# DIFFERENCE BETWEEN CONTOUR-IMPROVED AND FIXED-ORDER PERTURBATION THEORY IN $\tau$ DECAY

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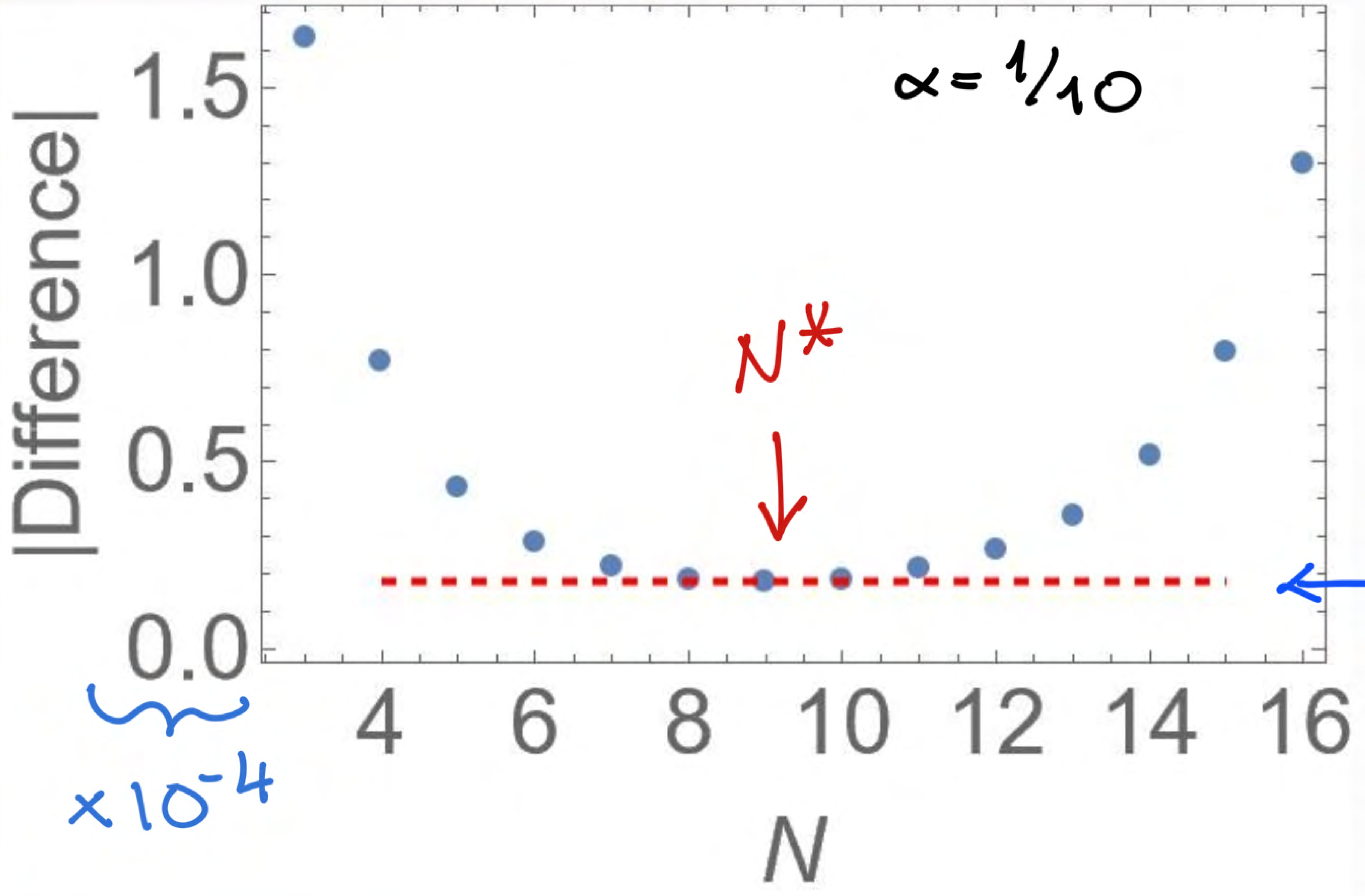
# Asymptotic Series

