Effective Lagrangian for the macroscopic motion of fermionic matter

Presentation at the 13th International Conference on New Frontiers in Physics, Crete, Greece

By

M.A. Zubkov

This work was carried out at Ariel University in collaboration with

Maik Selch, Dr. Ruslan Abramchuk

Zubarev statistical operator

relativistically covariant formulation of the statistical operator:

- spacetime possesses foliation into a family of spacelike hypersurfaces Σ_σ parametrized by "time" σ
- assumption : hydrodynamical approximation $\leftrightarrow$ local equilibrium in small vicinity of each point

The Zubarev statistical operator is constructed from conserved currents which characterize the system macroscopically. *D. N. Zubarev, A. V. Prozorkevich, and S. A. Smolyanskii, Derivation of nonlinear generalized equations of quantum relativistic hydrodynamics,"Theoretical and Mathematical Physics 40 no. 3, (Sep, 1979) 821-831. https://doi.org/10.1007/BF01032069.*

Density operator in equilibrium is expressed through the conserved quantities. Stress - energy tensor $\hat{T}^{\mu\nu} \equiv \hat{T}^{\mu\nu}_{BR}$ (Belifante - Rosenfeld, or symmetrized)

$$\hat{\rho} = \frac{1}{Z} exp(-\int_{\Sigma_\sigma} d\Sigma_\sigma n_\mu (\hat{T}^{\mu\nu}(x)\beta_\nu(x) - \hat{j}^\mu(x)\zeta(x)))$$

$$Tr(\hat{\rho}) = 1, \text{ timelike } \beta_\mu = \beta u_\mu, \; u_\mu u^\mu = 1, \; \zeta = \beta\mu$$

The partition function

$$Z = \mathrm{Tr}\, exp(-\int_{\Sigma_\sigma} d\Sigma_\sigma n_\mu (\hat{T}^{\mu\nu}(x)\beta_\nu(x) - \hat{j}^\mu(x)\zeta(x)))$$

In equilibrium $\frac{d\hat{\rho}}{d\sigma} = 0$.

Sufficient condition for equilibrium:

$$\frac{d\hat{\rho}}{d\sigma} = 0, \ \log \hat{\rho} = -log(Z) - \int d\Sigma_\sigma \beta n^\mu g_{\mu\nu} (\hat{T}^{\nu\rho} u_\rho - \sum_i \mu_i \hat{j}_i^\nu)$$

$$\Rightarrow 0 = \partial_\nu (\hat{T}^{\nu\rho} \beta u_\rho - \sum_i \hat{j}_i^\nu \beta \mu_i) = \hat{T}^{\nu\rho} \partial_\nu (\beta u_\rho) - \sum_i \hat{j}_i^\nu \partial_\nu (\beta \mu_i)$$

(operators vanish at spatial infinity, currents are conserved)
solution:

$$\beta \mu_i = \zeta_i = \mathrm{const.}, \ \beta_\rho = \beta u_\rho = b_\rho + \omega_{\rho\sigma} x^\sigma, \ b_\rho, \omega_{\rho\sigma} = \mathrm{const.}$$

Zubarev statistical operator in the presence of macroscopic motion

$$\hat{\rho} = \frac{1}{Z} exp(-\int_{\Sigma_\sigma} d\Sigma_\sigma n_\mu (\hat{T}^{\mu\nu}(x) \beta_\nu(x) - \hat{j}^\mu(x) \zeta(x)))$$

define the charge operators

$$\hat{P}^\mu = \int d\Sigma n_\nu \hat{T}^{\nu\mu}, \quad \hat{Q}_i = \int d\Sigma n_\nu \hat{j}_i^\nu,$$
$$\hat{M}^{\nu\mu} = \int d\Sigma n_\rho (\hat{T}^{\rho\mu} x^\nu - \hat{T}^{\rho\nu} x^\mu) = \epsilon^{\nu\mu\rho\sigma} \hat{J}^\rho u^\sigma + \hat{K}^\mu u^\nu - \hat{K}^\nu u_\mu$$

introduce the linear velocity, acceleration and vorticity

$$v_\mu = \frac{1}{\beta} b_\mu, \quad a_\mu = \frac{1}{\beta} \omega_{\mu\nu} u^\nu, \quad \omega_\mu = -\frac{1}{2\beta} \epsilon_{\mu\nu\rho\sigma} u^\nu \omega^{\rho\sigma}$$
$$\Leftrightarrow \omega_{\mu\nu} = \beta(\epsilon_{\mu\nu\rho\sigma} \omega^\rho u^\sigma + a_\mu u_\nu - a_\nu u_\mu)$$

the Zubarev statistical operator may be written as

$$\hat{\rho} = \frac{1}{Z} e^{-\beta(v_\mu \hat{P}^\mu + a_\mu \hat{K}^\mu - \omega_\mu \hat{J}^\mu - \sum_i \mu_i \hat{Q}_i)}$$

Fermions coupled to non-Abelian gauge bosons

QCD: fermions (quarks) coupled to $SU(3)$ gauge bosons (gluons) in the fundamental representation
$\rightarrow$derivation of the effective Lagrangian for general macroscopic motion in GTE for quarks and gluons
Dirac Lagrangian + gauge field Lagrangian:

$$\mathcal{L} = \overline{\Psi}(x)(\frac{i}{2}\gamma^\mu \overleftrightarrow{D}_\mu - m)\Psi(x) - \frac{1}{2}Tr(F_{\mu\nu}F^{\mu\nu})$$

$$D_\mu = \partial_\mu - igA_\mu,\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

