Renormalon subtraction in hadronic tau decays

Hiromasa Takaura (YITP, Kyoto University)

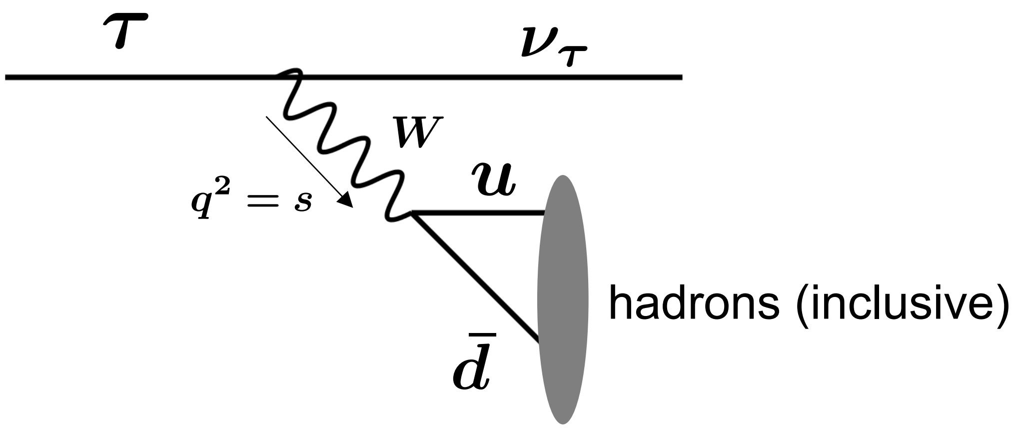
in collaboration with

Martin Beneke (TUM)

Work in progress

TAU2023

Hadronic tau decay



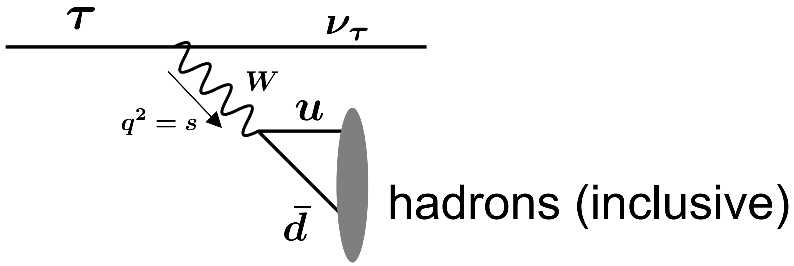
$$R_\tau = \frac{\Gamma(\tau \to \text{hadrons} + \nu_\tau)}{\Gamma(\tau \to e\bar{\nu}_e\nu_\tau)}$$