- Canonical example:

$$I(\alpha) = \int_0^\infty dt \, \frac{e^{-t/\alpha}}{1+t} \underset{\text{Borel}}{\overset{\text{Expand}}{\approx}} \sum_{n=0}^{N} (-)^n n! \, \alpha^n \quad ; \quad \alpha : \text{Expansion parameter}$$

- Series approaches $I(\alpha)$ until $N^* \sim 1/\alpha$, then diverges.
- $\exists$ minimum irreducible difference

$$I(\alpha) - \sum_{n=0}^{N} (-1)^n n! \, \alpha^{n+1} \equiv \Delta(N)$$



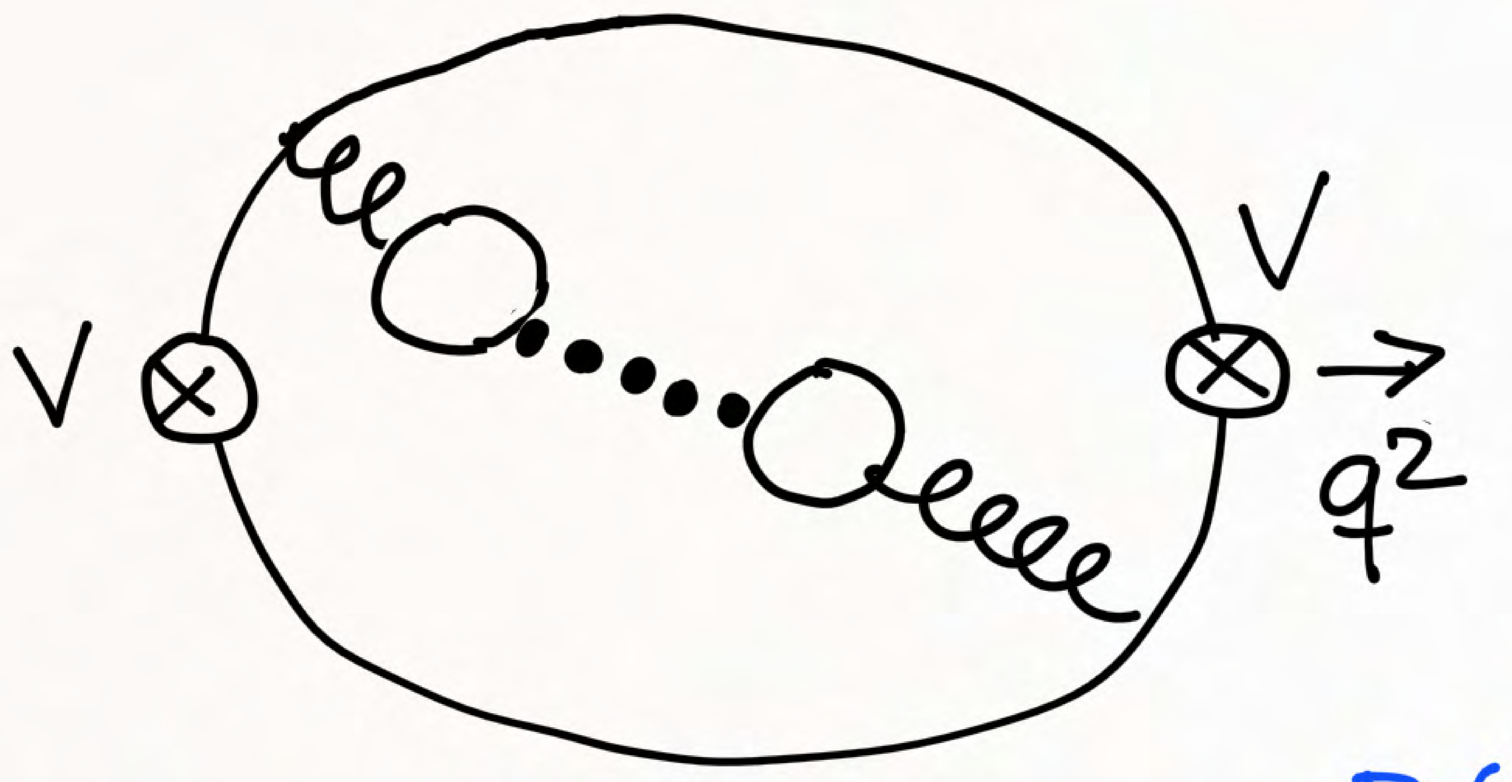

$\Delta(N^*)$

$$\Delta(N^*) = \sqrt{\frac{\pi\alpha}{2}} \, e^{-1/\alpha}$$

"non perturbative"

# Perturbative QCD is asymptotic : Renormalons

('t Hooft '77)