Dirac field and gauge field BR energy momentum tensors:

$$T_F^{\mu\nu}(x) = \frac{i}{4}\overline{\Psi}(x)(\gamma^\mu \overleftrightarrow{D}^\nu + \gamma^\nu \overleftrightarrow{D}^\mu)\Psi(x),\ \overleftrightarrow{D}_\mu = \overrightarrow{D}_\mu - \overleftarrow{D}_\mu$$

$$T_B^{\mu\nu}(x) = 2Tr(F^{\rho\mu}(x)F^\nu{}_\rho(x) + \frac{1}{4}g^{\mu\nu}(x)F_{\rho\sigma}(x)F^{\rho\sigma}(x))$$

Partition function expressed through the functional integral

We fix one particular hypersurface Σ at $t = 0$ with $n = (1, 0, 0, 0)$.
The partition function

$$\mathcal{Z}[1] = \mathrm{Tr}\exp\Big(- \int d\Sigma \beta n_\nu (\hat{T}^{\nu\rho} u_\rho - \sum_i \mu_i \hat{j}_i^\nu)\Big) = \mathrm{Tr}\exp\Big(- \mathbf{H}\Big)$$

define a Hamiltonian comprising macroscopic motion

$$\mathbf{H} := \int d\Sigma \beta n_\nu (\hat{T}^{\nu\rho} u_\rho - \sum_i \mu_i \hat{j}_i^\nu), \;\; d\Sigma = d^3 x$$

With the aid of the coherent state system (for fermions) and the systems of eigenstates of $\hat{A}$, $\hat{\Pi}$ (for gauge field) we obtain

$$\mathcal{Z}[h] = \mathrm{Tr}\exp\Big(- \mathbf{H}\, h\Big) = \int D\bar{\phi} D\phi D\mathcal{A}_\mu \, e^{\int_\Sigma d^3x \, \int_0^h d\tau L(\bar{\phi}(\vec{x},\tau),\phi(\vec{x},\tau),\mathcal{A}(\vec{x},\tau))}$$

Non - interacting fermions

Divide the interval $(0, h)$ into $N \to \infty$ pieces with length δ

$$\exp\Big(- \mathbf{H}\, h\Big) = \exp\Big(- \mathbf{H}\, \delta\Big) \times 1_\phi \times \exp\Big(- \mathbf{H}\, \delta\Big) \times 1_\phi \times ... \times \exp\Big(- \mathbf{H}\, \delta$$

with

$$1_\phi = \int D\bar{\psi} D\psi e^{-\int d\Sigma n_\mu \bar{\psi}\gamma^\mu \psi} |\psi\rangle\langle\psi|$$

Here

$$|\phi\rangle = e^{\int d\Sigma n_\mu \hat{\Psi}^\dagger \gamma^0 \gamma^\mu \phi} |\Omega\rangle, \quad \langle\tilde{\phi}| = \langle\Omega|\, e^{\int d\Sigma n_\mu \tilde{\phi}\gamma^0\gamma^\mu \hat{\Psi}}$$

$|\Omega\rangle$ is the Fock space state, in which all one - particle states are vacant, while all anti - particle states are occupied.

$$\mathcal{Z}[h] = \mathrm{Tr}\exp\Big(- \mathbf{H}\, h\Big) = \int D\bar{\phi} D\phi \, e^{\int_\Sigma d^3x \, \int_0^h d\tau L_\phi(\bar{\phi}(\vec{x},\tau),\phi(\vec{x},\tau))}$$

Non - interacting fermions

$$\mathcal{Z}[h] = \mathrm{Tr}\exp\Big(- \mathbf{H}\, h\Big) = \int D\bar{\phi} D\phi\, e^{\int_\Sigma d^3x\, \int_0^h d\tau L_\phi(\overline{\phi}(\vec{x},\tau),\phi(\vec{x},\tau))}$$

The Euclidean lagrangian

$$\begin{aligned} L(\overline{\phi},\phi) = {} & -\overline{\phi}\frac{1}{2}\gamma^0 \overleftrightarrow{\partial}_\tau \phi + \beta(0,\vec{x})\Big(\sum_i \mu_i j^0_i \\ & - u_0(0,\vec{x}) m \overline{\phi}\phi - u^k(0,\vec{x})\overline{\phi} i \gamma^j \overleftarrow{\partial}_j \gamma^0 \frac{1}{4}\gamma_k \phi \\ & + u^k(0,\vec{x})\overline{\phi}\frac{1}{4}\gamma_k\gamma^0 i\gamma^j \overrightarrow{\partial}_j \phi - \overline{\phi}\frac{i}{4}\gamma_0 \overleftrightarrow{\partial}_j \phi u^j(0,\vec{x}) \\ & - u^0(0,\vec{x})\overline{\phi} i \gamma^j \overleftarrow{\partial}_j \frac{1}{2}\phi + u^0(0,\vec{x})\overline{\phi}\frac{1}{2} i\gamma^j \overrightarrow{\partial}_j \phi\Big) \end{aligned}$$

Pure gauge field

Hamiltonian density in the gauge $\hat{\mathcal{A}}_0 = 0$:

$$\hat{\mathcal{H}} = \frac{1}{2}\left(\hat{\Pi}_i^a \hat{\Pi}_i^a + \hat{\mathcal{B}}_i^a \hat{\mathcal{B}}_i^a\right)$$

vectors of physical Hilbert space are identified with the complex - valued wave functionals $\Phi[A]$: $\hat{\mathcal{A}}^{ai}(x)\Phi_\sigma[A] = A^{ai}(x)\Phi_\sigma[A]$, $\hat{\Pi}_i^a(x)\Phi_\sigma[A] = -i\frac{\delta}{\delta A^{ai}(x)}\Phi_\sigma[A]$, $x \in \Sigma_\sigma$. Operator $\hat{\Pi}_i$ obeys canonical commutation relations with $\hat{\mathcal{A}}$:

$$[\hat{\Pi}_i^a(x), \hat{\mathcal{A}}^{jb}(y)] = -i\delta^{ab}\delta_i^j\delta^{(3)}(x-y)\Big|_{x,y\in\Sigma_\sigma}$$

We identify $\Phi_\sigma[A^i(t,\vec{x})] \equiv \Phi_{\sigma'}[A^i(t',\vec{x})]$. Gauss constraint

$$\hat{G}^a\Phi[A] = D_i\hat{\Pi}^{ai}\Phi[A] = 0$$

is generator of space - dependent gauge transformations.

