The determination of the strong coupling from τ decays: facts vs. myths

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Motivation

- Two different approaches to the determination of α_s from hadronic τ decays:
  “truncated OPE’’ (tOPE) strategy (Pich and others) *vs.*
  “duality-violation’’ (DV) strategy (Boito *et al.*)

- Many unsubstantiated criticisms of the DV strategy
  Now finally a paper providing details (Pich & Rodríguez-Sánchez, JHEP 07 (2022) 145 (PRS))

- This talk: *all criticisms of PRS in the JHEP paper are misleading or incorrect*
  DV strategy has the tools to check this (and thus validate this strategy)

- In contrast, several criticisms of tOPE remain unanswered (Boito *et al.*, PRD 95 (2017); PRD 100 (2019))

Finite energy sum rules

1) Both methods are based on FESRs: $\int_0^{s_0} ds\, w(s)\, \rho(s) = -\frac{1}{2\pi i}\oint_{|z|=s_0} dz\, w(z)\, \Pi(z)$

$\rho(s)$ is experimentally measured spectral function (here, vector, non-strange, isospin 1)

$\Pi(z)$ is (scalar) current two-point function, $\Pi_{\mu\nu}(q^2) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2)$ from theory (QCD)

$w(z)$ is polynomial in z/s_0

2) Approximate right-hand side by:

$$-\frac{1}{2\pi i}\oint_{|z|=s_0} dz\, w(z)\left(\Pi_{\text{pert.th.}}(z;\alpha_s) + \Pi_{\text{OPE}}(z)\right) + \int_{s_0}^{\infty} ds\, w(s)\, \rho_{\text{DV}}(s)$$

$\Pi_{\text{pert.th.}}(z;\alpha_s)$ known to α_s^4 (Baikov *et al.* '08, Herzog *et al.* '17)

OPE: expansion in powers of $1/z$ – known to be an asymptotic expansion (at best)

DV: resonance oscillations seen in spectral functions – not captured by pert.th. + OPE

3) Choose strategy, tOPE (Pich and others) or DV (Boito *et al.*), next slide

Strategies

FESR: $$\int_0^{s_0} ds\, w(s\, \rho(s) = -\frac{1}{2\pi i}\oint_{|z|=s_0} dz\, w(z)\left(\Pi_{\rm pert.th.}(z;\alpha_s)+\Pi_{\rm OPE}(z)\right)+\int_{s_0}^{\infty} ds\, w(s)\, \rho_{\rm DV}(s)$$

tOPE: - set DV part equal to zero (this is a model for duality violations!)
- include high-degree polynomials (with DVs suppressed via zeros at $z = s_0$)
- use a single s_0 value, as close as possible to m_τ^2 , dropping OPE parameters until # fit parameters < # FESRs; OPE treated as if convergent to very high order (up to $1/z^8$)

