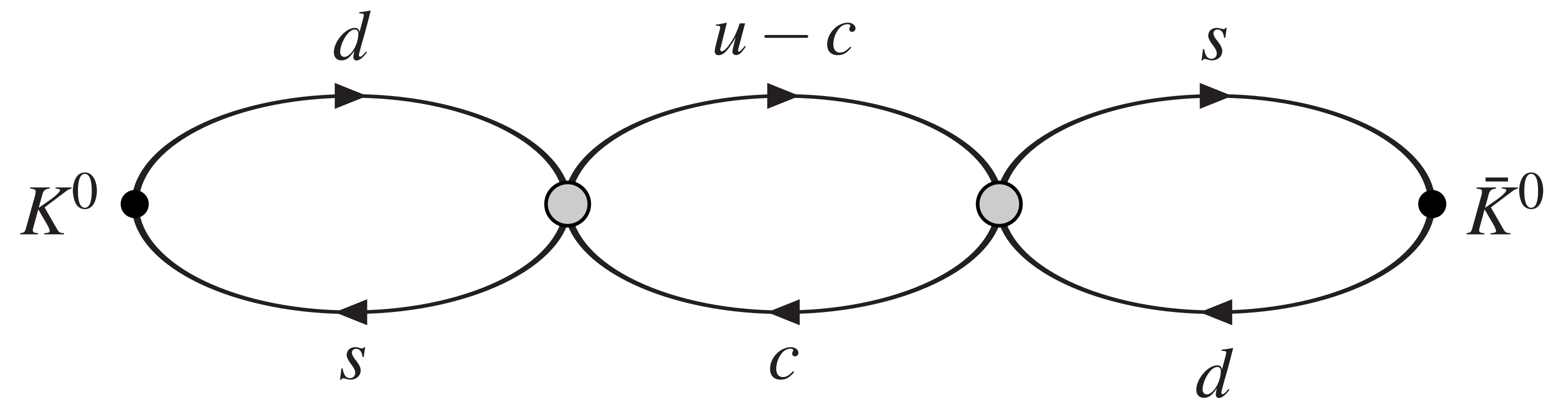
1. λ_u^2 is real and hence the corresponding term does not contribute to Im $M_{\bar{0}0}$.
2. The term proportional to λ_t^2 can be evaluated in perturbation theory.
   (Term with two charm quark propagators CKM suppressed relative to that in the $\lambda_u \lambda_t$ term.)
3. Thus the only term which requires a lattice computation is the one proportional to $\lambda_u \lambda_t$, reducing the cost.

Long distance contribution to ϵ_K (cont.)

- QCD penguins ⇒ additional topology (type 5 diagrams)



- Renormalisation: As an example consider the diagram:

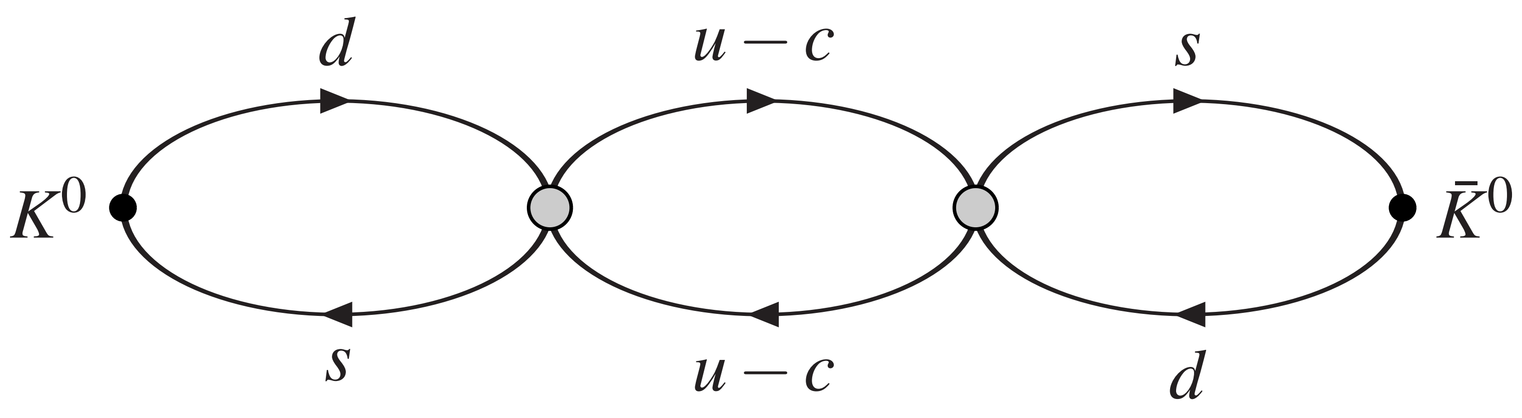


- We start by writing the effective weak Hamiltonian in the $\overline{\mathrm{MS}}$ scheme:

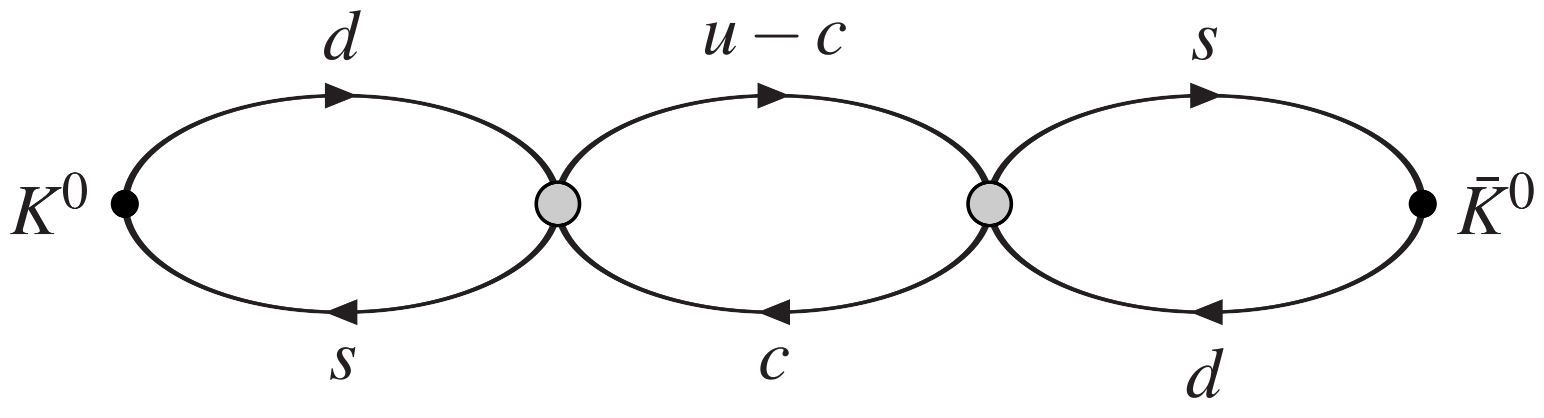
$$H_{ut}^{\Delta S=2} = \frac{G_F^2}{2}\lambda_u\lambda_t \sum_{i=1}^{2}\left\{\sum_{j=1}^{6}\int d^4x\; C_i^{\overline{\mathrm{MS}}} C_j^{\overline{\mathrm{MS}}}\left[\left[\tilde{Q}_i^{\overline{\mathrm{MS}}}(x)\,\tilde{Q}_j^{\overline{\mathrm{MS}}}(0)\right]\right]^{\overline{\mathrm{MS}}} + C_{7i}^{\overline{\mathrm{MS}}}\,O_{LL}^{\overline{\mathrm{MS}}}(0)\right\}$$

- The challenge is to rewrite this expression in terms of matrix elements which can be computed in Lattice QCD.

Long distance contribution to ϵ_K (cont.)



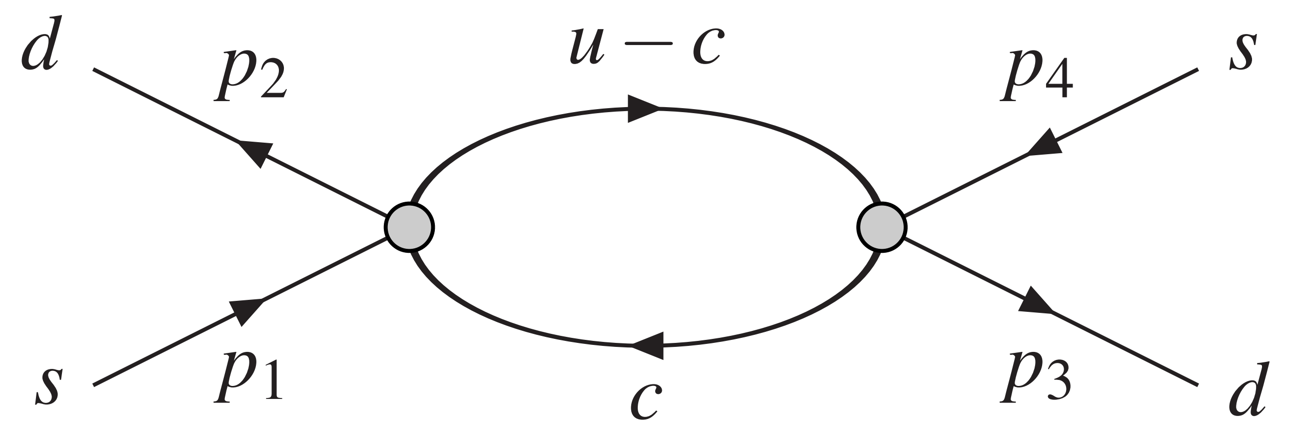
Contribution to Δm_K



Contribution to ϵ_K

- Δm_K: At large loop momentum p we have $\int d^4p \, \frac{1}{p^3} \frac{1}{p^3} \Rightarrow$ convergence.
- ϵ_K: At large loop momenta p we have $\int d^4p \, \frac{1}{p^3} \frac{1}{p} \Rightarrow$ logarithmic divergence.
- This is not surprising since at short distances ϵ_K is dominated by the operator $O_{LL} = (\bar{s}d)_{V-A} (\bar{s}d)_{V-A}$, which has an anomalous dimension.