Pure gauge field

Eigenstates of $\hat{\mathcal{A}}$ and $\hat{\Pi}$: $\langle A|A'\rangle = \delta[A - A']$, $\langle \Pi|\Pi'\rangle = \delta[\Pi - \Pi']$

$$\langle A|\Pi\rangle = \frac{1}{\sqrt{\mathrm{Vol}}} e^{i\int d\Sigma A^{ai}(x)\Pi^a_i(x)}$$

Here Vol is divergent constant from

$$\frac{1}{\mathrm{Vol}} \int D_{x\in\Sigma}\Pi \, e^{i\int d\Sigma \Pi^a_i(x) A^{ai}(x)} = \delta[A]$$

The functional delta - function: for any functional Φ

$$\int D_{x\in\Sigma} A_1 \, \Phi[A_1] \, \delta(A_1 - A_2) = \Phi[A_2]$$

Next, we define

$$\langle \Pi|\hat{\mathcal{H}}|A\rangle = h(\Pi, A) \, \langle \Pi|A\rangle \quad (1)$$

Direct calculation gives $h(\Pi, A) = \frac{1}{2}\left(\Pi^a_i \Pi^a_i + B^a_i B^a_i\right)$.

Pure gauge field

Divide the interval $(0, h)$ into $N \to \infty$ pieces with length δ

$$\exp\Big(- \mathbf{H}\, h\Big) = 1_\Pi \times \exp\Big(- \mathbf{H}\, \delta\Big) \times 1_\mathcal{A} \times 1_\Pi \times \exp\Big(- \mathbf{H}\, \delta\Big) \times 1_\mathcal{A} \times$$
$$...1_\mathcal{A} \times 1_\Pi \times \exp\Big(- \mathbf{H}\, \delta\Big) \times 1_\mathcal{A}$$

with

$$1_\mathcal{A} = \int D\mathcal{A}\, |\mathcal{A}\rangle\, \langle \mathcal{A}|\,, \quad 1_\Pi = \int D\Pi\, |\Pi\rangle\, \langle \Pi|$$

Then

$$\mathcal{Z}[h] = \mathrm{Tr}\, \exp\Big(- \mathbf{H}\, h\Big) = \int D\mathcal{A} D\Pi\, e^{\int_\Sigma d^3x\, \int_0^h d\tau L_{\Pi,\mathcal{A}}(\mathcal{A}(\vec{x},\tau),\Pi(\vec{x},\tau))}$$

Pure gauge field

$$\mathcal{Z}[h] = \mathrm{Tr}\exp\Big(- \mathbf{H}\, h\Big) = \int D\mathcal{A} D\Pi \, e^{\int_\Sigma d^3x \, \int_0^h d\tau L_{\Pi,\mathcal{A}}(\mathcal{A}(\vec{x},\tau),\Pi(\vec{x},\tau))}$$

$$L_{\Pi,\mathcal{A}}(A,\Pi) = \Pi^{ak} i \frac{\partial}{\partial \tau} A^a_k - \beta(0,\vec{x}) T_{0\mu}[A,\Pi] u^\mu(0,\vec{x})$$

Here

$$T_{\mu\nu}(x) = F^{a\rho}_{\ \ \mu}(x) F^a_{\nu\rho}(x) + \frac{1}{4} g_{\mu\nu}(x) F^a_{\rho\sigma}(x) F^{a\rho\sigma}(x)$$
$$= \delta^0_\mu \delta^0_\nu (\frac{1}{2}\Pi^{ai}\Pi^{ai} + \frac{1}{2} B^a_i B^a_i) + (\delta^j_\mu \delta^0_\nu + \delta^0_\mu \delta^j_\nu)\epsilon_{ijk}\Pi^{ai} B^a_k$$
$$+ \frac{1}{2}(\delta^i_\mu \delta^j_\nu + \delta^j_\mu \delta^i_\nu)(\frac{1}{2}\delta_{ij} B^a_k B^a_k - B^a_i B^a_j + \frac{1}{2}\delta_{ij}\Pi^{ak}\Pi^{ak} - \Pi^{ai}\Pi^{aj}).$$

with identification $\Pi^{ai} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^a_i)} = \partial_0 A^a_i$ and $B^a_i = -\frac{1}{2}\epsilon_{ijk} F^a_{jk}$.