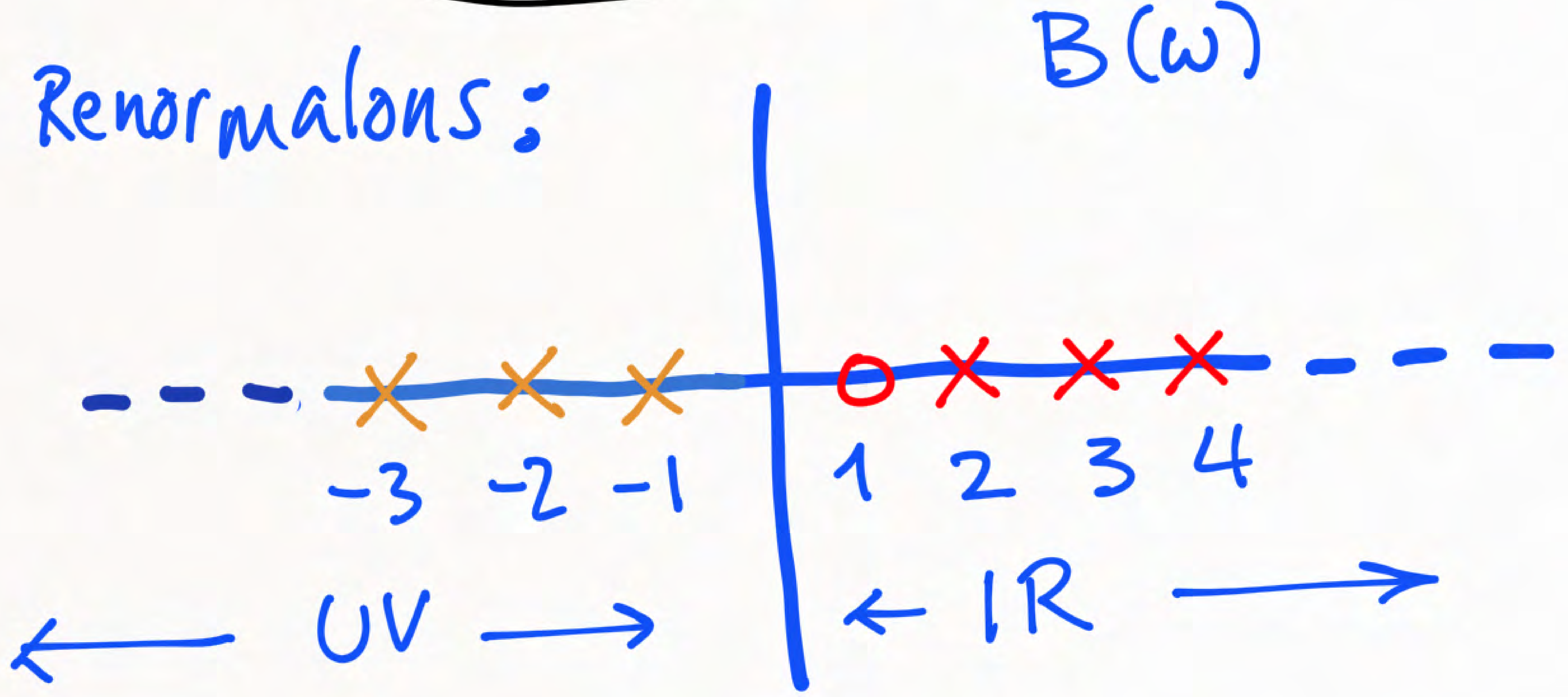



Adler

$$D(\alpha) = -z\frac{d\Pi}{dz} = \int_0^\infty dw\, B(\omega)\, e^{-\omega/\alpha(z)}\ , \quad \alpha(z) \equiv \frac{\beta_0 \alpha_s(z)}{2}$$

$(z = q^2)$

Renormalons:




- large $\beta_0$ approximation $\Longleftrightarrow$ $\beta(\alpha_s) = -\beta_0 \alpha_s^2$
  (simpler math, same physics)

$$B_{IR}(\omega) \sim \frac{1}{(p-\omega)^\gamma}\ , \quad \begin{array}{l} p=2,\ \gamma=1 \\ p>2,\ \gamma=2 \end{array}$$

- (Asymptotic) Perturbation Theory $\Longleftrightarrow$ $\left[\frac{1}{(p-\omega)^\gamma}\right]_T$ , $[\ ]_T$ : Taylor expansion

- Ambiguity @ $\omega = 2$ $\Longleftrightarrow$ OPE $\dfrac{\langle \alpha_s G^2 \rangle}{z^2}$ (Mueller '85)

# CONTOUR INTEGRALS

- $|z| = s_0 \lesssim m_\tau^2$, FESR radius $\Longrightarrow$ $A_m(s_0) \equiv -\frac{1}{2i\pi} \oint_{|z|=s_0} \frac{dz}{z} \left(\frac{z}{s_0}\right)^m D(z)$

(m selects term in OPE: Residue Theorem)

$$A_m(s_0) = -\frac{1}{2i\pi} \oint_{|z|=s_0} \frac{dz}{z} \left(\frac{z}{s_0}\right)^m \int_0^\infty dw \left[\frac{1}{(p-w)^\gamma}\right]_T \boxed{e^{-w/\alpha(z)}}$$

