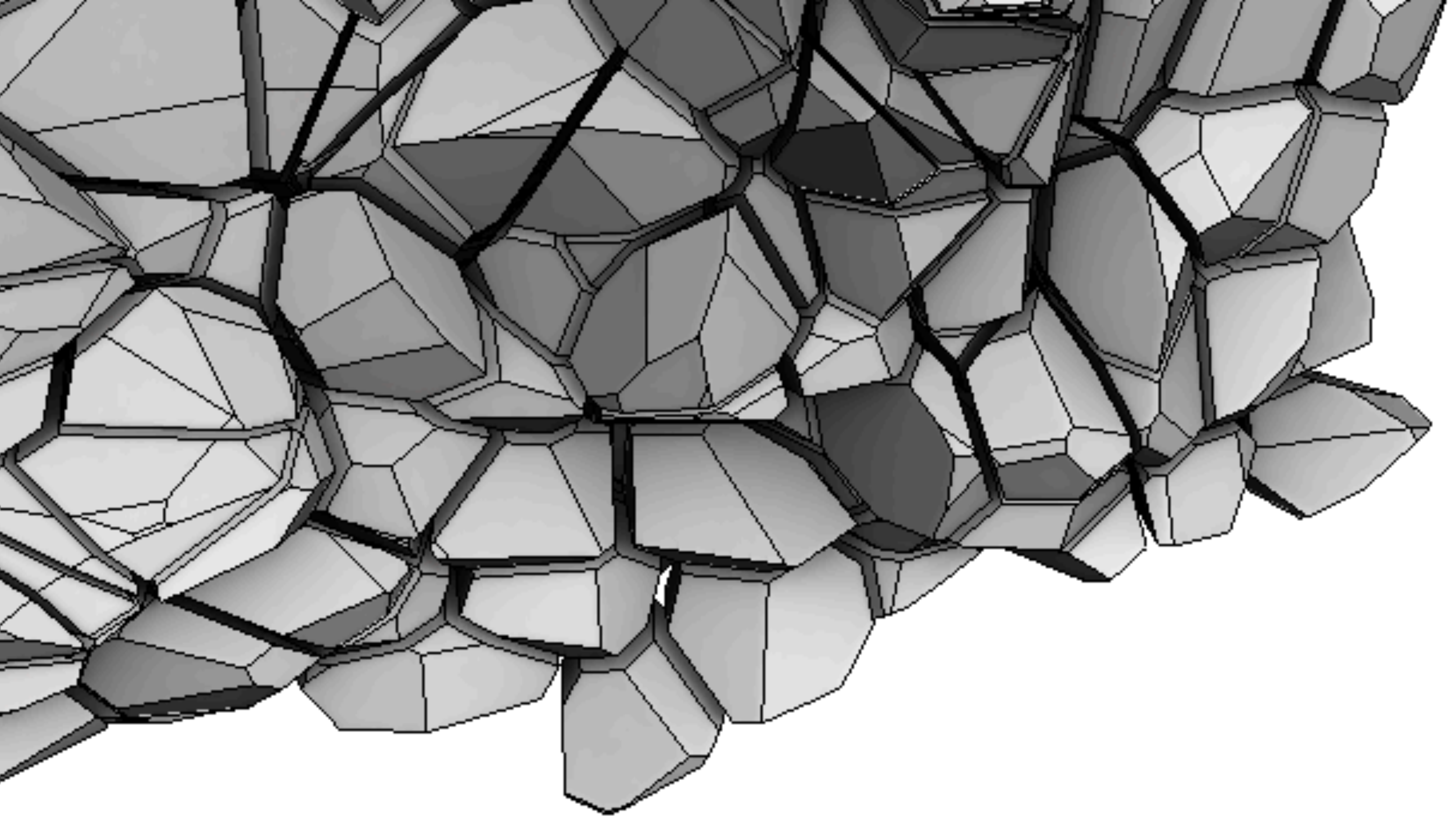
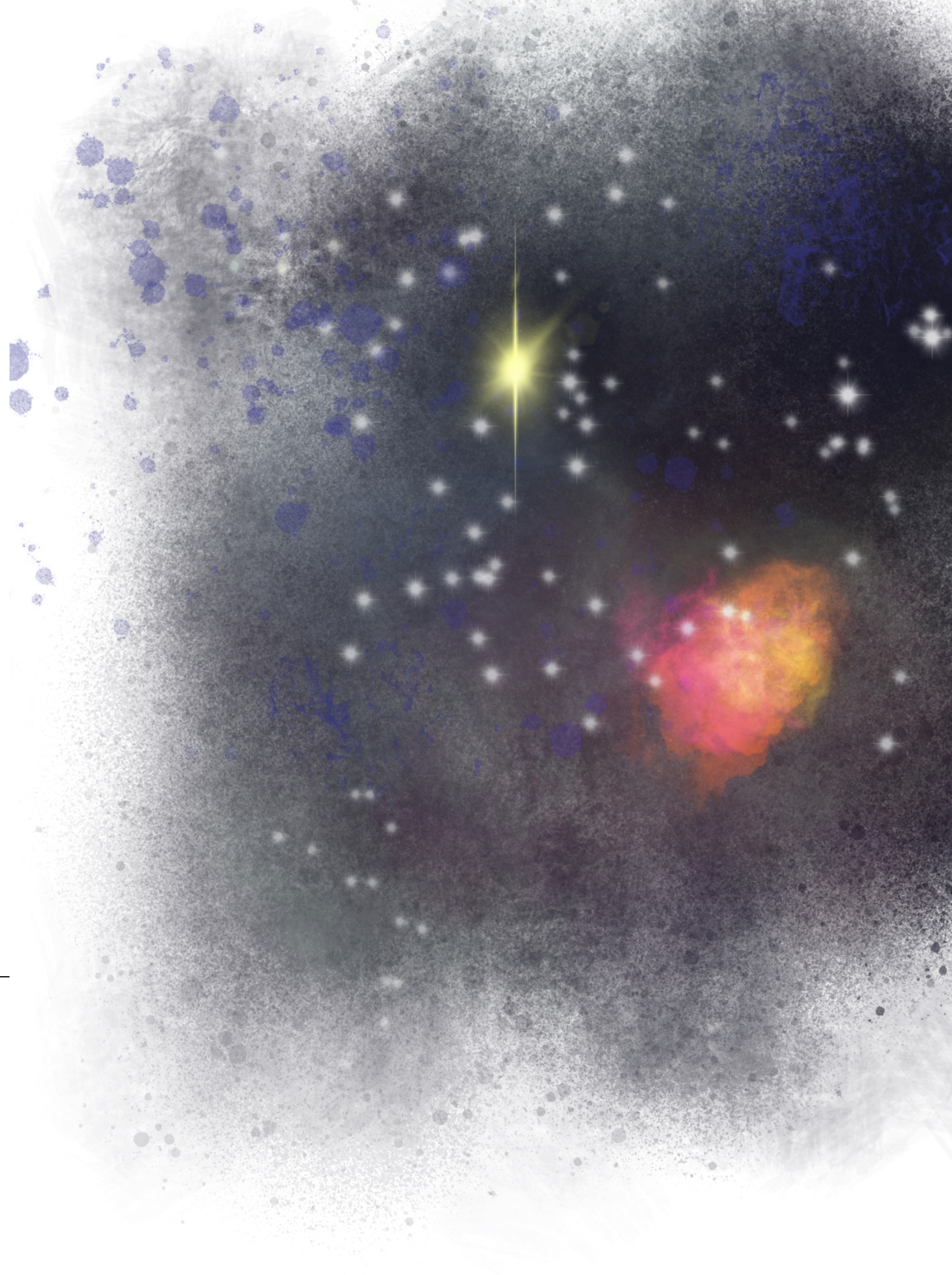




Subir Fest — September 12, 2023
University of Oxford

Reconstruction in Cosmology

Sebastian von Hausegger
Beecroft Fellow, Oxford Astrophysics

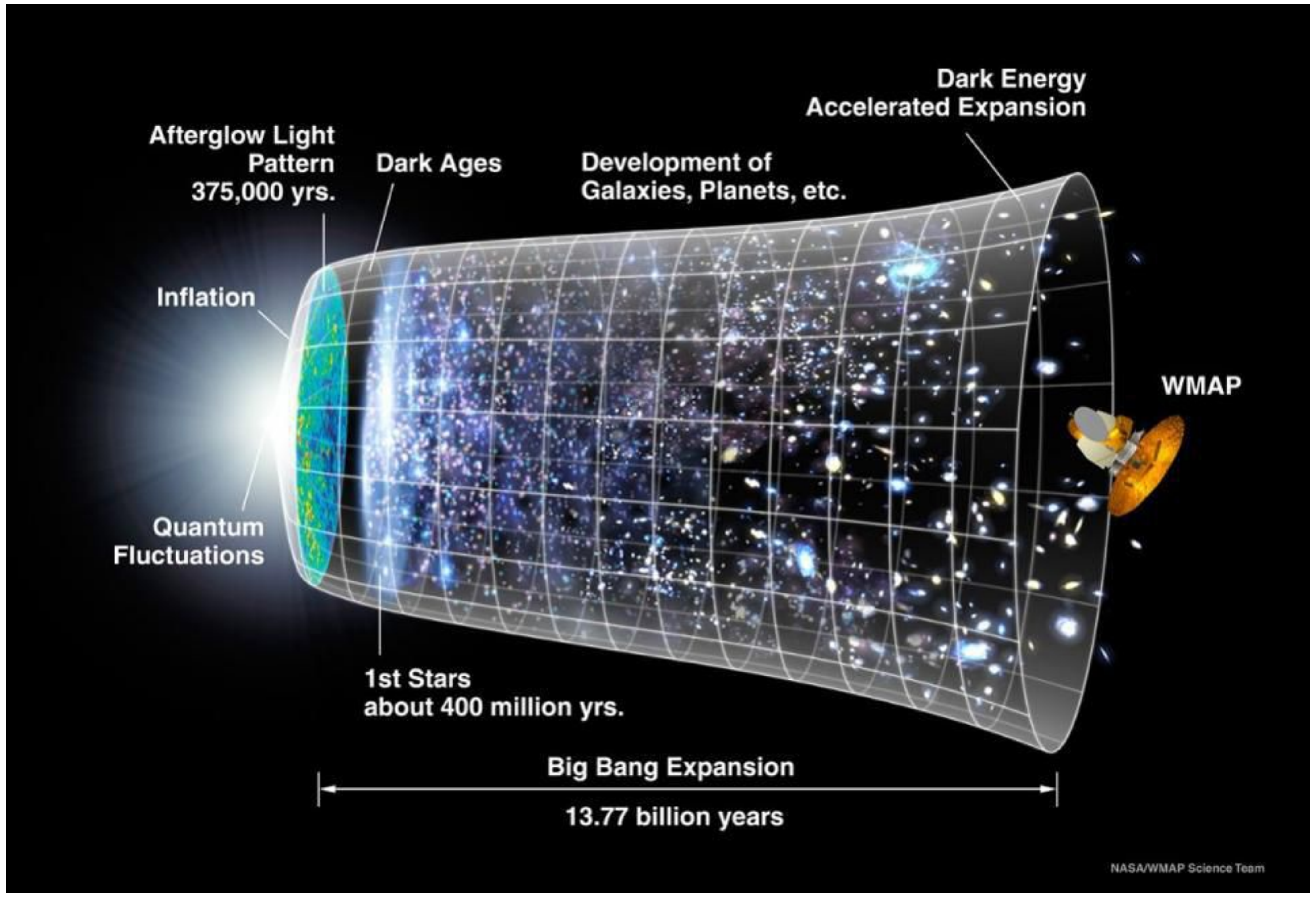
UNIVERSITY OF OXFORD

CARLSBERG FOUNDATION


NIELS BOHR INSTITUTET
1920

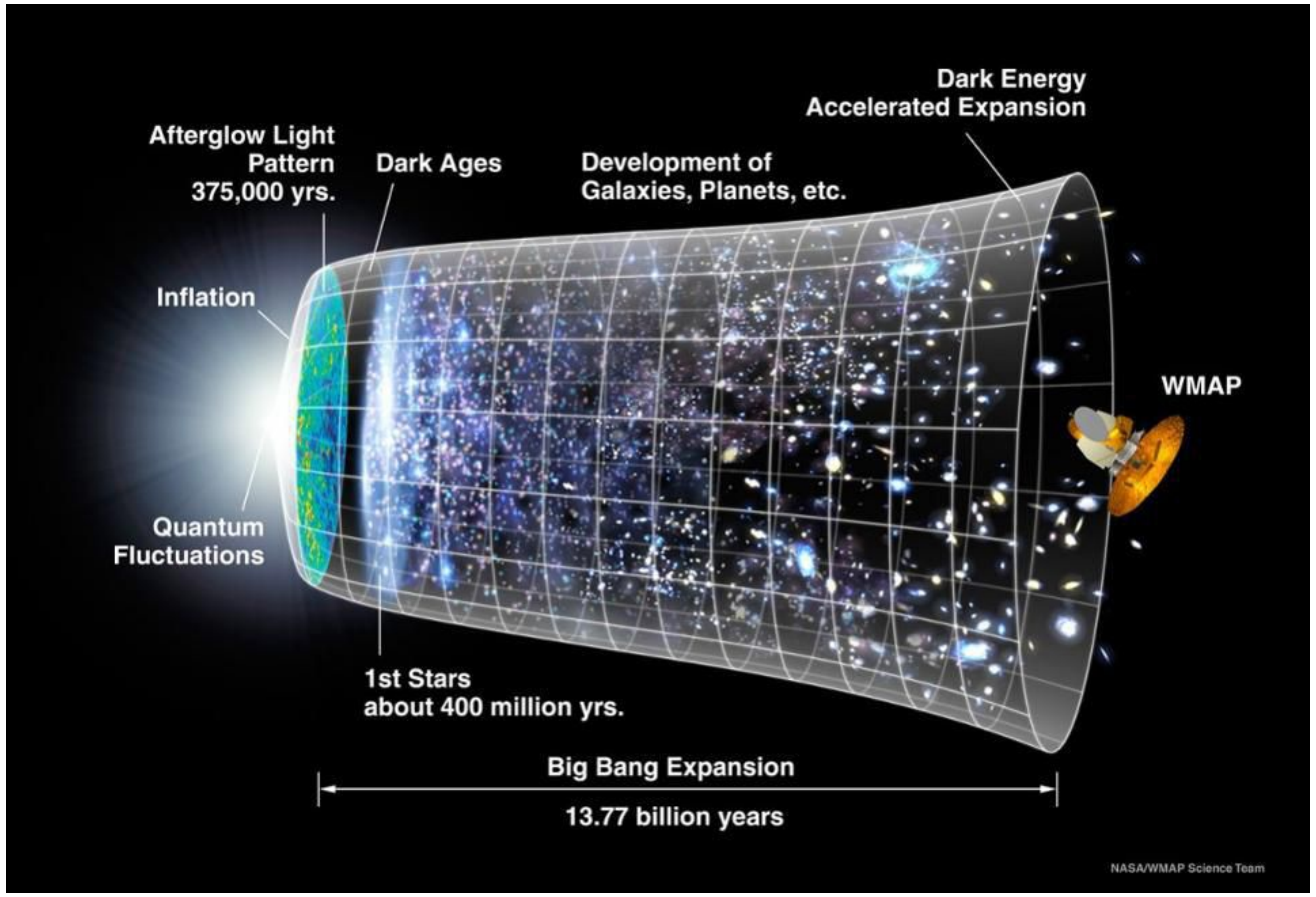
“Freedom is the consciousness of necessity”