Measured at sub-percent level at ALEPH

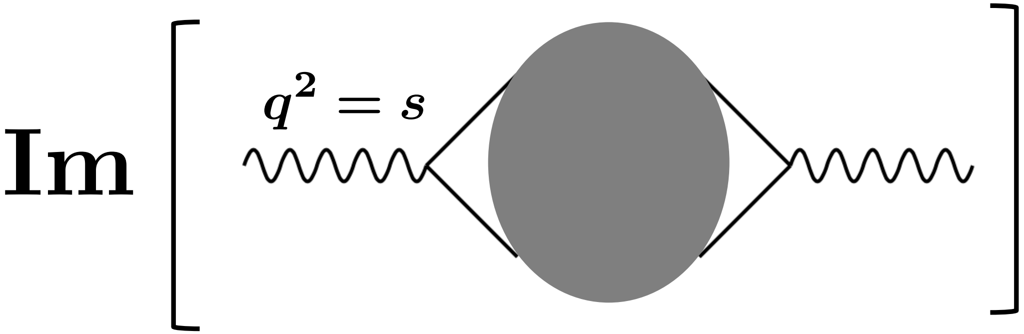
↓

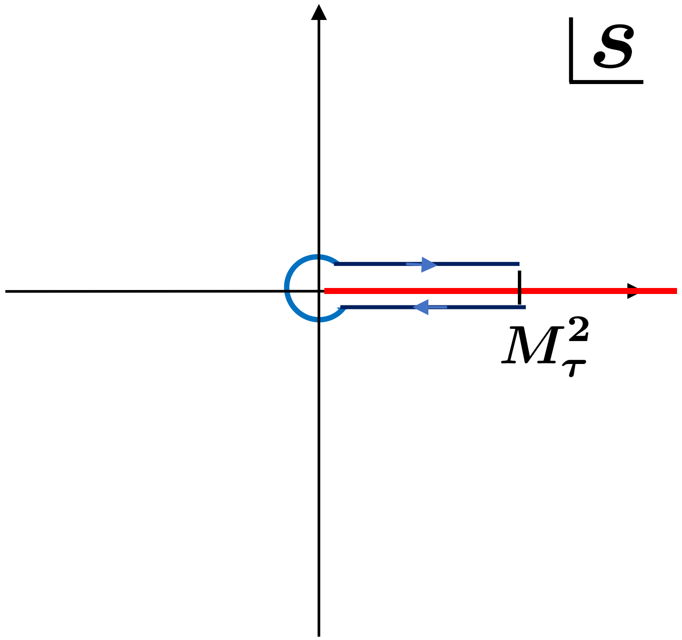
Precise α_s determination is possible
if precise theoretical prediction is obtained.

Theoretical calculation



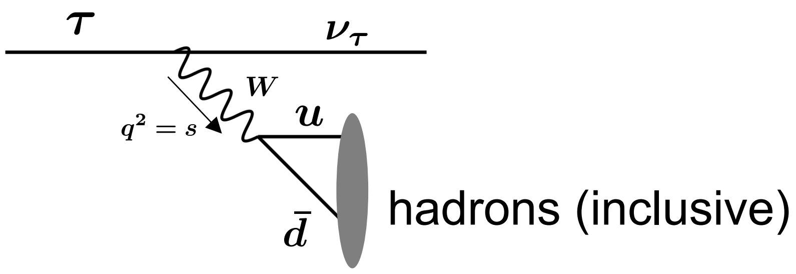
Optical theorem



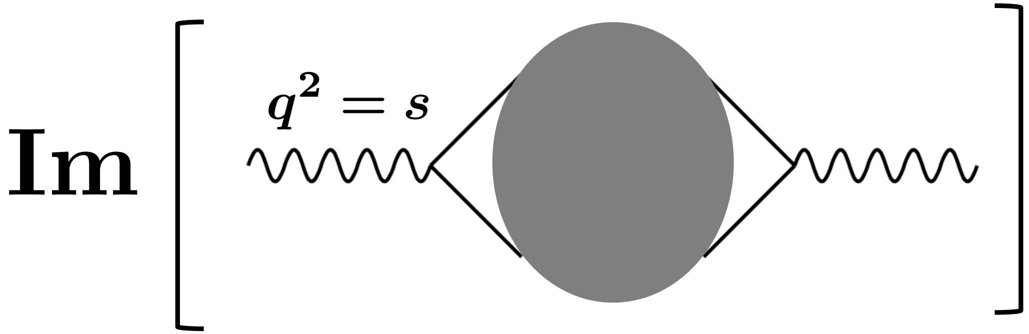


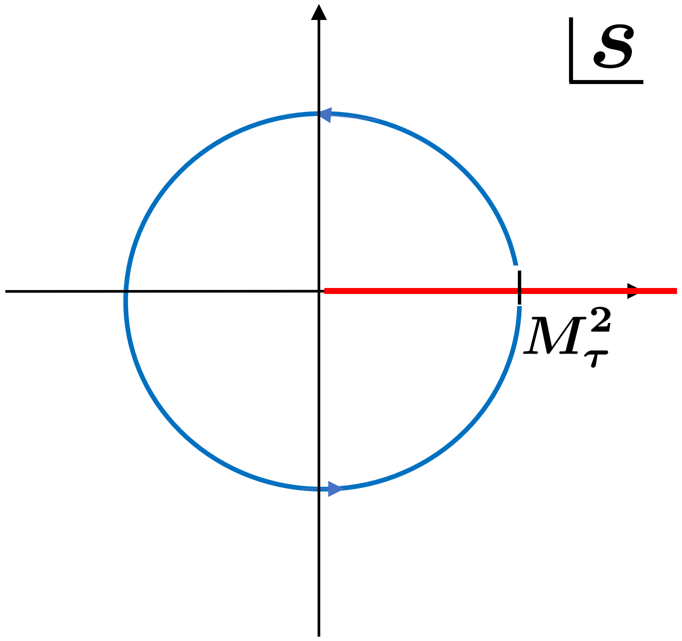
$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2\frac{s}{M_\tau^2}\right) \mathrm{Im}\Pi(s)$$