Long distance contribution to ϵ_K — Renormalisation



- Step 1: Calculate the diagrams non-perturbatively at some chosen kinematics; e.g.

$$\sqrt{2}p_1 = (\mu_{\mathrm{RI}}, \mu_{\mathrm{RI}}, 0, 0), \quad \sqrt{2}p_2 = (\mu_{\mathrm{RI}}, 0, \mu_{\mathrm{RI}}, 0) \quad \sqrt{2}p_3 = (0, \mu_{\mathrm{RI}}, 0, \mu_{\mathrm{RI}}) \quad \sqrt{2}p_4 = (0, 0, \mu_{\mathrm{RI}}, \mu_{\mathrm{RI}})$$

- Step 2: From the answer subtract the matrix element of $O_{LL}^{\mathrm{Lat}}(1/a)$ with the coefficient $X^{\mathrm{Lat}}(1/a, \mu_{\mathrm{RI}})$ chosen such that the difference is zero.
- Step 3: Perform the corresponding calculation perturbatively in $\overline{\mathrm{MS}}$ subtracting the matrix element of $O_{LL}^{\overline{MS}}(\mu_{\overline{\mathrm{MS}}})$ at the same kinematics with a coefficient $Y^{\overline{\mathrm{MS}}}(\mu_{\overline{\mathrm{MS}}}, \mu_{\mathrm{RI}})$ such that the difference is zero.
- Step 4: This procedure is repeated for every pair of operators i, j resulting in the effective weak Hamiltonian:

$$H_{ut}^{\Delta S=2} = \frac{G_F^2}{2}\lambda_u\lambda_t \sum_{i=1}^{2}\left\{\sum_{j=1}^{6} C_i^{\mathrm{Lat}} C_j^{\mathrm{Lat}} \sum_x \left([[\tilde{Q}_i^{\mathrm{Lat}}(x)\tilde{Q}_j^{\mathrm{Lat}}(0)]]^{\mathrm{Lat}} - X_{ij}^{\mathrm{Lat}}(\mu_{\mathrm{RI}}) O_{LL}^{\mathrm{Latt}}(0)\right)\right.$$
$$\left. + \left(C_{7i}^{\overline{\mathrm{MS}}} + \sum_{j=1}^{6} C_i^{\overline{\mathrm{MS}}} C_j^{\overline{\mathrm{MS}}} Y_{ij}^{\overline{\mathrm{MS}}}(\mu_{\overline{\mathrm{MS}}}, \mu_{\mathrm{RI}})\right) Z_{LL}^{\mathrm{Lat}\to\overline{\mathrm{MS}}} O_{LL}^{\mathrm{Lat}}(0)\right\}.$$

Long distance contribution to ϵ_K — Numerical Study

Z.Bai, N.H.Christ, J.M.Karpie, CTS, A.Soni and B.Wang, arXiv:2309.01193

- We know how to compute the long-distance contribution to ϵ_K .
- As a "proof of principle" we have computed ϵ_K , including the long-distance contributions at unphysical kinematics:
  1. 200 gauge configurations on a $24^3 \times 64$ lattice
  2. Domain Wall Fermions and Iwasaki gauge action
  3. $a^{-1} = 1.78\,\text{GeV}$
  4. $m_\pi = 339\,\text{MeV},\ m_K = 592\,\text{MeV},\ m_c^{\overline{\text{MS}}}(2\,\text{GeV}) = 968\,\text{MeV}$
- We find $\epsilon_K^{\text{LD}}(\mu_{\text{RI}} = 2.11\,\text{GeV}) = 0.199\,(0.078)\,e^{i\phi_\epsilon} \times 10^{-3}$ at unphysical masses as above
- A recent result without long-distance corrections is $\epsilon_K^{\text{SD}} = 1.446\,(0.154)\,e^{i\phi_\epsilon} \times 10^{-3}$,

  S.Kim, S.Lee, W.Lee,J.Leem and S.Park, arXiv:2301.12375

  ($4.86\,\sigma$ from the standard model result of $|\epsilon_K| = 2.228\,(0.011) \times 10^{-3}$)
- To translate this result to the RI-SMOM SD one at $\mu_{\text{RI}} = 2.11\,\text{GeV}$ we should add $-0.085\,e^{i\phi_\epsilon} \times 10^{-3}$
- LD contributions appear to be about 5-10% as expected.
- Numerical studies of the long distance contributions just beginning.
- *"This calculation demonstrates that future work should be able to determine this long-distance contribution from first principles with a controlled error of 10 % or less."*
- Snowmass report: *"the discussion of the previous section [Δm_K] of errors and computational goals applies."*

RBC & UKQCD Collaborations, T.Blum et al., arXiv:2203.10998

4. $\mathbf{K^+ \to \pi^+ \ell^+ \ell^-}$ decays

$B(K^+ \to \pi^+ e^+ e^-) = 3.00(9) \times 10^{-7}$ $B(K^+ \to \pi^+ \mu^+ \mu^-) = 9.4(6) \times 10^{-8}$

New Result: $B(K^+ \to \pi^+ \mu^+ \mu^-) = 9.15(8) \times 10^{-8}$ NA62, arXiv:2209.05076

- The hadronic component of the decay amplitude is given by:

$$A^\mu(q^2) = \int d^4x \, \langle \pi^+(p) \,|\, T\,[j^\mu_{\rm em}(0)\, H_W(x)] \,|\, K^+(k) \rangle \,,$$

where $q = k - p$

$$= -i \frac{G_F}{16\pi^2} V(z) \left(q^2 (k+p)^\mu - (m_K^2 - m_\pi^2)\, q^\mu \right) \,,$$

where $z = \dfrac{q^2}{m_K^2}$.

- Analyticity $\Rightarrow$ the form factor V(z) takes the form:

$$V(z) = a_+ + b_+ z + V^{\pi\pi}(z)$$

Measurement	a_+	b_+
E865 - $K_{\pi ee}$	-0.587 ± 0.010	-0.655 ± 0.044
NA48/2 - $K_{\pi ee}$	-0.578 ± 0.016	-0.779 ± 0.066
NA48/2 - $K_{\pi\mu\mu}$	-0.575 ± 0.039	-0.813 ± 0.145
NA62 - $K_{\pi\mu\mu}$	-0.575 ± 0.013	-0.722 ± 0.043

$K^+ \to \pi^+\ell^+\ell^-$ decays (cont)

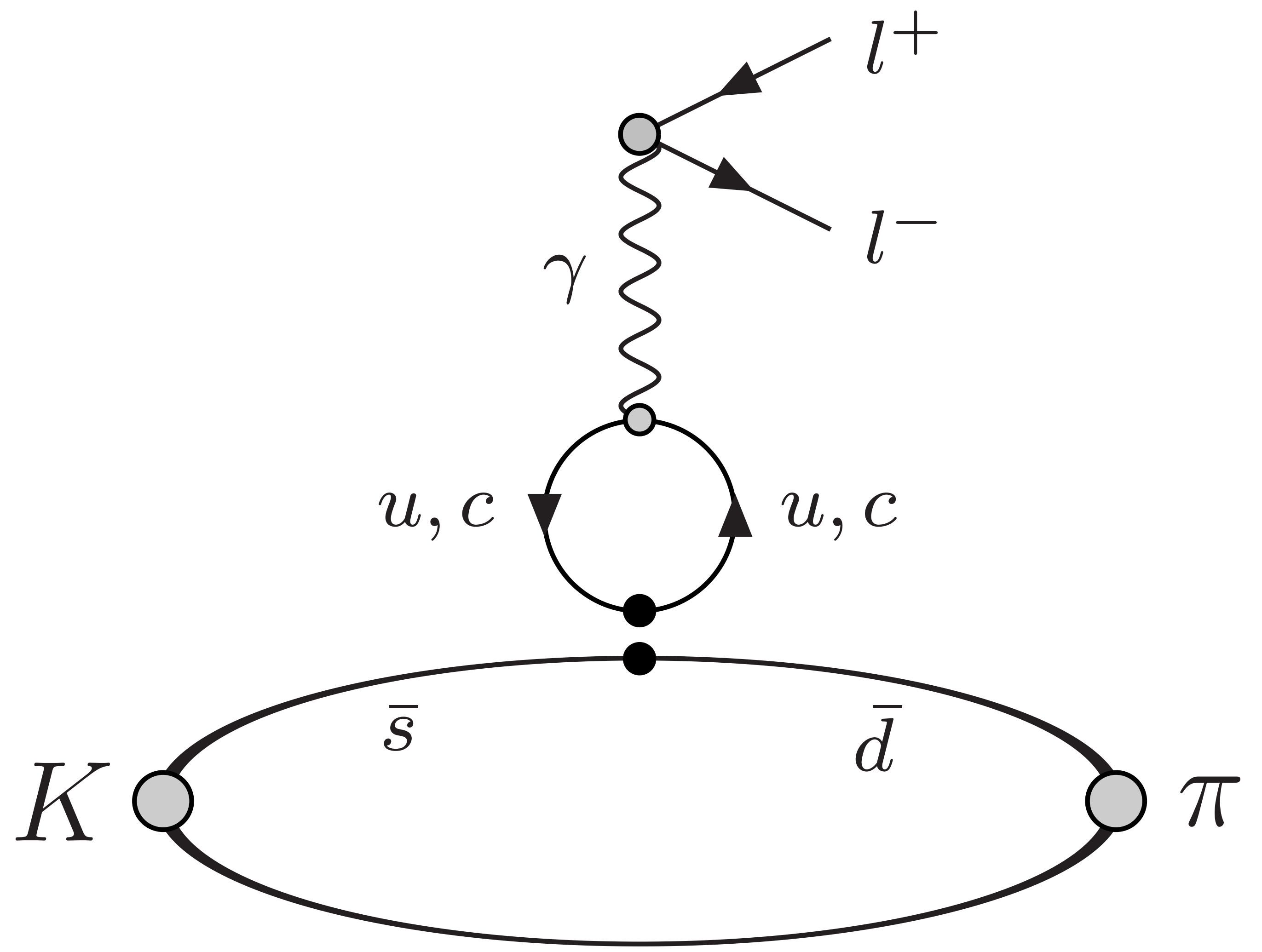
N.H.Christ, X.Feng, A.Portelli and CTS, arXiv:1507.03094

- There are no additional divergences as the two currents approach each other:

1. Quadratic divergences are absent due to gauge invariance ⇒ logarithmic divergence
G.Isidori, G.Martinelli and P.Turchetti, hep-lat/0506026

2. Checked explicitly at one-loop order for Wilson and Clover fermions.

3. Logarithmic divergence cancelled by GIM.



- Thus we understand, in principle, how to evaluate the amplitude

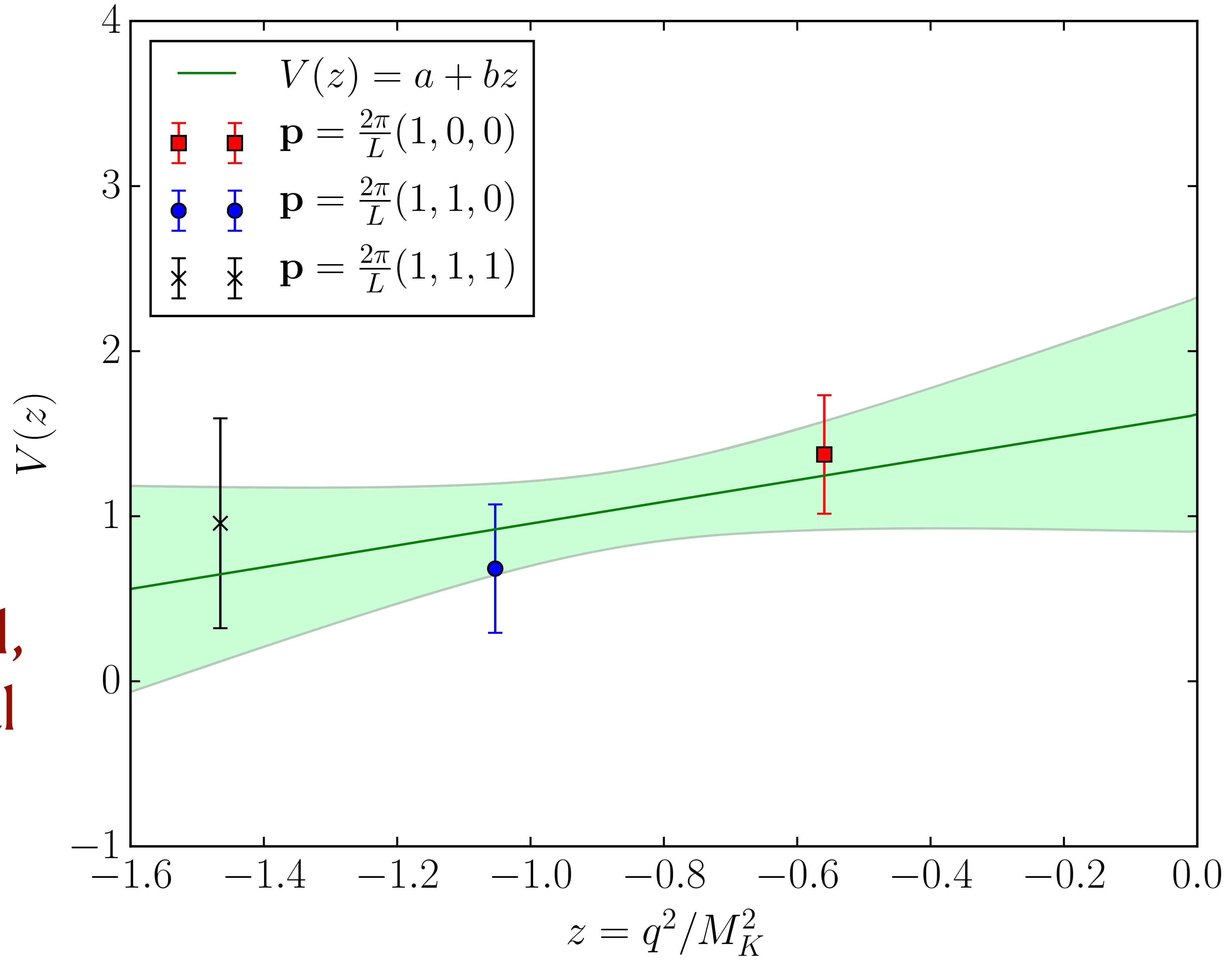
$K^+ \to \pi^+\ell^+\ell^-$ decays — Numerical Studies