Cosmological evolution



Obligatory cosmology slide

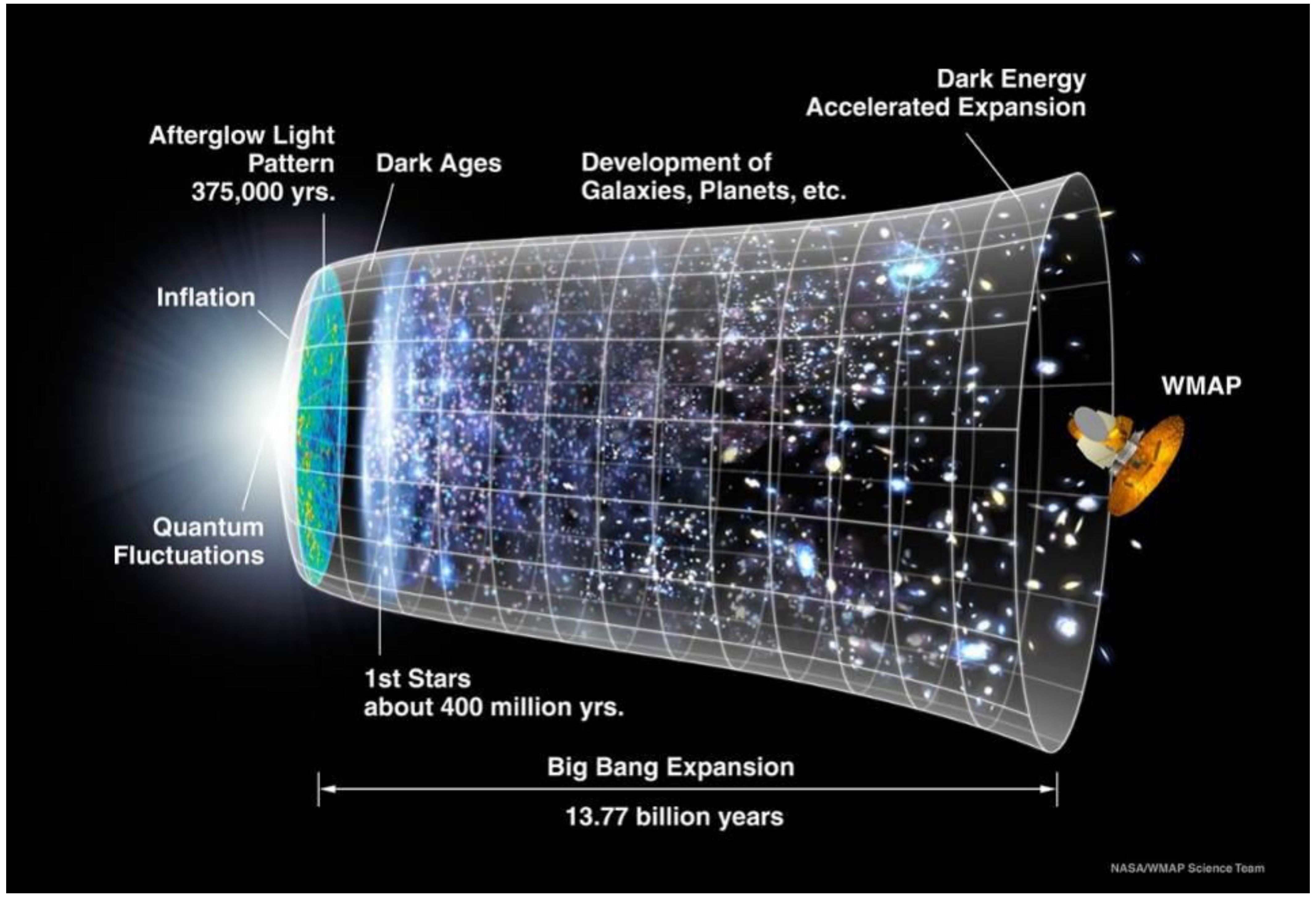
Cosmological evolution



Obligatory cosmology slide

Inflation ⟶ CMB ⟶ Matter distribution

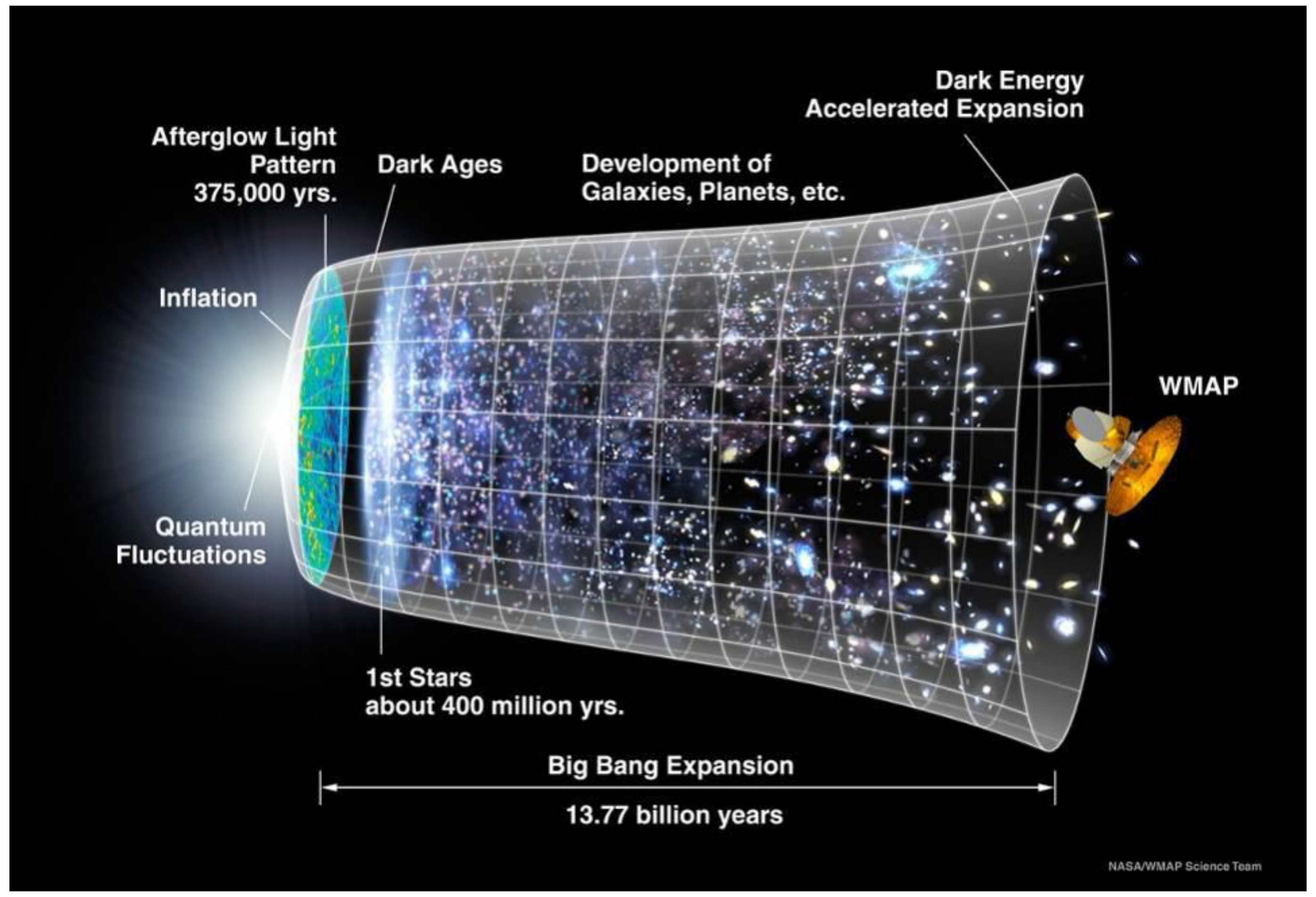
Cosmological evolution



Obligatory cosmology slide

Inflation → CMB → Matter distribution

Cosmological evolution



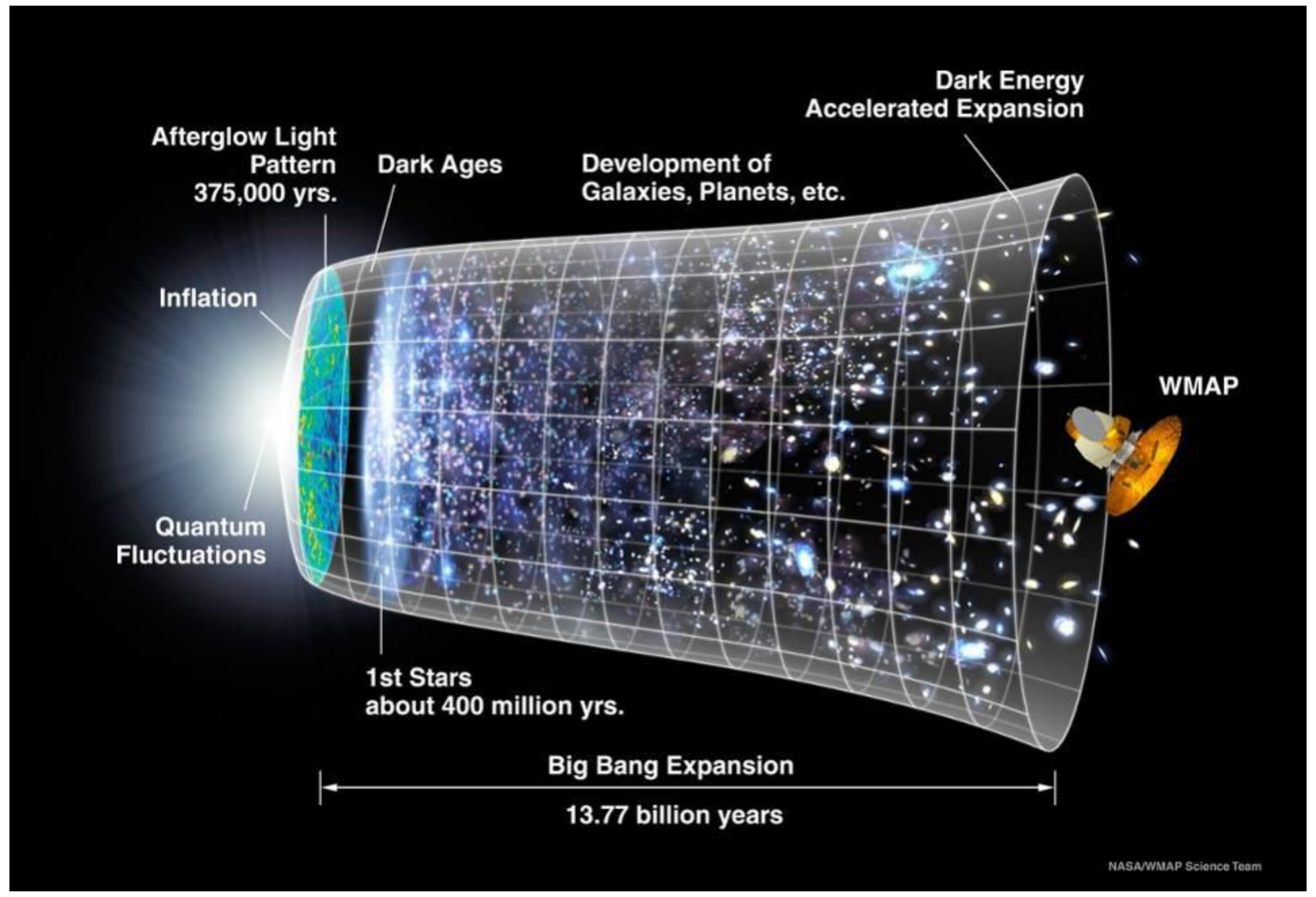
Obligatory cosmology slide

$$C_\ell = \langle \delta T_{\ell m} \delta T_{\ell m} \rangle$$

$P(k) \propto A_s k^{n_s - 1}$? $\quad P^m_{\text{lin}}(k) \quad P^m(k)$

Inflation ⟶ CMB ⟶ Matter distribution

Cosmological evolution



Obligatory cosmology slide

Evolution + projection

$C_\ell = \langle \delta T_{\ell m} \delta T_{\ell m} \rangle$

Non-linear gravitational clustering

$P(k) \propto A_s k^{n_s - 1}$?

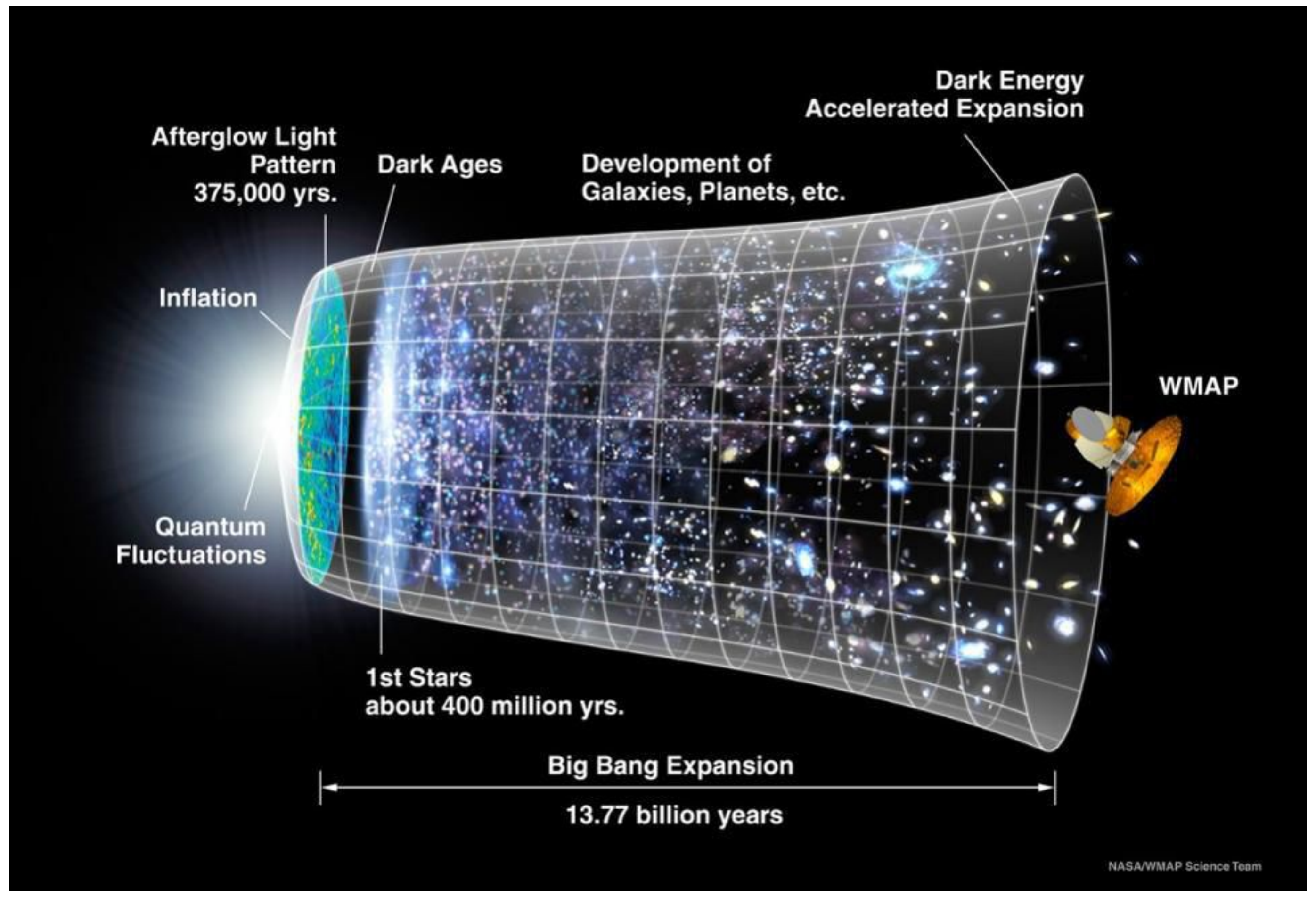
Evolution

$P^m_{\rm lin}(k)$

$P^m_{\rm nonlin}(k)$

Inflation ⟶ CMB ⟶ Matter distribution

Cosmological evolution



Obligatory cosmology slide

Evolution +
projection

ΛCDM
parameters?

$C_\ell = \langle \delta T_{\ell m} \delta T_{\ell m} \rangle$

Non-linear gravitational
clustering

$P(k) \propto A_s k^{n_s - 1}$?

Evolution

$P^m_{\rm lin}(k)$

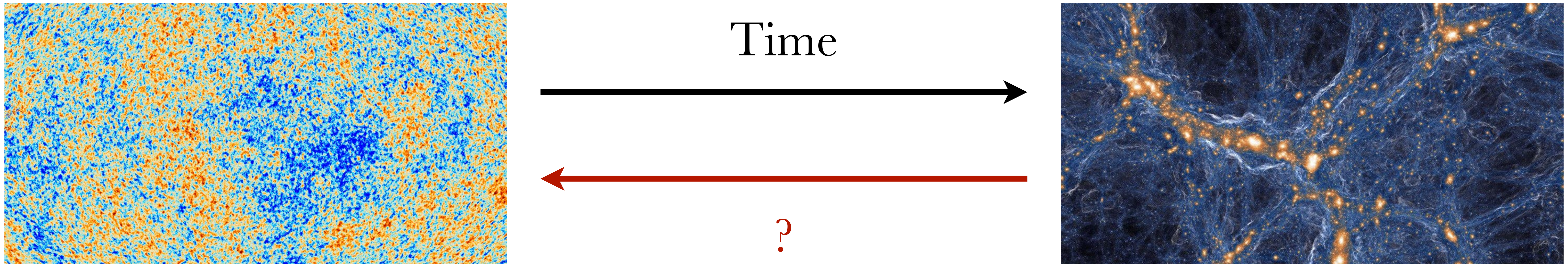
ΛCDM
parameters?

$P^m_{\rm nonlin}(k)$

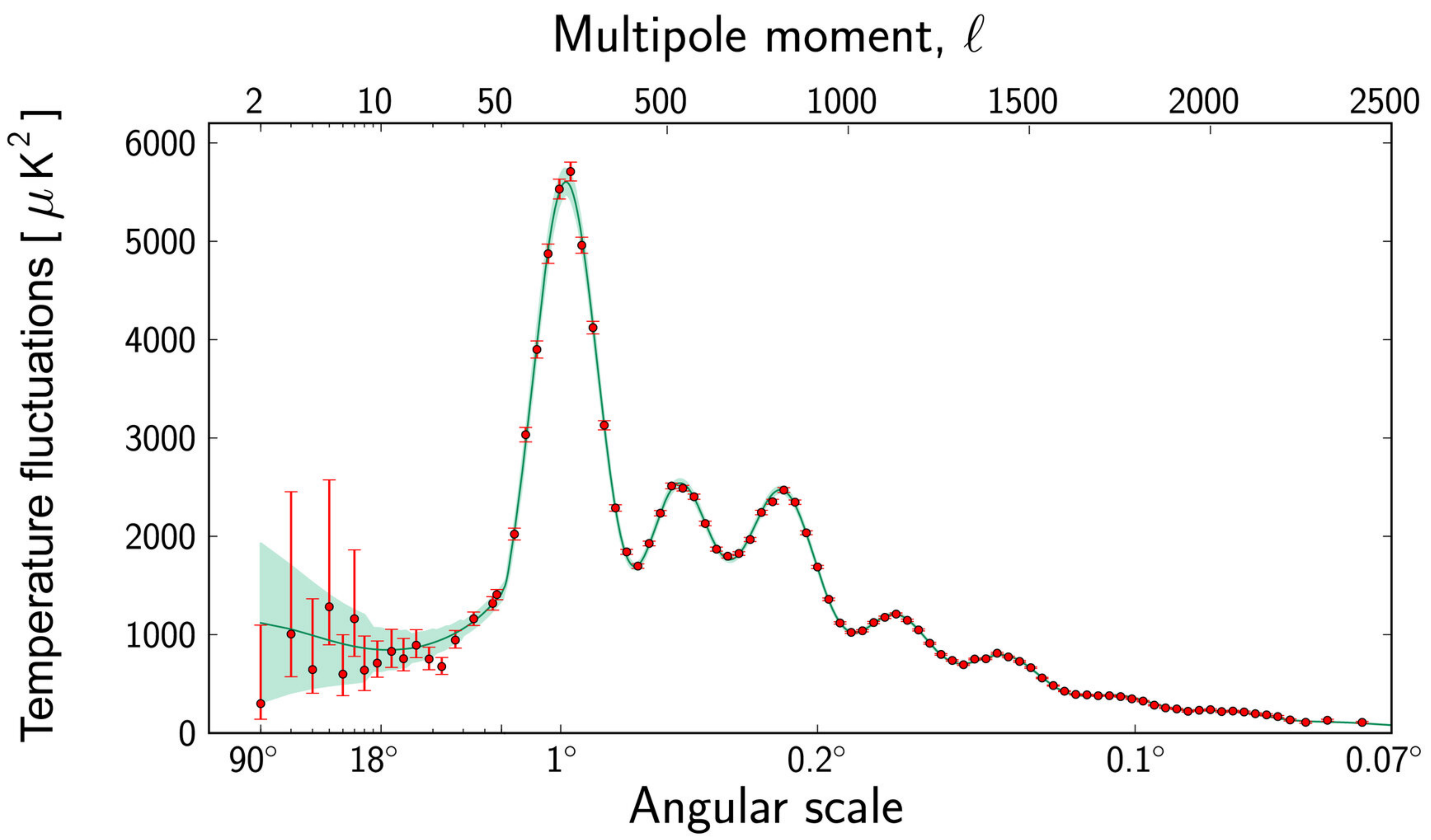
Many more
parameters!

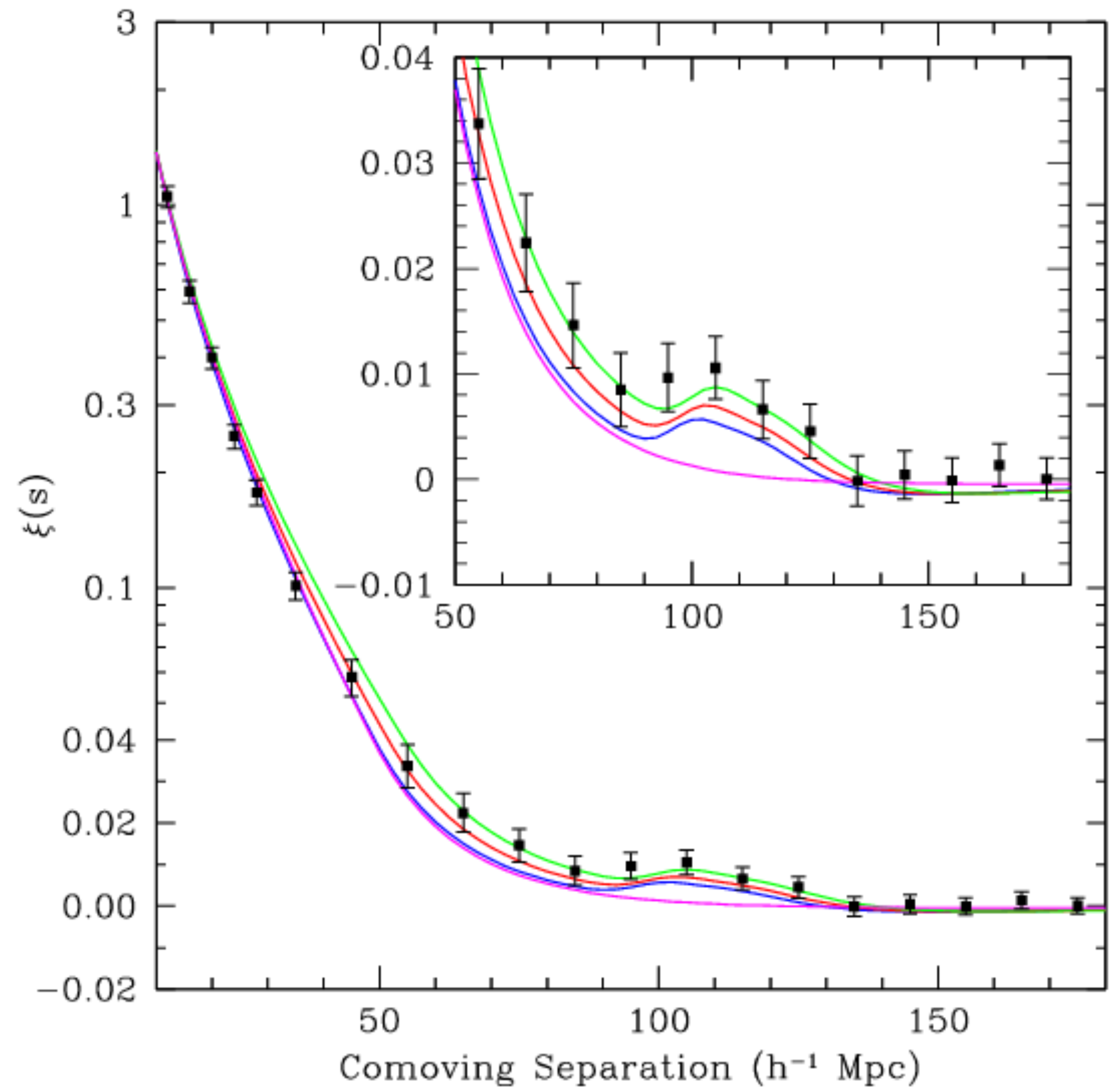
Inflation → CMB → Matter distribution

Cosmological reconstruction



Cosmic growth of structure
Late-time effects and Baryonic Acoustic Oscillations



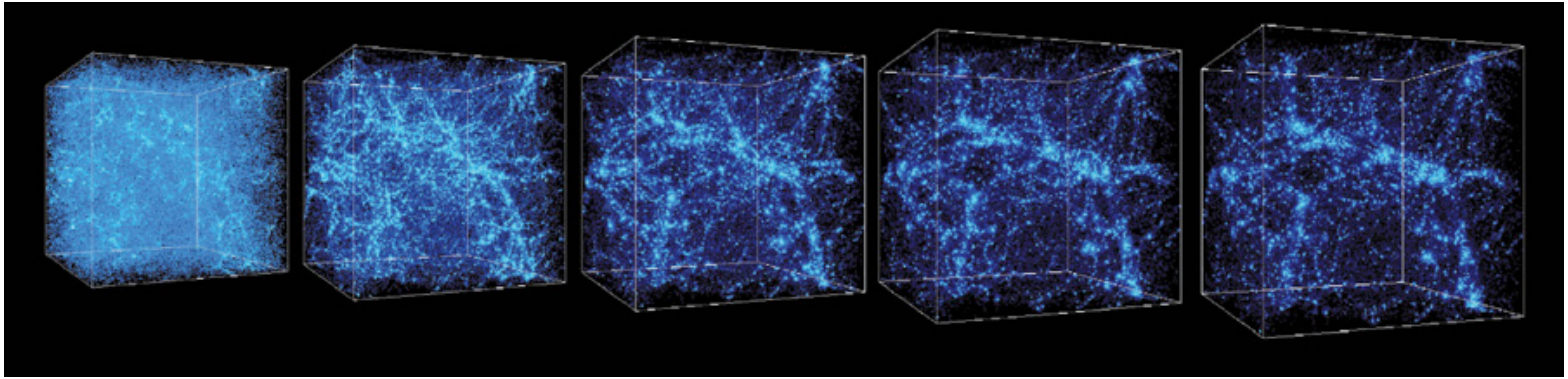


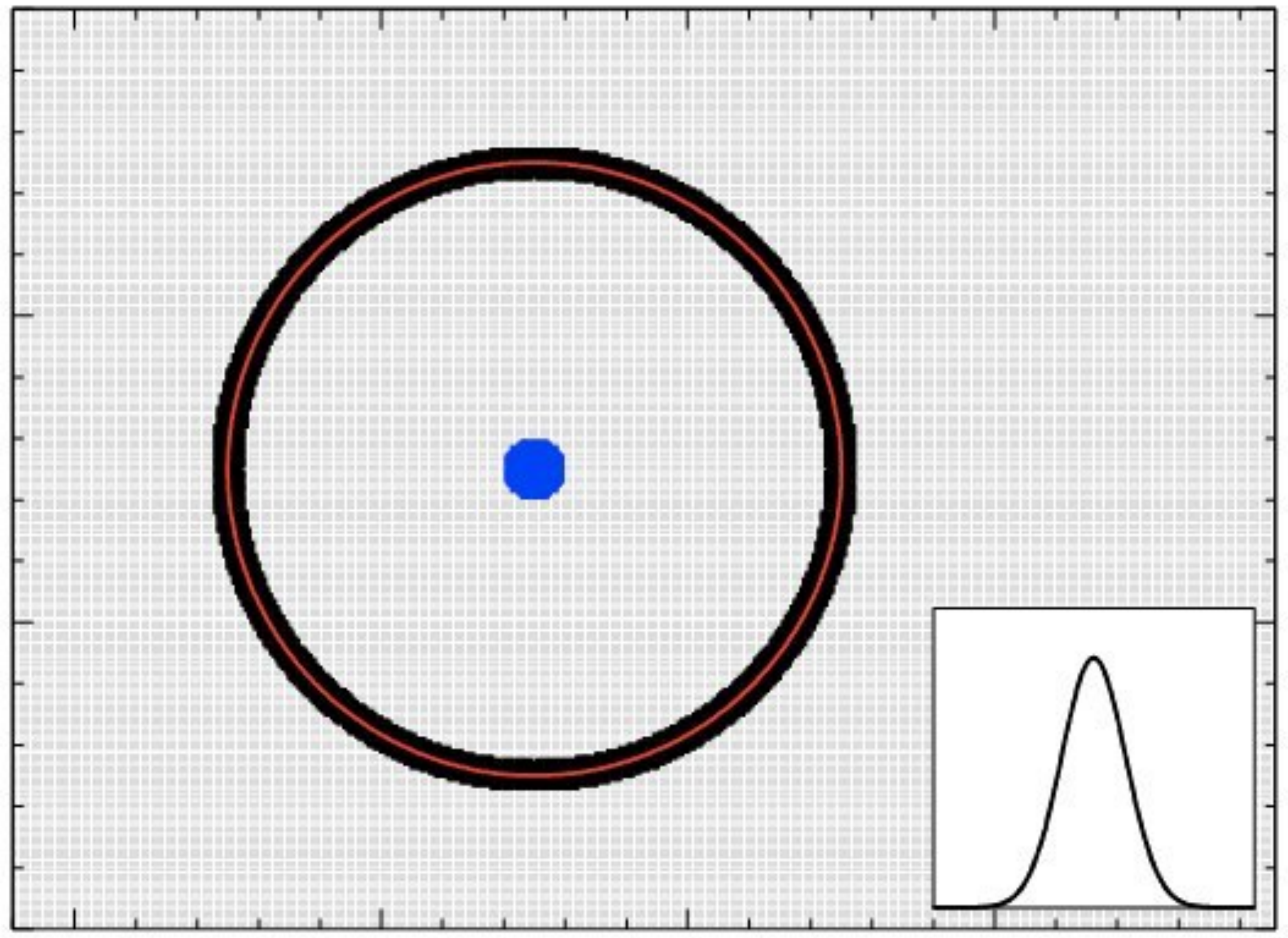
$$C_\ell = \langle \delta T_{\ell m} \delta T_{\ell m} \rangle$$