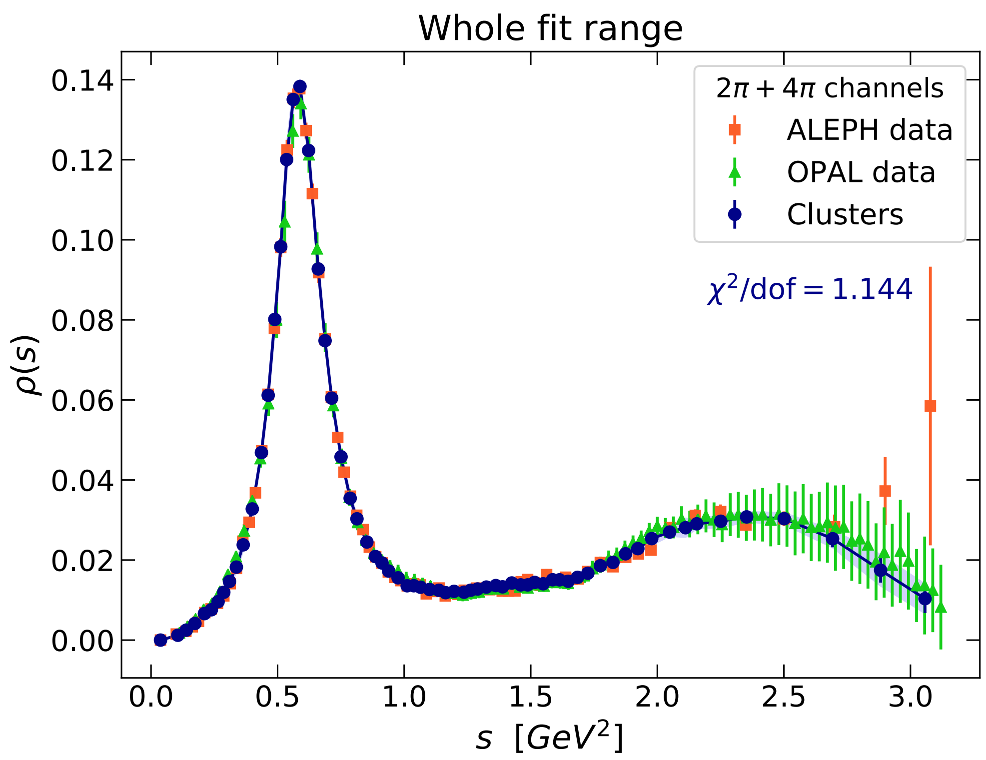
DV: Since OPE is asymptotic, use only to low orders (max $1/z^5$), don't drop OPE parameters ≥ 1 FESR with unsuppressed DVs, model with QCD-motivated *ansatz* (Regge theory and $1/N_c$)

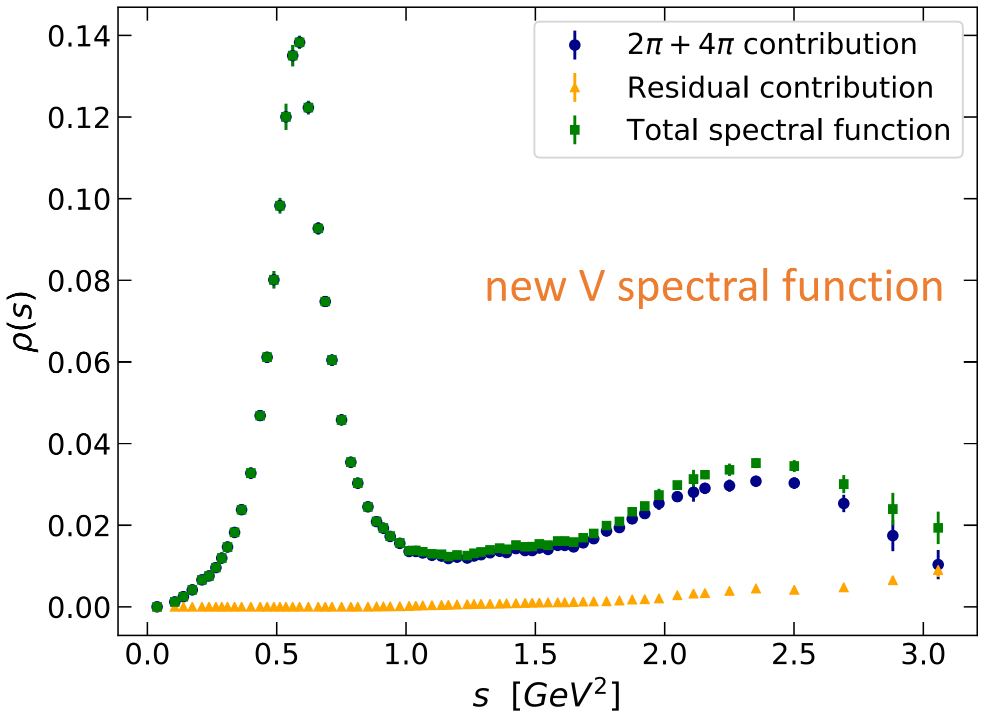
$$\rho_{\rm DV}(s) = e^{-\delta-\gamma s}\sin(\alpha+\beta s+\mathcal{O}(\log s))\left(1+\mathcal{O}\left(\frac{1}{s},\frac{1}{N_c},\frac{1}{\log s}\right)\right)$$

use, and test consistency of approach by varying, s_0 between $\sim 1.5\ {\rm GeV}^2$ and m_τ^2

