Hadronic vacuum polarization contribution to the muon $g-2$ from lattice QCD

Simon Kuberski

17th International Workshop
on Tau Lepton Physics
December 8, 2023

τ2023

Funded by
the European Union

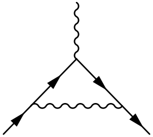
CERN

- Magnetic moment of charged leptons $l \in \{e, \mu, \tau\}$:

$$\vec{\mu}_l = g_l \cdot \frac{e}{2m_l} \cdot \vec{s}$$

- Quantum corrections lead to deviations from the classical value $g = 2$ (Dirac), the anomalous magnetic moment,

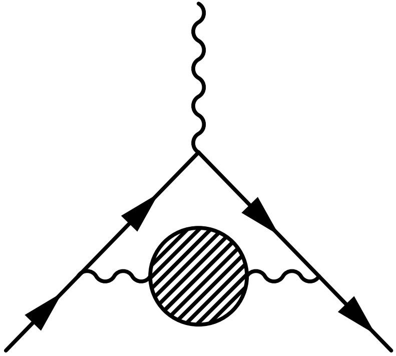
$$a_l = \frac{g_l - 2}{2} = \frac{\alpha}{2\pi} + \mathrm{O}(\alpha^2) \quad \text{(Schwinger)}$$



- Contributions from new physics at the scale Λ_{NP} enter a_l via

$$a_l - a_l^{\mathrm{SM}} \propto \frac{m_l^2}{\Lambda_{\mathrm{NP}}^2}$$

with $m_\mu / m_e \approx 207$.



(leading order)
hadronic vacuum polarization

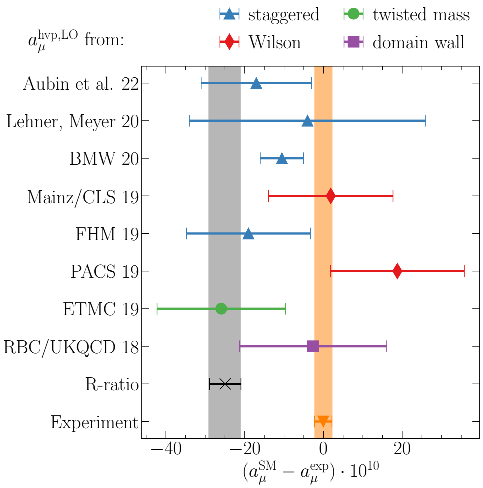
- Standard Model prediction from QED, electroweak and hadronic contributions:

$$a_l^{\mathrm{SM}} = a_l^{\mathrm{QED}} + a_l^{\mathrm{EW}} + a_l^{\mathrm{had}}$$

where $a_l^{\mathrm{had}} = a_l^{\mathrm{hvp}} + a_l^{\mathrm{hlbl}}$.

- $\Delta a_\mu^{\mathrm{SM}}$ is dominated by $\Delta a_\mu^{\mathrm{hvp}}$.

Compute the hadronic contributions to a_μ^{hvp} from lattice QCD.

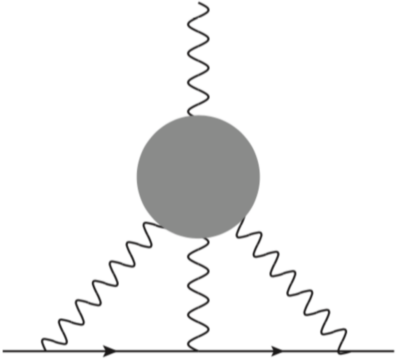


[BNL $g-2$, hep-ex/0602035]
[FNAL $g-2$, 2104.03281, 2308.06230]

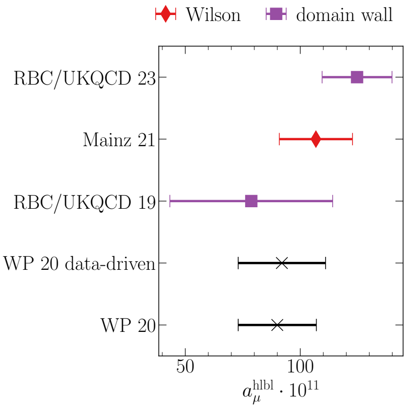
- $5.1\,\sigma$ discrepancy between the current experimental average and the White Paper average [2006.04822] (pre CMD-3).
- Average based on data-driven evaluation of the LO HVP contribution ("R-ratio") with 0.6% precision.
- One sub-percent determination of a_μ^{hvp} from the lattice [BMWc, 2002.12347]: In tension with the dispersive result.

Goal

Several lattice results at $< 0.5\%$ precision.