$$P^m_{\rm lin}(k)$$

Cosmic growth of structure

Late-time effects and Baryonic Acoustic Oscillations

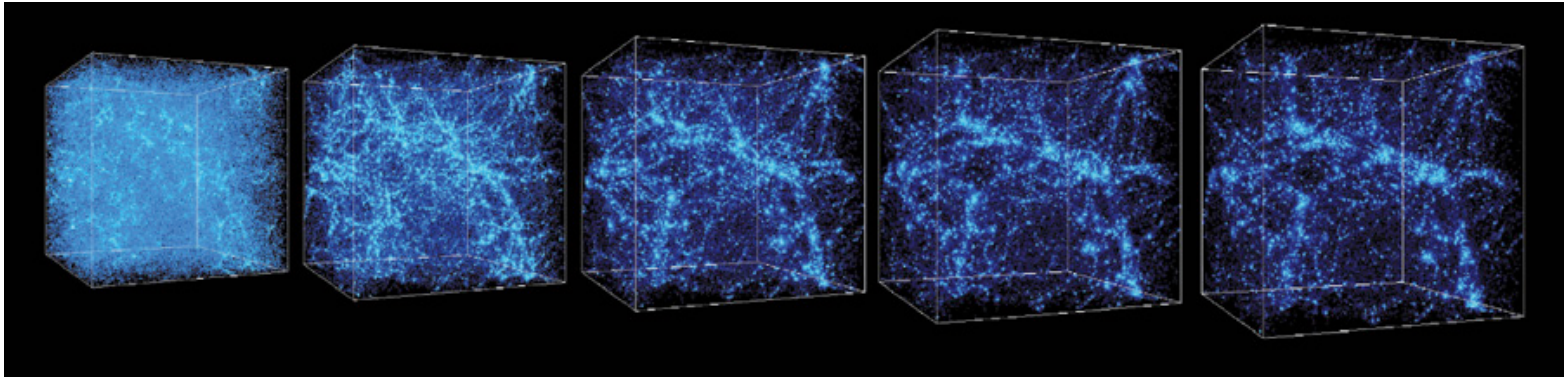


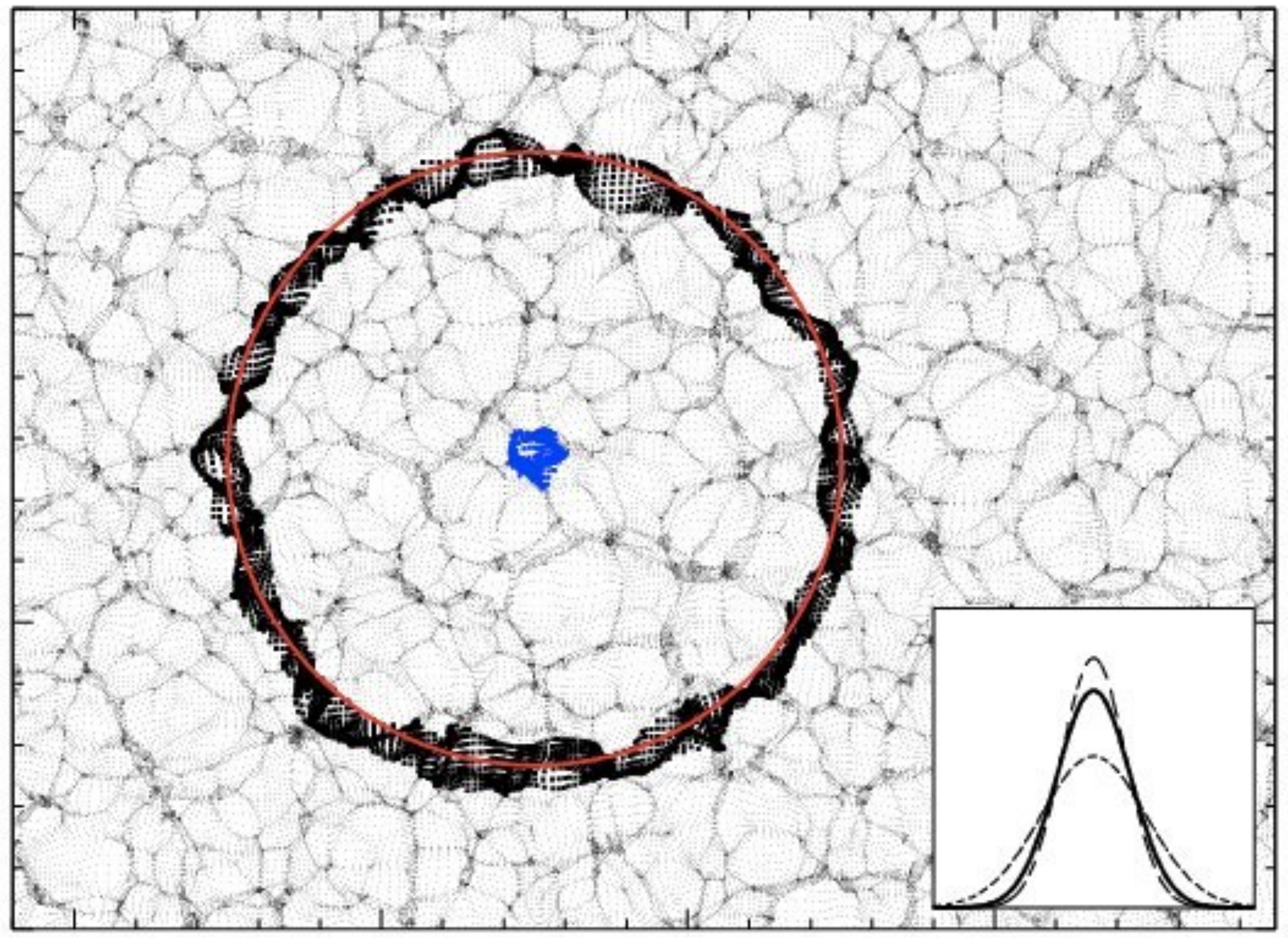


Padmanabhan+ (2012) [1202.0090]
Mon.Not.Roy.Astron.Soc., 427, 3

Cosmic growth of structure

Late-time effects and Baryonic Acoustic Oscillations

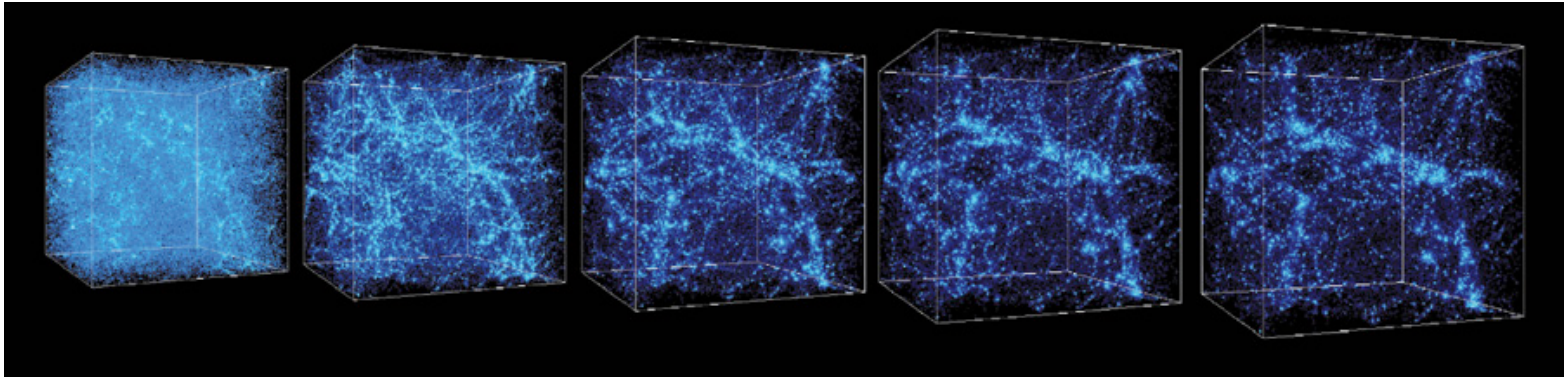


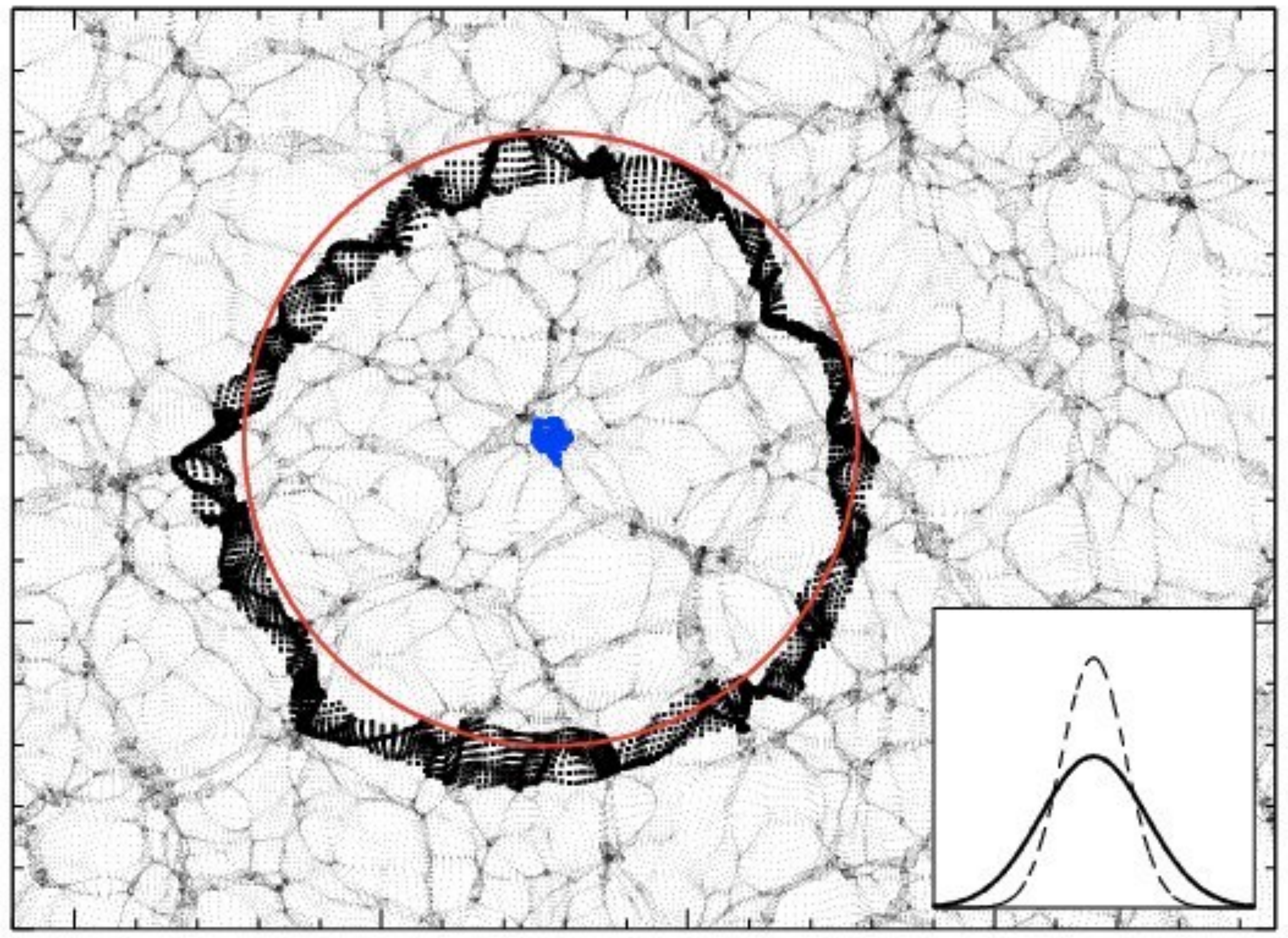


Padmanabhan+ (2012) [1202.0090]
Mon.Not.Roy.Astron.Soc., 427, 3

Cosmic growth of structure

Late-time effects and Baryonic Acoustic Oscillations

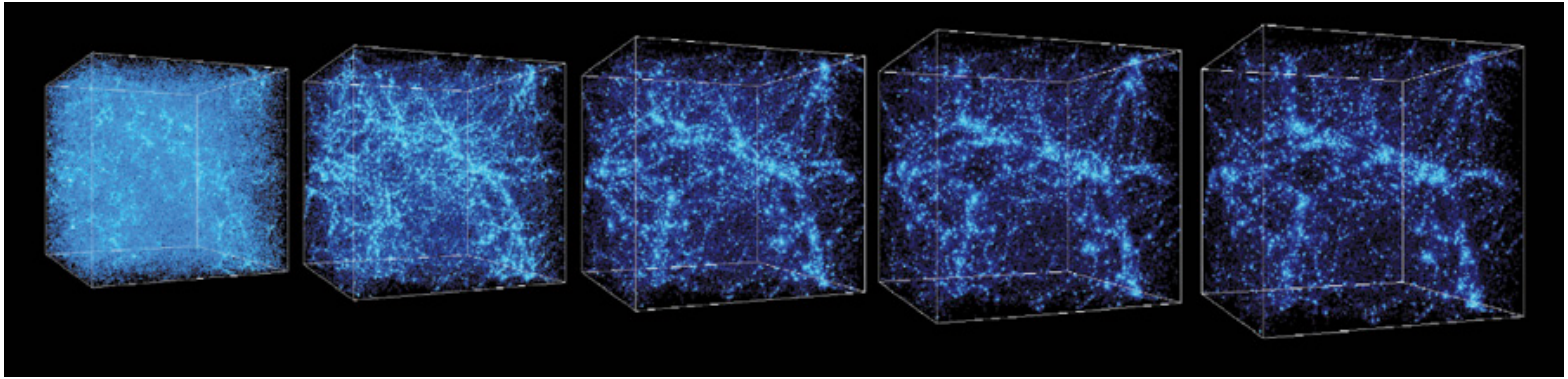


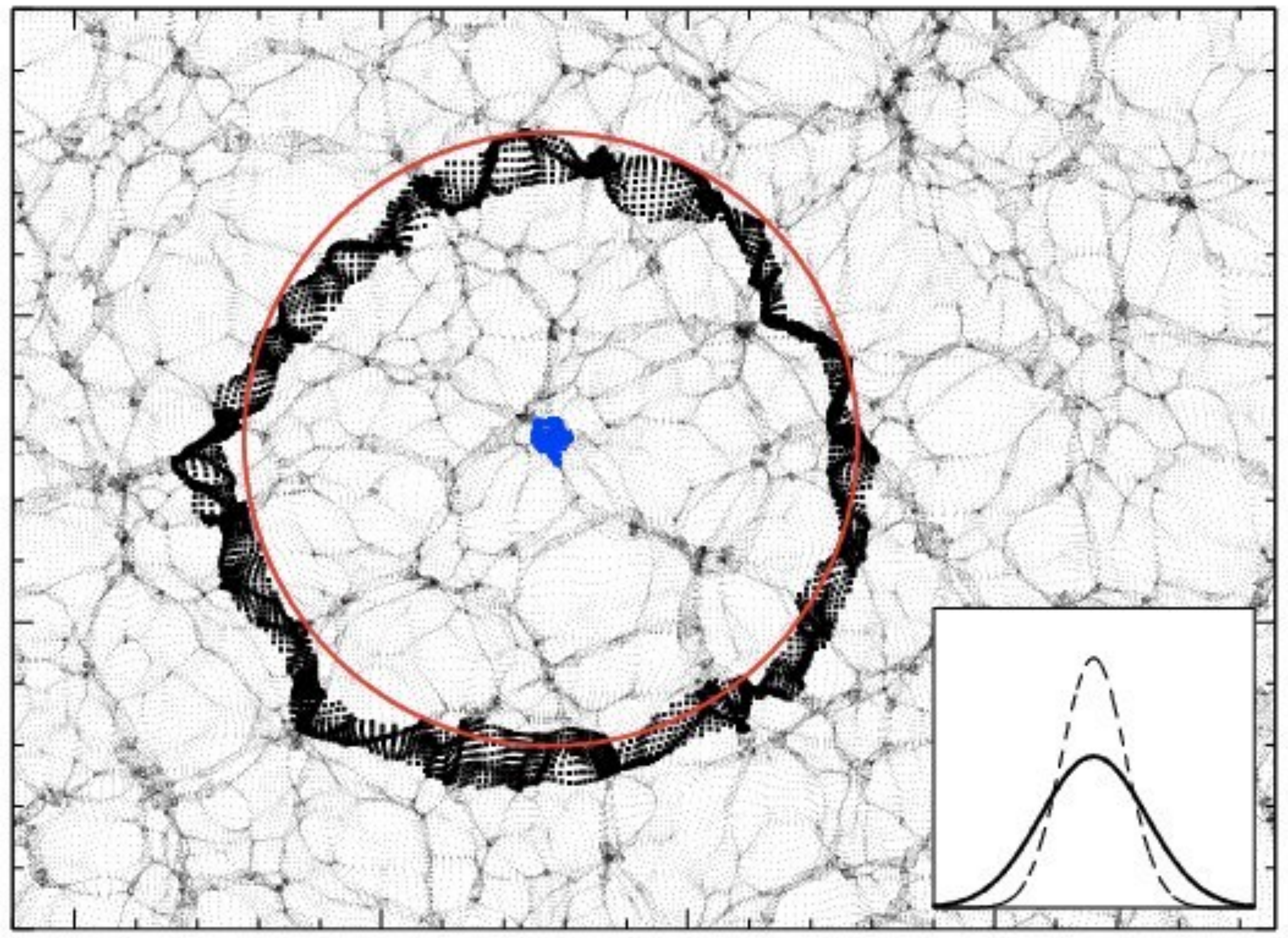


Padmanabhan+ (2012) [1202.0090]
Mon.Not.Roy.Astron.Soc., 427, 3

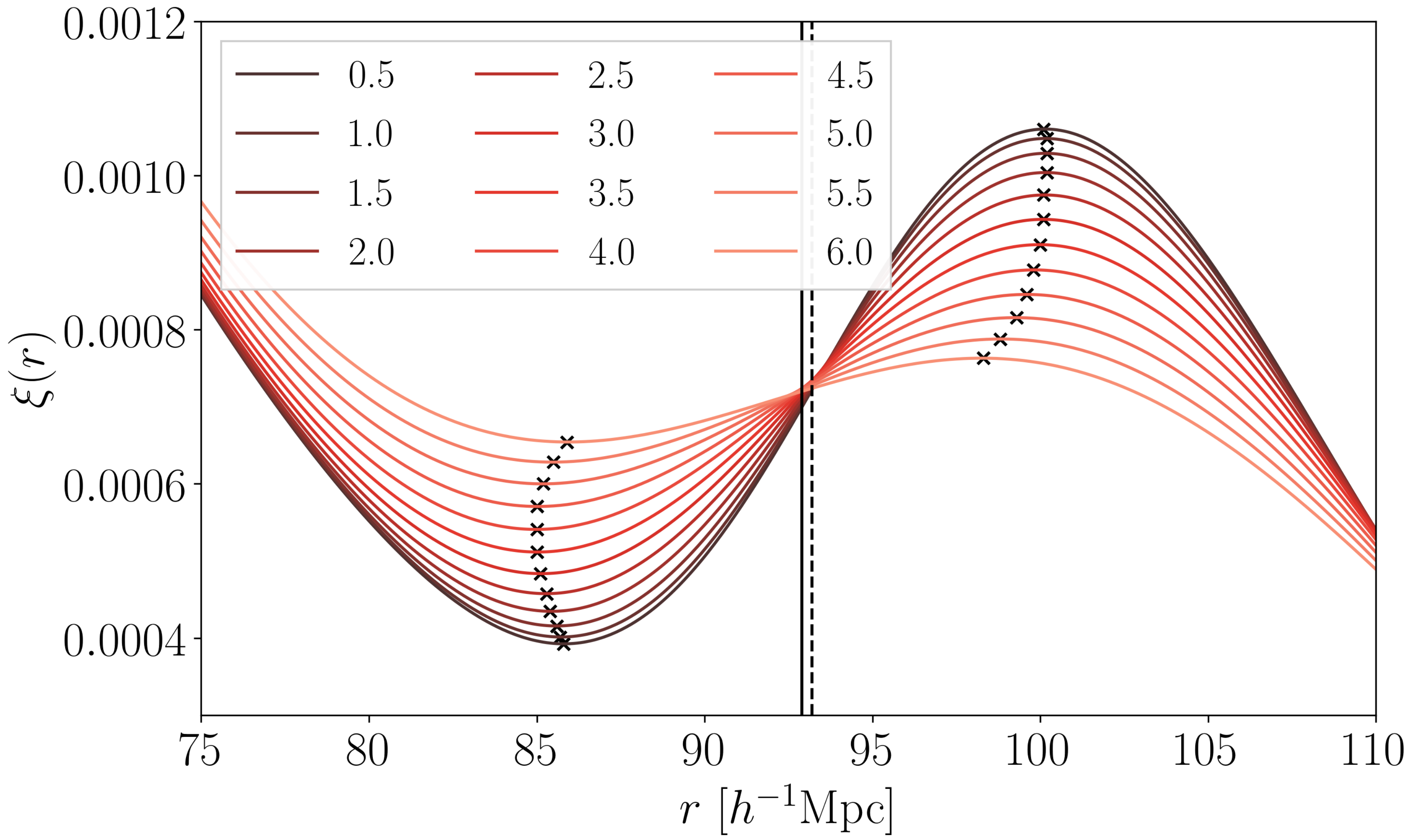
Cosmic growth of structure

Late-time effects and Baryonic Acoustic Oscillations





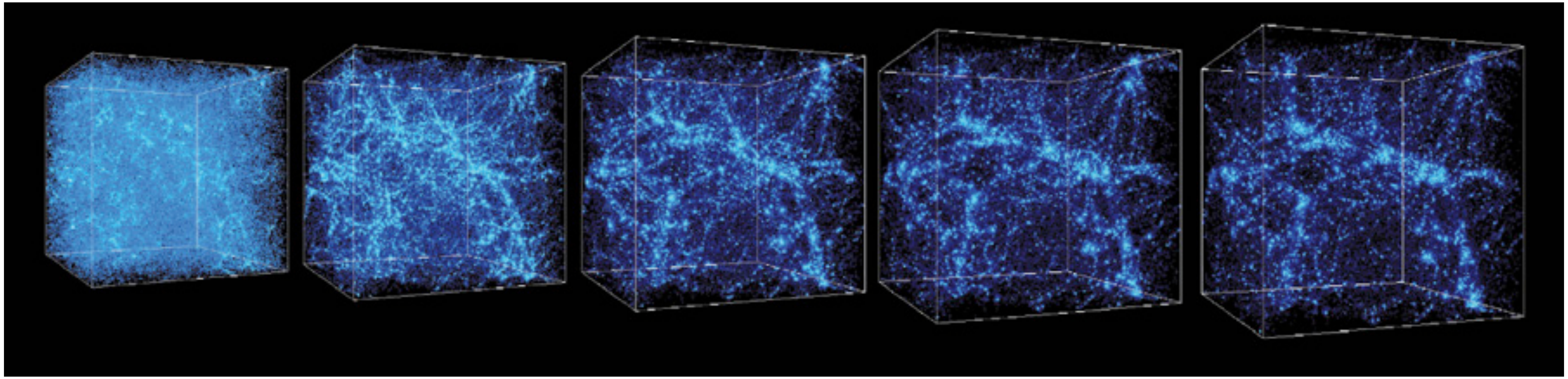
Padmanabhan+ (2012) [1202.0090]
Mon.Not.Roy.Astron.Soc., 427, 3