$$\frac{1}{\alpha(z)} = \frac{1}{\alpha(s_0)} + \log\left(\frac{-z}{s_0}\right)$$

(good old $\overline{MS}$) $\left[\left(\frac{-z}{s_0}\right)^{-w}\right]_T e^{-w/\alpha(s_0)}$

$$\left(\frac{-z}{s_0}\right)^{-w} e^{-w/\alpha(s_0)}$$

(Pivovarov '91)

(Le Diberder, Pich '92)

$$A_m(s_0) = \int_0^\infty \frac{dw}{\pi} \left[\frac{\sin(\pi w)}{(p-w)^\gamma (m-w)}\right]_T e^{-w/\alpha(s_0)}$$

$$A_m(s_0) = \int_0^\infty \frac{dw}{\pi} \left[\frac{1}{(p-w)^\gamma}\right]_T \frac{\sin(\pi w)}{(m-w)} e^{-\frac{w}{\alpha(s_0)}}$$

$\longleftarrow$ FIXED-ORDER PERT. THEORY $\longrightarrow$

$\longleftarrow$ CONTOUR-IMPROVED PERT. THEORY $\longrightarrow$

# FOPT vs. CIPT (I)

- Instructive case : simple pole ($\gamma=1$) & $p \neq m$.

Look @ integrand

$$A_m(s_0) = \int_0^\infty \frac{d\omega}{\pi} \left[ \frac{\sin(\pi\omega)}{(p-\omega)^{\gamma=1}\,(m-\omega)} \right]_T e^{-\frac{\omega}{\alpha(s_0)}} \quad \text{vs.} \quad \int_0^\infty \frac{d\omega}{\pi} \left[ \frac{1}{p-\omega} \right]_T \frac{\sin(\pi\omega)}{(m-\omega)} e^{-\omega/\alpha(s_0)}$$