Fermions + gauge field

Divide the interval $(0, h)$ into $N \to \infty$ pieces with length δ

$$\exp\Big(- \mathbf{H}\, h\Big) = 1_\phi \otimes 1_\Pi \times \exp\Big(- \mathbf{H}\, \delta\Big) \times 1_\phi \otimes 1_{\mathcal{A}} \times 1_\Pi \times$$

$$\times \exp\Big(- \mathbf{H}\, \delta\Big) \times 1_\phi \otimes 1_{\mathcal{A}} \times \ldots \times 1_\phi \otimes 1_{\mathcal{A}} \times 1_\Pi \times \exp\Big(- \mathbf{H}\, \delta\Big) \times 1_{\mathcal{A}}$$

with

$$1_{\mathcal{A}} = \int D\mathcal{A}\, |\mathcal{A}\rangle \, \langle \mathcal{A}| \,, \; 1_\Pi = \int D\Pi\, |\Pi\rangle \, \langle \Pi|$$

$$1_\phi = \int D\bar{\psi} D\psi e^{-\int d\Sigma n_\mu \bar{\psi}\gamma^\mu \psi} |\psi\rangle\langle\psi|$$

Then

$$\mathrm{Tr} \exp\Big(- \mathbf{H}\, h\Big) = \int D\bar{\phi} D\phi \, D\mathcal{A} D\Pi \, e^{\int_\Sigma d^3x \int_0^h d\tau L(\bar{\phi}(\vec{x},\tau),\phi(\vec{x},\tau),\mathcal{A}(\vec{x},\tau),\Pi(\vec{x},\tau))}$$

Fermions + gauge field

$$\mathrm{Tr}\exp\Big(-\mathbf{H}\,h\Big) = \int D\bar{\phi}D\phi\, D\mathcal{A}D\Pi\, e^{\int_\Sigma d^3x\ \int_0^h d\tau L(\bar{\phi}(\vec{x},\tau),\phi(\vec{x},\tau),\mathcal{A}(\vec{x},\tau),\Pi(\vec{x},\tau))}$$

$$\begin{aligned} L(\bar{\phi},\phi,A,\Pi) &= -\bar{\phi}\frac{1}{2}\gamma^0\overleftrightarrow{\partial}_\tau\phi + \beta(0,\vec{x})\Big(\sum_i \mu_i j^0_i - u_0(0,\vec{x})m\bar{\phi}\phi \\ &- u^k(0,\vec{x})\bar{\phi}i\gamma^j\overleftarrow{D}_j\gamma^0\frac{1}{4}\gamma_k\phi + u^k(0,\vec{x})\bar{\phi}\frac{1}{4}\gamma_k\gamma^0 i\gamma^j\overrightarrow{D}_j\phi - \bar{\phi}\frac{i}{4}\gamma_0\overleftrightarrow{D}_j\phi u^j(0,\vec{x}) \\ &- u^0(0,\vec{x})\bar{\phi}i\gamma^j\overleftarrow{D}_j\frac{1}{2}\phi + u^0(0,\vec{x})\bar{\phi}\frac{1}{2}i\gamma^j\overrightarrow{D}_j\phi\Big) \\ &+ \Pi^{ak}i\frac{\partial}{\partial\tau}A^a_k - \beta(0,\vec{x})\,T_{0\mu}[A,\Pi]u^\mu(0,\vec{x}) \end{aligned}$$

The covariant derivatives here act on ϕ and $\bar{\phi}$ only.

From Euclidean functional integral to the functional integral in Minkowski space - time

$$\mathcal{Z}[h] = \int D\bar{\phi} D\phi D\mathcal{A} D\Pi \, e^{\int_\Sigma d^3x \int_0^h d\tau L(\bar{\phi}(\vec{x},\tau),\phi(\vec{x},\tau),\mathcal{A}(\vec{x},\tau),\Pi(\vec{x},\tau))}$$

In the same way we obtain also $\mathcal{Z}[ih] = \exp\Big(- i\mathbf{H}\, h\Big)$:

$$\exp\Big(- i\mathbf{H}\, h\Big) = \int D\bar{\phi} D\phi D\mathcal{A} D\Pi \, e^{i \int_\Sigma d^3x \int_0^h dw \mathcal{L}(\bar{\phi}(\vec{x},w),\phi(\vec{x},w),\mathcal{A}(\vec{x},w),\Pi(\vec{x},w))}$$

Rescaling of *emergent* "time"variable w:

$$t = w\mathfrak{B}(\vec{x}), \;\; \psi(\vec{x},t) = \phi(\vec{x}, t/\mathfrak{B}(\vec{x})), \;\; A(\vec{x},t) = \mathcal{A}(x, t/\mathfrak{B}(\vec{x}))$$

In the new "time"variable

$$\mathcal{Z}[ih] = \int D\bar{\psi} D\psi D A D\Pi \, e^{i \int_\Sigma d^3x \int_0^{h\mathfrak{B}(\vec{x})} dt \mathcal{L}(\bar{\psi}(\vec{x},t),\psi(\vec{x},t),A(\vec{x},t),\Pi(\vec{x},w))}$$

From Euclidean functional integral to the functional integral in Minkowski space - time

$$\mathcal{Z}[ih] = \int D\bar{\psi} D\psi DAD\Pi \, e^{i\int_\Sigma d^3x \int_0^{h\mathfrak{B}(\vec{x})} dt \mathcal{L}(\overline{\psi}(\vec{x},t),\psi(\vec{x},t),A(\vec{x},t),\Pi(\vec{x},w))}$$

with $\mathfrak{U}(\vec{x}) = \frac{\beta(0,\vec{x})}{\mathfrak{B}(\vec{x})} u(0,\vec{x})$:

$$\mathcal{L}(\overline{\psi},\psi,A,\Pi) = \mathfrak{U}^0(\overline{\psi}(\gamma^\mu \frac{i}{2}\overset{\leftrightarrow}{D}_\mu - m)\psi) + (1-\mathfrak{U}_0)(\overline{\psi}\gamma^0 \frac{i}{2}\overset{\leftrightarrow}{D}_0\psi)$$
$$+ \sum_i \mu_i j^0_i + \mathfrak{U}_k(\overline{\psi}\gamma^0(\frac{i}{8}[\gamma^j,\gamma^k](\overset{\leftarrow}{D}_j + D_j) - \frac{i}{2}\overset{\leftrightarrow}{D}^k)\psi) +$$
$$+ \Pi^{ak}E^a_k - \mathfrak{U}_0(\frac{1}{2}\Pi^{ai}\Pi^{ai} + \frac{1}{2}B^a_i B^a_i) - \epsilon_{ijk}\Pi^{ai}B^a_k\mathfrak{U}^j$$