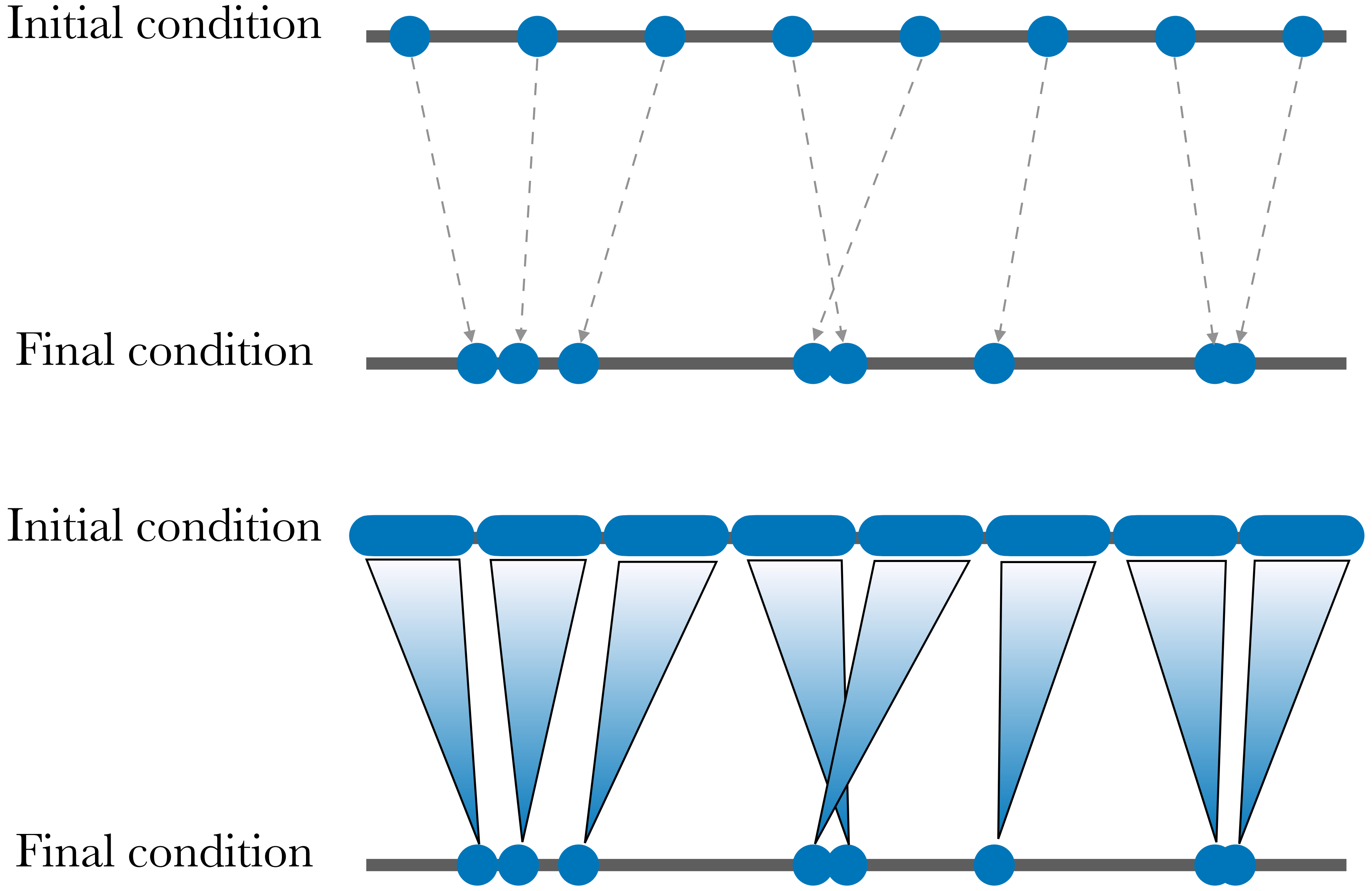


Nikakhtar+ (2021) [2101.08376] *Phys.Rev.D, 104,* 4

Cosmic growth of structure

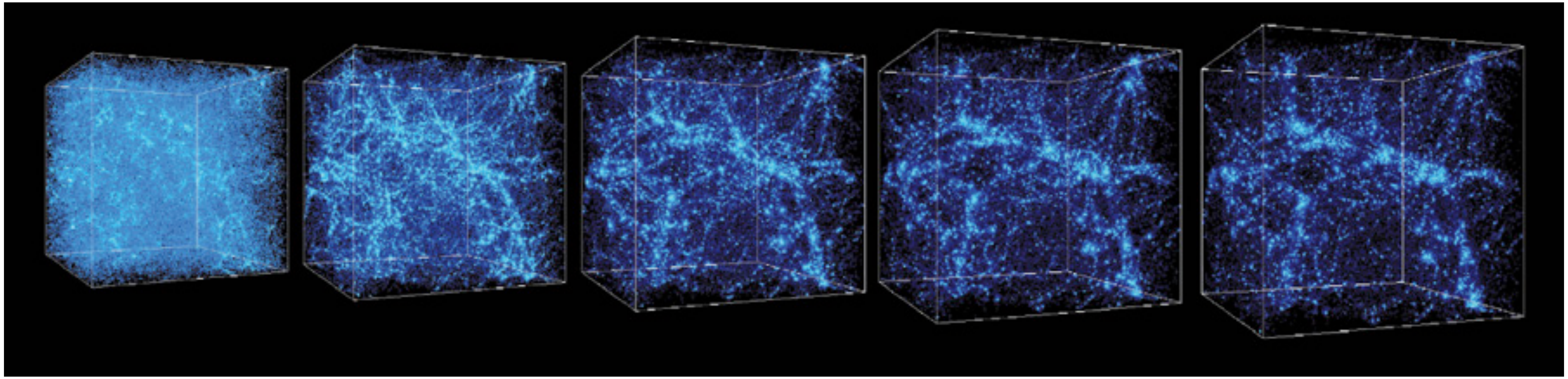
Late-time effects and Baryonic Acoustic Oscillations

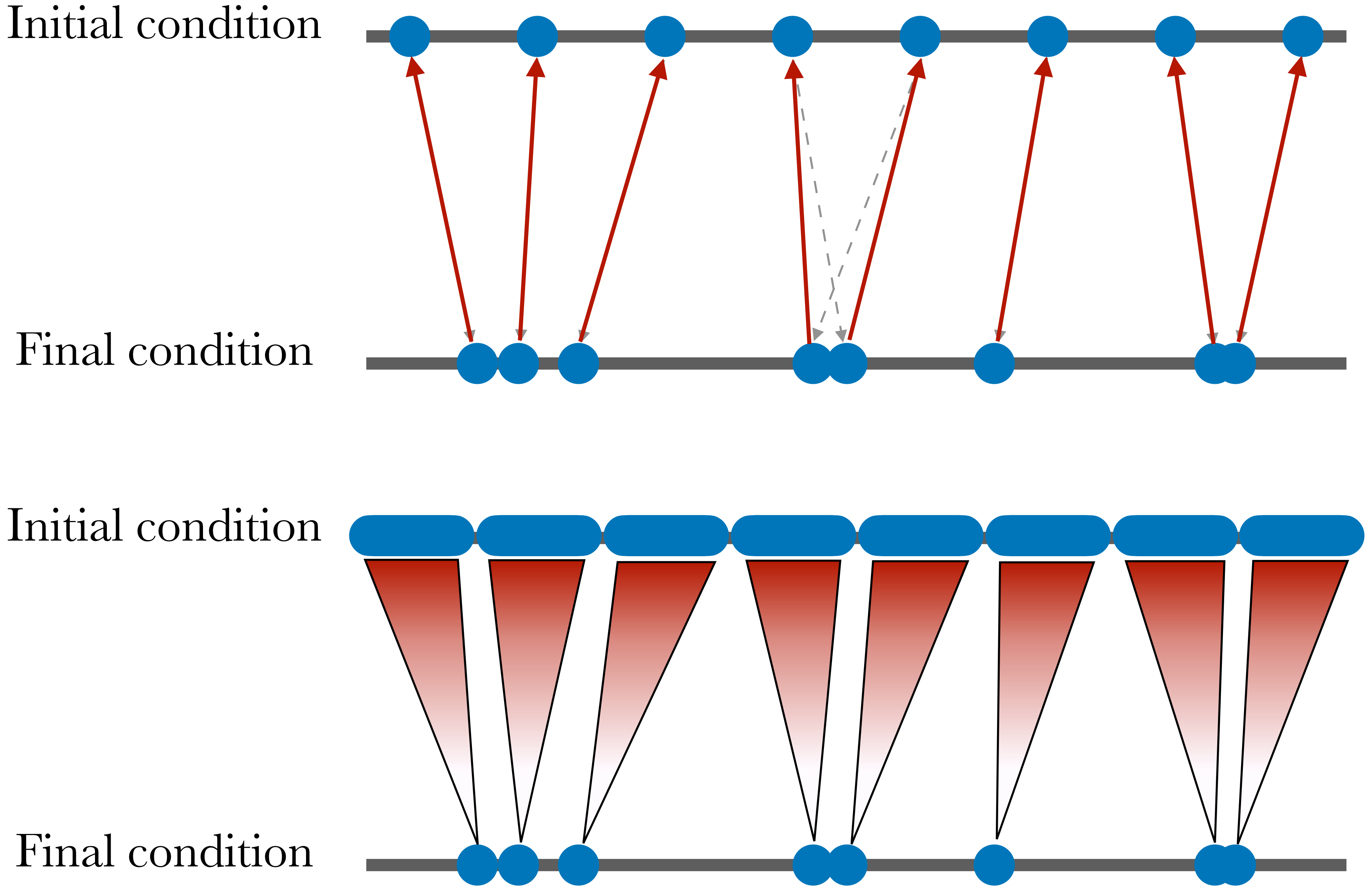




Cosmic growth of structure

Late-time effects and Baryonic Acoustic Oscillations

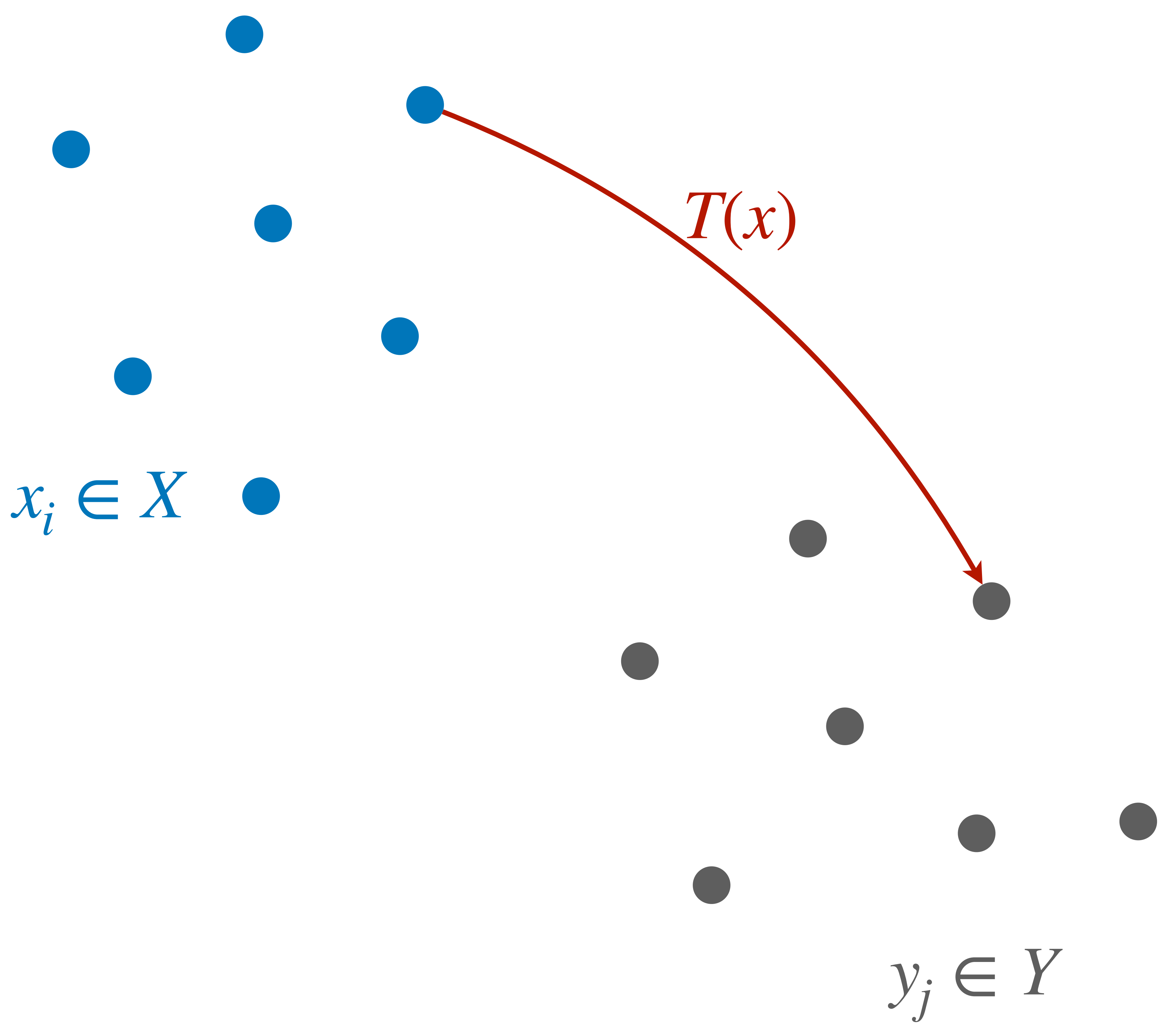




Introduction to Optimal Transport

Monge's Optimal Transport (1/3)

Discrete setting



Transport x_i to y_j at a cost $c(x_i, y_j)$ without loss via a 'transport map' $T(x)$.

$$T : X \to Y$$

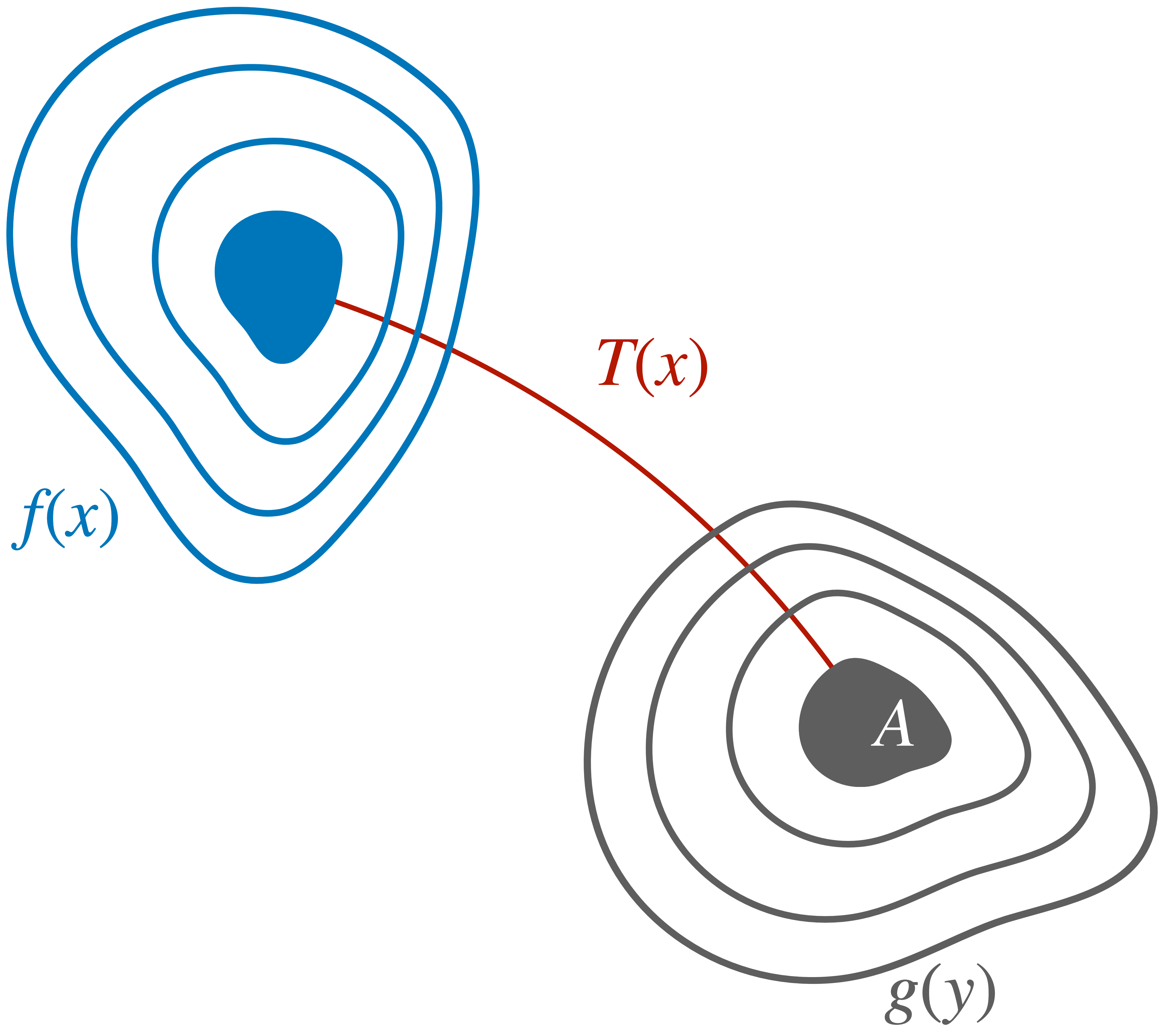
Define 'source measure' μ and 'target measure' ν. Then 'no loss' means

$$\mu(X) = \nu(Y)$$

i.e. mass balance.

Monge's Optimal Transport (1/3)

Continuous setting



Transport x_i to y_j at a cost $c(x_i, y_j)$ without loss via a 'transport map' $T(x)$.

$$T : X \to Y$$

Define 'source measure' μ and 'target measure' ν. Then 'no loss' means

$$\mu(X) = \nu(Y)$$

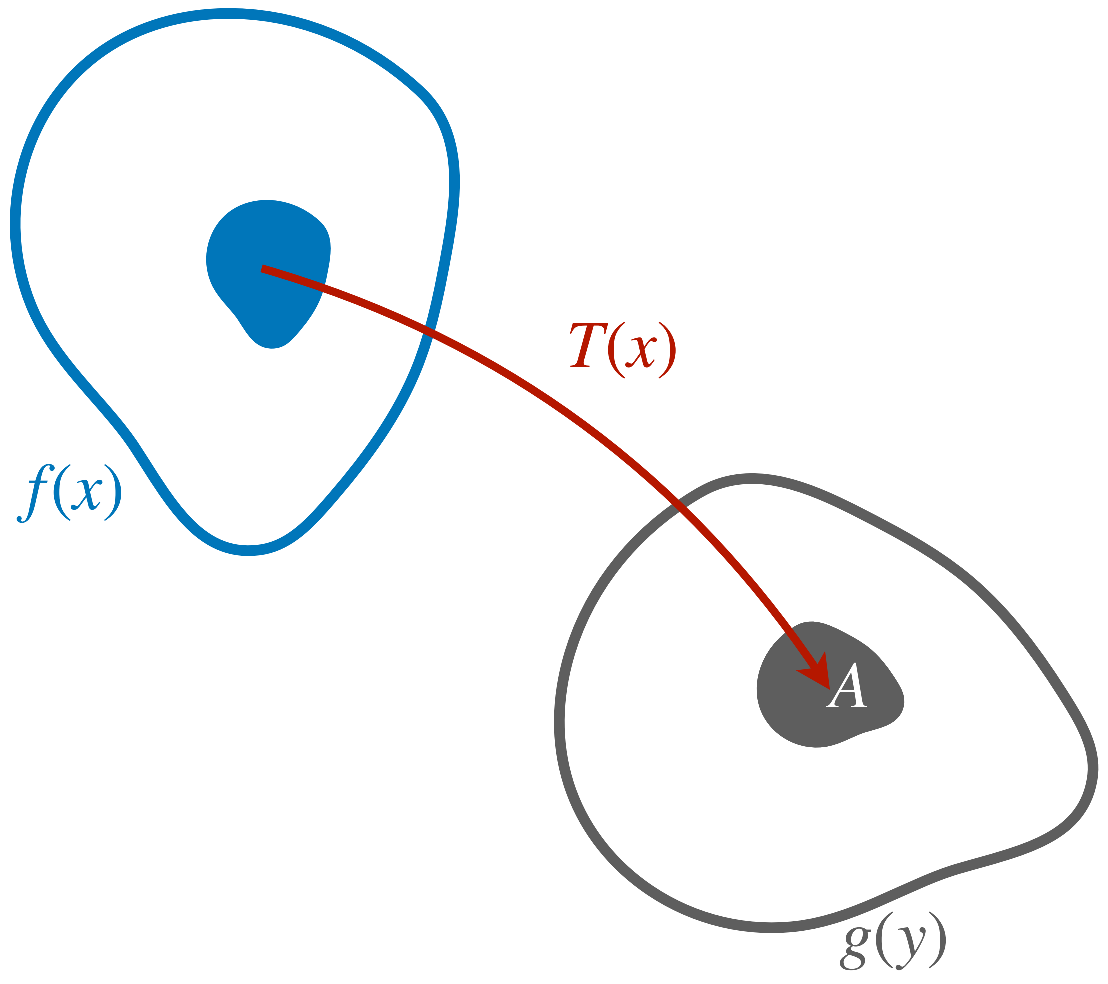
i.e. mass balance. In particular,

$$\mu(T^{-1}(A)) = \nu(A) \quad , \quad \forall A \subset Y$$

or, $T\#\mu = \nu$ ('push-forward'), ensures conservation of mass.