Theoretical calculation



Optical theorem



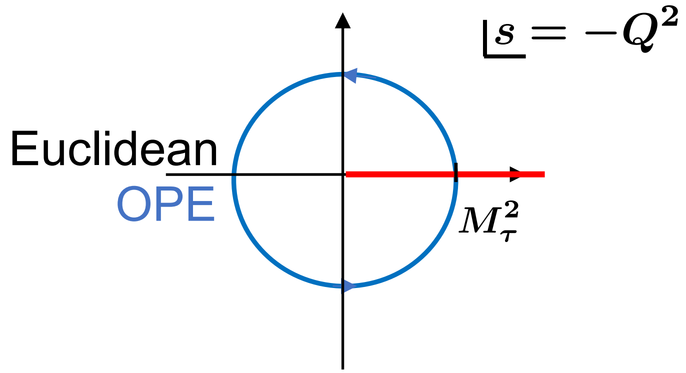


$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2\frac{s}{M_\tau^2}\right) \mathrm{Im}\Pi(s)$$

$$= -i\pi \oint_{|x|=1} \frac{dx}{x} (1-x)^3 3(1+x) D(M_\tau^2 x)$$

$$\boxed{D(-s = Q^2) = -s\frac{d}{ds}\Pi(s)}$$ Adler function

Short distance expansion



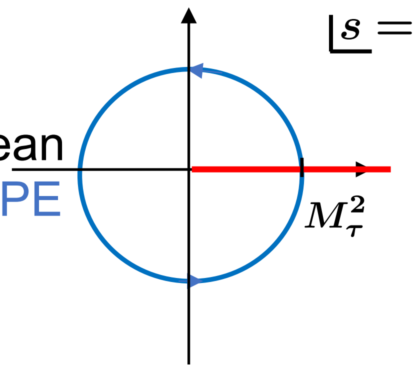
$$M_\tau(\approx 1.777\,\mathrm{GeV}) \gg \Lambda_{\overline{\mathrm{MS}}}(\approx 0.332\,\mathrm{GeV})$$

$$D(Q^2) = C_1^D(Q^2) + C_{FF}^D(Q^2)\frac{\langle F^2\rangle}{Q^4} + \mathcal{O}(\Lambda_{\overline{\mathrm{MS}}}/Q)^6$$

leading — suppressed as $(\Lambda_{\overline{\mathrm{MS}}}/Q)^4$

$\langle F^2\rangle = -\left\langle 0\left|\frac{\beta(\alpha_s)}{\pi b_0 \alpha_s} F^a_{\mu\nu}F^{a\mu\nu}\right|0\right\rangle$: Scale invariant gluon condensate (GC)

Short distance expansion



$$M_\tau(\approx 1.777\,\mathrm{GeV}) \gg \Lambda_{\overline{\mathrm{MS}}}(\approx 0.332\,\mathrm{GeV})$$

$$D(Q^2) = \underbrace{C_1^D(Q^2)}_{\text{leading}} + \underbrace{C_{FF}^D(Q^2)\frac{\langle F^2\rangle}{Q^4}}_{\text{suppressed as }(\Lambda_{\overline{\mathrm{MS}}}/Q)^4} + \mathcal{O}(\Lambda_{\overline{\mathrm{MS}}}/Q)^6$$

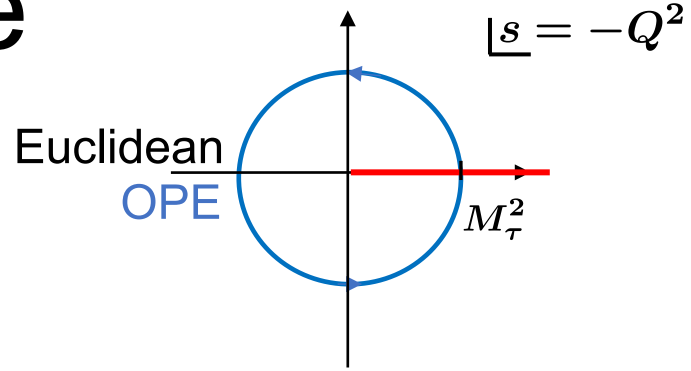
$\langle F^2\rangle = -\left\langle 0\left|\frac{\beta(\alpha_s)}{\pi b_0 \alpha_s} F^a_{\mu\nu}F^{a\mu\nu}\right|0\right\rangle$: Scale invariant gluon condensate (GC)

