Impact on EWPOs of the $\alpha_{QED}(m^2_Z)$ uncertainty
and possible mitigation

context: on 14 July 2022 there was **MiniWorkshop: parametric uncertainties: α_em**
https://indico.cern.ch/event/1173700/
where the possible calculations and improvements on this important effect were discussed.

It is notable that $\alpha_{QED}(m^2_Z)$ can be measured in FCC-ee from the slope of A_{FB} ($\mu\mu$) vs $\sqrt{s}$ around the Z pole (Janot) This is new and specific to the circular Higgs/EW factory projects.
NB Further possibilities of direct meausrements of $\alpha_{QED}(m^2_Z)$ in TeraZ data should be investigated.

In view of the meeting I had prepared a table of the effect of $\Delta\alpha$ on important EW observables, compared with the expected statistical precisions of FCC-ee, which were presented
by Juan Alcaraz, at the FCC physics workshop of 2022.
https://indico.cern.ch/event/1066234/contributions/4708127/

Aim: understand which experimental tests and observables are sensitive to $\Delta\alpha$, and which are not.
While of course this information is included in global fits, it is useful to extract it when preparing an optimization of the run plan (assignment of integrated luminosity delivery among energy points)
e.g. how much data should be dedicated to W mass vs Z width scan etc...

Table 1. Dependence of selected precision measurements at FCC-ee upon the uncertainty on $\alpha_{QED}(m_Z^2)$, or on $\sin^2\theta_W^{eff}$. Experimental data have been compiled from [3].

$\sin^2\theta_w^{eff}$ from A_{FB} leptons and A_{FB}^{Pol} (τ)

Observable	present value ± error	FCC-ee Stat.	from $\alpha_{QED}(m_Z^2)$	Comments
m_Z (keV)	91186700 ± 2200	4	N.A.	Input
$G_F(\times 10^{-5})$	1.166378 ± 0.000006	N.A	N.A.	Input
$1/\alpha_{QED}(m_Z^2)(\times 10^3)$	128952 ± 14	3	N.A.	Input; from $A_{FB}^{\mu\mu}$ off peak
$\sin^2\theta_W^{eff}(\times 10^6)$	231480 ± 160	1.5	10.5	from $A_{FB}^{\mu\mu}$ and $A_{FB}^{pol,\tau}$ at Z peak Possible alternative input
m_W (MeV)	80350 ± 15	0.250	0.547	
Γ_ℓ (keV)	83985 ± 86	0.2	0.53	stat. based on muon pair statistics ρ parameter
R_ℓ^Z $(\times 10^3)$	20767 ± 25	0.06	0.17	ratio of hadrons to leptons quark and lepton universality determination of $\alpha_{QCD}(m_Z^2$
R_b $(\times 10^6)$	216290 ± 660	0.3	0.42	ratio of $b\bar{b}$ to hadrons N.P. coupled to 3d generation
Γ_Z (keV)	2495200 ± 2300	4	27	From Z line shape scan Beam energy calibration

How this was calculated

History: at the time of LEP preparation (1986-1989) we worked on the definition of 'effective' parameters representing definite blocks of corrections **(4)** in collaboration with e.g. Bryan Lynn and others.
Basically LEP corrections sensitive to heavy physics were driven by these terms:

$$
\begin{aligned}
M_Z^2 &= \frac{\pi\alpha(M_Z^2)}{\sqrt{2}G_F(1+\Delta\rho)(1+\Delta_{3Q})\sin^2\theta_W^{\text{eff}}\cos^2\theta_W^{\text{eff}}};\\
\Gamma_\ell &= \frac{G_F M_Z^3}{24\sqrt{2}\pi}[1+\Delta\rho]\left[1+\left(\frac{g_{V\ell}}{g_{A\ell}}\right)^2\right]\left(1+\frac{3}{4}\frac{\alpha}{\pi}\right);\\
\Gamma_b &= \Gamma_d(1+\delta_{vb});\\
M_W^2 &= \frac{\pi\alpha(M_Z^2)}{\sqrt{2}G_F(1-\Delta r^{ew})(1-\frac{M_W^2}{M_Z^2})}.
\end{aligned}
$$

$$g^f_{L,R} = \sqrt{\rho^f}\,[I^f_{L,R} - Q^f\kappa^f\,sin^2\theta_W^{\text{eff}}]$$

2

related to S and T parameters
(see Peskin Takeuchi '90, also Altarelli, Barbieri, Jadach 91)
$S \propto (\Delta_{3Q} - \Delta_{3Q}^{SM})/\alpha$
$T = (\Delta\rho - \Delta\rho^{SM})/\alpha$
δ_{vb} also important!
running of α_{QED} explicit, same for $\sin^2\theta_w^{eff}$ and m_W

old stuff, see e.g. my ICHEP Warsaw presentation in 1994
https://inspirehep.net/literature/377153

A very nice set of equations appears in the 2005 LEP physics report arXiv:0509008