Monge's Optimal Transport (1/3)

Continuous setting



Find optimal transport map T by

$$\inf_T \left\{ \int_{\mathbb{R}^n} c(x, T(x))\, d\mu \mid T\#\mu = \nu \right\}$$

See also "Earth-movers distance (EMD)"

Transport x_i to y_j at a cost $c(x_i, y_j)$ without loss via a 'transport map' $T(x)$.

$$T : X \to Y$$

Define 'source measure' μ and 'target measure' ν. Then 'no loss' means

$$\mu(X) = \nu(Y)$$

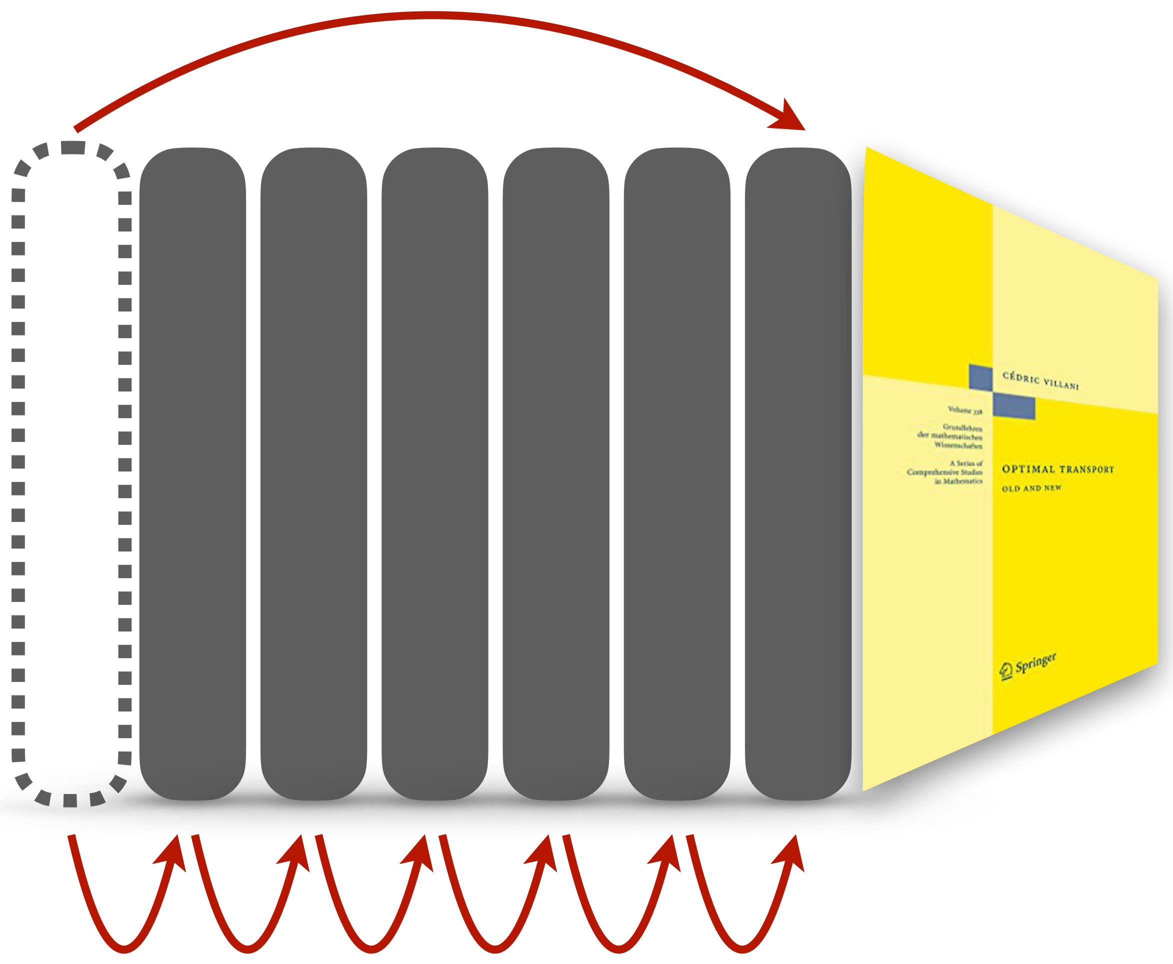
i.e. mass balance. In particular,

$$\mu(T^{-1}(A)) = \nu(A) \quad , \quad \forall A \subset Y$$

or, $T\#\mu = \nu$ ('push-forward'), ensures conservation of mass.

Book-moving problem (1/3)

A 1-dimensional, discrete example



Transport distribution of books $f(x)$ to form other distribution $g(y)$ at a cost $c(x, y)$ without loss via a 'transport map' $T(x)$.

Consider $c_1(x, y) = |x - y|$ and $c_2(x, y) = (x - y)^2$:

$$\hat{d} = \inf_T \left\{ \sum_i c(x_i, T(x_i)) \right\}$$

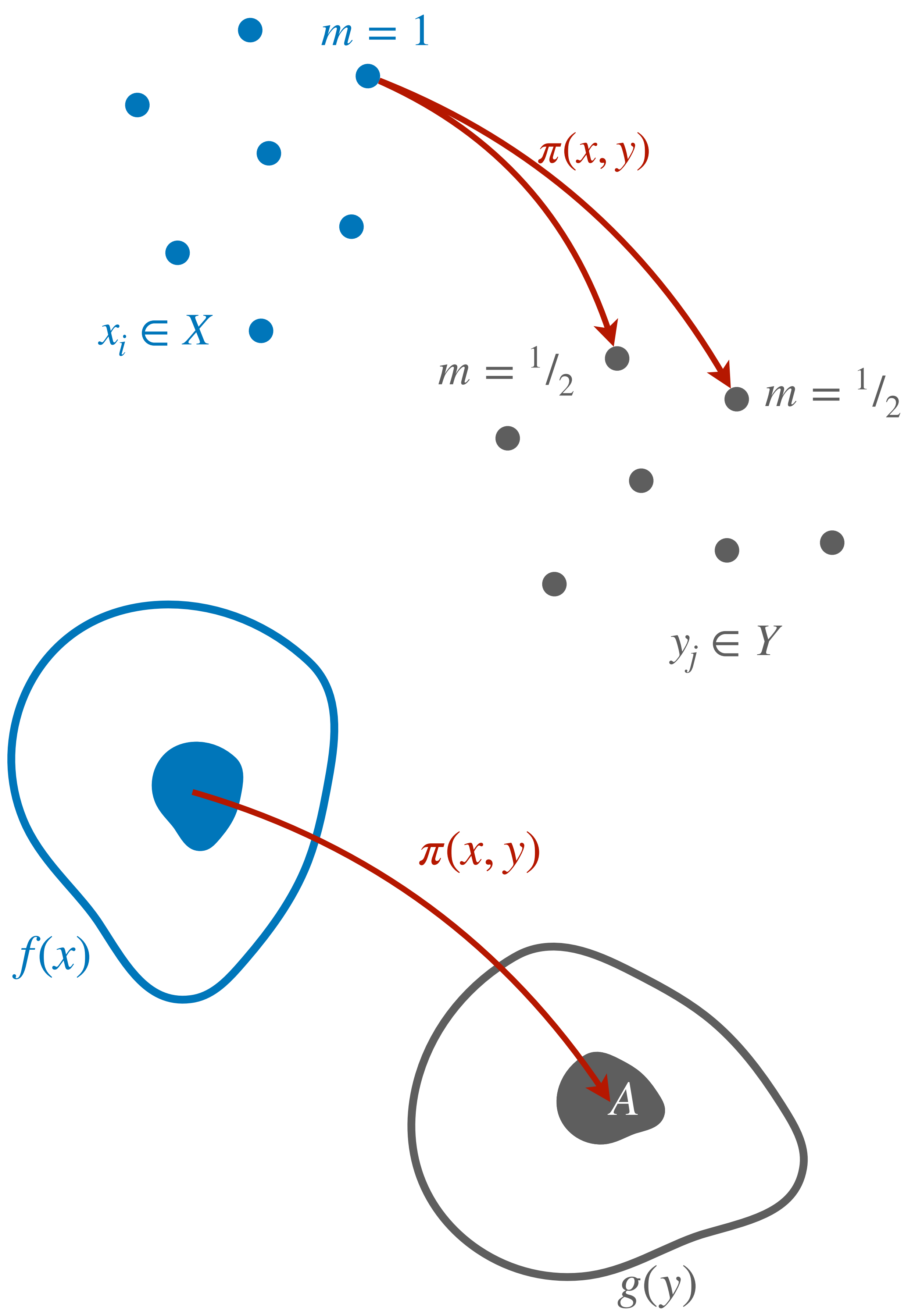
Solution 1: Move 1 book by N separations.

Solution 2: Move all N books by 1 separation each.

Optimal transport plan depends on cost!

Kantorovich's Optimal Transport (2/3)

Considering weights and 'splitting mass'



How much mass is transported from x_i to y_j can be stored in another measure $\pi(x, y)$

e.g. $\pi(B, A)$ documents how much mass moves from B to A, $\forall\, B \subset X$ and $A \subset Y$.

Conservation of mass:

$$\pi(B, Y) = \mu(B)\ \forall B \subset X$$
$$\pi(X, A) = \nu(A)\ \forall A \subset Y$$

Optimal transport:

$$\inf_{\pi} \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y)\, d\pi(x, y) \mid \pi \in \Pi(\mu, \nu) \right\}$$

Quadratic cost (2/3)

This ensures convexity, *cf.* the cosmological setting

Brenier (1991) Communications on Pure and Applied Mathematics, 44, 375

Find optimal transport map T by

$$\inf_T \left\{ \int_{\mathbb{R}^n} c(x, T(x))\, f(x)\, dx \mid T\#\mu = \nu \right\}$$

Subject to

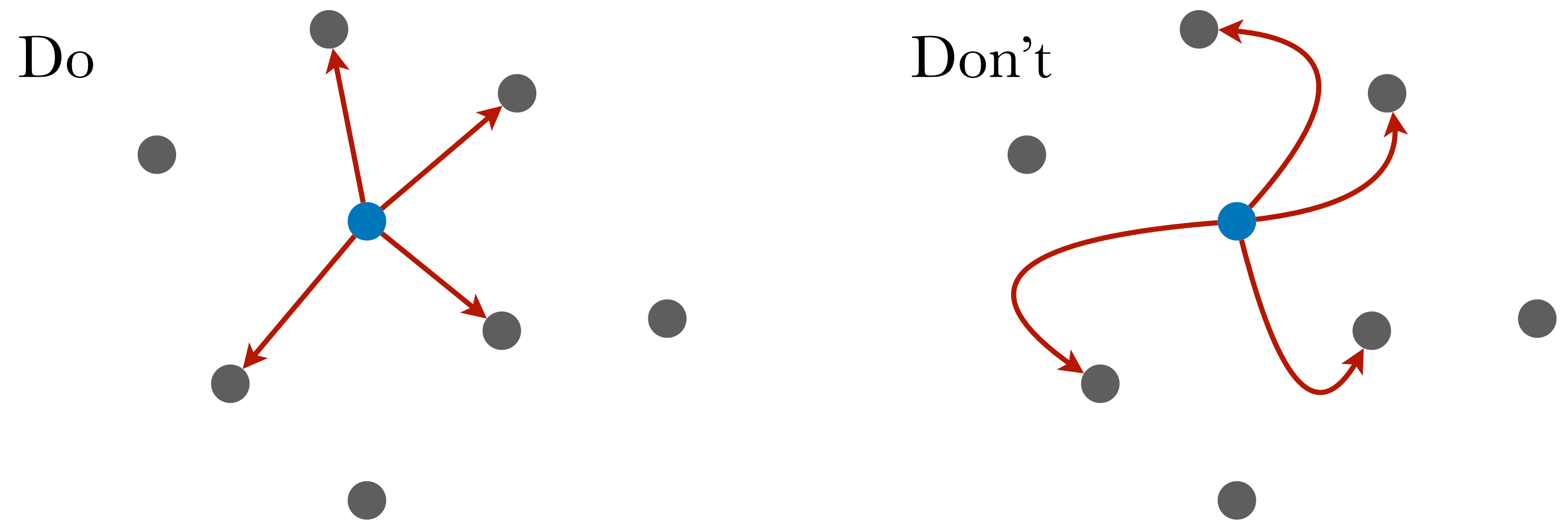
$$f(x) = g(y) \det \left(\nabla T(x)\right)$$

Constraint can be non-linear

Brenier's theorem:

A cyclically monotone map exists that can be expressed as a gradient of a convex function (potential)

$$T(x) = \nabla p(x)$$



Transport goods along direct ways, or don't move in circles! (Note relation to curl-free fields in physics)

Quadratic cost (2/3)

This ensures convexity, *cf.* the cosmological setting

Brenier (1991) Communications on Pure and Applied Mathematics, 44, 375

Find optimal transport map T by

$$\inf_T \left\{ \int_{\mathbb{R}^n} c(x, T(x))\, f(x)\, dx \mid T\#\mu = \nu \right\}$$

Subject to

$$f(x) = g(y) \det\left(\nabla T(x)\right)$$

Brenier's theorem:

A cyclically monotone map exists that can be expressed as a gradient of a convex function (potential)

$$T(x) = \nabla p(x)$$

With mass conservation this becomes:

$$\det\left(D^2 p(x)\right) g(T(x)) = f(x)$$

Monge-Ampére equation

Gradient flow in two slides (1/2)

Consider $F : \mathbb{R}^n \to \mathbb{R}$ convex

$$x'(t) = -\nabla F(x(t))$$
$$x(0) = 0$$

Backward Euler scheme (discrete)

$$\frac{x^{n+1} - x^n}{\tau} = -\nabla F(x^{n+1})$$

Or:

$$\nabla \left[\frac{1}{2\tau} |x - x^n|^2 + \nabla F(x) \right]_{x = x^{n+1}} = 0$$

Gradient flow in two slides (1/2)

Consider $F : \mathbb{R}^n \to \mathbb{R}$ convex

$$x'(t) = -\nabla F(x(t))$$
$$x(0) = 0$$

Backward Euler scheme (discrete)

$$\frac{x^{n+1} - x^n}{\tau} = -\nabla F(x^{n+1})$$

Or:

$$\nabla \left[\frac{1}{2\tau} |x - x^n|^2 + \nabla F(x) \right]_{x = x^{n+1}} = 0$$

$$x_\tau^{n+1} \in \operatorname{argmin} \left\{ \frac{1}{2\tau} |x - x_\tau^n|^2 + \nabla F(x) \right\}$$

Gradient flow in two slides (1/2)

Consider $F : \mathbb{R}^n \to \mathbb{R}$ convex

$$x'(t) = -\nabla F(x(t))$$
$$x(0) = 0$$

Backward Euler scheme (discrete)

$$\frac{x^{n+1} - x^n}{\tau} = -\nabla F(x^{n+1})$$

Or:

$$\nabla \left[\frac{1}{2\tau} |x - x^n|^2 + \nabla F(x) \right]_{x = x^{n+1}} = 0$$

$$x_\tau^{n+1} \in \operatorname{argmin} \left\{ \frac{1}{2\tau} |x - x_\tau^n|^2 + \nabla F(x) \right\}$$

More generally, on metric space (X, d)
(and some conditions on F)

$$x_\tau^{n+1} \in \operatorname{argmin} \left\{ \frac{1}{2\tau} d(x, x_\tau^n)^2 + \nabla F(x) \right\}$$

Gradient flow in two slides (2/2)

More generally, on metric space (X, d) (and some conditions on F)

$$x_\tau^{n+1} \in \operatorname{argmin}\left\{\frac{1}{2\tau}d(x, x_\tau^n)^2 + \nabla F(x)\right\}$$

In $\mathbb{W}_2$ metric (Wasserstein gradient flows), and in continuous limit, one finds the PDE:

$$\rho_t - \nabla\left(\rho\frac{\delta F}{\delta\rho}\right) = 0$$

Gradient flow in two slides (2/2)

More generally, on metric space (X, d) (and some conditions on F)

$$x_\tau^{n+1} \in \operatorname{argmin} \left\{ \frac{1}{2\tau} d(x, x_\tau^n)^2 + \nabla F(x) \right\}$$

In $\mathbb{W}_2$ metric (Wasserstein gradient flows), and in continuous limit, one finds the PDE:

$$\rho_t - \nabla \left(\rho \frac{\delta F}{\delta \rho} \right) = 0$$

Example:

$$F(\rho) = \int \rho \log \rho \, \mathrm{d}x$$

leads to the PDE:

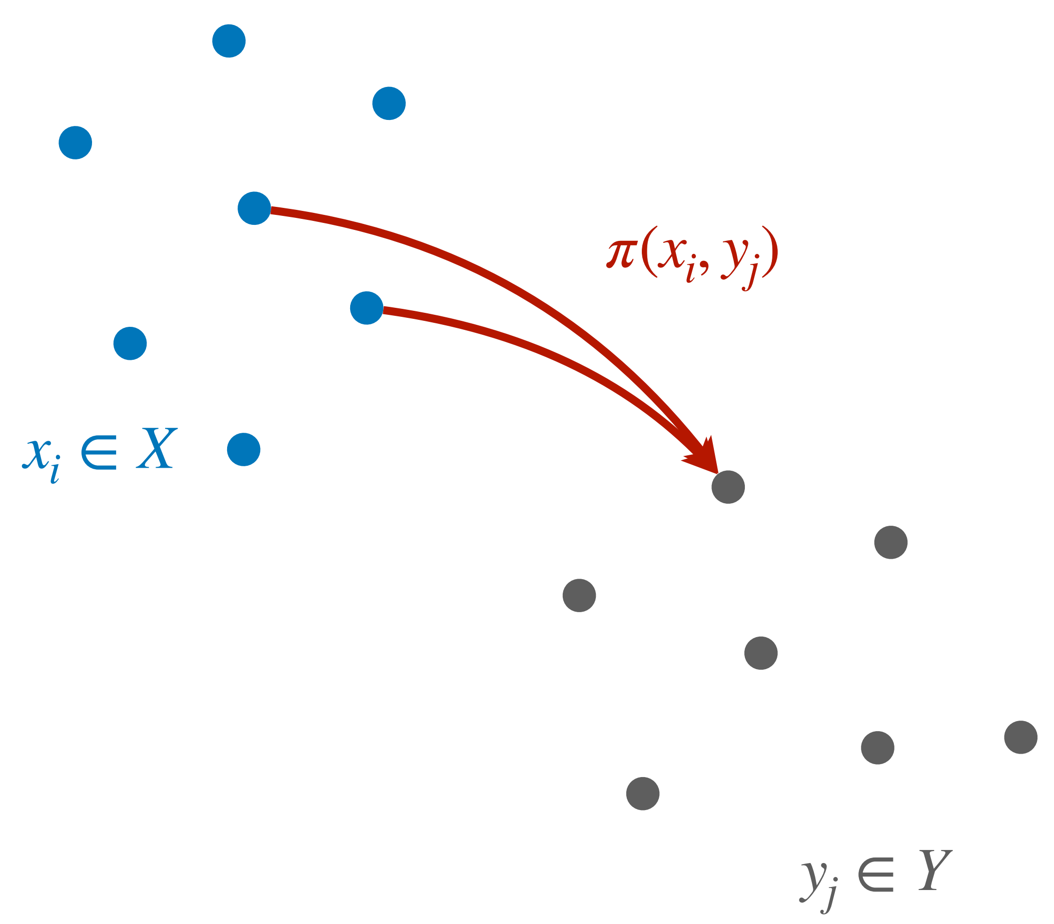
$$\rho_t - \nabla^2 \rho = 0$$

Heat equation from optimal transport!