Integration over Π may be performed.

$$\mathcal{Z}[ih] = \int D\bar{\psi} D\psi DA_\mu \, e^{i \int d^4x \, \mathcal{L}(\overline{\psi},\psi,A)}$$

integration region: $\Sigma_0(hyperplane) \to \Sigma_h = \{(h\mathfrak{B}(\vec{x}), \vec{x}) | \vec{x} \in \Sigma\}$
introduce the new variable $\mathfrak{U}^\mu(\vec{x}) = \frac{\beta(0,\vec{x})}{\mathfrak{B}(\vec{x})} u^\mu(0, \vec{x})$
two convenient choices for the scaling function $\mathfrak{B}(\vec{x})$:

- $\mathfrak{B}(\vec{x}) = \beta(0, \vec{x}) \Rightarrow \mathfrak{U}^\mu(\vec{x}) = u^\mu(0, \vec{x})$
- $\mathfrak{B}(\vec{x}) = \beta(0, \vec{x}) u^0(0, \vec{x}) \Rightarrow \mathfrak{U}(\vec{x}) = u^\mu(0, \vec{x})/u^0(0, \vec{x})$

boundary conditions:

fermions: $\psi(\vec{x}, \mathfrak{B}(\vec{x})h) = -\psi(\vec{x}, 0), \;\; \overline{\psi}(\vec{x}, \mathfrak{B}(\vec{x})h) = -\overline{\psi}(\vec{x}, 0)$
gauge bosons: $A_\mu(\vec{x}, \mathfrak{B}(\vec{x})h) = A_\mu(\vec{x}, 0)$

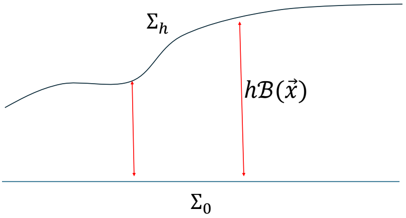
Temporal gauge $A_0 = 0$

Effective Lagrangian comprising macroscopic motion:

$$\begin{aligned}\mathcal{L}(\overline{\psi}, \psi, A) =& \overline{\psi}(\gamma^\mu \frac{i}{2} \overleftrightarrow{D}_\mu - m)\psi + \sum_i \mu_i j_i^0 \\ &+ \mathfrak{U}_k \overline{\psi} \gamma^0 (\frac{i}{8}[\gamma^j, \gamma^k](\overleftarrow{D}_j + D_j) - \frac{i}{2} \overleftrightarrow{D}^k)\psi \\ &+ \frac{1}{2}\Big(E^{ai} E^{ai} - B^{ai} B^{ai}\Big) \\ &+ \frac{1}{2}(\mathfrak{U}^k \mathfrak{U}_k \delta^{ij} - \mathfrak{U}^i \mathfrak{U}^j) B_i^a B_j^a - \epsilon_{ijk} E_i^a B_j^a \mathfrak{U}^k\end{aligned}$$

$$\mathfrak{B}(\vec{x}) = \beta(t_0, \vec{x}) u^0(t_0, \vec{x}) \Leftrightarrow \mathfrak{U}^\mu(\vec{x}) = \frac{u^\mu(t_0, \vec{x})}{u^0(t_0, \vec{x})}$$

$E_i^a = F_{\mu i}^a n^\mu, \quad B_i^a = -\frac{1}{2} \epsilon_{\mu ijk} F^{ajk} n^\mu, \quad V^0 = V^\mu n_\mu, \ i, j$ spacelike



Integration region

$$\mathcal{Z}[ih] = \int D\bar{\psi} D\psi DA_\mu \, e^{i \int d^4x \, \mathcal{L}(\overline{\psi}, \psi, A)}$$

Gauge invariant formulation

$$\mathcal{L}(\bar{\psi}, \psi, A) = (\mathfrak{U}n)\bar{\psi}(\gamma^{\mu}\frac{i}{2}\overleftrightarrow{D}_{\mu} - m)\psi + (1 - (\mathfrak{U}n))(\bar{\psi}(\gamma n)\frac{i}{2}(n\overleftrightarrow{D})\psi)$$
$$+ \sum_i \mu_i(j_i n) + \mathfrak{U}^{\mu}\Delta_{\mu\rho}\Delta_{\nu\sigma}(\bar{\psi}(\gamma n)\frac{i}{8}[\gamma^{\sigma}, \gamma^{\rho}](\overleftarrow{D}^{\nu} + D^{\nu})\psi)$$
$$- \mathfrak{U}^{\mu}\Delta_{\mu\rho}\bar{\psi}(\gamma n)\frac{i}{2}\overleftrightarrow{D}^{\rho}\psi - \frac{1}{4\mathfrak{U}n}F^{a\mu\nu}F^{a}_{\mu\nu} - \frac{\mathfrak{U}^2 - 1}{4\mathfrak{U}n}F^{a\rho\sigma}F^{a\mu\nu}\Delta_{\mu\rho}\Delta_{\nu\sigma}$$
$$- \frac{1}{8\mathfrak{U}n}(n^{\mu}\epsilon_{\mu\nu\rho\sigma}F^{a\rho\sigma}\mathfrak{U}^{\nu})(n^{\bar{\mu}}\epsilon_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}}F^{a\bar{\rho}\bar{\sigma}}\mathfrak{U}^{\bar{\nu}})$$
$$- \frac{1}{\mathfrak{U}n}F^{a}_{\nu\mu}F^{a\nu\rho}\mathfrak{U}_{\rho}n^{\mu} + F^{a}_{\nu\mu}F^{a\nu\rho}n_{\rho}n^{\mu}$$
$$\mathfrak{U}^{\mu}(\vec{x}) = (\beta(0, \vec{x})/\mathfrak{B}(\vec{x}))u^{\mu}(0, \vec{x}), \quad \Delta_{\mu\nu} = g_{\mu\nu} - n_{\mu}n_{\nu}$$