Electroweak predictions for Z → ff observables can be obtained by using the coupling formulae

$$g^f_{L,R} = \sqrt{\rho^f}\,[I^f_{L,R} - Q^f \kappa^f \sin^2\theta^{\text{eff}}_{\text{W}}]$$

(I_R = 0 for right-handed fermions)
$g_L = ½ (g_V + g_A)$ $g_R = ½ (g_V - g_A)$
$g_V = (g_L + g_R)$ $g_A = (g_L - g_R)$

for practical reasons the effective weak mixing angle is defined from the leptonic couplings:

$\sin^2\theta^{\text{lept}}_{\text{eff}} = ¼\,(1 - g_V/g_A)^{\text{charged lepton}}$ and $\rho^{\text{charged lepton}} \equiv (1 + \Delta\rho)$

For neutrinos and quarks small correction factors are necessary for non-universal corrections (vertex and box diagrams).
In the SM, these additional corrections bear no sensitivity to running alphaQED.

No sensitivity to heavy physics in SM either, except for the b-quark where the correction is sensitive to top and Higgs boson masses.
They bear sensitivity to other new physics such as new "partners", e.g. as stop mass or Heavy Neutrinos. coupled or mixed

It is important to note that the running of α_{QED} appears at this order
only in the prediction for **$\sin^2\theta_w^{\text{eff}}$ and m_W, and with the same correction factor $\Delta\alpha$**

$$\alpha_{\text{QED}}(m_Z^2) = \frac{\alpha_{\text{QED}}(0)}{1 - \Delta\alpha}$$

it is also possible to establish a relation that eliminates $\Delta\alpha$ between $\sin^2\theta_w^{\text{eff}}$ and m_W

$$\sin^2\theta^{\text{eff}}_{\text{W}} = \mathcal{K}_w\left(1 - \frac{m_{\text{W}}^2}{m_{\text{Z}}^2}\right)$$

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m_W (MeV)	80350 ± 15	**0.250**	**0.547**		
Γ_ℓ (keV)	83985 ± 86	**0.2**	**0.53**		stat. based on muon pair statistics ρ parameter
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m_W (MeV)	80350 ± 15	0.250	0.547	0.078	
Γ_ℓ (keV)	83985 ± 86	0.2	0.53	0.076	stat. based on muon pair statistics ρ parameter
R_ℓ^Z $(\times 10^3)$	20767 ± 25	0.06	0.17	0.025	ratio of hadrons to leptons quark and lepton universality determination of $\alpha_{QCD}(m_Z^2$
R_b $(\times 10^6)$	216290 ± 660	0.3	0.42	0.06	ratio of $b\bar{b}$ to hadrons N.P. coupled to 3d generation
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from inspecting the table one can observe the following:

1. Even with the direct measurement of α_{QED} **(m_Z)** at FCC-ee ($3\ 10^{-5}$) the resulting uncertainty on the prediction of all observables (except the neutrino partial widths) is larger than the expected statistical uncertainty.

2. The observable that is most affected is **$\sin^2\theta_w^{eff}$**

3. Since all sensitivity to $\Delta\alpha$ is contained in the correction to **$\sin^2\theta_w^{eff}$**

it is conceivable to eliminate most of the $\Delta\alpha$ uncertainty in the prediction of the other observables by using the measured value of $\sin^2\theta_w^{eff}$ in the SM prediction, thus sharpening the predictive power of these other measurements.

➔ **By using the measured value of $\sin^2\theta_w^{eff}$ rather $\Delta\alpha$ in for the predictions, the $\Delta\alpha$ parametric errors on all quantities becomes smaller than the expected FCC-ee accuracy.**

for some observables (such as R_{lepton}) this procedure would also removes the top mass sensitivity.
but not for others such as Z width and m_W

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Conclusions

1. The impact of Δ $\boldsymbol{\alpha_{QED}(m^2_Z)}$ on FCC-ee EWPOs can be concentrated on $\boldsymbol{\sin^2\theta_w^{eff}}$ and eliminated by using $\boldsymbol{\sin^2\theta_w^{eff}}$ instead as input parameter in the predictions of the others EWPOs.

2. Of course is is extremely important to calculate and measure experimentally $\boldsymbol{\alpha_{QED}(m_Z)}$ as precisely as possible.
   ➔ further investigations should be encouraged (other leptons than muons, other processes than A_{FB} etc...)

... and to measure $\boldsymbol{\sin^2\theta_w^{eff}}$ as accurately as possible.

3. In order to enhance the precision of the other EWPOs (including the W mass) it would be very useful to be able to implement the SM prediction of the other EWPOs using the measured value of $\boldsymbol{\sin^2\theta_w^{eff}}$, since this is an efficient way to eliminate their dependence on $\Delta\alpha$

4. in any case it is absolutely worthwhile to measure the W mass down to ~200 keV precision, the Z width down to 4 keV