- Hadronic light-by-light scattering: $\mathrm{O}(\alpha^3)$, target precision: 10%.

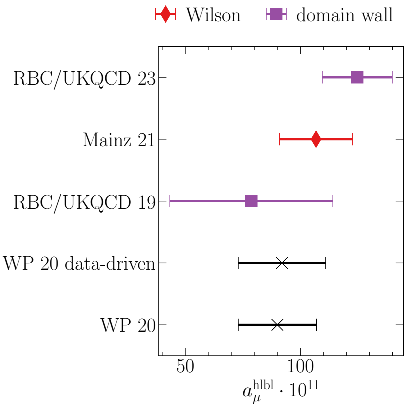


- Hadronic light-by-light scattering: $O(\alpha^3)$, target precision: 10%.
- White paper recommended value:

$$a_\mu^{\text{hlbl}} = (92 \pm 18) \cdot 10^{-11}$$

- Two lattice calculations since then, [Mainz 21, 2104.02632, 2204.08844] and [RBC/UKQCD 23, 2304.04423].
- Lattice and data-driven computations are an outstanding success.

Hadronic light-by-light scattering



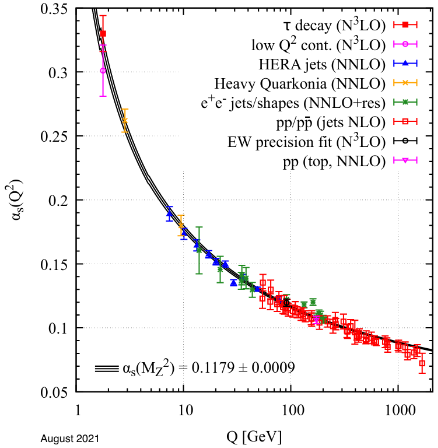
- Hadronic light-by-light scattering: $\mathrm{O}(\alpha^3)$, target precision: 10%.
- White paper recommended value:

$$a_\mu^{\mathrm{hlbl}} = (92 \pm 18) \cdot 10^{-11}$$

- Two lattice calculations since then, [Mainz 21, 2104.02632, 2204.08844] and [RBC/UKQCD 23, 2304.04423].
- Lattice and data-driven computations are an outstanding success.
- Probably not the reason for tensions between SM and experiment.
- Data-driven and lattice predictions are compatible.

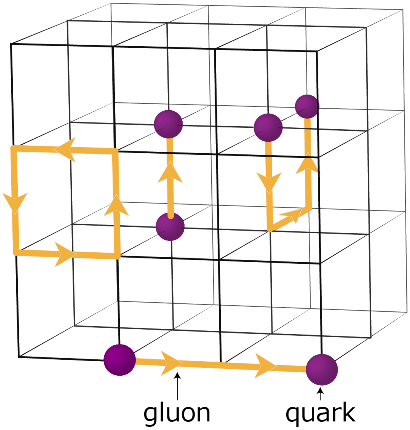
a_μ^{hvp} ON THE LATTICE (LEADING ORDER)

LATTICE QCD



- QCD is a strongly coupled theory in the hadronic regime at $Q \sim 300\,\mathrm{MeV}$.
- Perturbative expansion fails below $1\,\mathrm{GeV}$.

[1] R. L. Workman *et al.* [Particle Data Group], PTEP **2022** (2022), 083C01 doi:10.1093/ptep/ptac097



- QCD is a strongly coupled theory in the hadronic regime at $Q \sim 300\,\mathrm{MeV}$.
- Perturbative expansion fails below $1\,\mathrm{GeV}$.
- Formulate the theory
  - on a finite grid $\rightarrow$ regulator Λ_{UV}.
  - in finite volume $\rightarrow \Lambda_{\mathrm{IR}}$.
  - in Euclidean space-time
  - as a Boltzmann distribution
- Compute expectation values $\langle \mathrm{O} \rangle$ by sampling the QCD path integral with Markov Chain Monte Carlo methods.

[2] http://www.jicfus.jp/en/promotion/pr/mj/guido-cossu/

The QCD Lagrange density

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{f=1}^{N_f} \bar{\psi}_f (\not{D} + m_f)\psi_f + \frac{1}{4}\mathrm{tr} F_{\mu\nu} F^{\mu\nu}$$

- Contains $N_f + 1$ **bare** parameters (gauge coupling and N_f quark masses)
- Renormalize the theory from hadronic input, e.g., m_Ω, m_π, m_K, m_{D_s}, m_{B_s}.
  $\rightarrow$ All other observables are **predictions**.
- Freedom of choice on how to discretize $\mathcal{L}_{\mathrm{QCD}}$: Wilson, twisted mass, staggered, domain wall, overlap, $\ldots$
- *Ab initio* predictions after lifting the cutoffs:
  - Λ_{IR}: Infinite-volume limit.
  - Λ_{UV}: Continuum limit.