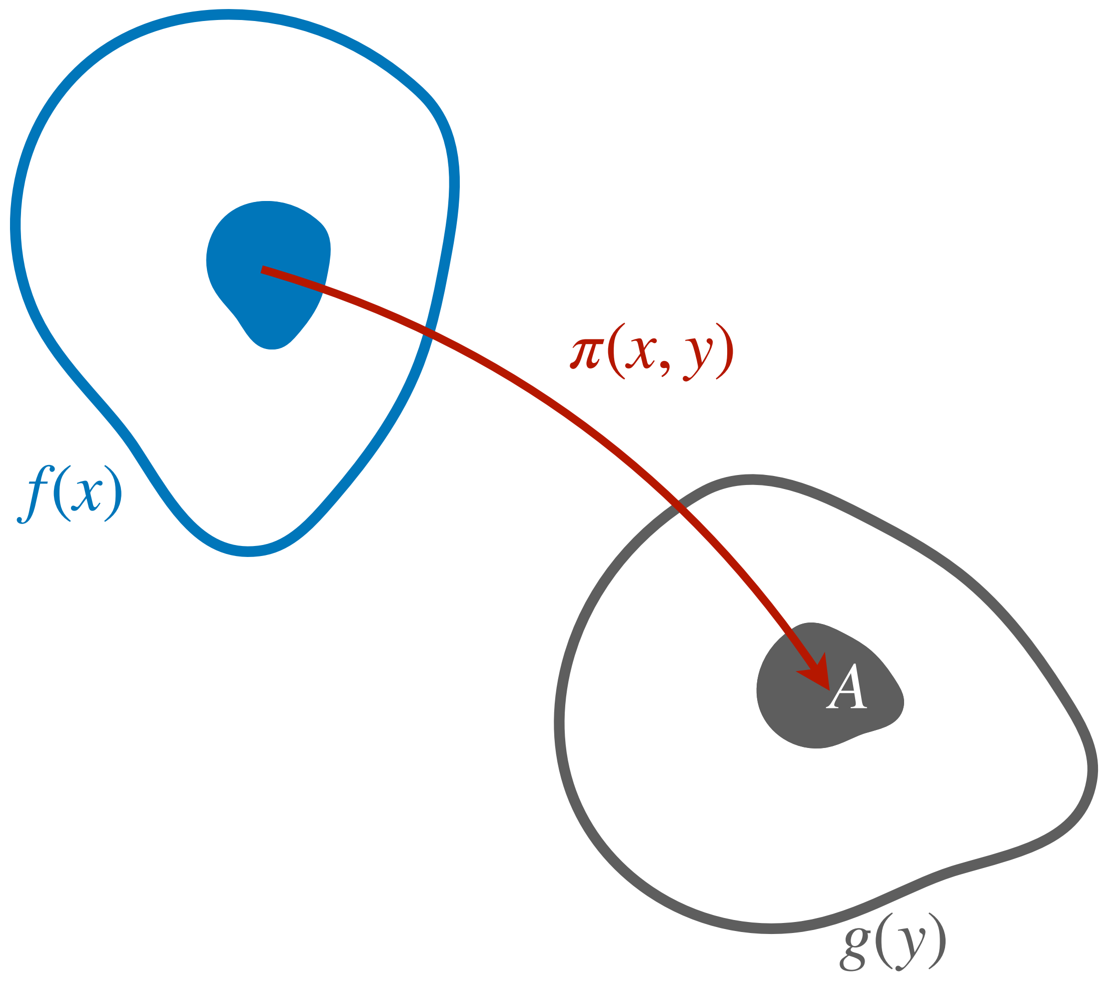
Jordan, Kinderlehrer, Otto (JKO), SIAM Journal on Mathematical Analysis, 1998, 29, 1

see also, Santambrogio (2015), Optimal transport for applied mathematicians. Birkhäuser/Springer

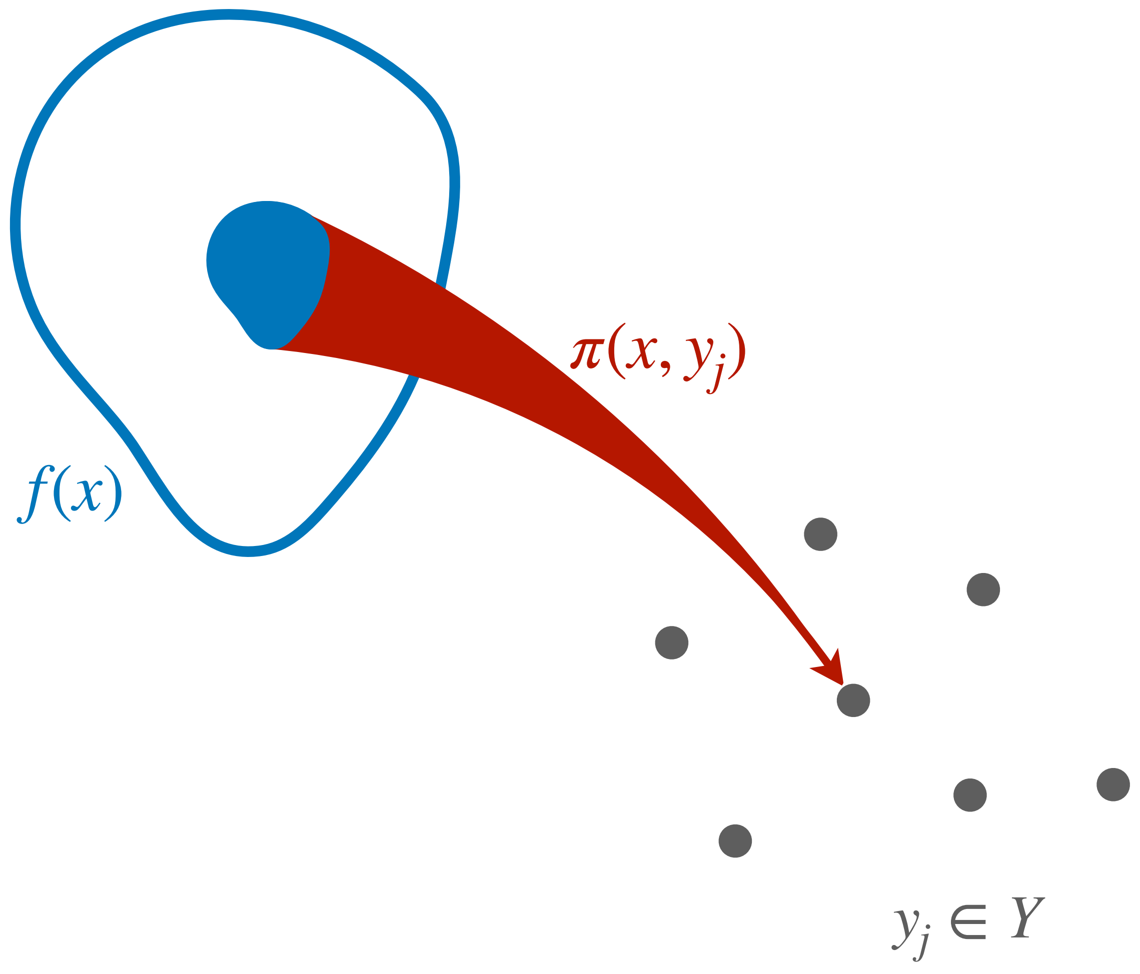
Semi-discrete Optimal Transport (3/3)



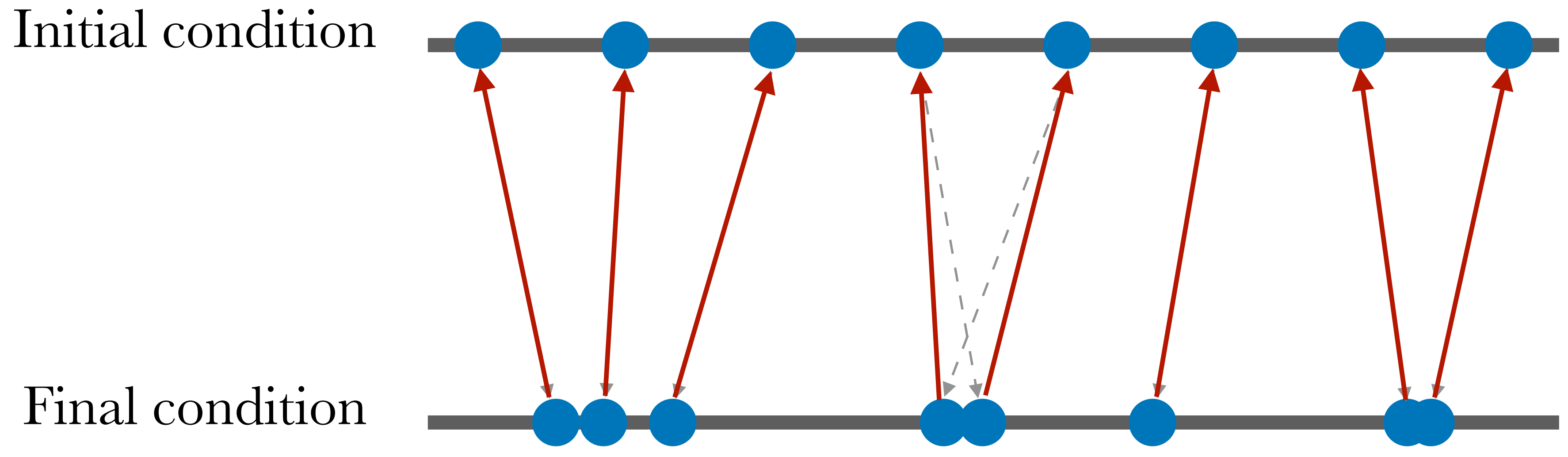
Discrete OT

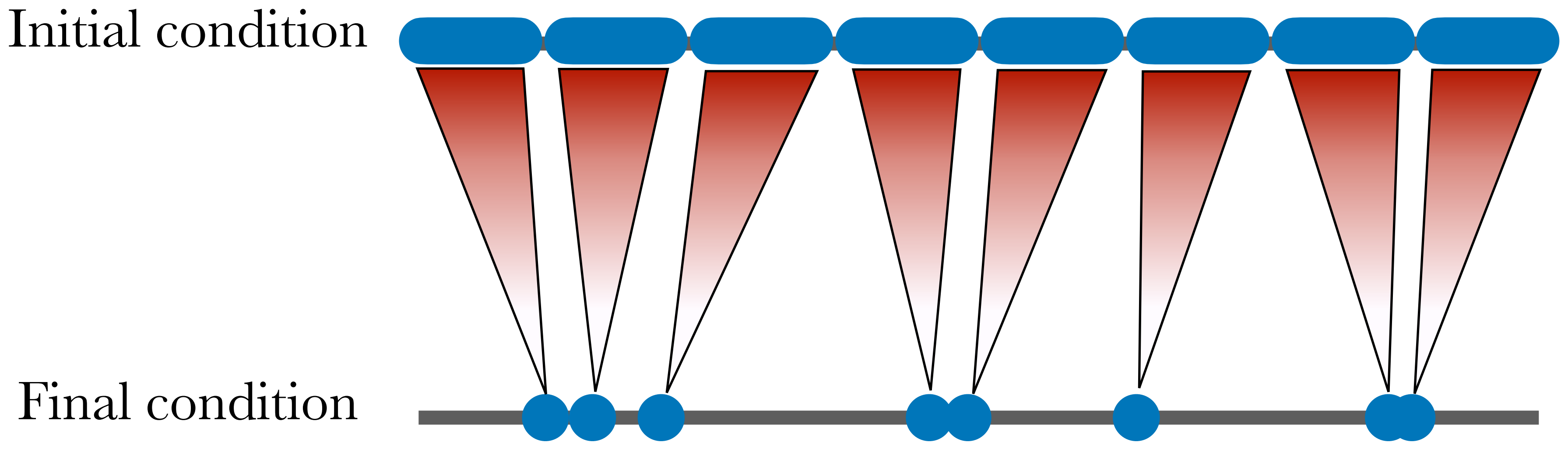


Continuous OT

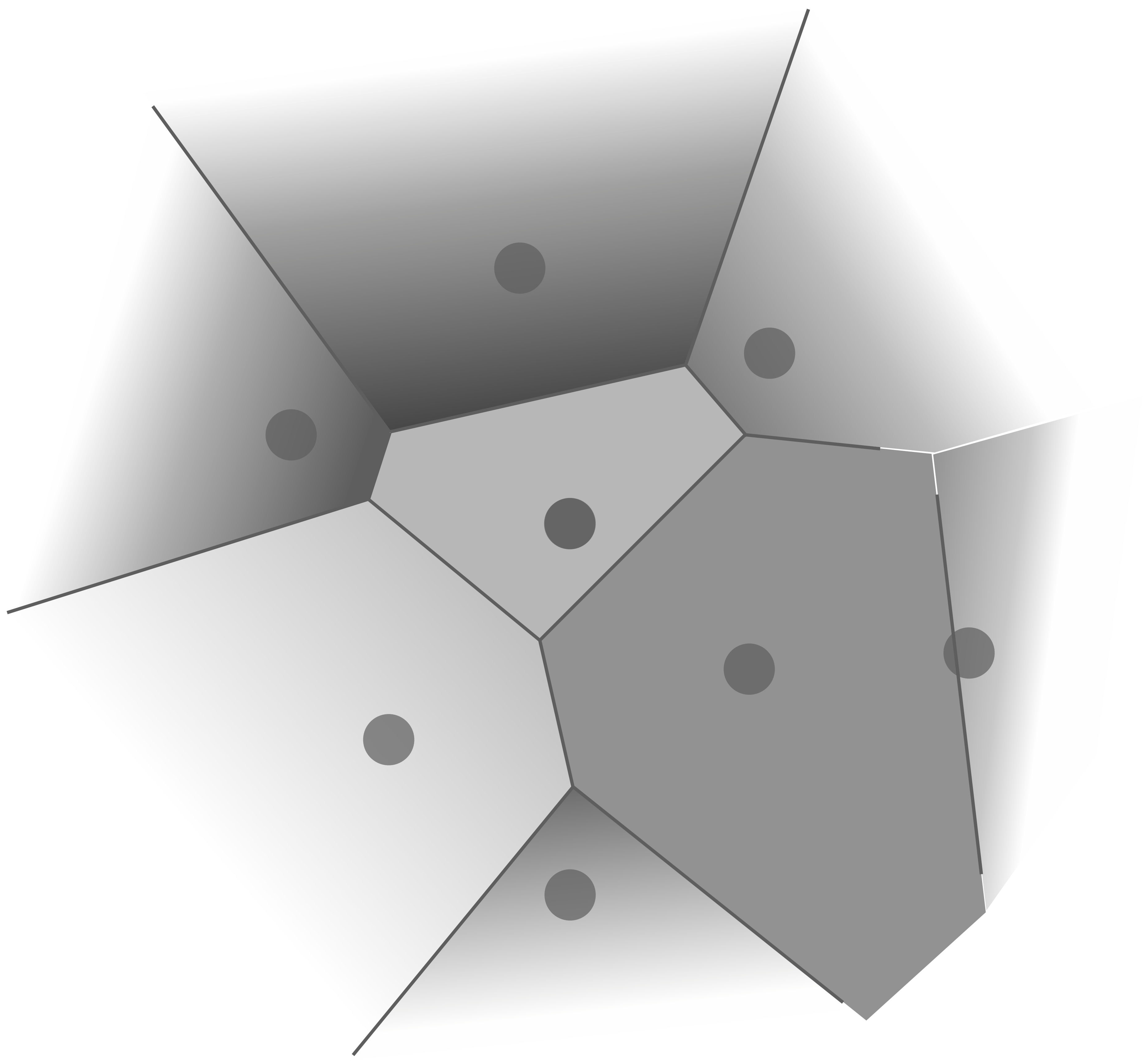


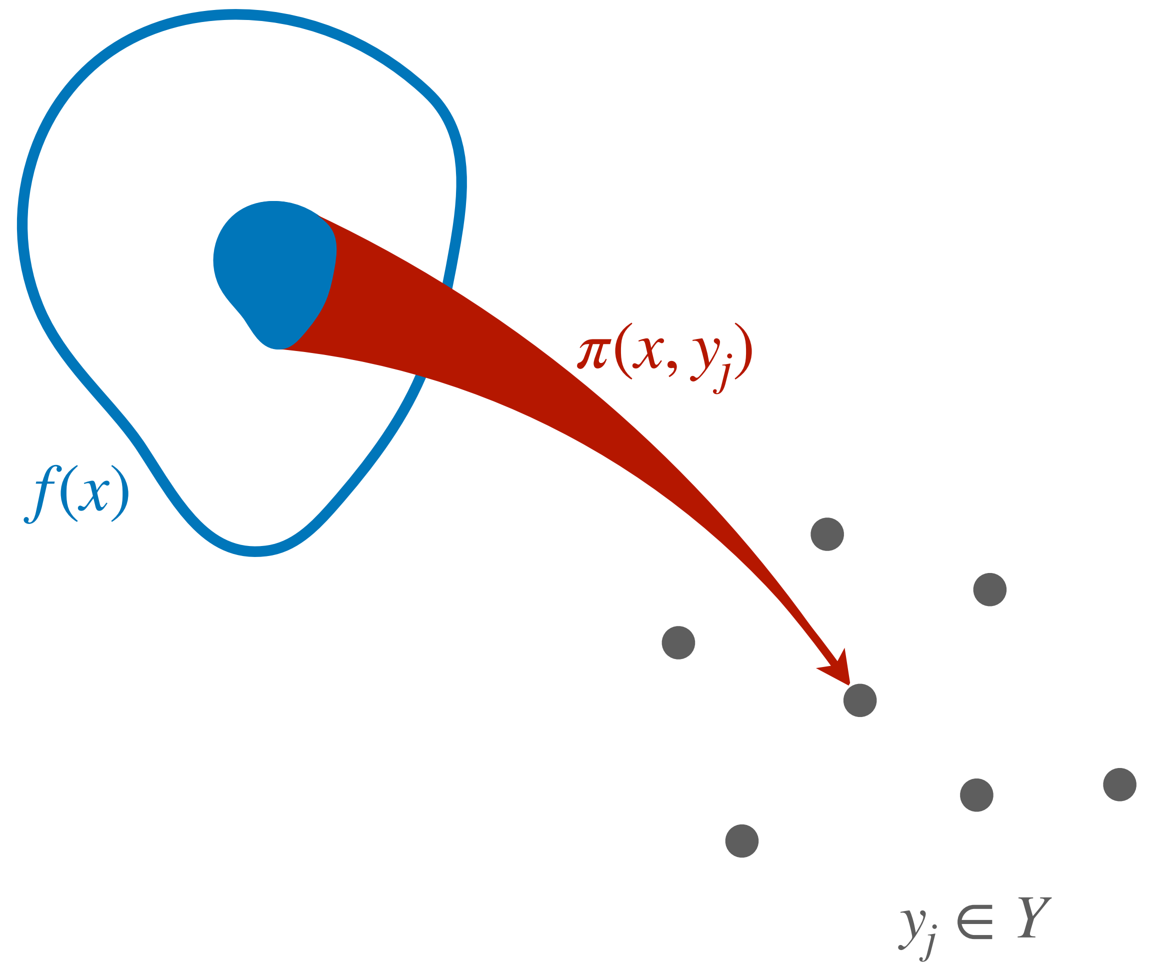
Semi-discrete OT



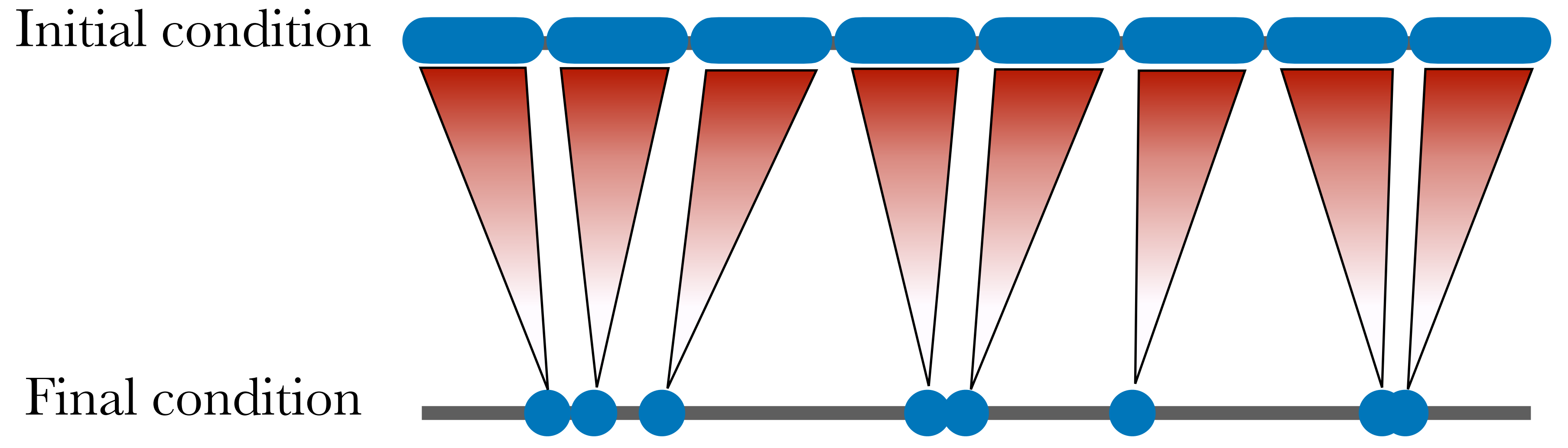


Semi-discrete Optimal Transport (3/3)