Rotation (pure gauge field)

1. "Passive" rotation: the system is at rest in the rotating reference frame:

$$S(g^R_{\mu\nu}, A_\mu) = \int d^4x \mathcal{L}(g^R_{\mu\nu}, A_\mu),\ \mathcal{L}(g^R_{\mu\nu}, A_\mu) = -\frac{1}{4} F^a_{\mu\rho} g^{\mu\nu}_R g^{\rho\sigma}_R F^a_{\nu\sigma}$$

2. "Active" rotation: Zubarev operator with corresponding β, u:

$$S(\eta_{\mu\nu}, A_\mu, u^\mu) = \int d^4x \mathcal{L}(\eta_{\mu\nu}, A_\mu, u^\mu),$$

$$\begin{aligned}\mathcal{L}(\eta_{\mu\nu}, A_\mu, u^\mu) = &- \frac{1}{4\mathfrak{U}n} F^{a\mu\nu} F^a_{\mu\nu} - \frac{\mathfrak{U}^2 - 1}{4\mathfrak{U}n} F^{a\rho\sigma} F^{a\mu\nu} \Delta_{\mu\rho}\Delta_{\nu\sigma} \\ &- \frac{1}{8\mathfrak{U}n} (n^\mu \epsilon_{\mu\nu\rho\sigma} F^{a\rho\sigma} \mathfrak{U}^\nu)(n^{\bar\mu} \epsilon_{\bar\mu\bar\nu\bar\rho\bar\sigma} F^{a\bar\rho\bar\sigma} \mathfrak{U}^{\bar\nu}) \\ &- \frac{1}{\mathfrak{U}n} F^a_{\nu\mu} F^{a\nu\rho} \mathfrak{U}_\rho n^\mu + F^a_{\nu\mu} F^{a\nu\rho} n_\rho n^\mu\end{aligned}$$

Rotation (pure gauge field)

Insert the following background data:

1. "Passive"rotation:

$$ds^2 = g^R_{\mu\nu} dx^\mu dx^\nu$$
$$= (1 - \omega^2(x^2 + y^2))dt^2 + 2\omega y dt dx - 2\omega x dt dy - dx^2 - dy^2 - dz^2$$

2. "Active" rotation:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2$$

$$\mathfrak{U}^\mu(\vec{x}) = \frac{\beta(0,\vec{x})}{\mathfrak{B}(\vec{x})} u^\mu(0,\vec{x}), \quad \mathfrak{B}(\vec{x}) = \beta(0,\vec{x}) u^0(0,\vec{x})$$

$$u^\mu = \frac{1}{\sqrt{1 - \omega^2(x^2 + y^2)}}(1, -\omega y, \omega x, 0) \Rightarrow \mathfrak{U}^\mu = (1, -\omega y, \omega x, 0)$$

$$\Rightarrow S(g^R_{\mu\nu}, A_\mu) \equiv S(\eta_{\mu\nu}, A_\mu, u^\mu)$$

Accelerated motion

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2$$

$$\mathfrak{U}^\mu(\vec{x}) = \frac{\beta(t_0, \vec{x})}{\mathfrak{B}(\vec{x})} u^\mu(t_0, \vec{x}), \quad \mathfrak{B}(\vec{x}) = \beta(t_0, \vec{x}) u^0(t_0, \vec{x})$$

$$u^\mu(t_0, \vec{x}) = \frac{1}{\sqrt{(1+ax)^2 - at^2}} (1 + ax, at, 0)\Big|_{t=t_0} \Rightarrow \mathfrak{U}^\mu = (1, 0, 0, 0)$$

In the effective Lagrangian only the spatial components of $\mathfrak{U}$ enter non-trivially! (for this choice of $\mathfrak{B}$)
$\Rightarrow$ the only effect if acceleration is manifested through the scaling function ($\leftrightarrow$ spacetime dependent temperature)

$$\mathfrak{B}(\vec{x}) = \beta(0, \vec{x}) u^0(0, \vec{x}) = \beta(0, \vec{x}) = \beta_0(1 + ax)$$

Freedom in the choice of the function $\mathfrak{B}(\vec{x})$

Freedom in the choice of the function $\mathfrak{B}(\vec{x}) \Leftrightarrow$ dependence of the integration measure on reparametrizations $t \to t\mathfrak{B}(\vec{x}) \Leftrightarrow$ anomaly in $D_\mu T^{\mu\nu}$ (absent in $4D$)