$C_1^D(Q^2) = D(Q^2)|_{\mathrm{PT}}$ (PT: perturbation theory)

$$= d_0 + d_1\alpha_s(\mu^2) + (d_2 + d_1 b_0 \log(\mu^2/Q^2))\alpha_s^2(\mu^2)$$
$$+(d_3 + 2d_2 b_0 \log(\mu^2/Q^2) + d_1 b_1 \log(\mu^2/Q^2) + d_1 b_0^2 \log^2(\mu^2/Q^2))\alpha_s^3(\mu^2) + \cdots$$

α_s^4 result is available. 12 Baikov, Chetyrkin, Kuhn, Rittinger

Choices of renormalization scale



Contour integral on $Q^2 = M_\tau^2 e^{i\theta} \quad (-\pi < \theta < \pi)$

Two seemingly reasonable choices of the renormalization scale

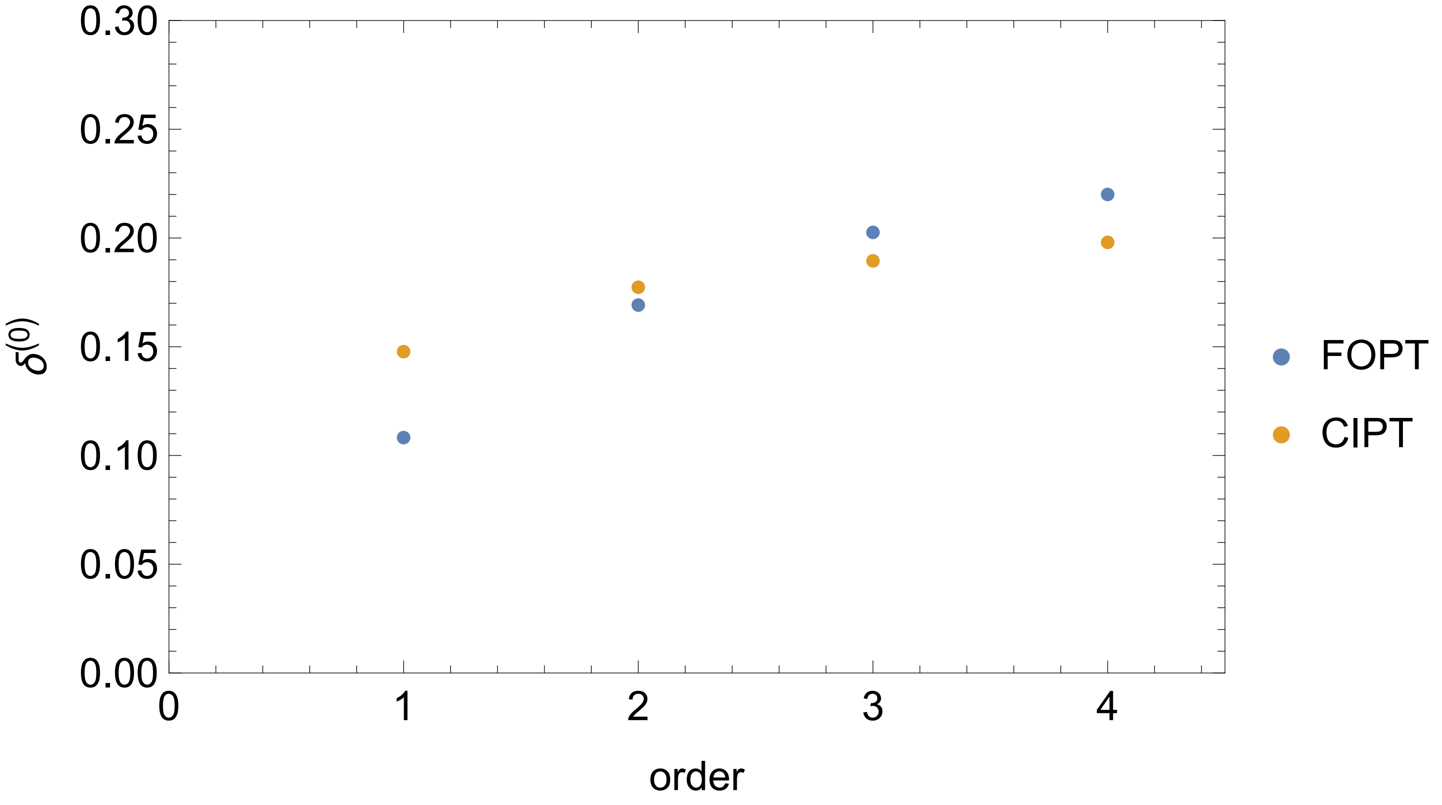
$\cdot\, \mu^2 = M_\tau^2$: FOPT (Fixed order perturbation theory)