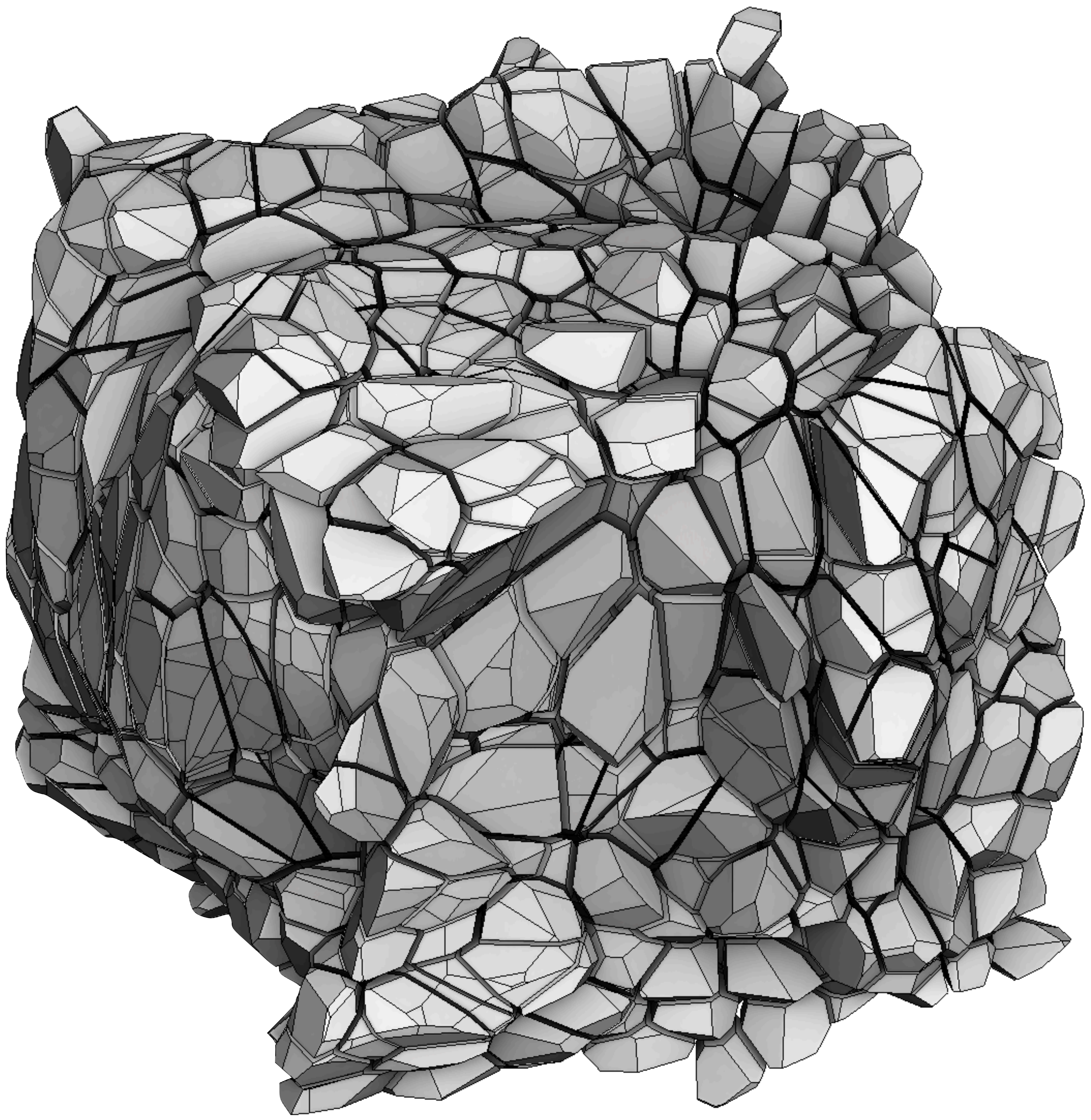


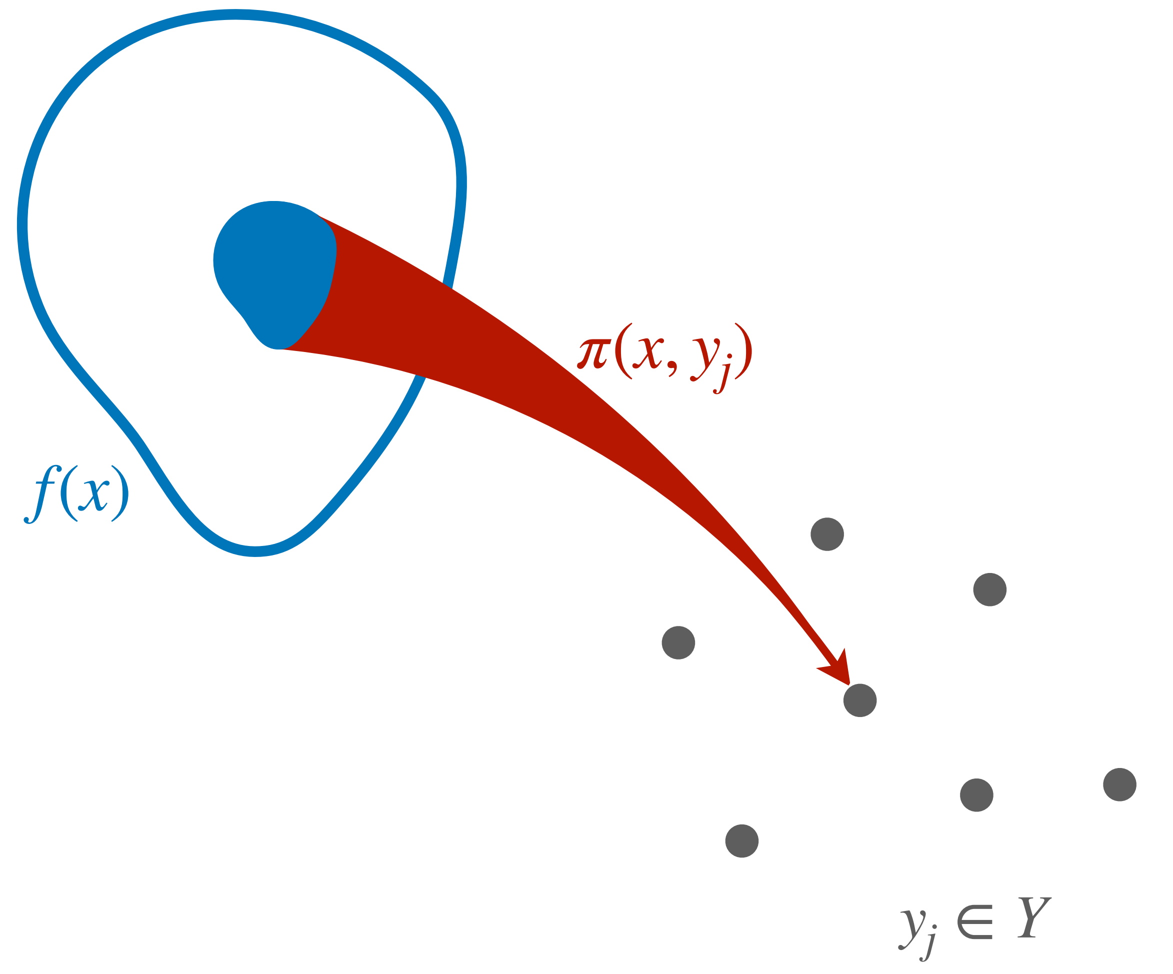


Semi-discrete OT

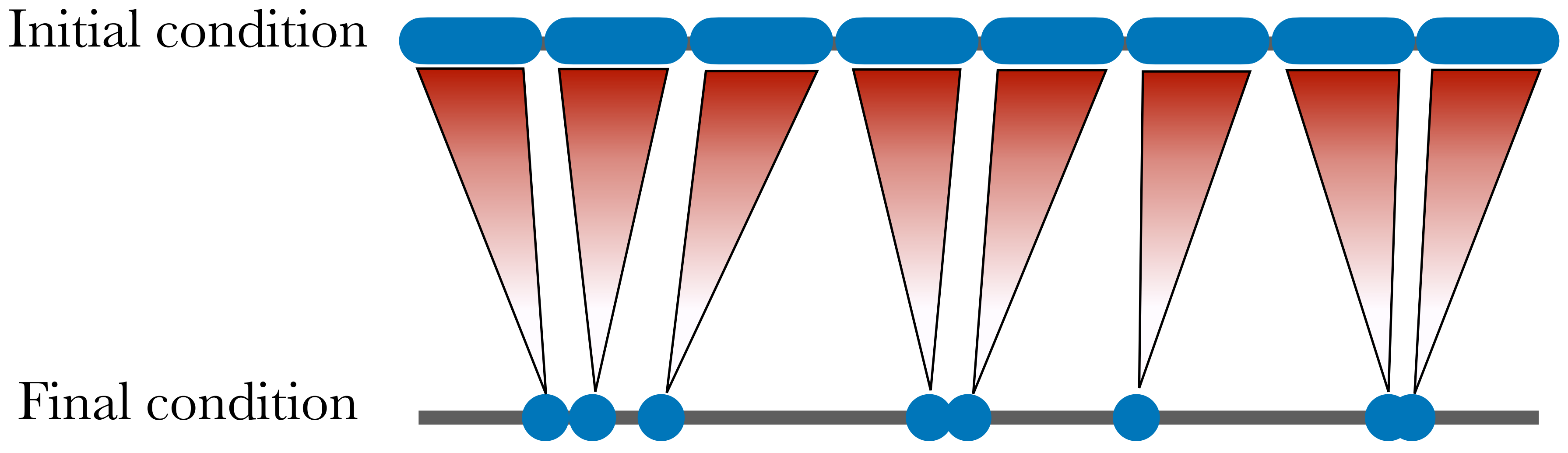


Semi-discrete Optimal Transport (3/3)





Semi-discrete OT



Cosmological growth of matter

Euler-Poisson system:

$$\partial_t \rho + \nabla(\rho \mathbf{v}) = 0$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \rho^{-1} \nabla p + \nabla \phi = 0$$

$$\Delta \phi = 4\pi G \rho$$

But in expanding background, $\mathbf{v} = H(t)\,\mathbf{x}$, in comoving coordinates, $\mathbf{x} = a(t)\,\mathbf{q}$, and proper time, $dt = a(t)d\tau$:

$$\partial_\tau \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0$$

$$\partial_\tau \mathbf{v} + (\mathbf{v} \cdot \nabla_{\mathbf{x}})\mathbf{v} = -\frac{3}{2\tau}(\nabla_{\mathbf{x}} \phi + \mathbf{v})$$

$$\Delta_{\mathbf{x}} \phi = \frac{\rho - 1}{\tau}$$

Cosmic growth of structure

$$\partial_\tau \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0$$
$$\partial_\tau \mathbf{v} + (\mathbf{v} \cdot \nabla_{\mathbf{x}})\mathbf{v} = -\frac{3}{2\tau}(\nabla_{\mathbf{x}}\phi + \mathbf{v})$$
$$\Delta_{\mathbf{x}}\phi = \frac{\rho - 1}{\tau}$$

Define initial and final times

$\tau \in [\tau_I, \tau_F] = [0,1]$

Poisson eq.: Initial density $\rho(\mathbf{x}, \tau_I) = 1$
Euler eq.: $\mathbf{v}(\mathbf{x}, \tau_I) = -\nabla_{\mathbf{x}}\phi(\mathbf{x}, \tau_I)$

Consider Lagrangian coordinates $\mathbf{q}$ and Euler equation becomes

$\mathbf{v}(\mathbf{q}, \tau_I) \approx -\nabla_{\mathbf{q}}\phi(\mathbf{q}, \tau_I) = -\nabla_{\mathbf{q}}\phi_I(\mathbf{q})$

"Zel'dovich approximation"

→Solves Poisson equation and results in uniform, rectilinear motion:
$\mathbf{x}_F(\mathbf{q}) = \mathbf{q}_I + \tau_F \mathbf{v}_I(\mathbf{q}) = \mathbf{q}_I - \tau_F \nabla_{\mathbf{q}}\phi_I(\mathbf{q})$

Cosmic growth of structure

$$\partial_\tau \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0$$
$$\partial_\tau \mathbf{v} + (\mathbf{v} \cdot \nabla_{\mathbf{x}})\mathbf{v} = -\frac{3}{2\tau}(\nabla_{\mathbf{x}}\phi + \mathbf{v})$$
$$\Delta_{\mathbf{x}}\phi = \frac{\rho - 1}{\tau}$$

Define initial and final times

$\tau \in [\tau_I, \tau_F] = [0,1]$

Poisson eq.: Initial density $\rho(\mathbf{x}, \tau_I) = 1$
Euler eq.: $\mathbf{v}(\mathbf{x}, \tau_I) = -\nabla_{\mathbf{x}}\phi(\mathbf{x}, \tau_I)$

Alternatively considering $\rho(\mathbf{x}, \tau)$ to be the $\mathbb{W}_2$ geodesic between $\rho_I = 1$ and ρ_F, and if ϕ is the Kantorovich potential then

$$\det\left(I + \tau D^2\phi(\mathbf{x}, \tau)\right) = \rho(\mathbf{x}, \tau)$$

→Solves the Monge-Ampére equation and also results in uniform, rectilinear motion: $\mathbf{x}_F(\mathbf{q}) = \mathbf{q}_I + \tau_F \mathbf{v}_I(\mathbf{q}) = \mathbf{q}_I - \tau_F \nabla_{\mathbf{q}}\phi_I(\mathbf{q})$

Cosmic growth and optimal transport

$$\partial_\tau \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0$$

$$\partial_\tau \mathbf{v} + (\mathbf{v} \cdot \nabla_{\mathbf{x}})\mathbf{v} = -\frac{3}{2\tau}(\nabla_{\mathbf{x}}\phi + \mathbf{v})$$

$$\Delta_{\mathbf{x}}\phi = \frac{\rho - 1}{\tau}$$

Benamou&Brenier (2000),
Numerische Mathematik, 84, 375

$$\mathbf{x}(\mathbf{q}, \tau) = \mathbf{q} + \frac{\tau}{\tau_F}\left(\mathbf{x}_F(\mathbf{q}) - \mathbf{q}\right)$$

Consider the action

$$I = \frac{1}{2}\int_{\tau_I}^{\tau_F} d\tau \int_V d^3x \, \rho \, |\mathbf{v}|^2$$

which (almost) gives the momentum equation when varied for given $\mathbf{x}(\mathbf{q}, \tau)$.

Now consider the equivalently minimised functional

$$\inf_{\mathbf{x}_F} \int_V d^3q \, \rho(\mathbf{q}) \, |\mathbf{q} - \mathbf{x}_F(\mathbf{q})|^2$$

Cosmic growth and optimal transport

$$\inf_{\mathbf{x}_F} \int_V d^3q\, \rho(\mathbf{q})\, |\,\mathbf{q} - \mathbf{x}_F(\mathbf{q})\,|^2$$

Dark matter mover's distance

Find optimal transport map T by

$$\inf_T \left\{ \int_{\mathbb{R}^n} c(x, T(x))\, d\mu \mid T\#\mu = \nu \right\}$$

Subject to mass conservation (continuity equation) and appropriate boundary conditions.

Mass conservation in Lagrangian coordinates:

$$\rho_F(\mathbf{x}_F(\mathbf{q}))\, \det(\nabla_{\mathbf{q}} \mathbf{x}_F(\mathbf{q})) = \rho_I(\mathbf{q})$$

Subject to

$$f(x) = g(y) \det\left(\nabla T(x)\right)$$

Brenier's theorem

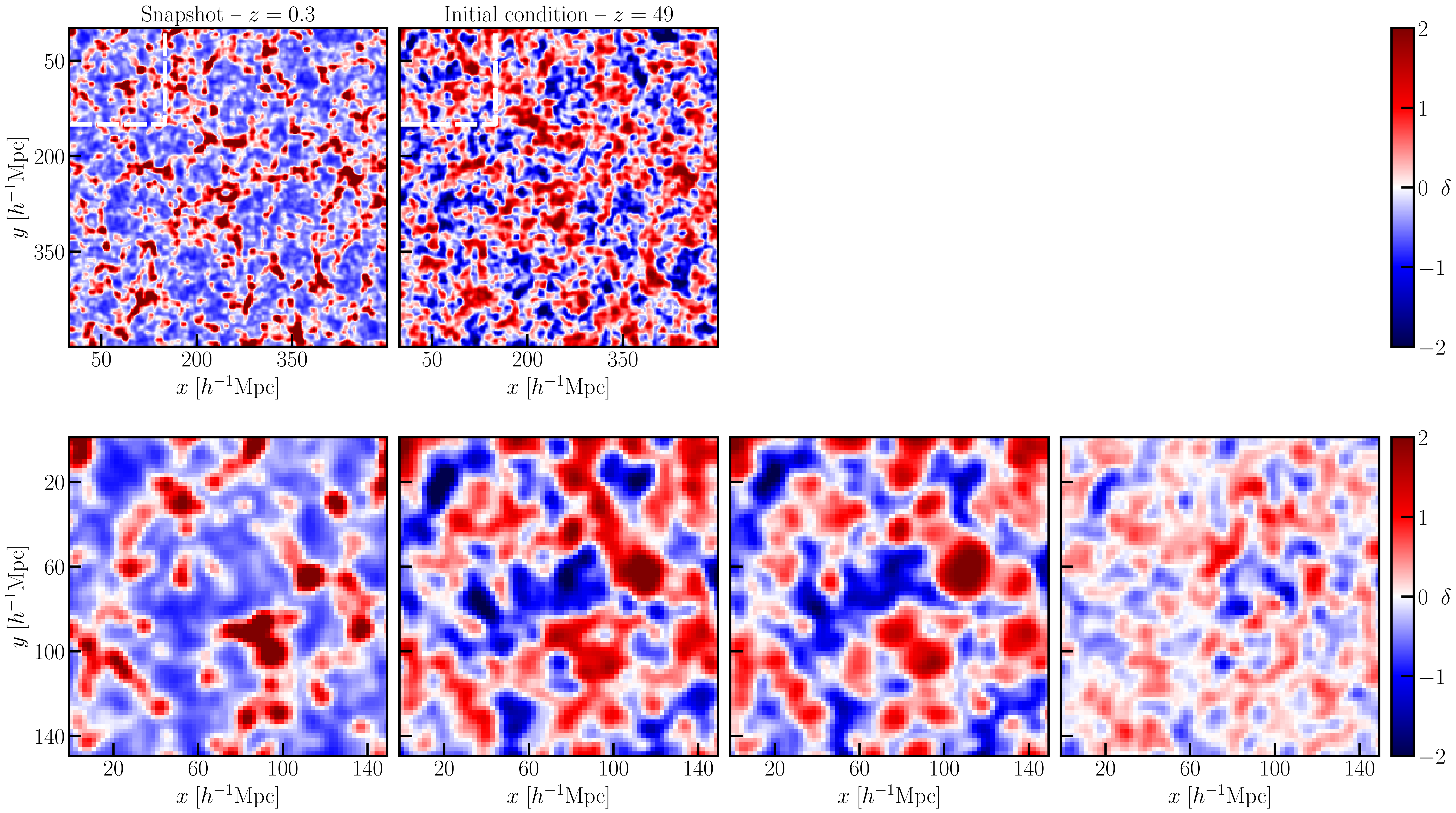
The final positions $\mathbf{x}_F(\tau)$ are the gradient of a convex potential

Frisch+ (2002) [astro-ph/0109483] *Nature*, 417

Brenier+ (2003), [astro-ph/0304214] *Mon. Not. R. Astron. Soc.*, 346

Qualitative comparison of reconstructed density field

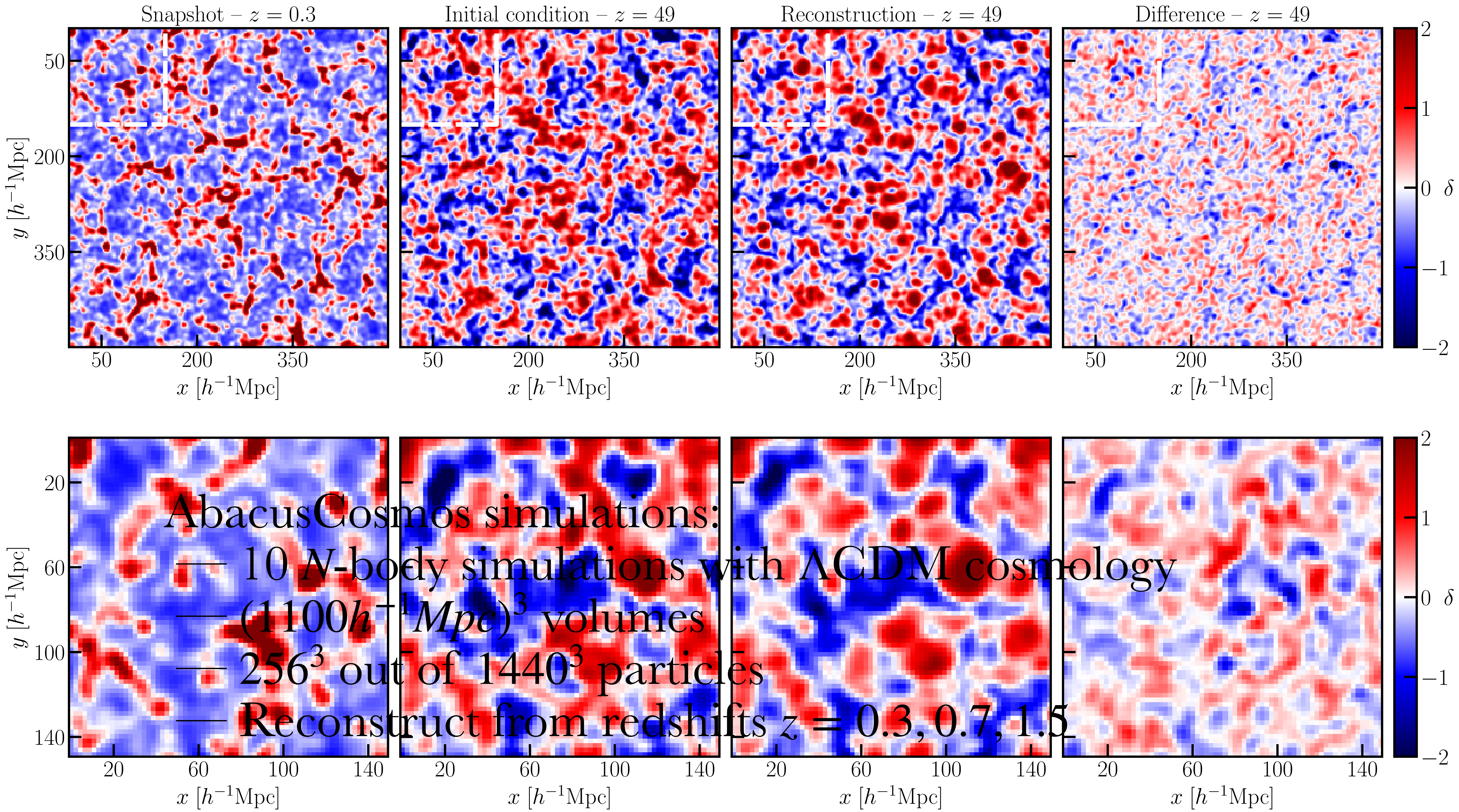
AbacusCosmos simulations — distributions



Lévy, Mohayaee, ***SvH***. [2012.09074] *Mon.Not.Roy.Astron.Soc. 506 (2021) 1, 1165*

Qualitative comparison of reconstructed density field

AbacusCosmos simulations — distributions



AbacusCosmos simulations:

— 10 N-body simulations with ΛCDM cosmology

— $(1100h^{-1}Mpc)^3$ volumes

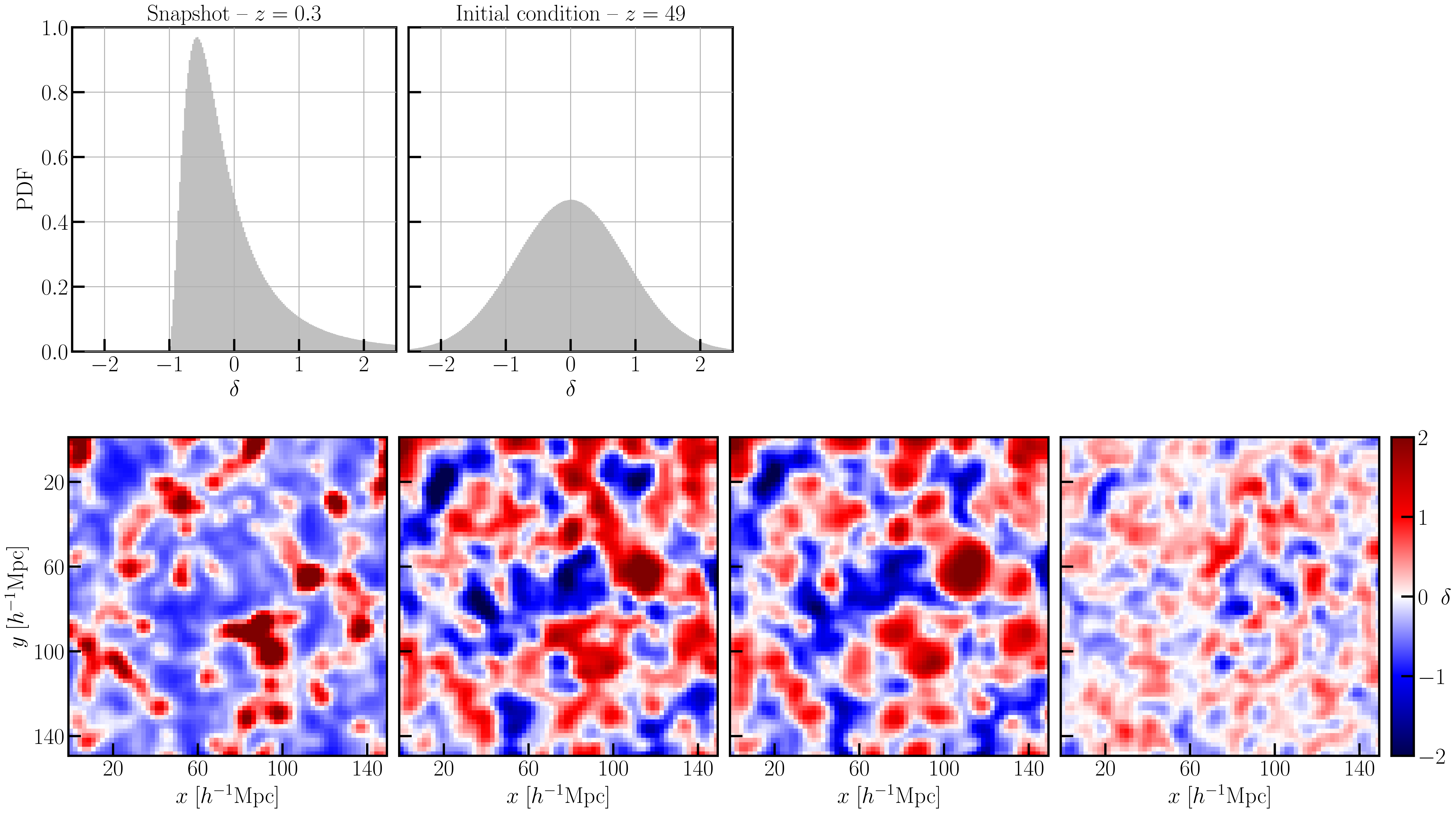
— 256^3 out of 1440^3 particles

— Reconstruct from redshifts $z = 0.3, 0.7, 1.5$

Lévy, Mohayaee, ***SvH***. [2012.09074] *Mon.Not.Roy.Astron.Soc. 506 (2021) 1, 1165*