(Catà *et al*. '05, Boito *et al*. '17)

Experimental data (non-strange vector spectral function):





OPAL: Ackerstaff *et al.* '98
ALEPH: Schael *et al.* '05, Davier *et al.* '14
Combination: Boito *et al.* '20

Residual modes from (mostly) electroproduction (instead of Monte-Carlo) Boito *et al.* '20

Criticism 1: DV model is not stable against variations

Try varying the DV model: $$\rho_{\rm DV}(s) = \left(1 + \frac{c}{s}\right) e^{-\delta-\gamma s} \sin(\alpha + \beta s)$$

$c = 0 \quad \Rightarrow \quad \alpha_s(m_\tau^2) = 0.296$

PRS '22: use $c = -1.35\ \mathrm{GeV}^2 \quad \Rightarrow \quad \alpha_s(m_\tau^2) = 0.319$

$c = -2\ \mathrm{GeV}^2 \quad \Rightarrow \quad \alpha_s(m_\tau^2) = 0.260$

(Note huge values of c , this is not a small correction for $s \sim 2\ \mathrm{GeV}^2$)

Obtained using non-strange ALEPH vector spectral function

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$c = 0 \quad \Rightarrow \quad \alpha_s(m_\tau^2) = 0.296 \pm 0.010$

PRS '22: use $c = -1.35\ \mathrm{GeV}^2 \quad \Rightarrow \quad \alpha_s(m_\tau^2) = 0.319 \pm 0.016$

$c = -2\ \mathrm{GeV}^2 \quad \Rightarrow \quad \alpha_s(m_\tau^2) = 0.260 \pm 0.089$

ALEPH vector spectral function not precise enough for this test!

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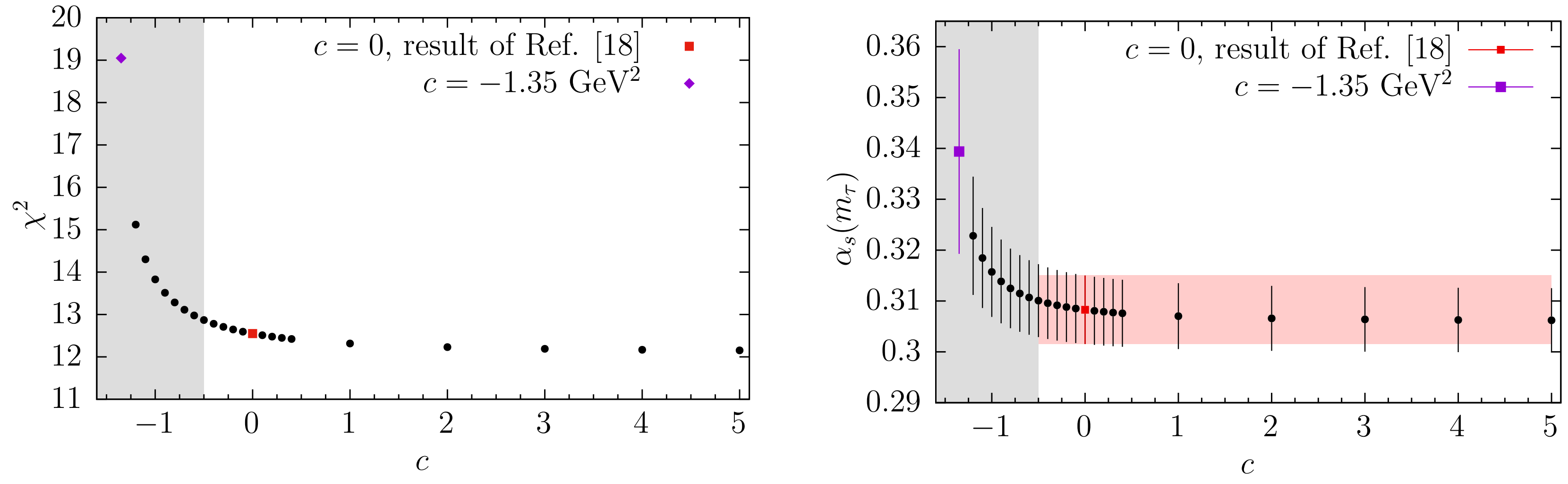
ALEPH vector spectral function not precise enough for this test! Use new vector spectral funtion:



rather spectacular stability!
(red point is our central result)

Criticism 2: Logarithmic corrections to OPE

FESR: $$\int_0^{s_0} ds\, w(s\,\rho(s) = -\frac{1}{2\pi i}\oint_{|z|=s_0} dz\, w(z)\,\left(\Pi_{\text{pert.th.}}(z;\alpha_s)+\Pi_{\text{OPE}}(z)\right) + \int_{s_0}^{\infty} ds\, w(s)\,\rho_{\text{DV}}(s)$$

$$\Pi(q^2) = \sum_{k=1}^{\infty}\frac{C_{2k}(q^2)}{(-q^2)^k} \qquad C_{2k}(q^2)$$ Wilson coefficients

Choose $C_{2k}(z)$ constant, then $-\frac{1}{2\pi i}\oint_{|z|=s_0} dz\left(\frac{z}{s_0}\right)^n \frac{C_{2k}}{(-z)^k} = \frac{C_{2(n+1)}}{(-s_0)^n}$: control which OPE terms contribute

Choose $C_{2k}(z) = C_{2k}(\mu^2)\left(1 + L_{2k}\log\frac{-s}{\mu^2}\right)$ then

$$-\frac{1}{2\pi i}\oint_{|z|=s_0} dz\left(\frac{z}{s_0}\right)^n \frac{C_{2k}}{(-z)^k} = \frac{C_{2(n+1)}}{(-s_0)^k}\left(\delta_{kn} + \frac{L_{2k}}{n-k}(1-\delta_{kn})\right)$$

⇒ higher-order logarithms (k) affect sum rules with low-degree polynomials (n)!

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⇒ higher-order logarithms (k) affect sum rules with low-degree polynomials (n)!

- Little is known about the values of L_{2k} other than that they are suppressed by powers of α_s
  PRS ‘22 choose (rather arbitrarily) $L_{2k} = 0.2$ – is this reasonable?
- What we know: *k* = 2 logs suppressed by two powers of α_s , very small
  Using large-*N* factorization for *k* = 3 four-fermion operators: $|L_{2k=6}| = \frac{19}{63\pi}\alpha_s(s_0) \simeq 0.03$
- These are an order of magnitude smaller than PRS chose, and have no effect on the DV strategy

Criticism 3: DV strategy is "redundant" – additional weights do not add information

PRS "Theorem": The weight $w(z) = 1$ in the DV strategy determines $\alpha_s(m_\tau^2)$ while the other weights, such as $w(z) = 1 - (z/s_0)^2$ are completely redundant, and only serve to determine higher-dimensional OPE coefficients, with no influence on $\alpha_s(m_\tau^2)$ and the fit quality.

- "Theorem" invalid due to logical/mathematical errors.
- No time for mathematics, but we'll demonstrate this through a few examples.
- This criticism does, in fact, apply to the tOPE strategy (backup slides).

Criticism 3: DV strategy is "tautological" – additional weights do not add information

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while the other weights, such as $w(z) = 1 - (z/s_0)^2$ are completely redundant,
and only serve to determine higher-dimensional OPE coefficients,
with no influence on $\alpha_s(m_\tau^2)$ and the fit quality.