N.H.Christ, X.Feng, A.Jüttner, A.Lawson, A.Portelli and CTS, arXiv:1608.07585

1. The first study was an exploratory one, on a RBC-UKQCD ensemble using 128 configurations on a $24^3 \times 64$ lattice with Shamir DWF and the Iwasaki gauge action, $a^{-1} = 1.78\,\text{GeV}$, $m_\pi \simeq 430\,\text{MeV}$ and $m_K \simeq 625\,\text{MeV}$.

   - The calculations were performed with the kaon at rest and $\vec{p}_\pi = 2\pi/L(1,0,0), 2\pi/L(1,0,0), 2\pi/L(1,1,1)$.

$\vec{p}_\pi$	$2\pi/L(1,0,0)$	$2\pi/L(1,1,0)$	$2\pi/L(1,1,1)$
$V(z)$	1.37(36)	0.68(39)	0.96(64)



- With these kinematics $q^2 < 0$.
- As a first exploratory study we considered this to be successful, but, of course, the results cannot be compared to experimental measurements.

$K^+ \to \pi^+\ell^+\ell^-$ decays — Numerical Studies (cont.)

- P.A.Boyle, F.Erben, J.M.Flynn, V.Gülpers, R.C.Hill, R.Hodgson, A.Jüttner, F.Ó hÓgáin, A.Portelli and CTS, arXiv:2202.08795

2. More recently we have performed a calculation of $V(z)$ on a RBC-UKQCD ensemble using 87 configurations on a $48^3 \times 96$ lattice with Möbius DWF and the Iwasaki gauge action, $a^{-1} = 1.73\,\text{GeV}$, $m_\pi = 139.2(4)\,\text{MeV}$, $m_K = 499(1)\,\text{MeV}$.

- Calculations were performed with 3 values of $m_c = 0.25, 0.30$ and 0.35 and the results were extrapolated to the physical value of $m_c = 0.510(1)$.
- The calculations were performed with the kaon at rest and $\vec{p}_\pi = 2\pi/L(1,0,0) \simeq 225\,\text{MeV}$.
- At $z = 0.013(2)$, we obtained $V(z) \simeq a_+ = -0.87 \pm 4.44$.
- Large uncertainty largely due to lack of correlation in the GIM subtraction.
    - Potentially reduced by optimising stochastic estimator for $u - c$ loops.
    - Can also explore 3 flavour theory - corresponding renormalisation to be implemented.
- *"In conclusion, despite obtaining a first physical result with a large uncertainty, we believe that the optimisation of the methodology, combined with the increased capabilities of future computers, should allow for a competitive prediction of the $K^+ \to \pi^+\ell^+\ell^-$ amplitude within the next years."*

$K^+ \to \pi^+\ell^+\ell^-$ decays - Prospects

- Based on these exploratory studies the authors of the Snowmass report conclude that:

"We believe that over the next 5-10 years, lattice QCD will be in a position to produce predictions of a_s, a_+, b_s, b_+ with uncertainties below the 10 % level."

"Discovering new physics in rare kaon decays"
RBC & UKQCD Collabotations
T.Blum et al., arXiv:2203.10998

5. $K^+ \to \pi^+\nu\bar{\nu}$ rare decays

- The decays is dominated by the top quark, and is therefore sensitive to V_{ts} and V_{td} .

- Experimental result:
  NA62 (2016-2018 runs): $B(K^+ \to \pi^+\nu\bar{\nu}) = \left(10.6\,^{+4.0}_{-3.4}\,|_{\text{stat}} \pm 0.9\,|_{\text{syst}}\right) \times 10^{-11}$ NA62, E.Cortina et al., arXiv:2103.15389

- Theoretical Prediction: $B(K^+ \to \pi^+\nu\bar{\nu}) = \left(9.11 \pm 0.72\right) \times 10^{-11}$

  A.J.Buras, D.Buttazzo, J.Girrbach-Noe and R.Knegjens, arXiv:1503.02693

  $B(K^+ \to \pi^+\nu\bar{\nu}) = \left(8.60 \pm 0.42\right) \times 10^{-11}$ A.J.Buras and E.Venturini, arXiv:2109.11032

- To what extent can lattice computations of the long-distance contributions reduce the theoretical uncertainty?

To what extent can lattice calculations reduce the theoretical uncertainty?

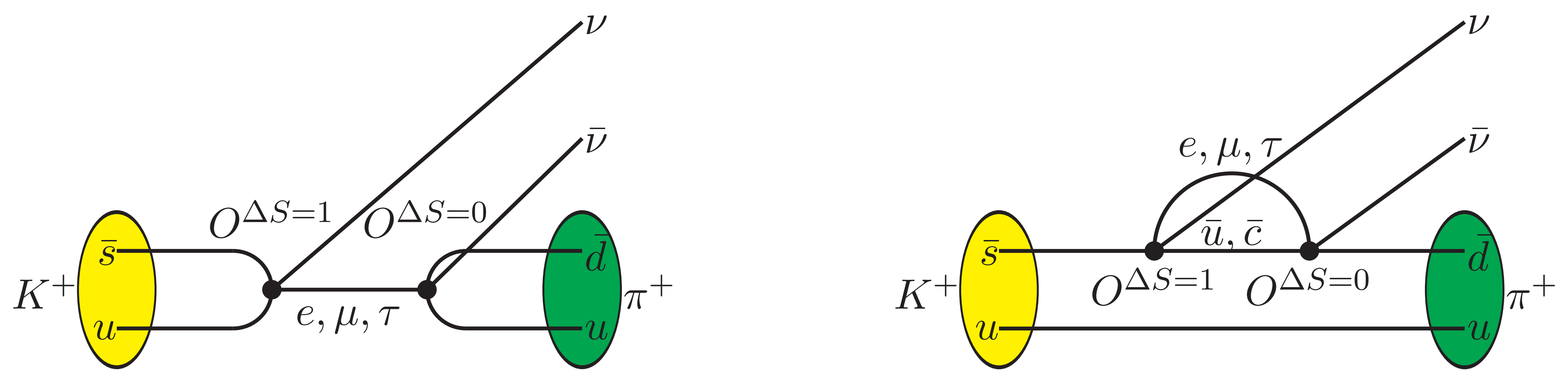
- $K \to \pi\nu\bar{\nu}$ decays are short-distance dominated and the hadronic effects can therefore be determined from CC semileptonic decays such as $K^+ \to \pi^0 e^+ \nu$.
- Long-distance contributions, i.e. contributions from distances greater than $1/m_c$ are negligible for K_L decays and are expected to be of $O(5\%)$ for K^+ decays.
    - K_L decays are therefore among the cleanest places to search for the effects of New Physics.
    - The aim of our studies continues to be the computation of the long-distance effects. (These provide a significant, if probably subdominant contribution to the uncertainty, which is dominated by this on the CKM matrix elements.)
    - Similar techniques to those used for ϵ_K are used to renormalise the additional divergences when the two weak currents approach each other.

- Lattice QCD can provide a first principles determination of the long-distance effects with controlled errors.

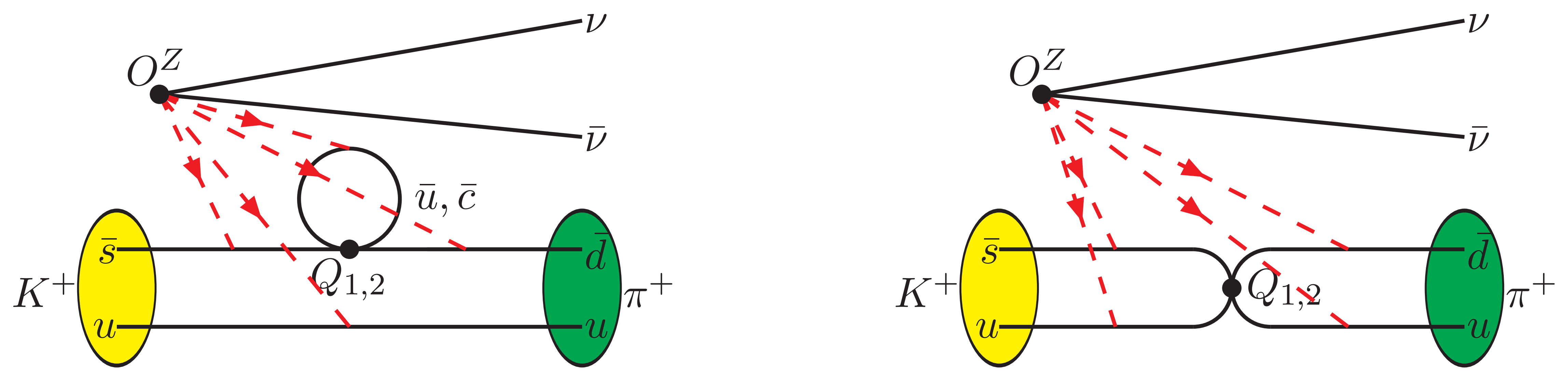
- The theoretical framework has been developed. N.H.Christ, X.Feng, A.Portelli and CTS, arXiv:1605.04442

- It has been implemented in a number of exploratory studies with unphysical quark masses.