$a_\mu^{\rm hvp}$ ON THE LATTICE

- Compute $a_\mu^{\rm hvp}$ via [Laurup et al.] [Blum, hep-lat/0212018]

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}Q^2 f(Q^2)\hat{\Pi}(Q^2)\,, \qquad \text{with} \quad \hat{\Pi}(Q^2) = 4\pi^2\left[\Pi(Q^2) - \Pi(0)\right]$$

from a known QED kernel function $f(Q^2)$ and the polarization tensor

$$\Pi_{\mu\nu}(Q) = \int \mathrm{d}^4x\, \mathrm{e}^{iQ\cdot x} \langle j_\mu^{\rm em}(x)\, j_\nu^{\rm em}(0)\rangle = (Q_\mu Q_\nu - \delta_{\mu\nu}Q^2)\Pi(Q^2)\,.$$

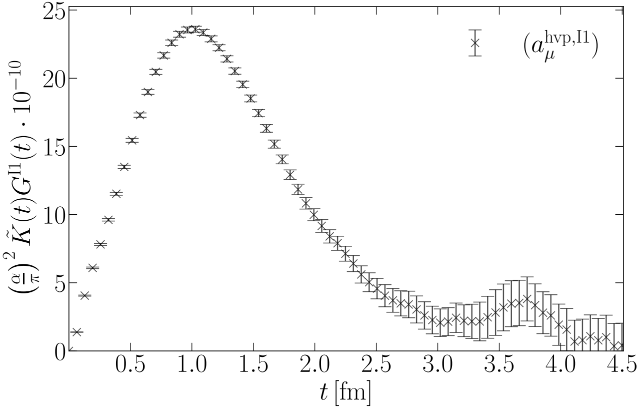
- $a_\mu^{\rm hvp}$ in the time-momentum representation (TMR) [Bernecker, Meyer, 1107.4388],

$$a_\mu^{\rm hvp} := \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\widetilde{K}(t) \quad \text{with the known QED kernel function } \widetilde{K}(t)\,,$$

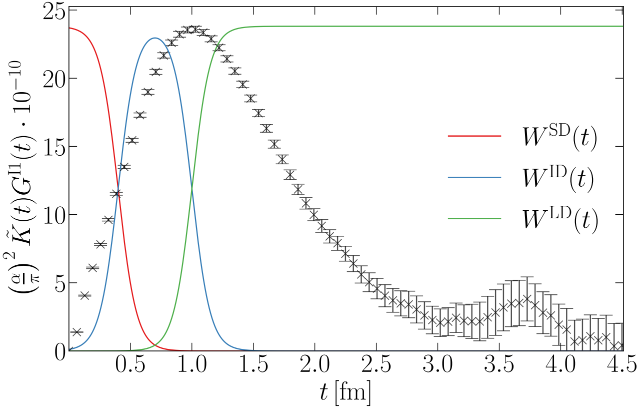
in terms of the zero-momentum vector correlator $G(t)$ (de facto standard).

- Alternative: coordinate space method [Meyer, 1706.01139] [Chao et al., 2211.15581].

$$(a_\mu^{\mathrm{hvp}}) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\widetilde{K}(t), \qquad G(t) = -\frac{a^3}{3}\sum_{k=1}^{3}\sum_{\vec{x}} \langle j_k^{\mathrm{em}}(t,\vec{x})\, j_k^{\mathrm{em}}(0)\rangle$$

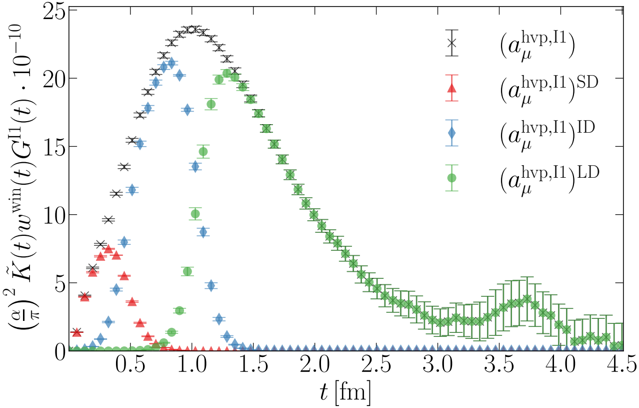


$$(a_\mu^{\text{hvp}})^i = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t) \widetilde{K}(t) \;\; W^i(t; t_0; t_1)\,, \qquad G(t) = -\frac{a^3}{3} \sum_{k=1}^{3} \sum_{\vec{x}} \langle j_k^{\text{em}}(t, \vec{x})\, j_k^{\text{em}}(0) \rangle$$