FOPT

$$= \frac{\sin(\pi\omega)}{(p-\omega)^{\gamma=1}\,(m-\omega)} e^{-\omega/\alpha(s_0)}$$

$\Downarrow$

Convergent series in $[\alpha(s_0)]^n$ : radius $1/\pi$.
Defines analytic function $A_m(s_0)$
(For $\gamma=2$ or $p=m$ $\rightarrow$ Ppal Value)

CIPT

$$\neq \frac{\sin(\pi\omega)}{(p-\omega)^{\gamma=1}\,(m-\omega)} e^{-\omega/\alpha(s_0)}$$

$\Downarrow$

Divergent (Asymptotic) series not in $\alpha^n$

but in $\Phi_n(\alpha) \equiv \int_0^\infty \frac{d\omega}{\pi} \frac{\omega^n \sin(\pi\omega)}{(m-\omega)} e^{-\omega/\alpha}$

(Minimum irreducible difference ; Borel ?)

# FOPT vs. CIPT / MINIMUM DISTANCE

$$\Delta_{\text{SERIES}}^{(n)} \equiv A_m(s_0)\Big|_{\text{FOPT}}^{\text{BOREL}} - \sum_{k=0}^{n-1} c_k \left[\alpha(s_0)\right]^k \Big|_{\text{CIPT}}^{\text{SERIES}}$$

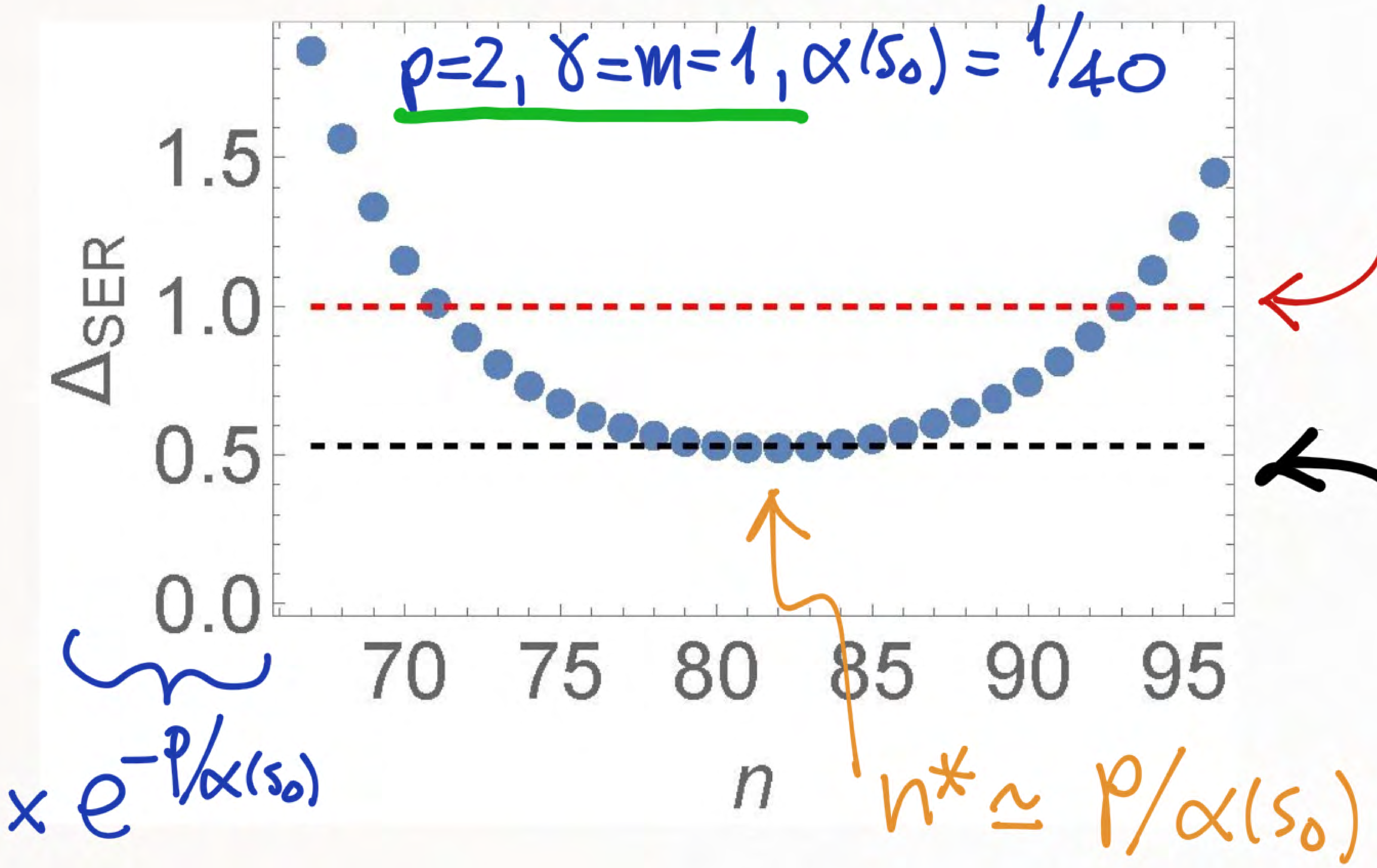
$$= \int_0^\infty \frac{dw}{\pi} \, \frac{e^{-w/\alpha(s_0)} \sin(\pi w)}{(p-w+i\epsilon)^\gamma \, (m-w+i\epsilon)} \left(\frac{w}{p}\right)^n \left(\frac{p+n(p-w)}{p}\right)^{\gamma-1}$$