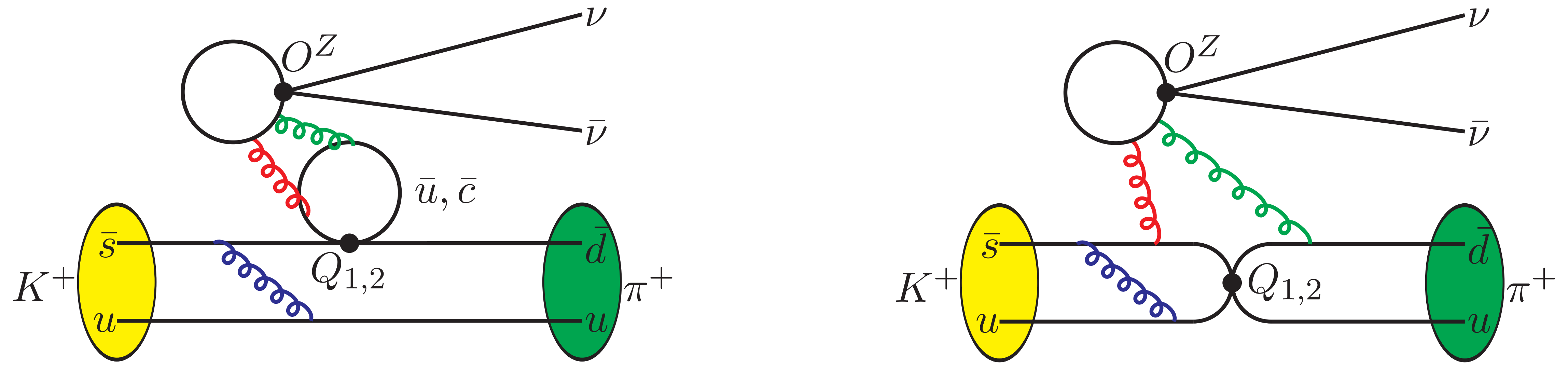
Z.Bai, N.H.Christ, X.Feng, A.Lawson, A.Portelli and CTS, arXiv:1701.02858 & 1806.11520
N.H.Christ, X.Feng, A.Portelli and CTS, arXiv:1910.10644



WW diagrams



Connected Z-exchange diagrams



Disconnected Z-exchange diagrams

The Diagrams

- The latest study was performed with "near-physical" pions, $m_\pi = 172(1)\,\text{MeV}$, $m_K = 493(3)\,\text{MeV}$ on a $32^3 \times 64$ lattice and with $m_c(3\,\text{GeV}) \simeq 750\,\text{MeV}$.

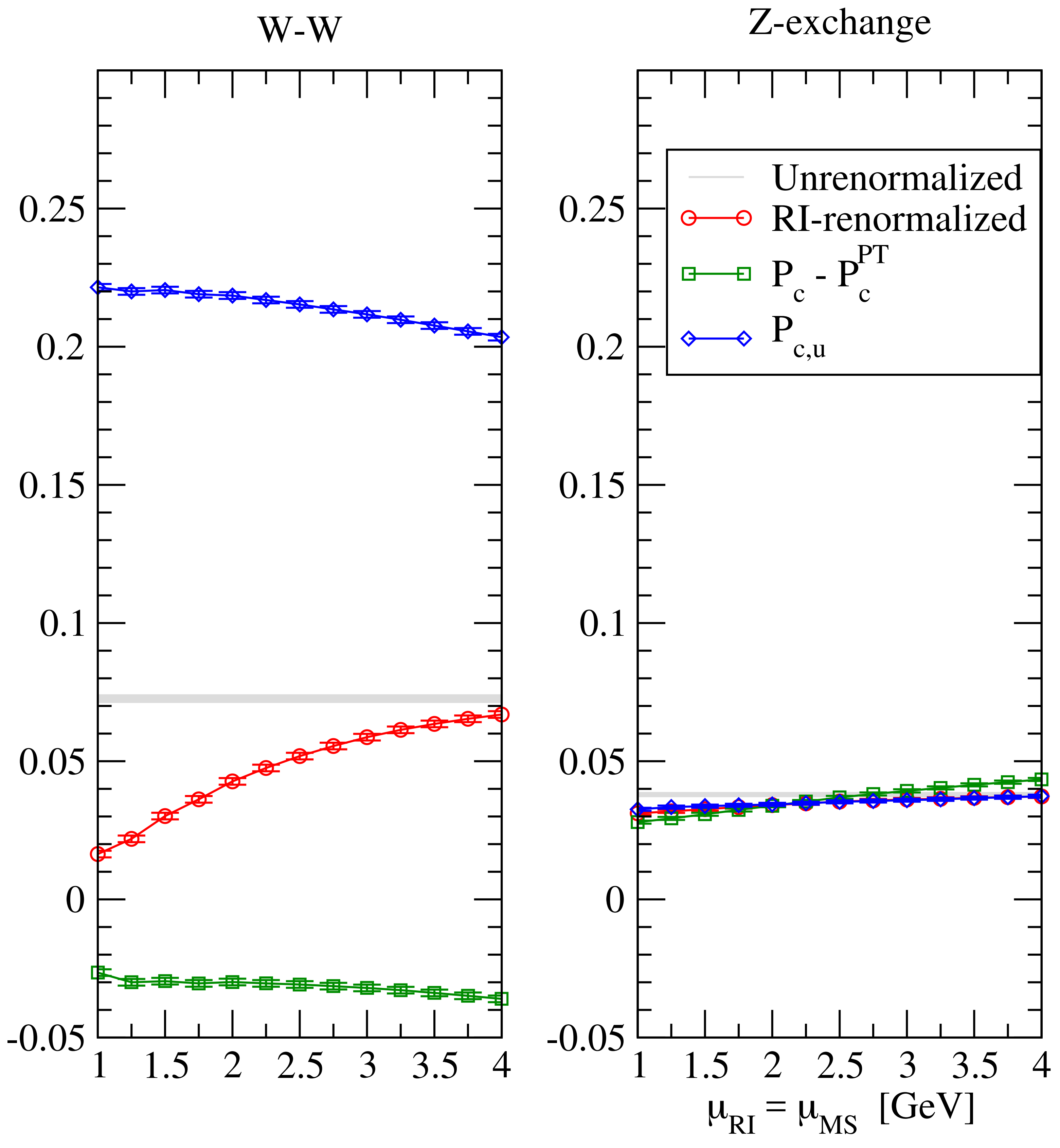
  N.H.Christ, X.Feng, A.Portelli and CTS, arXiv:1910.10644

- The aim of this study was two-fold. Firstly to study the momentum dependence which was found to be very mild $\Rightarrow$ we will not have to perform the computations at many kinematic points.

- Secondly, we found that we could manage the contribution from two-pion intermediate states, including the finite-volume corrections, and found that the contribution is less than $1\,\%$.

- Following these preparatory studies we now have ensembles on a 64×128 lattice, with $m_\pi = 135.9(3)\,\text{MeV}$, $m_K = 496.9(7)\,\text{MeV}$, and a series of charm quark masses spanning the physical one $\Rightarrow$ $30\,\%$ uncertainties.

$K^+ \to \pi^+\nu\bar{\nu}$ - Results from a previous exploratory calculation



- Results from 800 configurations on a $16^3 \times 32$ lattice with $N_F = 2 + 1$ DWF, $a^{-1} = 1.73\,\text{GeV}$, $m_\pi \simeq 420\,\text{MeV}$, $m_K \simeq 563\,\text{MeV}$ and $m_c^{\overline{\text{MS}}}(2\,\text{GeV}) \simeq 863\,\text{MeV}$.

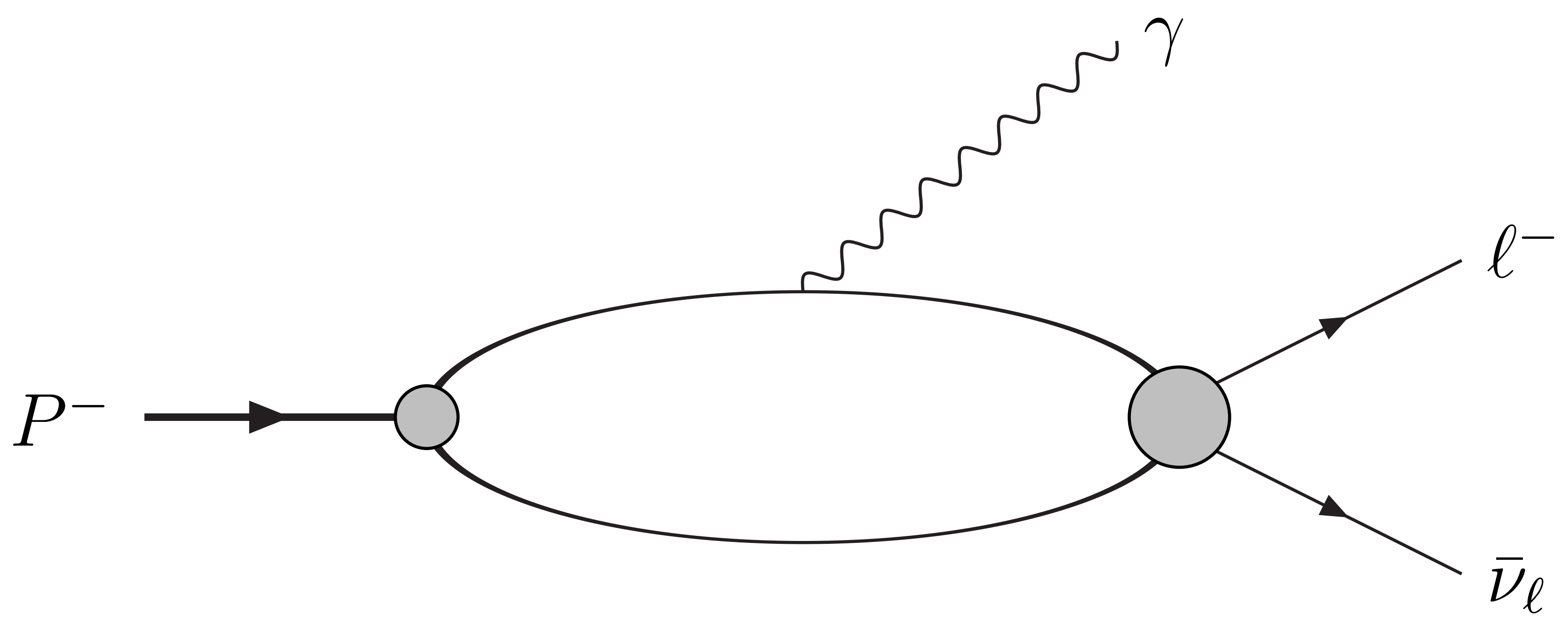
- For this unphysical kinematics we find

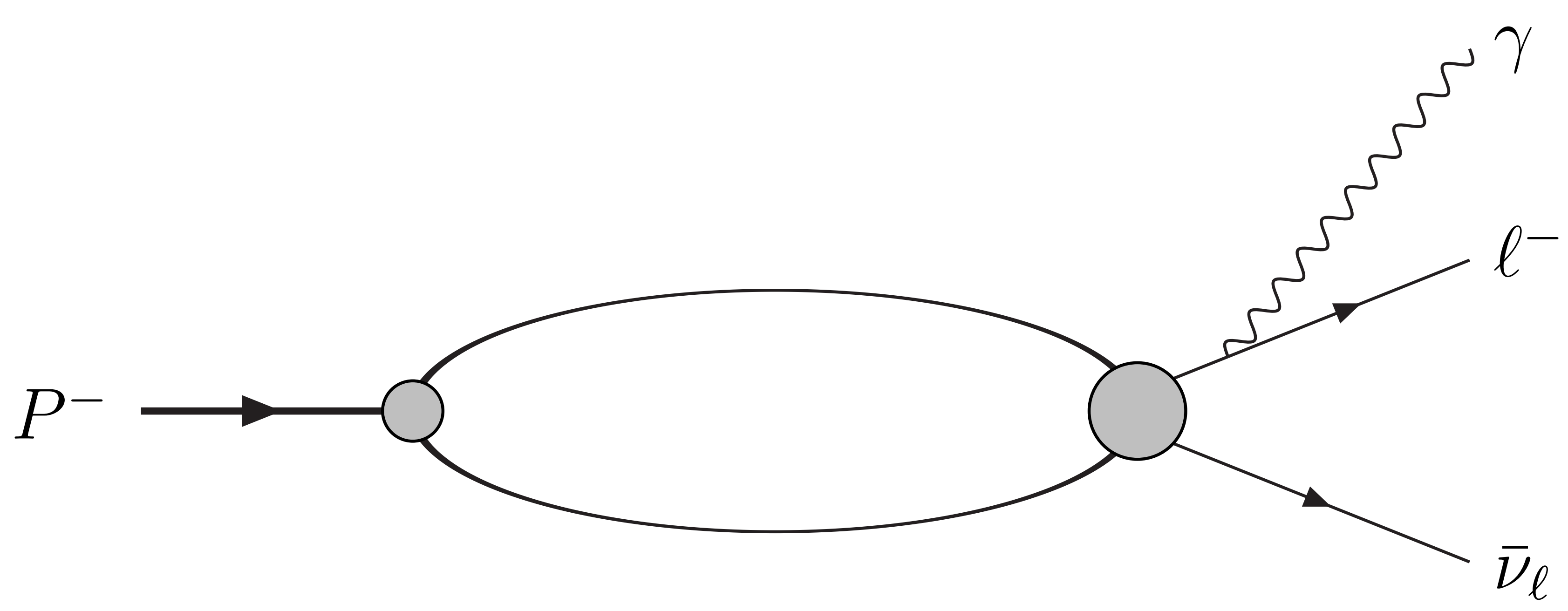
$$P_c = 0.2529(\pm 13)(\pm 32)(-45)_{\text{FV}}$$
$$\Delta P_c = 0.0040(\pm 13)(\pm 32)(-45)_{\text{FV}}\,.$$

- Large cancellation between WW and Z-exchange contributions.