- Windows in the TMR: separate short- from long-distance effects [RBC/UKQCD, 1801.07224].

$$(a_\mu^{\rm hvp})^i = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t) \widetilde{K}(t)\; W^i(t; t_0; t_1)\,, \qquad G(t) = -\frac{a^3}{3}\sum_{k=1}^{3}\sum_{\vec{x}} \langle j_k^{\rm em}(t,\vec{x})\, j_k^{\rm em}(0)\rangle$$



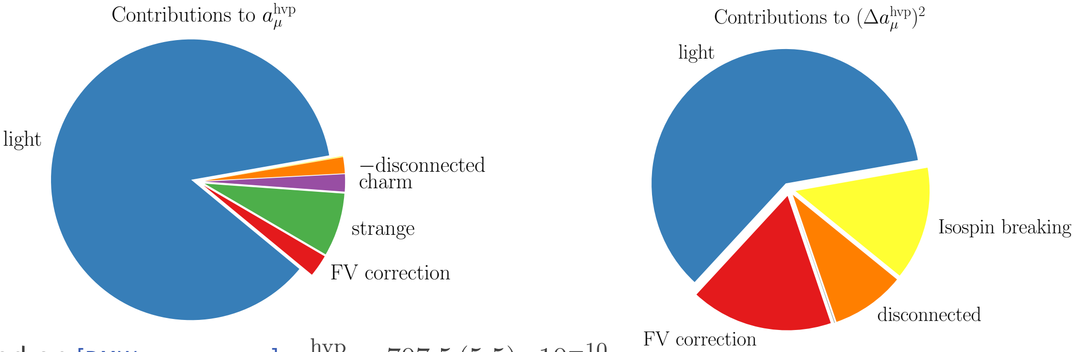
- Windows in the TMR: separate short- from long-distance effects [RBC/UKQCD, 1801.07224].
- Intermediate window $a_\mu^{\rm win}$:
  - Cutoff effects suppressed.
  - No signal-to-noise problem.
  - Finite-volume effects small.

The electromagnetic current

$$j_\mu^{\rm em} = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \ldots = j_\mu^{I=1} + j_\mu^{I=0}$$

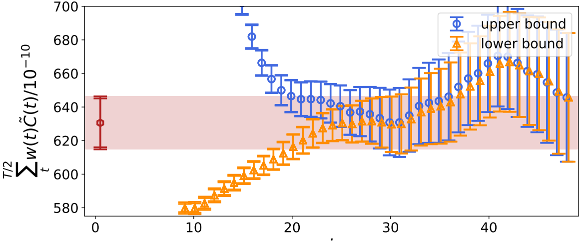
from zero-momentum vector-vector correlation functions

$$G^{\rm isoQCD}(t) = \frac{5}{9}G^{\rm light}(t) + \frac{1}{9}G^{\rm strange}(t) + \frac{4}{9}G^{\rm charm}(t) + G^{\rm disc}(t) + \ldots$$



Based on [BMWc, 2002.12347]: $a_\mu^{\rm hvp} = 707.5\,(5.5)\cdot 10^{-10}$

Dominant sources of uncertainty

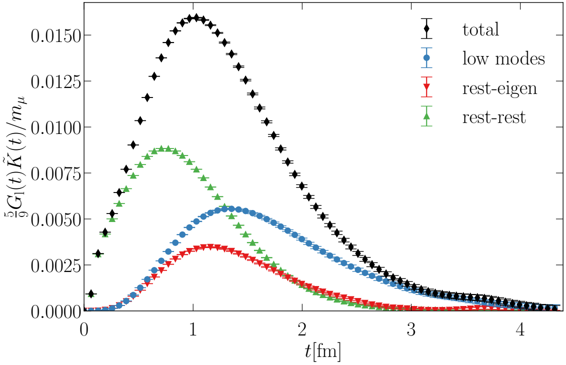


Exponential deterioration of the signal-to-noise ratio.