$$C_1^D(Q^2 = M_\tau^2 e^{i\theta})$$
$$= d_0 + d_1 \alpha_s(M_\tau^2) + (d_2 + d_1 b_0 \log(e^{-i\theta}))\alpha_s^2(M_\tau^2)$$
$$+(d_3 + 2d_2 b_0 \log(e^{-i\theta}) + d_1 b_1 \log(e^{-i\theta}) + d_1 b_0^2 \log^2(-i\theta))\alpha_s^3(M_\tau^2) + \cdots$$

$\cdot\, \mu^2 = M_\tau^2 e^{i\theta}$: CIPT (Contour improved perturbation theory)

$$C_1^D(Q^2 = M_\tau^2 e^{i\theta})$$
$$= d_0 + d_1 \alpha_s(M_\tau^2 e^{i\theta}) + d_2 \alpha_s^2(M_\tau^2 e^{i\theta}) + d_3 \alpha_s^3(M_\tau^2 e^{i\theta}) + \cdots$$

Discrepancy between FOPT and CIPT



$\delta^{(0)}$: QCD correction to R_τ

The difference does not reduce by including higher order!

Long standing problem

Understanding of the discrepancy

- The problem is attributed to the divergent behavior of the perturbative series (renormalon).

08 Beneke, Jamin
13 Beneke, Boito, Jamin

$$d_n \sim (n-1)!(b_0/u)^n \qquad [C_1^D(Q^2) = \sum_{n\geq 0} d_n \alpha_s^n(Q^2)]$$

Divergent series require a careful treatment of the interchange of the resummation (RG improvement) and the contour integral.

Only FOPT can approximate the correct answer, while CIPT cannot.

- A mathematical explanation why CIPT fails was given.

20 Hoang, Regner

- CIPT is also shown to work once the most serious renormalon is removed!

22 Benitez-Rathgeb, Boito, Hoang, Jamin

In this talk, I introduce a new method to remove the renormalon.

Renormalons of the Adler function

$$d_n \sim K_{-1} n! (-b_0)^n + K_2 n! (b_0/2)^n + K_3 n! (b_0/3)^n + \cdots$$

u=-1 renormalon

u=2 renormalon
most serious

u=3 renormalon

Positive (IR) renormalons induce inevitable uncertainties of nonperturbative form:

$$\delta C_1^D \sim K_2 (\Lambda_{\overline{\rm MS}}/Q)^4 + K_3 (\Lambda_{\overline{\rm MS}}/Q)^6 + \cdots$$

Renormalons of the Adler function

$$d_n \sim K_{-1} n! (-b_0)^n + K_2 n! (b_0/2)^n + K_3 n! (b_0/3)^n + \cdots$$

u=-1 renormalon u=2 renormalon u=3 renormalon

most serious

Positive (IR) renormalons induce inevitable uncertainties of nonperturbative form:

$$\delta C_1^D \sim K_2 (\Lambda_{\overline{\rm MS}}/Q)^4 + K_3 (\Lambda_{\overline{\rm MS}}/Q)^6 + \cdots$$

OPE: $$D(Q^2) = C_1^D(Q^2) + C_{FF}^D(Q^2) \underbrace{\frac{\langle F^2 \rangle}{Q^4}}_{(\Lambda_{\overline{\rm MS}}/Q)^4} + \mathcal{O}(\Lambda_{\overline{\rm MS}}/Q)^6$$

The gluon condensate (GC) is also ambiguous (UV divergent) and cancels the u=2 renormalon.