Z.Bai, N.H.Christ, X.Feng, A.Lawson, A.Portelli and CTS, arXiv:1701.02858 & arXiv:1806.11520

6. K $\rightarrow \ell \nu_\ell \gamma$ radiative decays - the form factors.





Non-perturbative contribution to $P \rightarrow \ell \bar{\nu}_\ell \gamma$ is encoded in:

$$H_W^{\alpha r}(k, \overrightarrow{p}) = \epsilon_\mu^r(k)\, H_W^{\alpha\mu}(k, \overrightarrow{p}) = \epsilon_\mu^r(k) \int d^4y\, e^{ik\cdot y}\, \mathrm{T}\, \langle 0 |\, j_W^\alpha(0)\, j_{\rm em}^\mu(y)\, |\, K(\overrightarrow{p})\, \rangle$$

$$= \epsilon_\mu^r(k) \Bigg\{ \frac{H_1}{m_K} \left[k^2 g^{\mu\alpha} - k^\mu k^\alpha\right] + \frac{H_2}{m_K} \frac{\left[(p\cdot k - k^2)k^\mu - k^2(p-k)^\mu\right](p-k)^\alpha}{(p-k)^2 - m_K^2}$$

$$-i\frac{F_V}{m_K}\varepsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta + \frac{F_A}{m_K}\left[(p\cdot k - k^2)g^{\mu\alpha} - (p-k)^\mu k^\alpha\right] + f_P\left[g^{\mu\alpha} - \frac{(2p-k)^\mu (p-k)^\alpha}{(p-k)^2 - m_K^2}\right] \Bigg\}$$

- For decays into a real photon, $k^2 = 0$ and $\varepsilon \cdot k = 0$, only the decay constant f_K and the vector and axial form factors $F_V(x_\gamma)$ and $F_A(x_\gamma)$ are needed to specify the amplitude ($x_\gamma = 2p\cdot k/m_P^2$, $0 < x_\gamma < 1 - m_\ell^2/m_P^2$).

- In phenomenology $F^\pm \equiv F_V \pm F_A$ are more natural combinations.

$K \to \ell \nu_\ell \gamma$ radiative decays - the form factors (Cont.)

- We have computed $F_V(x_\gamma)$ and $F_A(x_\gamma)$ for $\pi, K, D_{(s)}$ mesons (and $H_{1,2}$ in an exploratory simulation for $K \to \pi \ell \nu_\ell \ell'^{+} \ell'^{-}$ decays). A.Desiderio et al. arXiv:2006.05358

- The computations were performed on 11 ETMC $N_f = 2+1+1$ ensembles with 0.062 fm < a <0.089 fm and 227 MeV< m_π <441 MeV and a range of volumes.
- Computations are performed in the electroquenched approximation.

- Our data is fully consistent with a parametrisation of the form :

$$F^P_{A,V}(x_\gamma) = C^P_{A,V} + D^P_{A,V} x_\gamma \, .$$

- Other parametrisation were also tried and presented.
- Values of the parameters are presented in the paper.

Comparison with Experimental Data

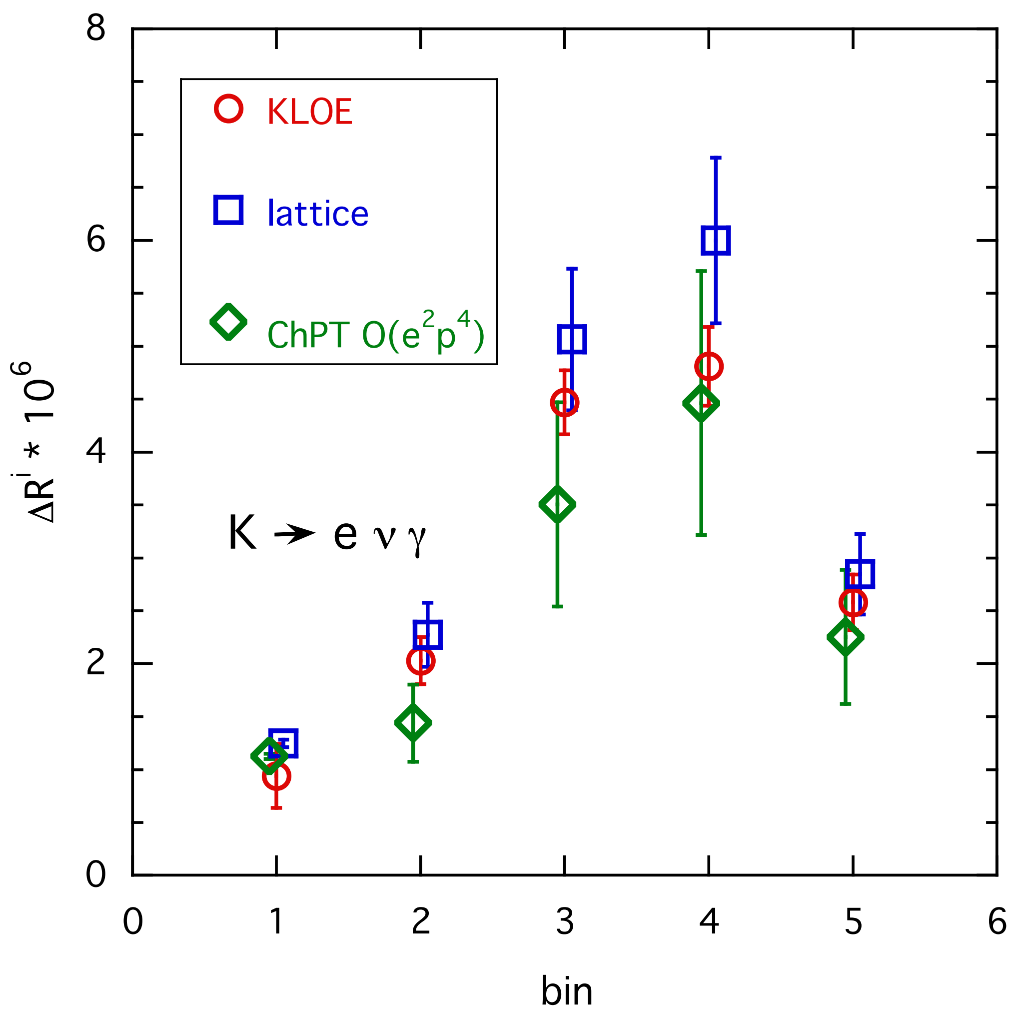
- $K \to e\nu_e\gamma$ KLOE, arXiv:0907.3594
  J-PARC E36, arXiv:2107.03583
  NA62, arXiv:2???.?????

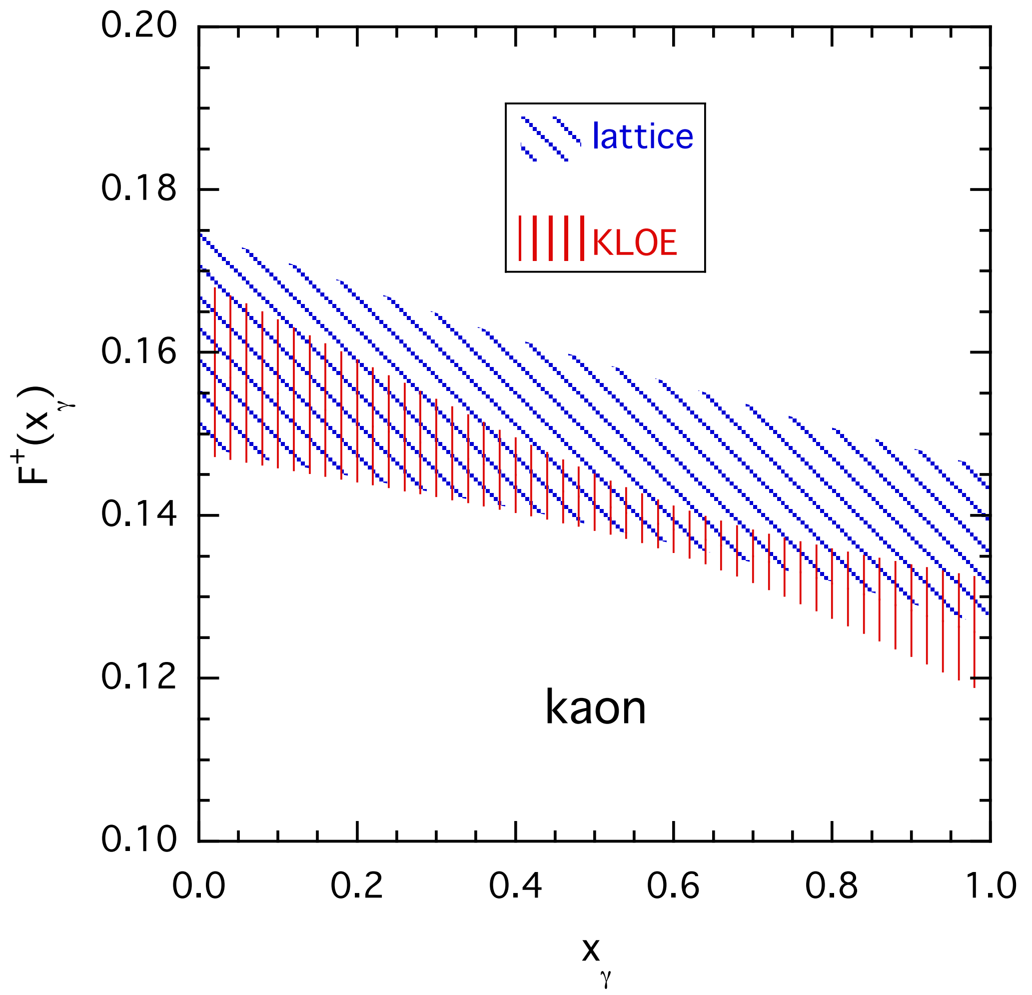
- $K \to \mu\nu_\mu\gamma$ E787@BNL AGS, hep-ex/0003019
  ISTRA+ @U-79 Protvino, arXiv:1005.3517
  OKA@U-79 Protvino, arXiv:1904.10078