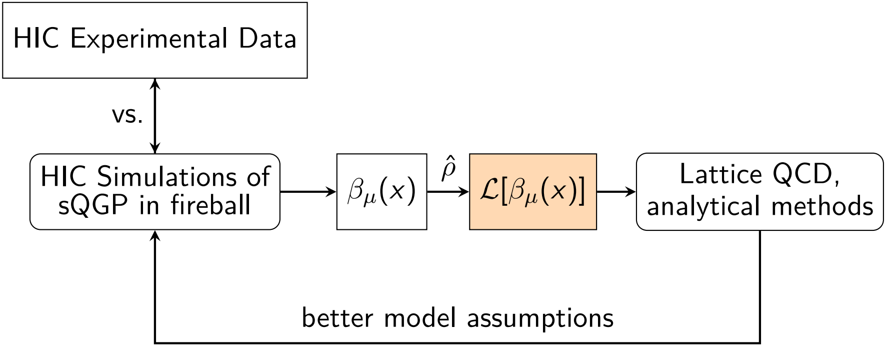
$$\mathcal{Z}[ih] = \int D\bar{\psi} D\psi DA_\mu \, e^{i\int d^4x \, \mathcal{L}(\bar{\psi},\psi,A)}$$

integration region: $\Sigma_0(hyperplane) \to \Sigma_h = \{(h\mathfrak{B}(\vec{x}), \vec{x}) | \vec{x} \in \Sigma\}$
introduce the new variable $\mathfrak{U}^\mu(\vec{x}) = \frac{\beta(0,\vec{x})}{\mathfrak{B}(\vec{x})} u^\mu(0, \vec{x})$

boundary conditions:

fermions: $\psi(\vec{x}, \mathfrak{B}(\vec{x})h) = -\psi(\vec{x}, 0)$, $\bar{\psi}(\vec{x}, \mathfrak{B}(\vec{x})h) = -\bar{\psi}(\vec{x}, 0)$
gauge bosons: $A_\mu(\vec{x}, \mathfrak{B}(\vec{x})h) = A_\mu(\vec{x}, 0)$

quark - gluon plasma in Heavy Ion Collisions



Extract $\beta_\mu(x)$ from the models of fireball $\rightarrow$ Lagrangian $\mathcal{L}[\beta_\mu(x)]$ $\rightarrow$ simulations and analytical calculations

Conclusions

- Zubarev statistical operator allows to describe the equilibrium in the presence of macroscopic motion (straight motion, rotation, acceleration)
- The effective lagrangian for description of such a system is derived
- In hydrodynamic approximation we can think of each coarse grain of the system as equilibrium. The obtained Lagrangian may then be used with arbitrary profile of macroscopic velocity and local temperature
- Applications to quark - gluon plasma simulations are possible

$^3He - A$

consider superfluid $^3He - A$ and repeat the just mentioned steps within the Zubarev statistical operator approach
preliminaries:

1. "normal" liquid ^3He is a Fermi liquid featuring a Fermi surface
2. it undergoes a phase transition (in case of appropriate external conditions - pressure and temperature) by spontaneous symmmetry breaking
3. the superfluid component of $^3He - A$ provides the vacuum background which is coupled to the normal component of excitations
4. difficulty: space and time dependent matrix-valued vierbein, the order parameter of the phase transition, in an emergent relativistic theory of chiral fermions

- starting point: effective Lagrangian of superfluid $^3He - A$ without macroscopic motion, slowly varying vierbein and thereby small values of the superfluid velocity

^3He without spin orbit interaction ($G = U(1) \times SO(3)^L \times SO(3)^S$):

$$S = \sum_{p,s} \bar{a}_s(p)\epsilon(p)a_s(p) - \frac{g}{\beta V} \sum_{p;i,\alpha=1,2,3} \bar{J}_{i\alpha}(p)J_{i\alpha}(p)$$

$$p = (\omega, k), \hat{k} = \frac{k}{|k|}, \ \epsilon(p) = i\omega - (\frac{k^2}{2M_3} - \mu) \approx i\omega - v_F(|k| - k_F),$$

$$J_{i\alpha}(p) = \frac{1}{2} \sum_{p_1+p_2} (\hat{k}_1^i - \hat{k}_2^i)a_A(p_2)[\sigma_\alpha]_B^C a_C(p_1)\epsilon^{AB}, \ \ \epsilon^{-+} = -\epsilon^{+-} = 1$$

requirement for validity of Taylor expansion around Fermi surface:

$$\frac{(\pm|k| \mp k_F)^2}{2M_3} \ll v_\perp |(\pm|k| \mp k_F)|$$

$\Rightarrow$ typical length scales a and time scales τ:

$$a \sim (|k| - k_F)^{-1} \gg \frac{v_F}{v_\perp k_F}, \quad \tau \gg \frac{1}{v_\perp k_F}$$

- carry out bosonization with complex bosonic fields $A_{i\alpha}$
- apply spontaneous symmetry breaking prescription for $^3He - A$ (specification of the form of $A_{i\alpha}$)

superfluid $^3He - A$-phase:

$$A_{i\alpha} = \sqrt{\beta V}\Delta_0(\mathbf{m}_i - i\mathbf{n}_i)d_\alpha = \sqrt{\beta V}k_F v_\perp(\mathbf{m}_i - i\mathbf{n}_i)d_\alpha, \quad i,\alpha = 1,2,3$$

with frame fields

$$\mathbf{d}\cdot\mathbf{d} = \mathbf{m}\cdot\mathbf{m} = \mathbf{n}\cdot\mathbf{n} = 1, \quad \mathbf{m}\cdot\mathbf{n} = 0, \quad \mathbf{l} = \mathbf{m}\times\mathbf{n}$$

introduce Nambu Gorkov spinors:

$$\Psi(p) = \begin{pmatrix} \chi^A(p) \\ \epsilon^{BA}\overline{\chi}^B(-p) \end{pmatrix} = \begin{pmatrix} a_+(p) \\ a_-(p) \\ \overline{a}_-(-p) \\ -\overline{a}_+(-p) \end{pmatrix}$$