Improve the signal at large t via:

- Bounds on the correlator.
- Noise reduction methods:
  - Truncated Solver Method
  - Low Mode Averaging
  - All Mode Averaging
- Spectral reconstruction of the $\pi\pi$ contributions.
- Multi-level integration.
  [Dalla Brida et al., 2007.02973]

[RBC/UKQCD, 1910.11745]

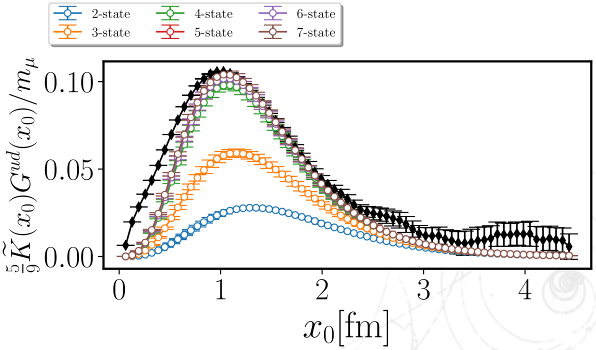


[Mainz, Lattice 2022]

Exponential deterioration of the signal-to-noise ratio.

Improve the signal at large t via:

- Bounds on the correlator.
- Noise reduction methods:
  - Truncated Solver Method
  - Low Mode Averaging
  - All Mode Averaging
- Spectral reconstruction of the $\pi\pi$ contributions.
- Multi-level integration.
  [Dalla Brida et al., 2007.02973]



Exponential deterioration of the signal-to-noise ratio.

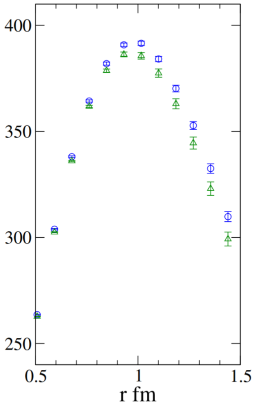
Improve the signal at large t via:

- Bounds on the correlator.
- Noise reduction methods:
  - Truncated Solver Method
  - Low Mode Averaging
  - All Mode Averaging
- Spectral reconstruction of the $\pi\pi$ contributions.
- Multi-level integration.
  [Dalla Brida et al., 2007.02973]

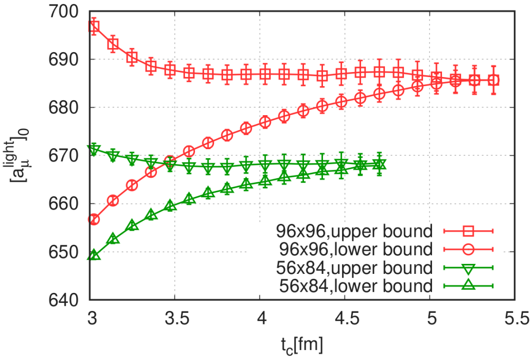
[Mainz]

3% finite-L corrections for $a_\mu^{\rm hvp}$ at $m_\pi L = 4$, mostly in the **isovector channel**.

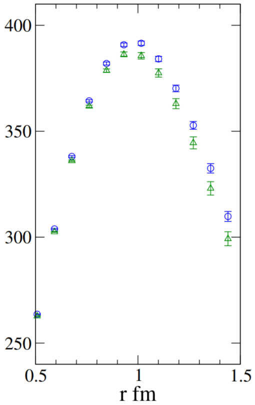
- EFT and model calculations.
  - NNLO χPT
  - Two-pion spectrum in finite-volume and the timelike pion form factor [Meyer, 1105.1892] [Lellouch and Lüscher, hep-lat/0003023] [Giusti et al., 1808.00887].
  - Pions winding around the torus and the electromagnetic pion form factor [Hansen, Patella, 1904.10010, 2004.03935].
  - Rho-pion-gamma model [Sakurai] [Jegerlehner, Szafron, 1101.2872] [HPQCD, 1601.03071].
- Simulations at $L > 10\,\mathrm{fm}$ [PACS, 1902.00885] [BMWc, 2002.12347].
  - Uncertainty statistics dominated.



3% finite-L corrections for a_μ^{hvp} at $m_\pi L = 4$, mostly in the **isovector channel**.



].



- Simulations at $L > 10\,\text{fm}$ [PACS, 1902.00885] [BMWc, 2002.12347].
  - Uncertainty statistics dominated.

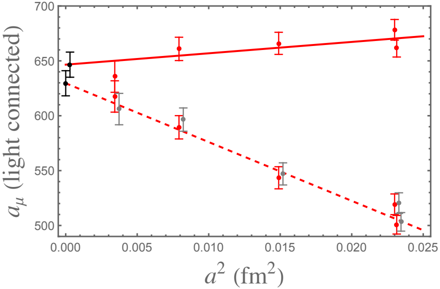
Cutoff effects

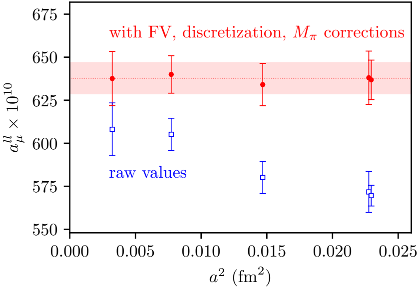
Systematic uncertainties from the continuum extrapolation may be dominant.