well defined $\forall \; \alpha(s_0), p, m, \gamma, n$.

Minimum Distance $\equiv$ minimum of $\Delta_{\text{SERIES}}^{(n)}$ @ $n = n^*$

# FOPT vs. CIPT / MINIMUM DISTANCE (II)

simple pole $p=2, \gamma=1$ $\Longleftrightarrow$ $D_{OPE}(z) \sim \frac{\langle \alpha G^2 \rangle}{z^2}$ $\Longleftrightarrow$ $\oint \frac{dz}{z}\left(\frac{z}{s_0}\right)^m D_{OPE}(z)\Big|_{m \neq 2} = 0$ !




Asymptotic Separation (Hoang, Regner '22)

$$\Delta_{AS} = (-)^p \frac{e^{-p/\alpha(s_0)}}{p-m}$$

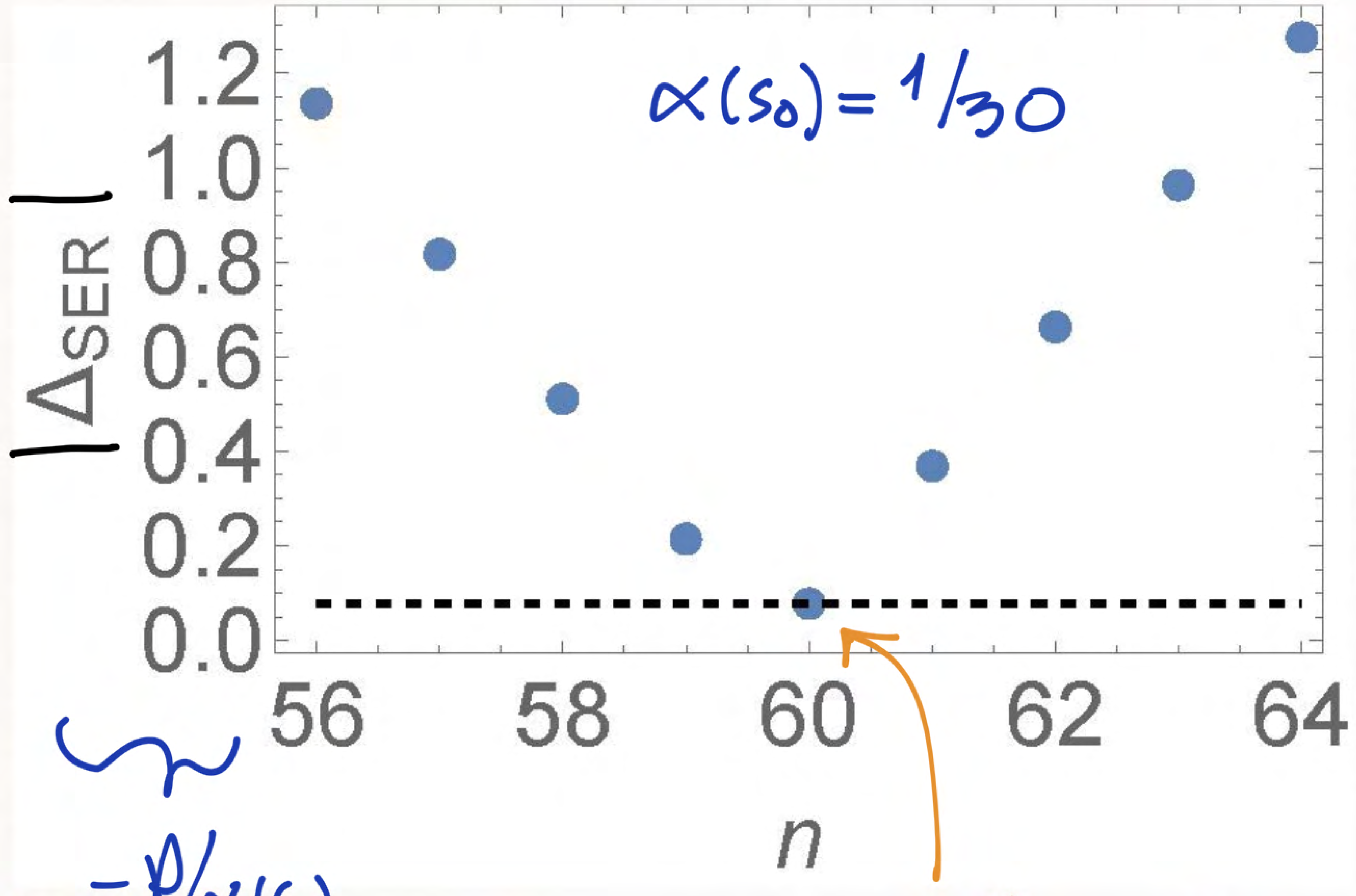
⊕ prescription to resum CIPT.

Minimum Distance: (Golterman, Maltman, SP '23)

$$\Delta^{\gamma=1}_{SERIES}(n^*) \simeq (-)^p \frac{\sqrt{2\pi p\, \alpha(s_0)}}{p-m}\, e^{-p/\alpha(s_0)} \quad \neq 0$$

$\Longrightarrow$ Nonperturbative but $\nexists$ OPE contribution!

- Asymptotic Separation ill-defined when $p=m$.
  (Hoang & Reguer define it as 0.)




$\sim \times e^{-p/\alpha(s_0)}$

$n^* \simeq p/\alpha(s_0)$

Example $p=m=2$, $\gamma=1$
(Minimum Distance well defined
but no simple analytic expression)

$$\ast \quad |\Delta_{SERIES}(n^*)| \lesssim e^{-p/\alpha(s_0)} \neq 0$$

(But very small)

# SUMMARY & CONCLUSIONS

- CIPT produces nonperturbative $O(e^{-p/\alpha})$ unrelated to OPE
  (see also Gracia, Hoang, Mateu '23)

- Analysis based on standard definition of Borel for FOPT ($\equiv \overline{MS}$) and the Minimum Distance wrt CIPT series. No need to resum CIPT series with a prescription, and also valid for $p=m$ (unlike Hoang & Regner's AS)

- Can CIPT be fixed? Yes, if renormalon associated with $\langle \alpha_s G^2 \rangle$ is subtracted (Benitez-Rathgeb et al. '22 ; Takaura's talk)

  RESULT : CIPT $\xrightarrow{\text{fix}}$ FOPT

- Simple solution : **FORGET ABOUT CIPT IN $\tau$ DECAY**