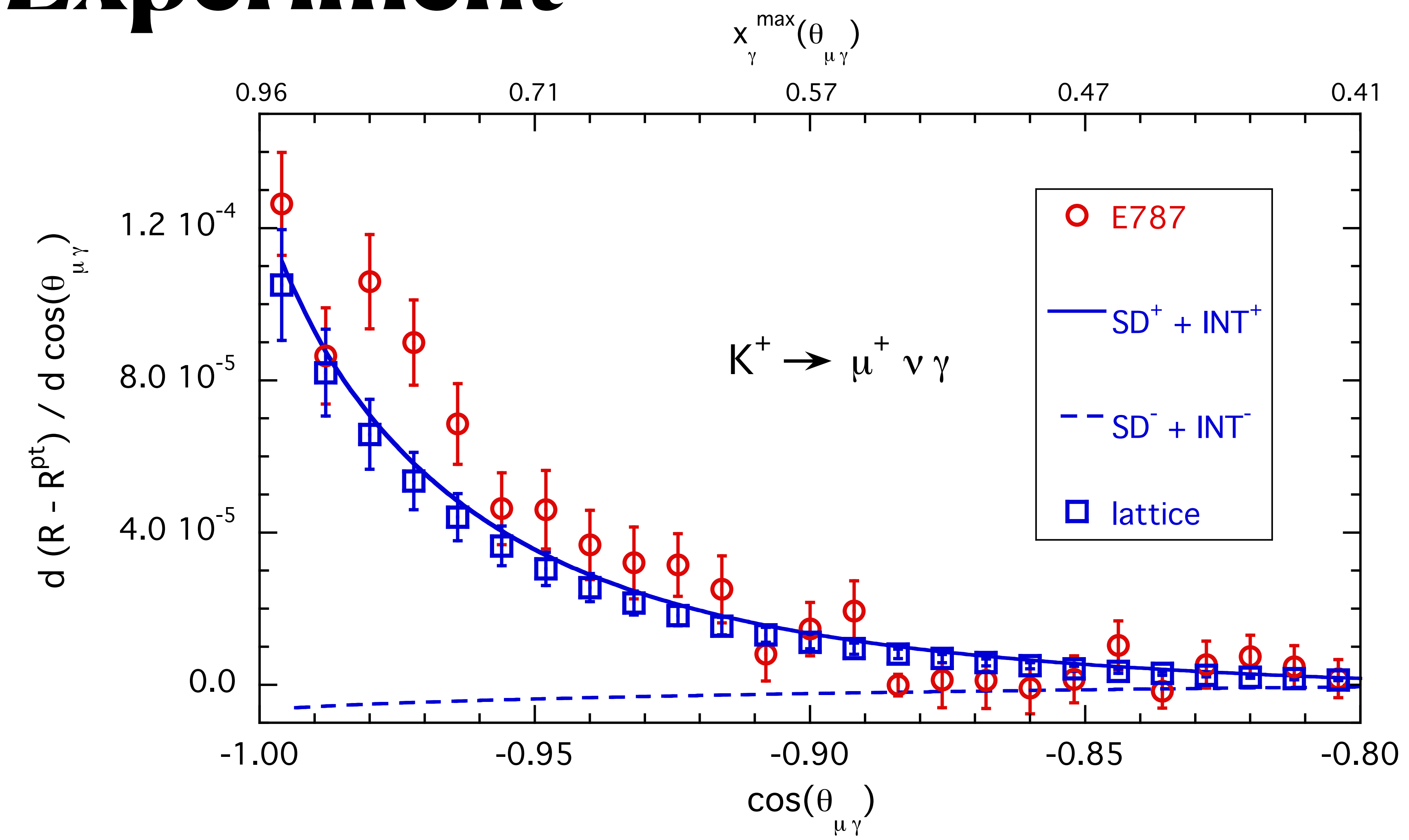
- $\pi \to e\nu_e\gamma$ PIBETA@πE1 beam line PSI, arXiv:0804.1815

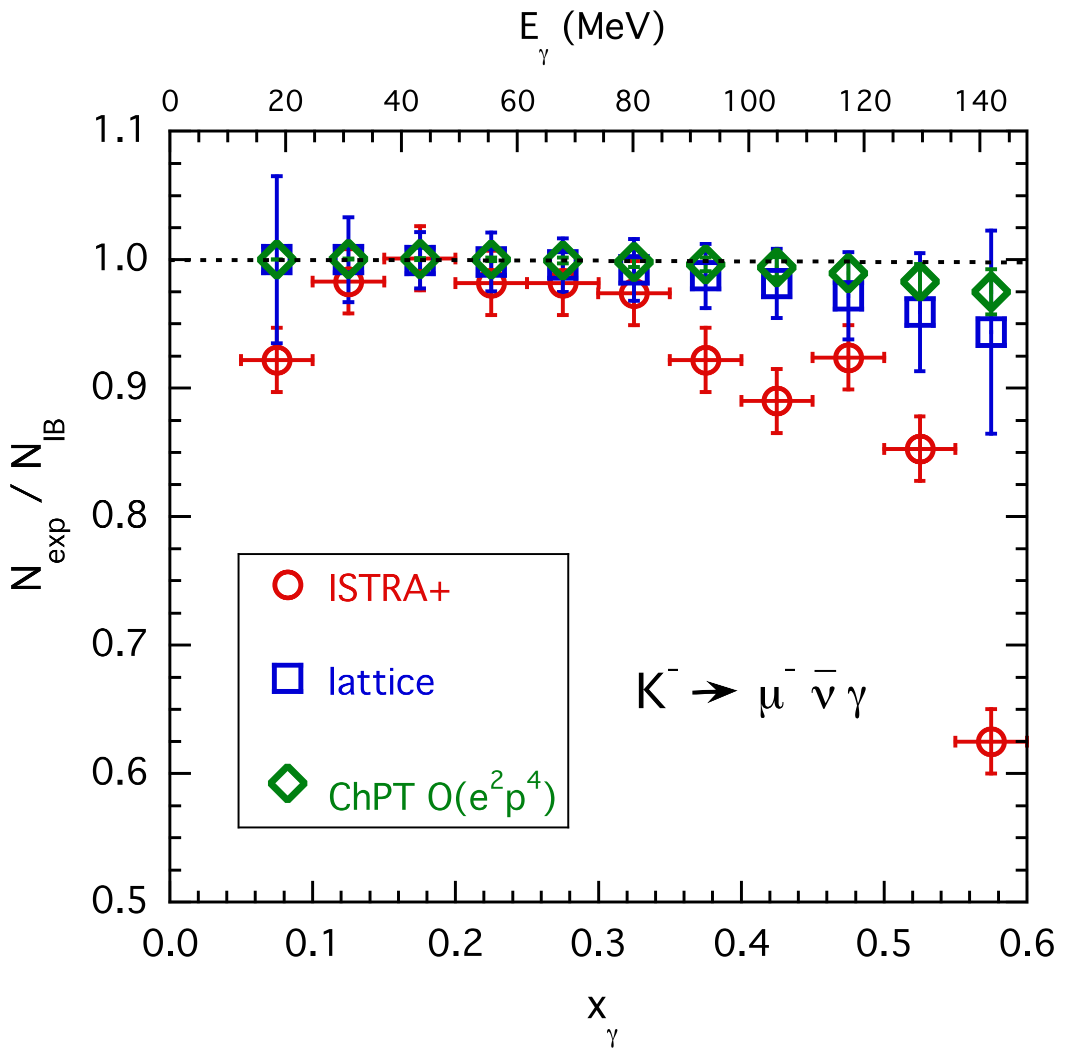
- The different experiments introduce different cuts on E_γ, E_ℓ and $\cos\theta_{\ell\gamma}$, resulting in sensitivities to different form factors.

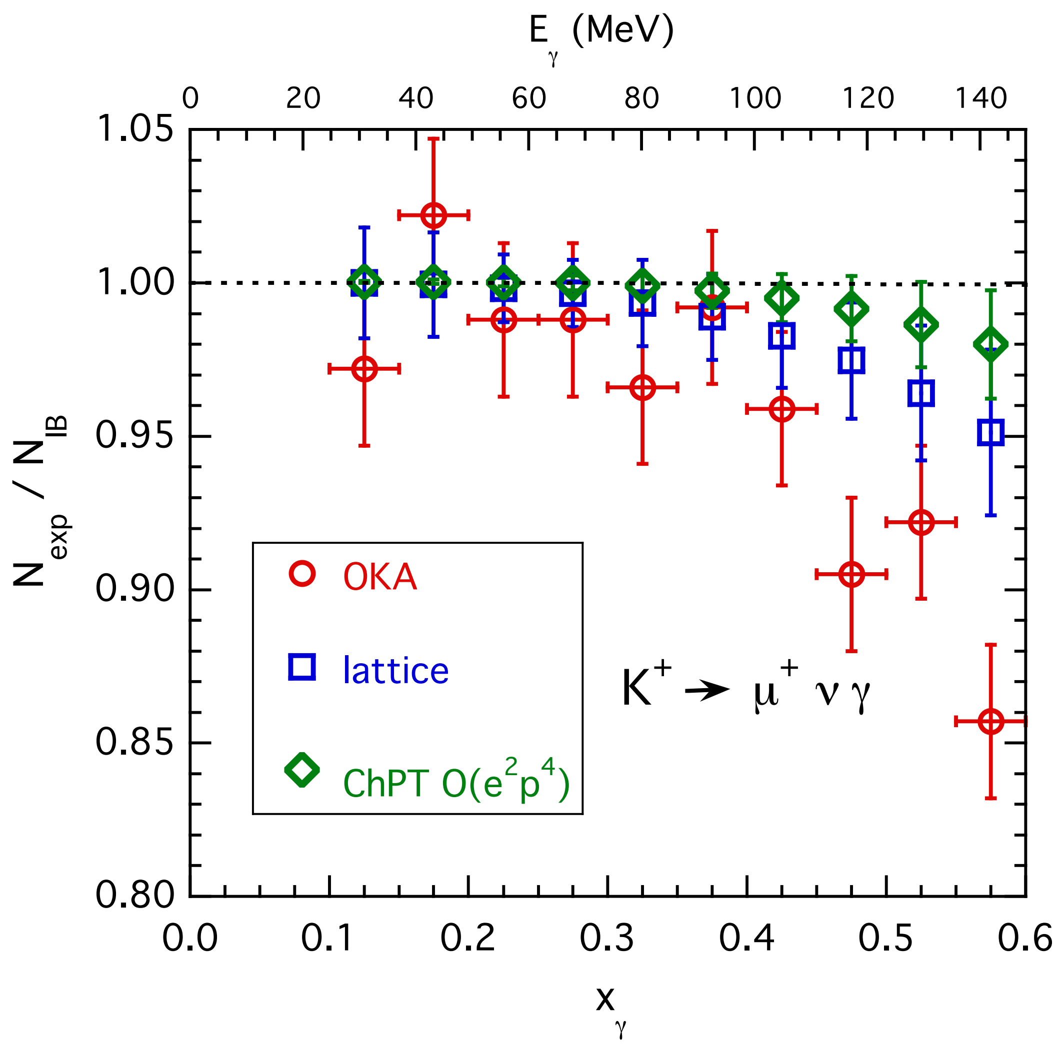
Comparison with Experiment











- Good Agreement with KLOE
- Significant tensions with $K \to \mu\nu_\mu\gamma$ experiments
- Unable to find a set of phenomenological form factors to account for all the data.
- NA62 will soon have the most precise results for $K \to e\nu_e\gamma$ decay rates.
- Is it conceivable that we have LFU-violation here?