- Consider an example: (i) fit $\alpha_s(m_\tau^2)$ and DV parameters to FESR with $w(z) = 1$

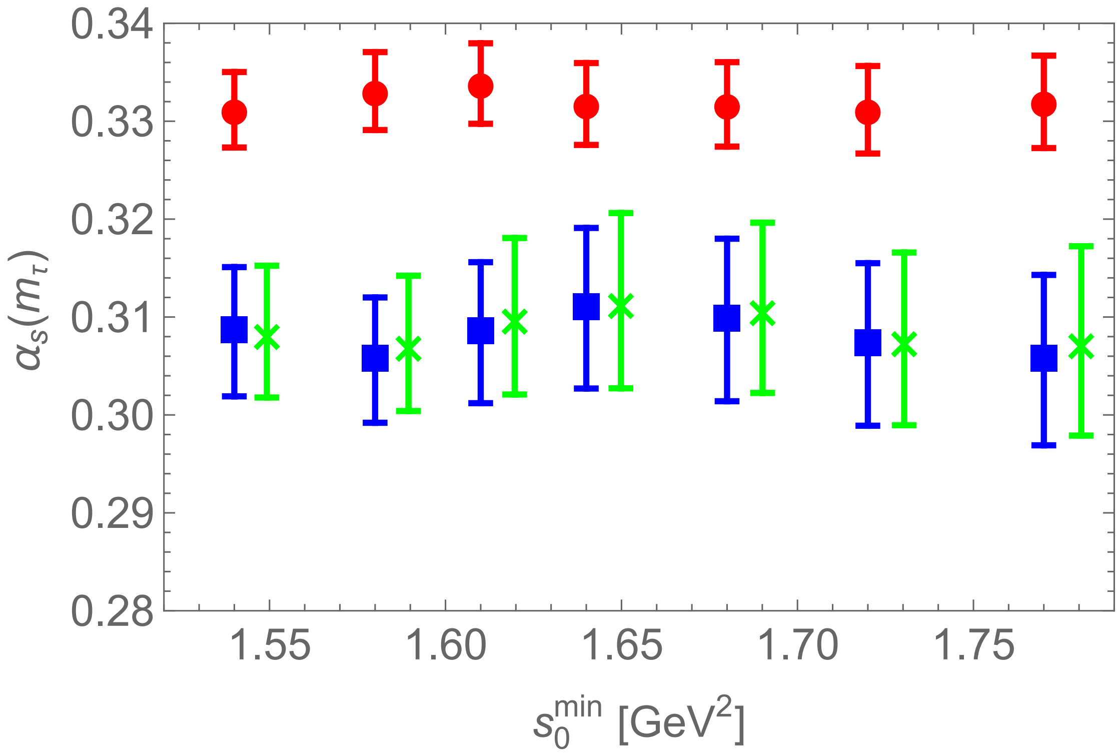
  Then: (ii) fit again, now with weights $w(z) = 1$ and $w(z) = 1 - (z/s_0)^2$ (add C_6)

  (a) with correct $1/s_0^3$ scaling for OPE term $\propto C_6$

  (b) with wrong $1/s_0^5$ scaling for OPE term $\propto C_6$

- According to this "theorem," fit should always give same value for $\alpha_s(m_\tau^2)$, adjusting C_6 …

Criticism 3: DV strategy is "tautological" – additional weights do not add information