Ambiguity of the GC closely relates to the u=2 renormalon
and hence the discrepancy b.t.w. FOPT and CIPT.

Our idea

We replace the ambiguous GC by a well-defined and measurable GC given through the gradient flow.

$$E(t) \equiv \frac{g_s^2}{4} \langle G^a_{\mu\nu}(t,x) G^a_{\mu\nu}(t,x) \rangle$$

This quantity is defined through the gradient flow equation (flow time t $\mathrm{dim}\,[t] = -2$)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\mu B_\nu(t,x), \quad B_\mu(t=0,x) = A_\mu(x)$$

$$G_{\mu\nu}(t,x) = \partial_\mu B_\nu(t,x) - \partial_\nu B_\mu(t,x) + g_s [B_\mu(t,x), B_\nu(t,x)]$$

Gradient-flow GC is UV finite 11 Luscher, Weisz
and can be measured on the lattice directly and accurately.

Renormalon cancellation

$$E(t) = \frac{1}{t^2} C_1^E(t) + C_{FF}^E(t) \langle F^2 \rangle + \mathcal{O}(t \Lambda_{\overline{\rm MS}}^6)$$

Small flow time expansion (~OPE)

leading (PT) ambiguous GC

Solve this equation for $\langle F^2 \rangle$

$$D(Q^2) = C_1^D(Q^2) + C_{FF}^D(Q^2) \frac{\langle F^2 \rangle}{Q^4} + \mathcal{O}(\Lambda_{\overline{\rm MS}}/Q)^6$$

$$= \left[C_1^D(Q^2) - \frac{C_{FF}^D(Q^2)}{C_{FF}^E(t)} \frac{1}{t^2 Q^4} C_1^E(t) \right] + \frac{C_{FF}^D(Q^2)}{C_{FF}^E(t)} \frac{E(t)}{Q^4} + \mathcal{O}(t \Lambda_{\overline{\rm MS}}/Q^4, \Lambda_{\overline{\rm MS}}^6/Q^6)$$

PT, renormalon free! Unambiguous GC!

- $1/\sqrt{t}$ can be regarded the IR cutoff. (gauge invariant!)
- The subtraction term behaves as $\mathcal{O}(t^0)$ at large order of PT because log(μ^2t) terms are effectively exponentiated.

How to choose t

- The IR cutoff $1/\sqrt{8t}$ (=the typical energy scale of E(t)) of the Adler function should be

$$1/\sqrt{8t} \ll Q \sim M_\tau$$

- Smaller $1/\sqrt{8t}$ realizes the renormalon cancellation at larger order of PT.
  Not too small $1/\sqrt{8t}$ should be chosen to realize the cancellation before the other renormalons become significant.

- $1/\sqrt{8t}$ should be a high enough energy so that the OPE is valid.

$$1/\sqrt{8t} \gg \Lambda_{\overline{\mathrm{MS}}}$$

- Nonperturbative lattice simulation can be accurately done.

$$a \ll \sqrt{8t} \ll L$$ (Loose condition)

Model of all-order series

To examine if this renormalon subtraction method works, we examine higher order behavior by modeling the all-order series (consistent with known coeffs.) based on the following ansatz:

- renormalons of the Adler function at u=-1, 2, 3 — cf. 08 Beneke, Jamin
- renormalons of E(t) at u=2, 3
- the theoretically revealed singularity structure for u=2 and rough singularity structures for u=-1 and u=3
- u=2 renormalon cancellation

Parameters in the all-order series model (Borel transforms) are fixed by known perturbative coefficients and a 5th order estimated coefficient for the Adler function.