- expansion around the two Fermi points yields an effective Lagrangian which is relativistically invariant near the Fermi points $K^i_{R,L} = K^i_{\pm} = \pm k_F l^i$ we define

$$\psi_R(p) = \Psi(K_+ + p) = \begin{pmatrix} \chi(K_+ + p) \\ -\chi^C(K_- - p) \end{pmatrix},$$

$$\psi_L(p) = \tau^3\Psi(K_- + p) = \begin{pmatrix} \chi(K_- + p) \\ \chi^C(K_+ - p) \end{pmatrix}, \ \mathbf{e}^\mu_a = \mathbf{e}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & v_\perp \mathbf{m}(\mathbf{d}\boldsymbol{\sigma}) \\ 0 & v_\perp \mathbf{n}(\mathbf{d}\boldsymbol{\sigma}) \\ 0 & v_\parallel \mathbf{l} \end{pmatrix}$$

with $\chi^C = -i\sigma^2\chi^* \Leftrightarrow \psi_R(p) = i\tau^1\sigma^2\psi^*_L(-p)$

$$S_{eff} = \frac{1}{4}\int d^4x\mathbf{e}[\overline{\psi}_L i\mathbf{e}^\mu_b(x)\overline{\tau}^b\nabla_\mu\psi_L - [\nabla_\mu\overline{\psi}_L]i\mathbf{e}^\mu_b(x)\overline{\tau}^b\psi_L + \overline{\psi}_R i\mathbf{e}^\mu_b(x)\tau^b\nabla_\mu\psi_R - [\nabla_\mu\overline{\psi}_R]i\mathbf{e}^\mu_b(x)\tau^b\psi_R]$$

split off the spin space dependence and introduce Dirac spinors:

$$\mathbb{P}^s = \frac{1 + s(\mathbf{d}\boldsymbol{\sigma})}{2} \text{ with eigenspinors } \eta^s = \frac{1}{\sqrt{2(1 - sd_3)}} \begin{pmatrix} -s(d_1 - id_2) \\ sd_3 - 1 \end{pmatrix}$$

$$\psi_{L/R} = \sum_{s=\pm} \psi^s_{L/R}, \quad \mathbb{P}^s \psi_{L/R} = \psi^s_{L/R} = \Psi^s_{L/R} \otimes \eta^s$$

$$\psi^s = (\psi^s_L, \psi^s_R)^T, \quad \Psi^s = (\Psi^s_L, \Psi^s_R)^T, \quad \mathbb{P}^s \psi = \psi^s = \Psi^s \otimes \eta^s$$

$$\Psi = \left(\Psi^+, \quad e^{\frac{\pi}{4}[\gamma^1, \gamma^2]} \Psi^-\right)^T, \quad e^\mu_a = e^{-1} \begin{pmatrix} 1 & 0 \\ 0 & v_\perp \mathbf{m} \\ 0 & v_\perp \mathbf{n} \\ 0 & v_\parallel \mathbf{l} \end{pmatrix}$$

$$S_{eff} = \frac{1}{4}\int d^4xe[\overline{\Psi}i\gamma^0\gamma^b e^\mu_b D_\mu\Psi - [\overline{\Psi}\overleftarrow{D}_\mu]i\gamma^0\gamma^b e^\mu_b\Psi]$$

$$D_\mu = \nabla_\mu - i\mathcal{B}_\mu, \quad \mathcal{B}^{rs}_\mu = i(\overline{\eta}^r\nabla_\mu\eta^s)(\delta^{rs}\mathbb{1} + i\epsilon^{rs}\frac{\pi}{8}[\gamma^1,\gamma^2])$$

$$\mathcal{B}_\mu = \begin{pmatrix} b^+_\mu & \frac{1}{8}\omega_{\mu 12}[\gamma^1,\gamma^2] \\ \frac{1}{8}\omega^*_{\mu 12}[\gamma^1,\gamma^2] & b^-_\mu \end{pmatrix}$$

Abelian Berry connections and non-Abelian spin connection:

$$b^s_\mu = i\overline{\eta}^s\nabla_\mu\eta^s, \quad \omega_{\mu 12} = 2\pi i\overline{\eta}^+\nabla_\mu\eta^-$$

- construct the Zubarev statistical operator from (conserved) current densities

$$\log \hat{\rho} = -\alpha - \int d\Sigma n_\mu (\hat{T}^\mu_a B^a - \frac{1}{2}\hat{M}^\mu_{ab}\Omega^{ab} - \zeta_A \hat{j}^\mu_A - \sum_i \zeta_i \hat{j}^\mu_i)$$
$$= -\alpha - \int d\Sigma n_\mu (\hat{T}^\mu_a \beta^a - \frac{1}{2}\hat{S}^\mu_{ab}\Omega^{ab} - \zeta_A \hat{j}^\mu_A - \sum_i \zeta_i \hat{j}^\mu_i)$$

for vanishing gauge field $\mathcal{B} \equiv 0$:

$$\beta^\mu = b^\mu + \omega^\mu{}_\nu x^\nu$$
$$\Omega_{\mu\nu} = \omega_{\mu\nu} + e^a_\mu (\nabla_\lambda e_{a\nu})\beta^\lambda = e^a_\mu (\mathcal{L}_\beta e)_{a\nu} = e^a_\mu [\beta^\lambda \nabla_\lambda e_{a\nu} + e_{a\lambda}\nabla_\nu \beta^\lambda]$$
$$\epsilon^{\mu\nu\rho\sigma}(\mathcal{L}_\beta T)_{\nu\rho\sigma} = 0, \quad T_{\mu\nu\rho} = e_{a\mu} T^a_{\nu\rho} = e_{a\mu}(\nabla_\nu e^a_\rho - \nabla_\rho e^a_\nu)$$

additionally for non-vanishing gauge field $\mathcal{B} \neq 0$:

$$\beta^a \hat{G}_a(\mathcal{B}) = 0, \quad \Omega^{ab}\hat{P}_{ab}(\mathcal{B}) = 0$$