blue points: fit with $w(z) = 1$

green points: fit with $w(z) = 1$
and $w(z) = 1 - (z/s_0)^2$
$1/s_0^3$ scaling

red points: fit with $w(z) = 1$
and $w(z) = 1 - (z/s_0)^2$
$1/s_0^5$ scaling

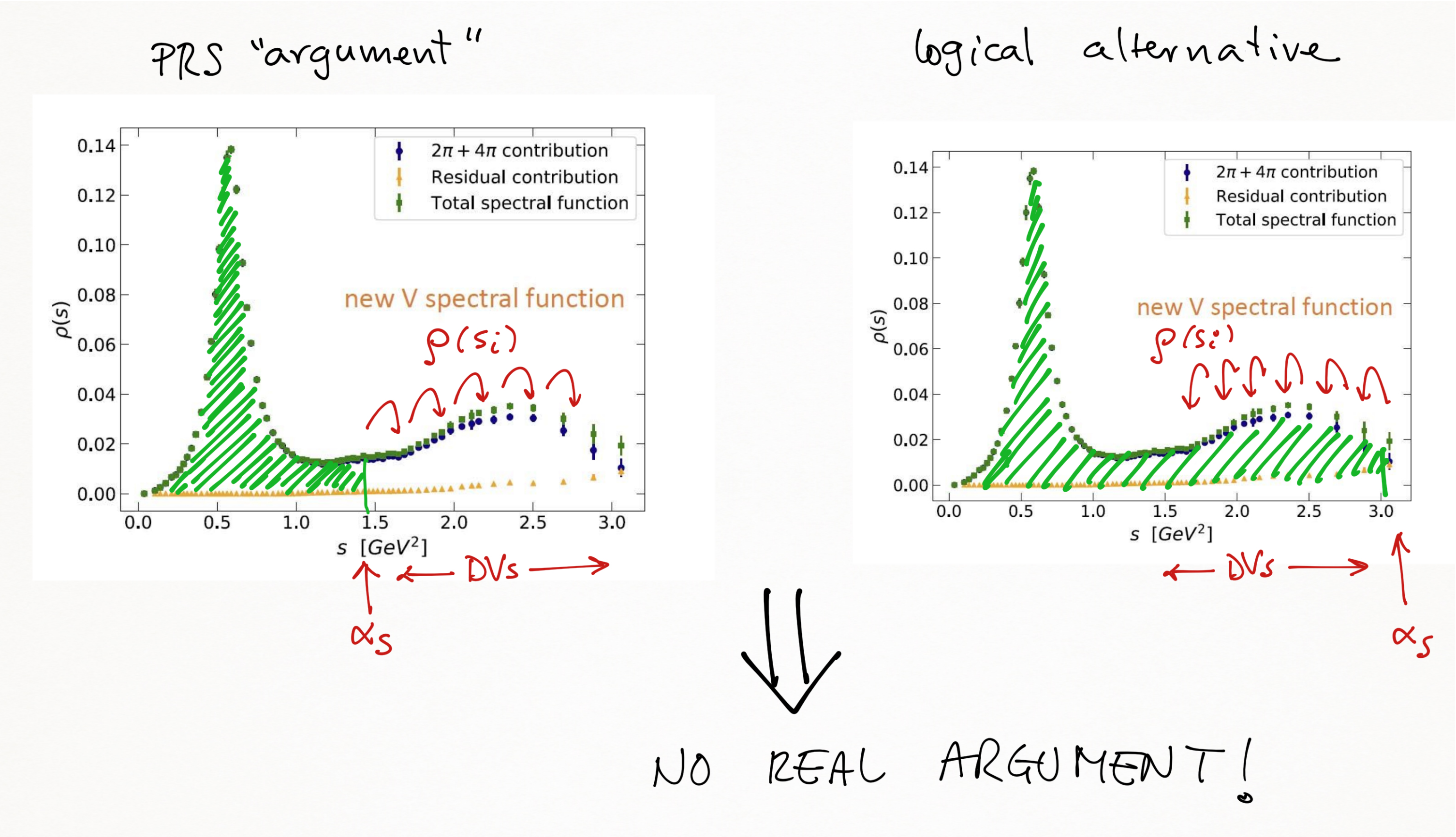
blue and green fits *not* the same,
but very consistent

red fit *very different* and *not* consistent!

- According to this "theorem," fit should always give same value for $\alpha_s(m_\tau^2)$: *based on math mistake!*

Criticism 4: DV strategy determines α_s at 1.55 GeV^2

PRS claim: Using the weight $w(z) = 1$ in the DV strategy determines $\alpha_s(m_\tau^2)$ at s_0^{min} and uses the spectral function for $s > s_0^{min}$ to fit the DV parameters.