Renormalon subtracted Adler function

$$C_1^D \propto 1 + 0.3183\alpha_s + 0.1661\alpha_s^2 + +0.2055\alpha_s^3 + 0.5038\alpha_s^4 + 0.92478\alpha_s^5 + 3.238\alpha_s^6 + \cdots$$

After renormalon subtraction

For $8tM_\tau^2 = 20$

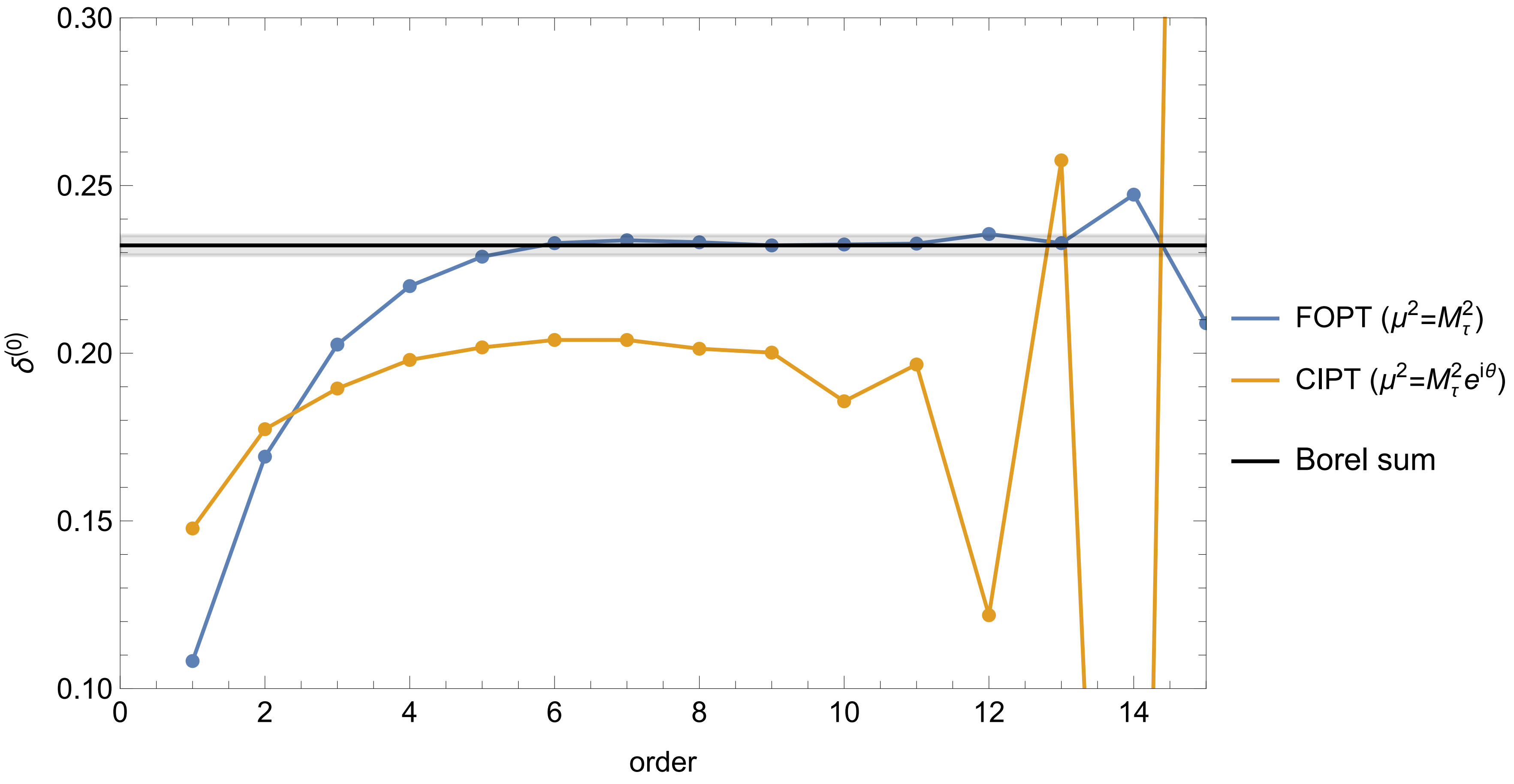
$$(C_1^D)^{\mathrm{RS}} \propto 1 + 0.2928\alpha_s + 0.08298\alpha_s^2 - 0.03517\alpha_s^3 + 0.05646\alpha_s^4 - 0.08525\alpha_s^5 + 0.5982\alpha_s^6 + \cdots$$