A.Desiderio, R.Frezzotti, M.Garofalo, D.Giusti, M.Hansen, V.Lubicz, G.Martinelli, CTS, F.Sanfilippo, S.Simula and N.Tantalo. arXiv:2006.05358
R.Frezzotti, M.Garofalo, V.Lubicz, G.Martinelli, CTS, F.Sanfilippo, S.Simula and N.Tantalo, arXiv:2012.02120

7. K $\to \pi\pi$ Decays

- $K \to \pi\pi$ decays are a very important class of processes with a long and noble history.
  - It is in these decays that both indirect and direct CP-violation was discovered.

- Bose symmetry $\Rightarrow$ the two-pion state has isospin 0 or 2 ,

$$_{I=2}\langle \pi\pi\,|\,H_W\,|\,K^0\rangle = A_2\,e^{i\delta_2}\,, \qquad _{I=0}\langle \pi\pi\,|\,H_W\,|\,K^0\rangle = A_0\,e^{i\delta_0}\,.$$

- Among the very interesting issues are the origin of the $\Delta I = 1/2$ rule ($\mathrm{Re}A_0/\mathrm{Re}A_2 \simeq 22.5$) and an understanding of the experimental value of ϵ'/ϵ, the parameter which was the first experimental evidence for direct CP-violation.

- The material here is taken from the following two RBC-UKQCD papers, which however represent the culmination of many years of preparatory work:

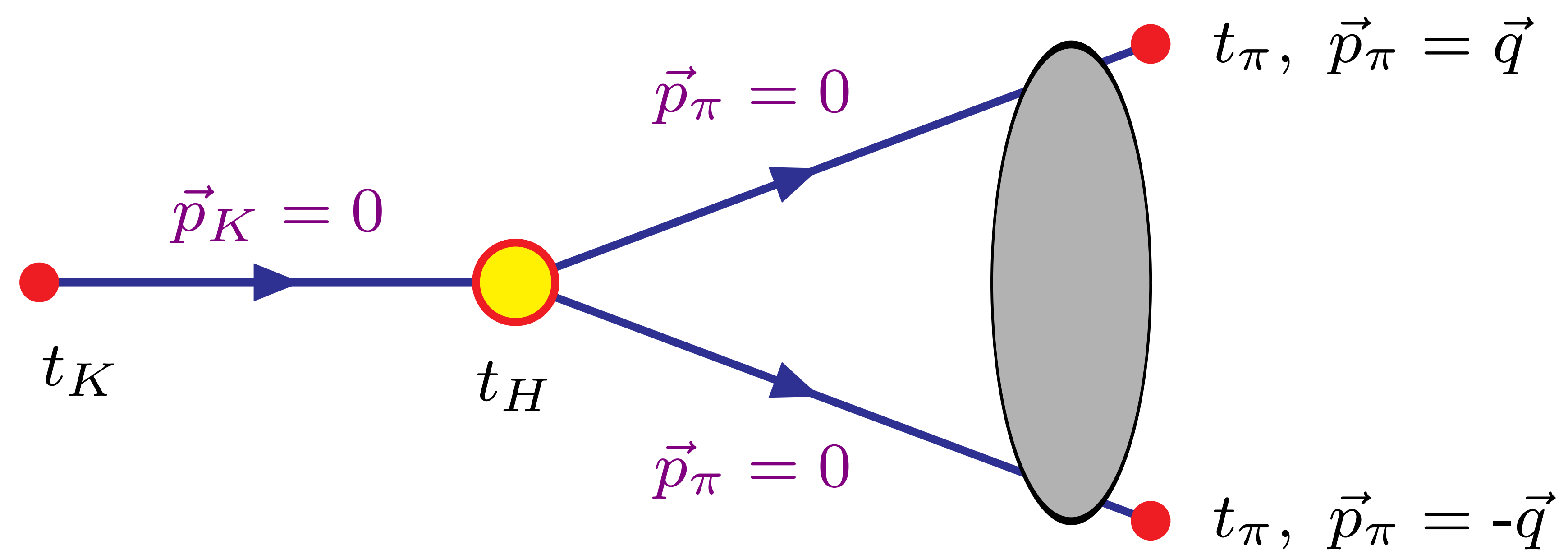
1. *"$K \to \pi\pi$ $\Delta I = 3/2$ decay amplitude in the continuum limit"*
T.Blum, P.A.Boyle, N.H.Christ, J.Frison, N.Garron, T.Janowski, C.Jung, C.Kelly, C.Lehner, A.Lytle, R.D.Mawhinney, CTS., A.Soni, H.Yin, and D.Zhang arXiv:1502.00263

2. *"Direct CP violation and the $\Delta I = 1/2$ rule in $K \to \pi\pi$ decay in the Standard Model"*
R.Abbott, T.Blum, P.A.Boyle, M.Bruno, N.H.Christ, D.Hoying, C.Jung, C.Kelly, C.Lehner, R.D.Mawhinney, D.J.Murphy, CTS, A. Soni, M.Tomii and T.Wang, arXiv:2004.09440

(Building on RBC-UKQCD, Z.Bai et al. arXiv:1505.07863)

- Detailed references to earlier work can be found in these papers.

Why are the amplitudes difficult to compute?



- $K \to \pi\pi$ correlation function is dominated by the lightest intermediate state. L.Maiani and M.Testa, Phys.Lett. B245 (1990) 585
  - With periodic boundary conditions this is the $\pi\pi$ state with both pions at rest for A_2 and the vacuum state for A_0.
  - We have chosen to use anti periodic boundary conditions for the d-quark for A_2 and G-parity boundary conditions for A_0.
  - Work is in progress to compute the amplitudes with periodic boundary conditions with excited $\pi\pi$ states. M.Tomii, Lattice 2023
- Volume must be tuned to ensure $E_{\pi\pi} = m_K$.
  - Moreover, the s-wave $I = 0$ and $I = 2$ channels are attractive and repulsive respectively and so the two cases must be treated separately.
- Finite-volume effects are not exponentially small and must be corrected.

L.Lellouch and M.Lüscher, hep-lat/00030023,
C-h.Kim, CTS and S.Sharpe, hep-lat/0507006

Results for A_2

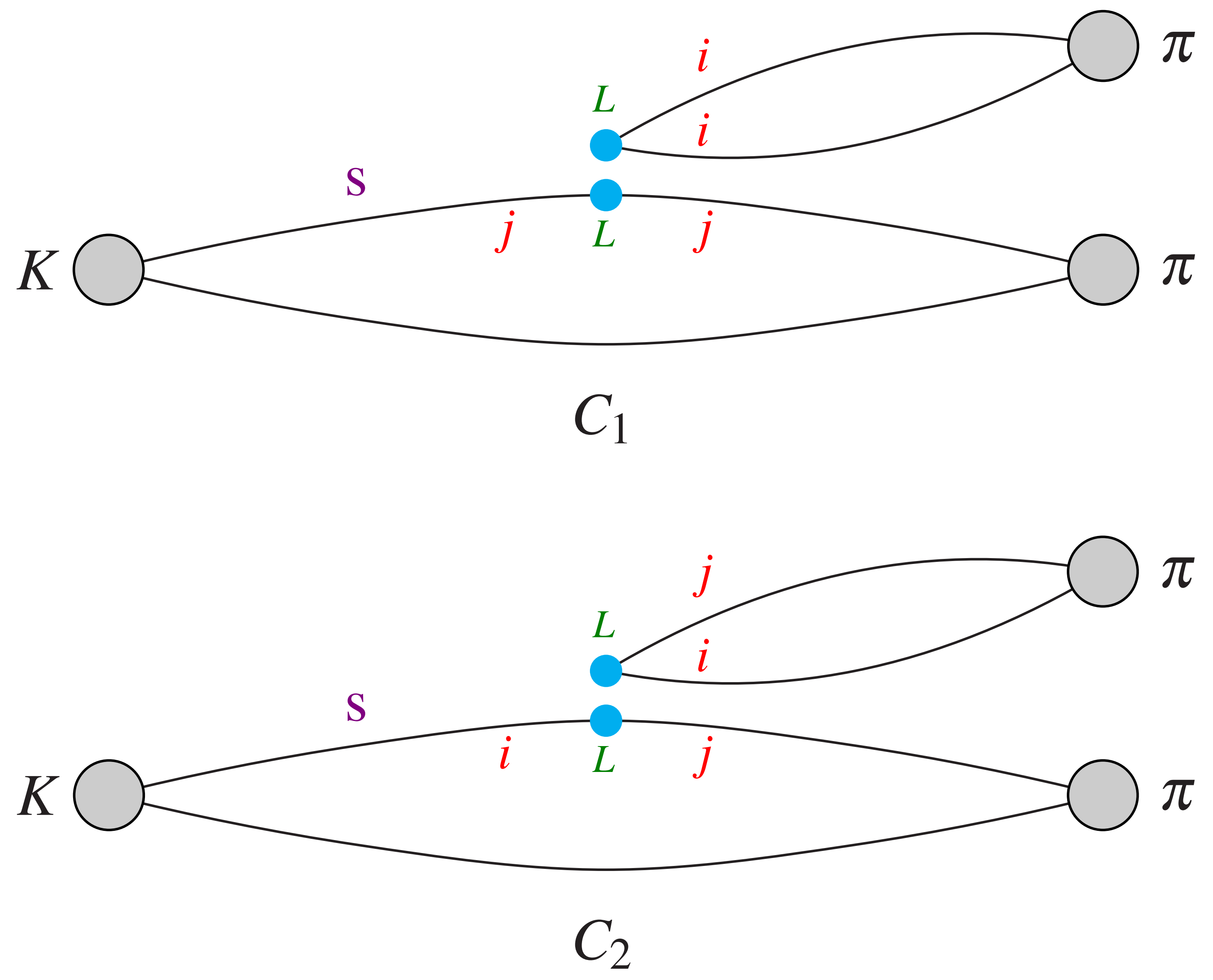
- A_2 is considerably easier to evaluate that A_0 .