Logical alternative:

DV strategy determines $\alpha_s(m_\tau^2)$ at s_0^{max} and uses the spectral function for $s < s_0^{max}$ to fit DV parameters

Reality: all parameters obtained from fits using all data

Conclusions

- Longstanding controversy between truncated OPE and DV strategies for determining $\alpha_s(m_\tau^2)$ from hadronic τ decays; new paper by PRS (finally) provides details of the criticisms

- All PRS criticism of DV strategy refuted – based on math mistakes/insufficient scrutiny of assumptions

- Answers to criticisms, however, do provide useful further tests of DV strategy

- Our result stands: $\alpha(m_\tau) = 0.3077 \pm 0.0075$ (Boito *et al.* '21)

  $\Rightarrow \quad \alpha(m_Z) = 0.1171 \pm 0.0010$

- Our criticism of the truncated OPE strategy remains unanswered (PRD 95 (2017); PRD 100 (2019))

BACKUP

Redundancy of tOPE strategy

- Consider "optimal weights": (PRS, '16)

$$w_{21}(y) = 1 - 3y^2 + 2y^3 \qquad \alpha_s,\ C_6,\ C_8$$

$$w_{22}(y) = 1 - 4y^3 + 3y^4 \qquad \alpha_s,\ C_8,\ C_{10}$$

$$w_{23}(y) = 1 - 5y^4 + 4y^5 \qquad \alpha_s,\ C_{10},\ C_{12}$$

$$w_{24}(y) = 1 - 6y^5 + 5y^6 \qquad \alpha_s,\ C_{12},\ C_{14}$$

$$w_{25}(y) = 1 - 7y^6 + 6y^7 \qquad \alpha_s,\ C_{14},\ C_{16}$$

- Set $C_{12} = C_{14} = C_{16} = 0$ and set $s_0 = 2.8\ \mathrm{GeV}^2$, hence $w_{24},\ w_{25}$ determine only $\alpha_s(m_\tau^2)$ (even with correlations), with C_{10} fixed by w_{23} , etc. Fit to ALEPH V+A non-strange data

Redundancy of tOPE strategy

(PRS '22)

irrelevant

- Consider "optimal weights": (PRS, '16)

$$w_{21}(y) = 1 - 3y^2 + 2y^3 \qquad \alpha_s,\ C_6,\ C_8 \text{ adds } C_6$$
$$w_{22}(y) = 1 - 4y^3 + 3y^4 \qquad \alpha_s,\ C_8,\ C_{10} \text{ adds } C_8$$
$$w_{23}(y) = 1 - 5y^4 + 4y^5 \qquad \alpha_s,\ C_{10} \text{ adds } C_{10}$$
$$w_{24}(y) = 1 - 6y^5 + 5y^6 \qquad \alpha_s$$
$$w_{25}(y) = 1 - 7y^6 + 6y^7 \qquad \alpha_s$$

(α_s, α_s) fixes $\alpha_s(m_\tau^2)$

- Results: $w_{24},\ w_{25}:\quad \alpha_s = 0.3146(28),\ \chi^2 = 2.376717$

$$w_{23}, \ldots w_{25}:\quad \alpha_s = 0.3146(28),\ \chi^2 = 2.376717,\ C_{10} = 0.00030(12)$$
$$w_{22}, \ldots w_{25}:\quad \alpha_s = 0.3146(28),\ \chi^2 = 2.376717,\ C_{10} = 0.00030(12),\ C_8 = -0.00078(21)$$
$$w_{21}, \ldots w_{25}:\quad \alpha_s = 0.3146(28),\ \chi^2 = 2.376717,\ C_{10} = 0.00030(12),\ C_8 = -0.00078(21),\ C_6 = 0.00125(24)$$

- $\alpha_s(m_\tau^2)$ purely from perturbation theory, *no* effect from OPE; OPE coefficients not fitted

  Can also get $\alpha_s(m_\tau^2)$ from *only* w_{25} (not a fit!): $\alpha_s = 0.3200(44)$ tests only pert.th., not the OPE!