For $8tM_\tau^2 = 10$

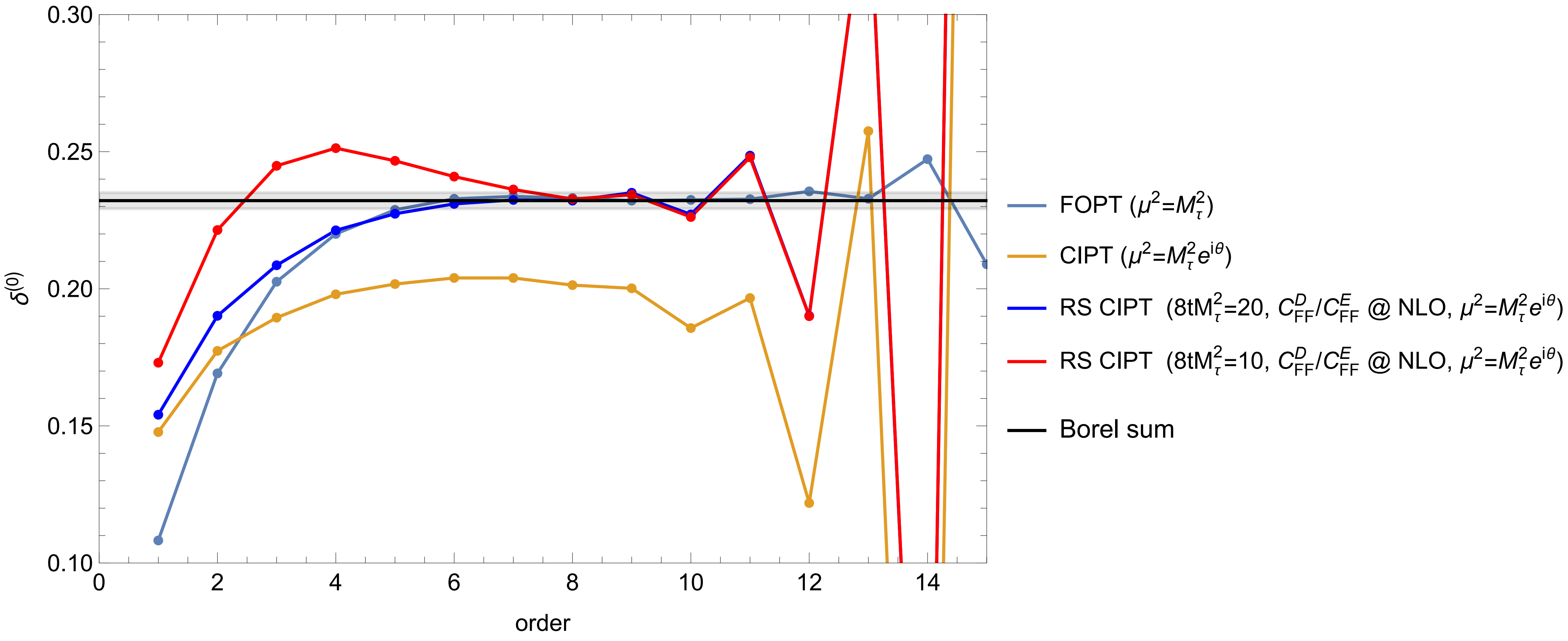
$$(C_1^D)^{\mathrm{RS}} \propto 1 + 0.2165\alpha_s - 0.1160\alpha_s^2 - 0.4233\alpha_s^3 + 0.08879\alpha_s^4 - 0.174074\alpha_s^5 + 0.3884\alpha_s^6 + \cdots$$

Perturbative coefficients are kept small

FOPT and CIPT



CIPT with renormalon subtraction



RS CIPT can successfully approximate the exact answer!

Summary and comments

・The hadronic tau decay width can be a useful observable for precisely extracting α_s.

・Recently, the long-standing discrepancy between FOPT and CIPT has been resolved by subtracting the u=2 renormalon.

・We propose a new renormalon subtraction method using the gradient flow and obtain a consistent result between FOPT and CIPT.

・Renormalon cancellation in our method is automatic (i.e. we do not need to estimate the size of the renormalon) and our nonperturbative GC can be measured accurately.

We'd like to perform a precise α_s determination using our method.