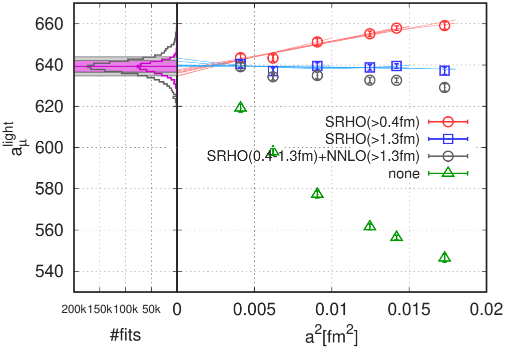
- Extrapolation to the continuum limit guided by Symanzik effective theory.
- Cutoff effects start at $\mathrm{O}(a^2)$ in modern lattice calculations.
- Mandatory to
  - include ≥ 4 resolutions to constrain higher order cutoff effects.
  - include fine resolutions $a \leq 0.05\,\mathrm{fm}$ for per-mil uncertainties.
- Staggered quarks: taste violations distort the pion spectrum.
  - This is a cutoff effect: Vanishes in the continuum limit.
  - Taste breaking may introduce non-linear effects (in a^2).

  $\rightarrow$ Corrections applied at finite lattice spacing.

The continuum limit: staggered quarks

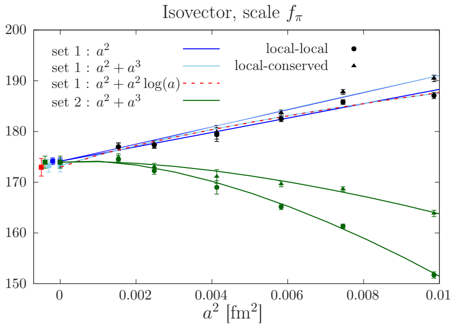


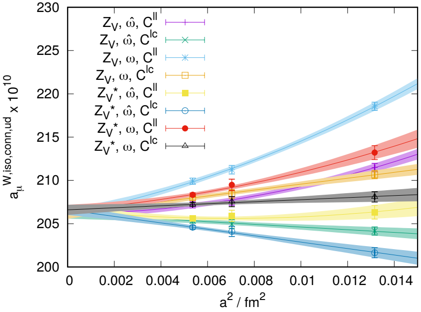


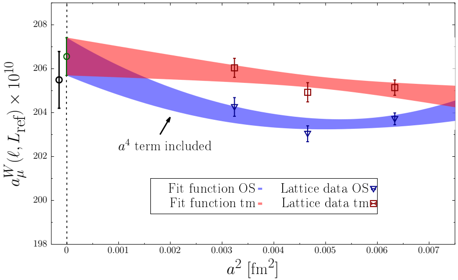


- Continuum extrapolations of a_μ^{hvp} computed with staggered quarks.
- Compare raw and corrected data.

[Aubin et al., 2204.12256] [BMWc, 2002.12347] [Fermilab, HPQCD, MILC, 1902.04223]







- Different discretization prescriptions have to agree in the continuum.
- Strong cross-check for **valence** cutoff effects.

[Mainz, 2206.06582] [RBC/UKQCD, 2301.08696] [ETMC, 2206.15084]

a_μ is dimensionless. Why do we need a precise scale setting?

- Scale enters via muon mass in $\tilde{K}(t)$. Determine the scale dependence via

$$\frac{\partial a_\mu^{\mathrm{hvp}}}{\partial_\Lambda} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t) \left(\frac{\partial}{\partial_\Lambda} \widetilde{K}(t)\right)$$

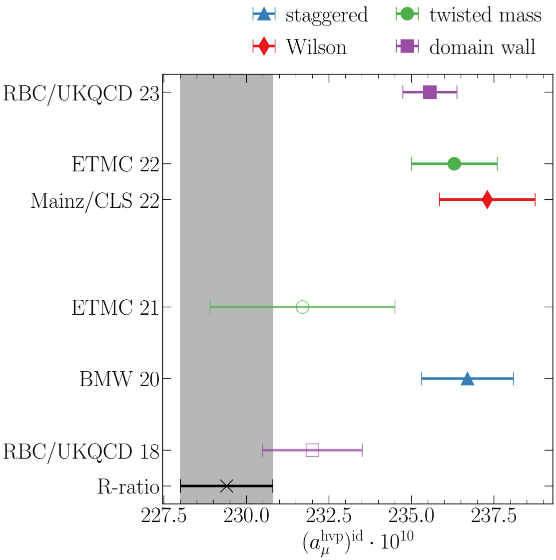
- Need (few) per-mill precision scale setting [Mainz, 1705.01775]:

$$\frac{\delta a}{a} = 1\,‰ \to \frac{\delta_a a_\mu^{\mathrm{hvp}}}{a_\mu^{\mathrm{hvp}}} = 1.8\,‰ \qquad \text{whereas} \qquad \frac{\delta_a a_\mu^{\mathrm{win}}}{a_\mu^{\mathrm{win}}} = 0.5\,‰$$

- Determine the lattice spacing a by matching to hadronic quantities:
  - Baryons (Ω, Ξ, ...)
  - Pseudoscalar decay constants, mostly f_π.

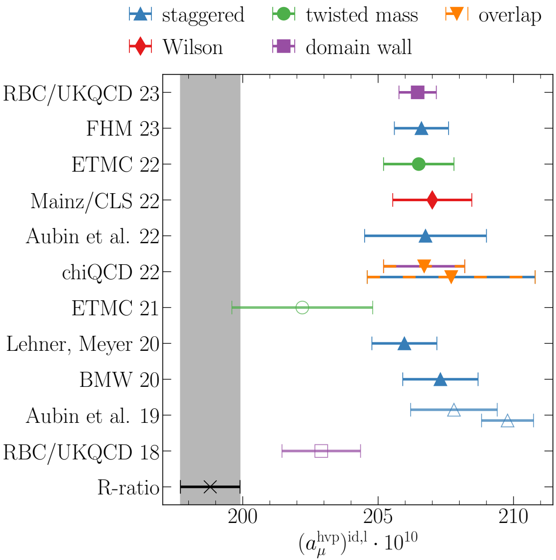
The intermediate-distance window

The intermediate-distance window



- $3.8\,\sigma$ tension between lattice QCD and data-driven evaluation [Colangelo et al., 2205.12963].
- This accounts for 50% of the difference between BMW 20 and the White Paper average for a_μ^{hvp}.

The intermediate-distance window



- $3.8\,\sigma$ tension between lattice QCD and data-driven evaluation [Colangelo et al., 2205.12963].
- This accounts for 50% of the difference between BMW 20 and the White Paper average for $a_\mu^{\rm hvp}$.
- Agreement across many actions for the light-connected contribution (87%).
- Data-driven estimate: [Benton et al., 2306.16808] [Maltman]

Isospin breaking effects

QED AND STRONG ISOSPIN BREAKING

Need to include $\mathrm{O}(\frac{m_\mathrm{u}-m_\mathrm{d}}{\Lambda_\mathrm{QCD}})$ and $\mathrm{O}(\alpha)$ effects for per-mil precision.

- Results in isospin symmetric QCD have to be compared in the same scheme.
  $\rightarrow$ Effort in FLAG to propose a common scheme [Tantalo, 2301.02097] [Portelli].

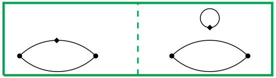
- Various ways to compute isospin breaking corrections:
  - **Perturbative expansion around isospin symmetric QCD** [RM123, 1303.4896].
  - Simulation of dynamical QCD+QED [CSSM/QCDSF/UKQCD] [RC*, 2212.11551].
  - Infinite volume QED [RBC/UKQCD, 1801.07224] [Biloshytskyi et al., 2209.02149]

- Major challenge: Formulation of QED in a finite box.

- QED_L: Finite-volume corrections scale as $\mathrm{O}(1/L^3)$ [Bijnens et al., 1903.10591]
  $\rightarrow$ sufficient for the precision goal.

Overview of published results - contributions to $a_\mu \times 10^{10}$

- Strong isospin breaking: Five groups agree within $1\,\sigma$.

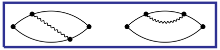


6.60(63)(53)		BMW
10.6(4.3)(6.8)		RBC/UKQCD
6.0(2.3)		ETM
7.7(3.7)	**9.0(2.3)**	FHM
9.0(0.8)(1.2)		LM

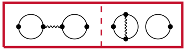
BMW [Nature 593 (2021) 7857, 51-55]
RBC/UKQCD [Phys.Rev.Lett. 121 (2018) 2, 022003]
ETM [Phys. Rev. D 99, 114502 (2019)]
FHM [Phys.Rev.Lett. 120 (2018) 15, 152001]
LM [Phys.Rev.D 101 (2020) 074515]