- Our latest result was obtained on two ensembles, $48^3 \times 96$ with $a = 0.11$ fm and $64^3 \times 128$ with $a = 0.084$ fm ,

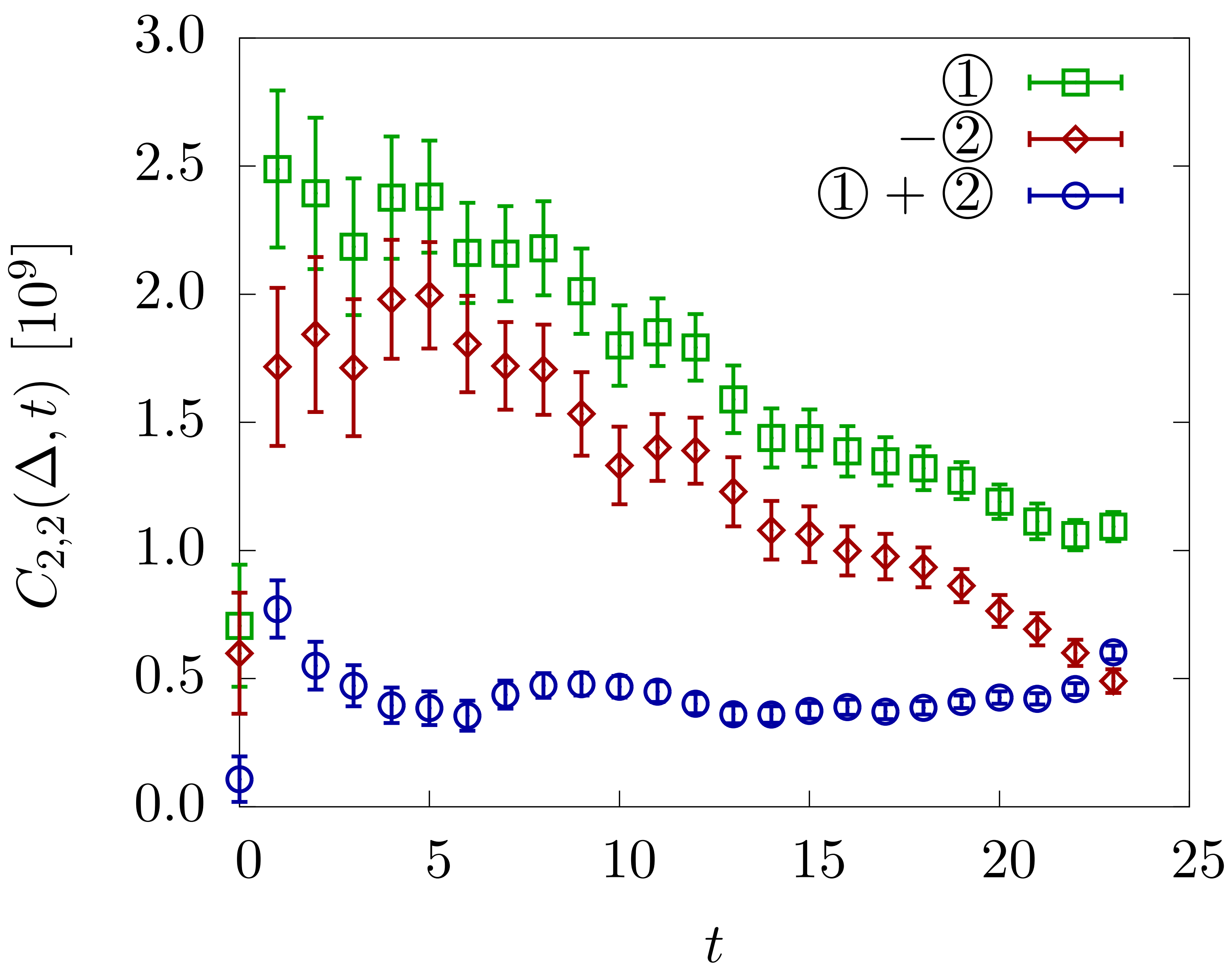
$$\mathrm{Re}\, A_2 = 1.50(4)_{\mathrm{stat}}(14)_{\mathrm{syst}} \times 10^{-8}\,\mathrm{GeV}\,, \qquad \mathrm{Im}\, A_2 = -\,6.99(20)_{\mathrm{stat}}(84)_{\mathrm{syst}} \times 10^{-13}\,\mathrm{GeV}\,.$$

  - Experimental value: $\mathrm{Re}\, A_2 = 1.497(4) \times 10^{-8}\,\mathrm{GeV}$.

- $\mathrm{Re}\, A_2$ is dominated by a single operator, $O^{3/2}_{(27,1)}$ and two diagrams



- Instead of $C_2 \simeq 1/3\, C_1$ as might be expected from colour counting, we find a large cancellation between the two, which is a significant contribution to the $\Delta I = 1/2$ rule.



Results for A_0

R.Abbott,T.Blum, P.A.Boyle, M.Bruno, N.H.Christ, D.Hoying, C.Jung, C.Kelly, C.Lehner, R.D.Mawhinney, D.J.Murphy, C.T.S, A. Soni, M.Tomii and T.Wang, arXiv:2004.09440 [hep-lat].

- Results were obtained from 741 configurations on a $32^3 \times 64$ lattice with $a^{-1} = 1.38\,\mathrm{GeV}$.
- $\mathrm{Re}\,A_0 = 2.99\,(0.32)\,(0.59) \times 10^{-7}\,\mathrm{GeV}$ (Experiment $3.3201(18) \times 10^{-7}\,\mathrm{GeV}$)

  $\mathrm{Im}\,A_0 = -\,6.98\,(0.62)\,(1.44) \times 10^{-11}\,\mathrm{GeV}$.
- Combining this result with our earlier ones for $\mathrm{Re}\,A_2$ we find

$$\frac{\mathrm{Re}\,A_0}{\mathrm{Re}\,A_2} = 19.9 \pm 2.3 \pm 4.4$$

  in good agreement with the experimental result of $22.45(6)$.
- Combining our result for $\mathrm{Im}\,A_0$ with our 2015 one for $\mathrm{Im}\,A_2$ and using the experimental results for the real parts we obtain

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) = 0.00217\,(26)_{\mathrm{stat}}\,(62)_{\mathrm{syst}}\,(50)_{\mathrm{IB}}\,.$$

  The result is consistent with the experimental value of $0.00166\,(23)$.

Isospin Breaking

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) = 0.00217\,(26)_{\rm stat}\,(62)_{\rm syst}\,(50)_{\rm IB}\,.$$

- At present we are not concerned with including $O(1\%)$ corrections.
  - However, because of the $\Delta I = 1/2$ rule, the isospin-breaking corrections are expected to be amplified.
- We use as our guide, the detailed updated study of IB corrections in the framework of ChPT and the large N_c approximation.

V.Cirigliano, H.Gisbert, A.Pich, A.Rodriguez-Sanchez, arXiv:1911.01359

  - A detailed discussion of these results, and the determination of the LECs at NLO in particular, is beyond the scope of our work and we include the central value as a further $23\,\%$ systematic error on our result.}
  - Note that if, instead of treating the isospin correction from this paper as a component of the systematic uncertainty, we were to implement on our result, we would obtain a central value $\epsilon'/\epsilon = 0.00167$, coincidentally identical to the experimental result.
- Work continues to control the IB corrections in $K \to \pi\pi$ decays.
- Prospects - Snowmass Report: *"It may not be unreasonable to expect that with continued effort a reduction in errors below the $30\,\%$ level in five years and below $10\,\%$ in ten years may be achieved.*

RBC & UKQCD Collaborations, T.Blum et al., arXiv:2203.10998

8. Summary and Conclusions

- I have sketched the very significant recent progress towards computing the long-distance hadronic effects in a selection of important quantities in kaon physics, including Δm_K, ϵ_K, the amplitudes for the rare kaon decays $K \to \pi \ell^+ \ell^-$ and $K^+ \to \pi^+ \nu \bar{\nu}$ as well as the direct CP-violating parameter ϵ'/ϵ and the $\Delta I = 1/2$ rule.
  - These rare FCNC processes and small quantities are excellent places to search for the effects of new physics Beyond the Standard Model.
  - I have also mentioned some discrepancies in the theoretical predictions and experimental measurements in radiative kaon decays $K \to \ell \nu_\ell \gamma$.

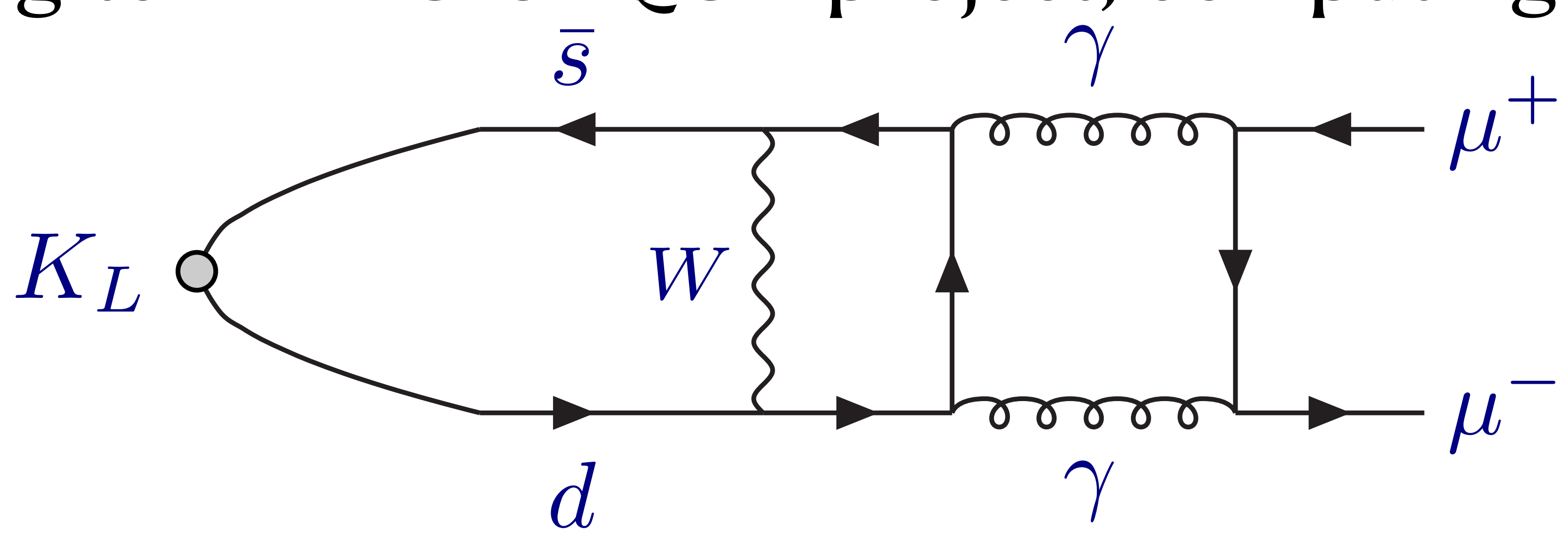
- In all cases the framework allowing lattice QCD computations of the non-perturbative hadronic effects to be possible has been developed.

- Work now continues to improve the precision, as sketched in the talk above.

- It would be great if other collaborations would join the effort to compute hadronic effects in rare kaon processes.

Summary and Conclusions (cont.)

- I have not been able to discuss the long-term RBC-UKQCD project, computing the two-photon contribution to the decay $K_L \to \mu^+\mu^-$ from diagrams such as



- Experimental result:
  $B(K_L \to \mu^+\mu^-) = (6.84 \pm 0.11) \times 10^{-9}$

- A number of preparatory/exploratory studies have been performed:

  - A calculation of the amplitude for the related, but simpler process $\pi^0 \to e^+e^-$ has been performed with physical pion masses, on a series of lattices so that the continuum limit can be taken:

$$\mathrm{Re}\,A = 18.60\,(1.19)_{\mathrm{stat}}\,(1.04)_{\mathrm{syst}}\,\mathrm{eV}, \quad \mathrm{Im}\,A = 32.59\,(1.50)_{\mathrm{stat}}\,(1.65)_{\mathrm{syst}}\,\mathrm{eV}\,, \quad \frac{\mathrm{Re}\,A}{\mathrm{Im}\,A} = 0.571\,(10)_{\mathrm{stat}}\,(4)_{\mathrm{syst}}$$

    to be compared to the experimental numbers:

$$\mathrm{Re}\,A = 24.10\,(2.0)\,\mathrm{eV}, \quad \mathrm{Im}\,A = 35.07\,(37)\,\mathrm{eV}$$

    N.H.Christ, X.Feng, L.Jin, C.Tu and Y.Zhao, 2208.03834

  - A strategy and exploratory calculation of the amplitude for the CP-concerning contribution to the amplitude for the $K_L \to \gamma\gamma$ decay was also presented. N.H.Christ and Y.Zhao, PoS (Lattice 2021) 2022 451

  - At Lattice 2023 an update on the project was presented by En-Hung Chao with a focus on the $\pi\pi\gamma$ intermediate state.