Qualitative comparison of reconstructed density field

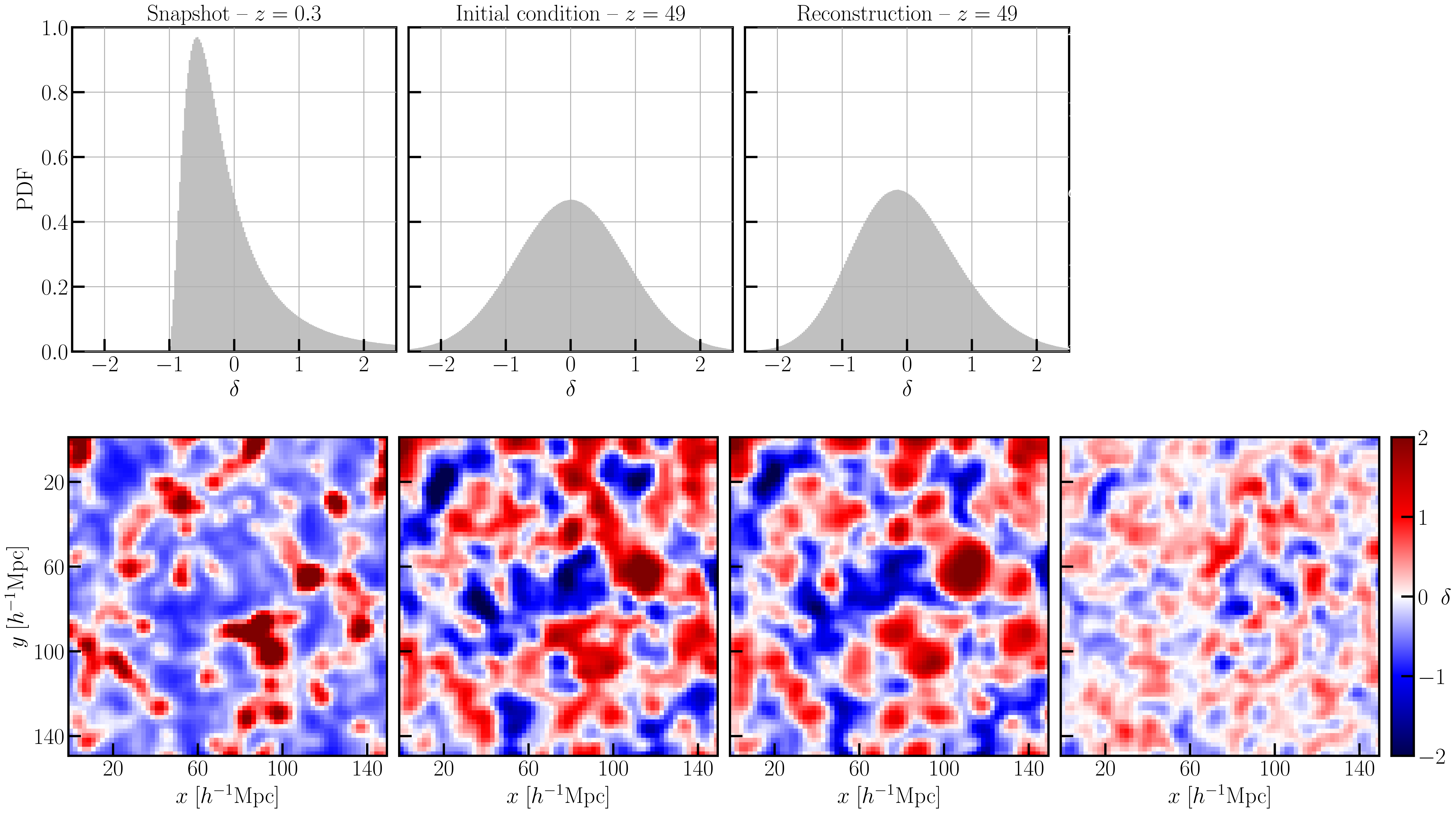
AbacusCosmos simulations — distributions and one-point functions



Lévy, Mohayaee, ***SvH***. [2012.09074] *Mon.Not.Roy.Astron.Soc. 506 (2021) 1, 1165*

Qualitative comparison of reconstructed density field

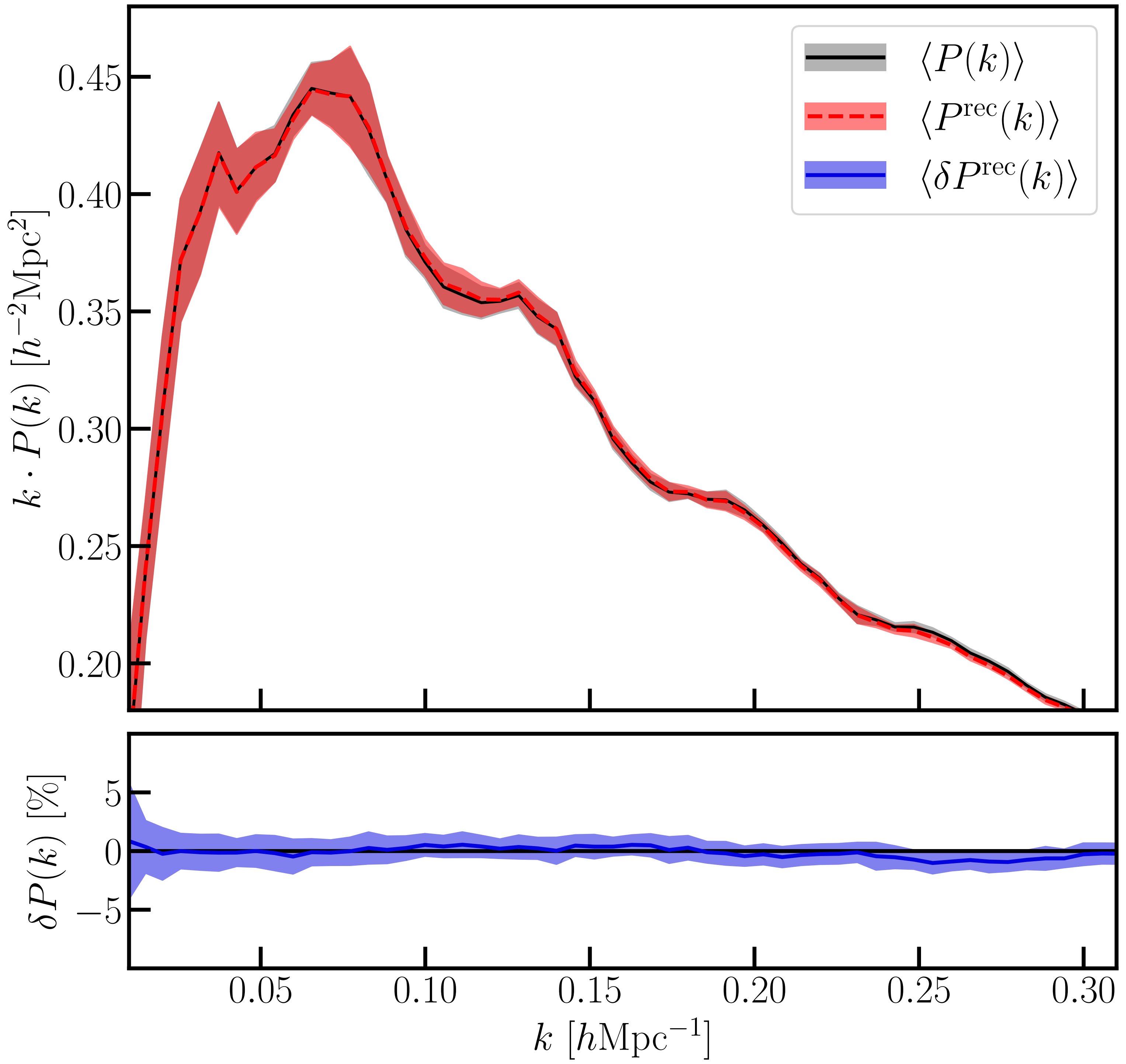
AbacusCosmos simulations — distributions and one-point functions



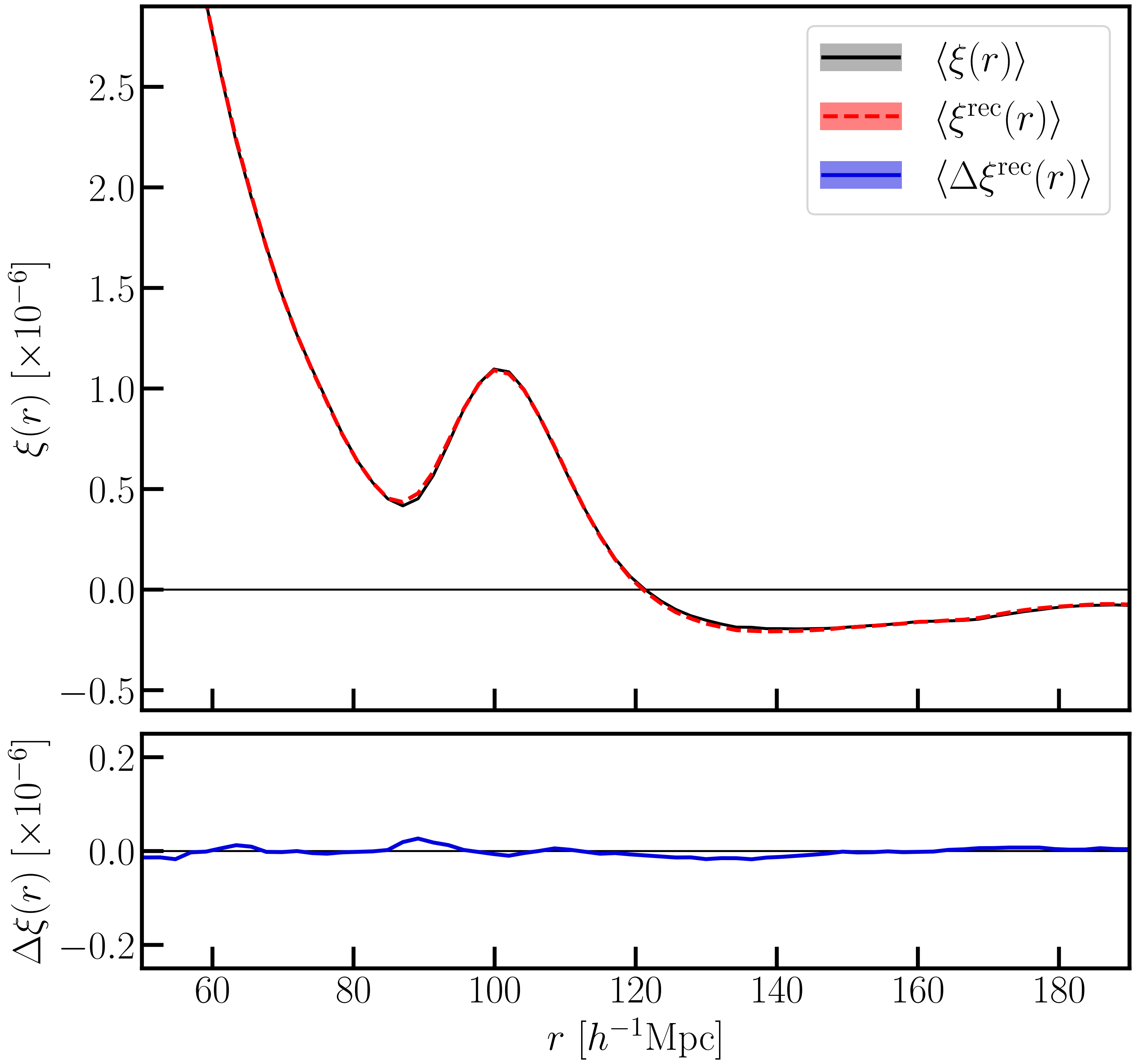
Lévy, Mohayaee, ***SvH***. [2012.09074] *Mon.Not.Roy.Astron.Soc. 506 (2021) 1, 1165*

Quantitative comparison of reconstructed density field

AbacusCosmos simulations — two-point functions



$z_F = 0.3$



$z_F = 0.3$

Lévy, Mohayaee, ***SvH***. [2012.09074] *Mon.Not.Roy.Astron.Soc. 506 (2021) 1, 1165*

Reconstruction of Baryonic Acoustic Oscillations

FastPM simulations and comparison with “standard reconstruction”

FastPM simulations:

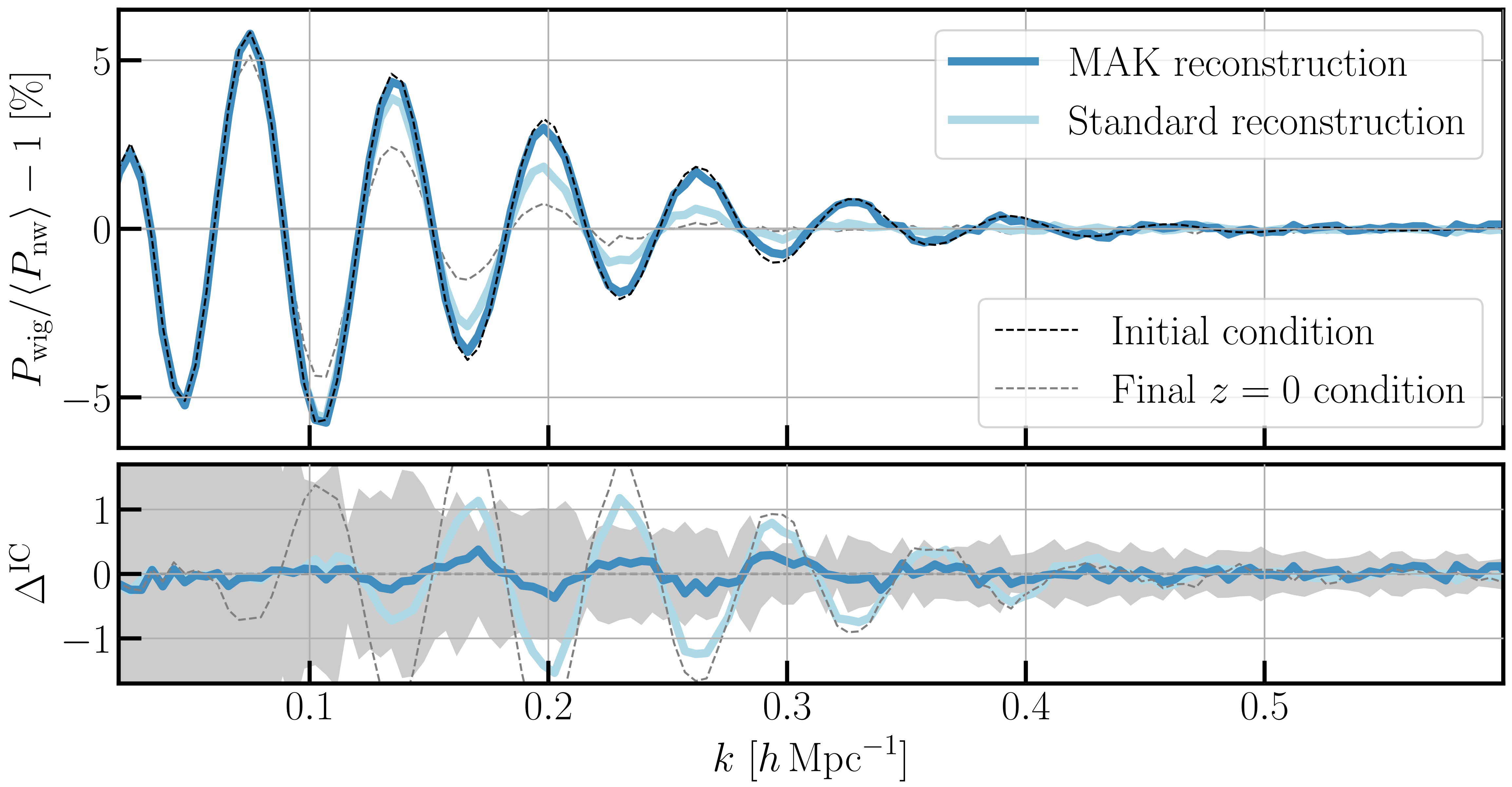
— 10 *pairs* of N-body simulations with ΛCDM cosmology

— with and without BAO

— $(1380h^{-1}Mpc)^3$ volumes

— ~1% out of 2048^3 particles (~85 million particles)

— Reconstruct from redshift $z = 0$



SvH, Mohayaee, Lévy. [2110.08868] *Phys.Rev.Lett. 128 (2022) 20, 201302*

Reconstruction in Cosmology

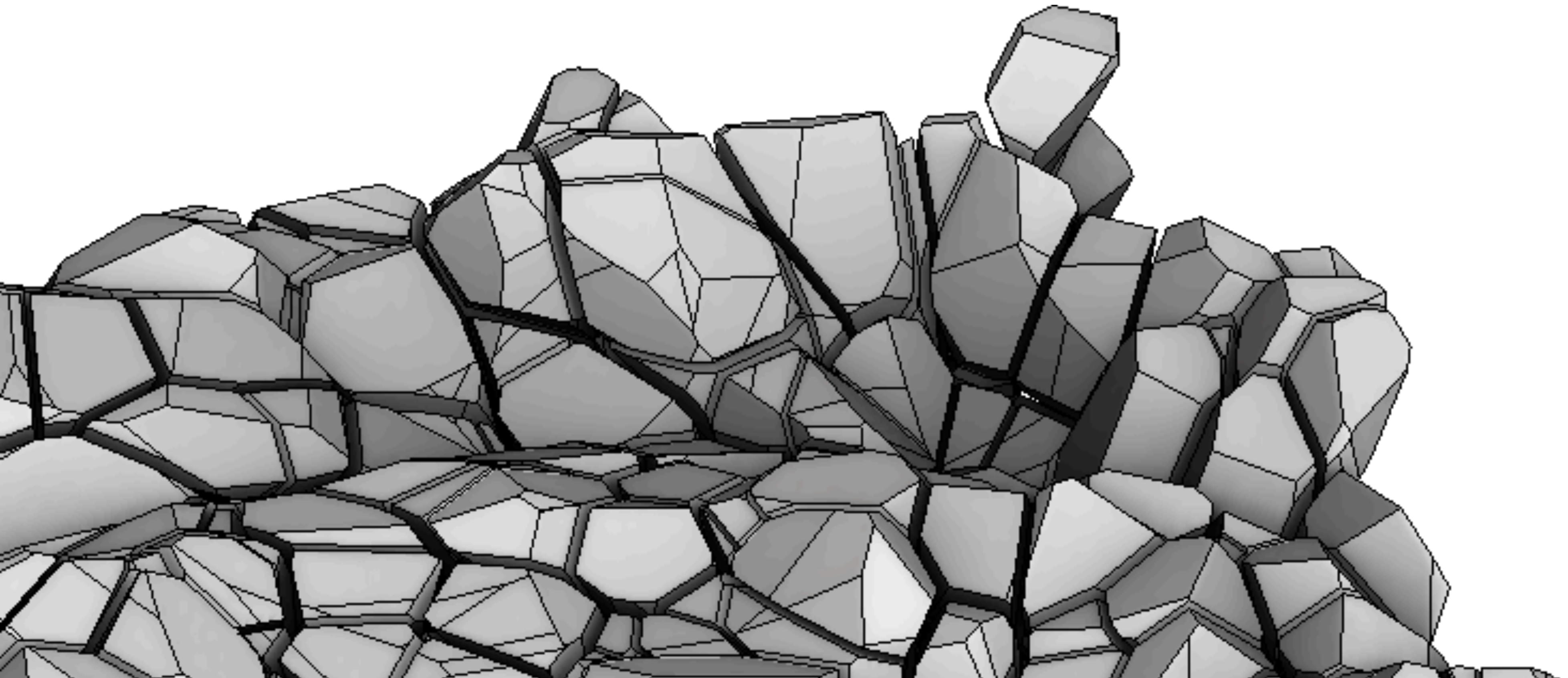
Optimal transport is a well-studied, versatile language applicable to a range of different problem settings in pure and applied mathematics + theoretical physics!

Cosmological reconstruction can be rephrased as an optimal transport problem

Efficient algorithms from computer science can make solving scalable

Results show high accuracy and promising prospects for upcoming astronomical and cosmological data

— Unveil relations to other problems!



Lévy, Mohayaee, ***SvH***. [2012.09074] *Mon.Not.Roy.Astron.Soc. 506 (2021) 1, 1165*

SvH, Mohayaee, Lévy. [2110.08868] *Phys.Rev.Lett. 128 (2022) 20, 201302*

Niels Bohr Lecture by Prof. Subir Sarkar

Title: Connecting inner space & outer space

Wednesday, September 4, 2013 at 15:15 in Aud. 3 at HCØ.

Prof. Subir Sarkar, University of Oxford and NBI

Abstract: We have just celebrated the centenary of the finding that the Earth is being constantly bombarded by high energy `cosmic rays' from space. This initiated a glorious era of discovery of many new particles (positron, muon, pion, ...) and developed into accelerator-based research into high energy physics. A century later this has given us the triumphant `Standard Model' of particle physics which provides a precise quantum description of all fundamental processes in terrestrial laboratories, including (with the recent discovery of "a Higgs boson") an understanding of how particles acquire mass.

Unfortunately the Standard Model does not explain any of the salient features of the universe as a whole - Why there is matter but no antimatter? Why there is so much more `dark matter' of unknown origin? Why is the expansion rate apparently accelerating, as if driven by a Cosmological Constant-like, dominant component of `dark energy'?

In this lecture I will describe how new kinds of experiments and theoretical developments at the rapidly growing interface of astro-particle physics are attempting to answer these cosmic questions, by linking them to possible new physics that lies beyond the Standard Model.

“Freedom is the consciousness of necessity”

Thank you! And happy birthday, Subir!