Adapted from [V. Gülpers @ Lattice HVP workshop 2020]

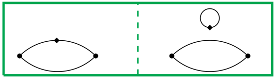
Overview of published results - contributions to $a_\mu \times 10^{10}$



BMW	$-1.23(40)(31)$
RBC/UKQCD	$5.9(5.7)(1.7)$
ETM	$1.1(1.0)$



$-0.55(15)(10)$	BMW
$-6.9(2.1)(2.0)$	RBC/UKQCD



$6.60(63)(53)$		BMW
$10.6(4.3)(6.8)$		RBC/UKQCD
$6.0(2.3)$		ETM
$7.7(3.7)$	$9.0(2.3)$	FHM
$9.0(0.8)(1.2)$		LM

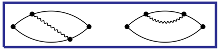
- Strong isospin breaking: Five groups agree within $1\,\sigma$.
- QED: agreement on the total valence contribution.

BMW [Nature 593 (2021) 7857, 51-55]
RBC/UKQCD [Phys.Rev.Lett. 121 (2018) 2, 022003]
ETM [Phys. Rev. D 99, 114502 (2019)]
FHM [Phys.Rev.Lett. 120 (2018) 15, 152001]
LM [Phys.Rev.D 101 (2020) 074515]

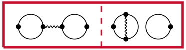
Adapted from [V. Gülpers @ Lattice HVP workshop 2020]

QED and strong isospin breaking: Results

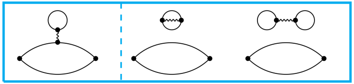
Overview of published results - contributions to $a_\mu \times 10^{10}$



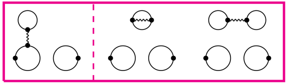
BMW	$-1.23(40)(31)$
RBC/UKQCD	$5.9(5.7)(1.7)$
ETM	$1.1(1.0)$



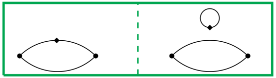
$-0.55(15)(10)$	BMW
$-6.9(2.1)(2.0)$	RBC/UKQCD



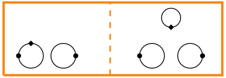
$-0.0093(86)(95)$ $0.37(21)(24)$ BMW



$0.011(24)(14)$ $-0.040(33)(21)$ BMW



$6.60(63)(53)$		BMW
$10.6(4.3)(6.8)$		RBC/UKQCD
$6.0(2.3)$		ETM
$7.7(3.7)$	$9.0(2.3)$	FHM
$9.0(0.8)(1.2)$		LM



$-4.67(54)(69)$ BMW

BMW [Nature 593 (2021) 7857, 51-55]
RBC/UKQCD [Phys.Rev.Lett. 121 (2018) 2, 022003]
ETM [Phys. Rev. D 99, 114502 (2019)]
FHM [Phys.Rev.Lett. 120 (2018) 15, 152001]
LM [Phys.Rev.D 101 (2020) 074515]

Adapted from [V. Gülpers @ Lattice HVP workshop 2020]

- Strong isospin breaking: Five groups agree within $1\,\sigma$.
- QED: agreement on the total valence contribution.
- One complete calculation [BMWc, 2002.12347]: $\delta a_\mu^{\text{hvp}} = 0.5(1.4)\cdot 10^{-10}$
- Work in progress:
  [Mainz, 2206.06582]
  [RBC/UKQCD, Lattice 2022]
  [BMWc, Lattice 2022]
  [FHM, 2212.12031]
  [Harris et al., 2301.03995]

- The discrepancy between lattice and data-driven calculations (White Paper average) in the **intermediate window** is firmly established.
- Further checks via $a_\mu^{\text{hvp,SD}}$ and $a_\mu^{\text{hvp,LD}}$ (to come).
- Other windows can be calculated to scrutinize the discrepancy [Lehner et al., 2003.04177] [Colangelo et al., 2205.12963] [FHM, 2207.04765] [Boito et al., 2210.13677]
- More insights from direct comparison with the smeared R-ratio? [EMTC, 2212.08467].
- Similar tension in $\Delta\alpha_{\text{had}}$ [BMWc, 1711.04980, 2002.12347] [Mainz, 2203.08676] [Davier et al., 2308.04221].

- Lattice QCD can provide SM predictions with sub-percent precision.
- More and more precise lattice results for a_μ^{hvp} urgently needed (and expected).
- Improvements: In the last years and ongoing
  - Isovector contribution with sub-percent precision.
  - EFT and data based finite-size corrections.
  - Finer lattices, more lattice spacings.
  - More precise scale setting.
  - Isospin breaking effects (beyond the electroquenched approximation).
  - **Blinded analyses**.