On Entropy Growth in Scattering

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Consider a bipartite quantum system:

Subsystem entropy:

Bipartite quantum system: Subsystem A , Subsystem B

Initial state = initial density matrix ρ

Subsystem (reduced) density matrices

$$\rho^A = \mathrm{Tr}_B[\rho] \ , \quad \rho^B = \mathrm{Tr}_A[\rho]$$

Subsystem entropy

$$\text{von Neumann :} \quad S = -\mathrm{Tr}[\rho^A \ln \rho^A]$$

$$\text{Tsallis :} \qquad S^{(n)} = \frac{1 - \mathrm{Tr}[(\rho^A)^n]}{n-1}$$

$$\lim_{n \to 1} S^{(n)} = S$$

Unitary time evolution:

Weak interaction (lowest orders of perturbation theory are valid)

Initial state: ρ

Final state : $\rho \;\rightarrow\; U\rho U^\dagger$

$$U = 1 + i\mathbf{T}$$

Assume unitarity $\mathbf{T} - \mathbf{T}^\dagger = i\mathbf{T}\mathbf{T}^\dagger$

Study change of subsystem entropy S or $S^{(n)}$

Depends on initial state, ρ and dynamics $\mathbf{T}$.

For which initial states ρ is the subsystem entropy ALWAYS non-decreasing?

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For which initial state ρ is $\delta S \geq 0$ for any possible T?

Clifford Cheung, Temple He, Allic Sivaramakrishnanar,
arXiv:2304.13052v2 [hep-th] Phys.Rev.D 108 (2023) 4, 045013

Prove that $\delta S^{(n)} \geq 0$ (with $S^{(n)}$ the Tsallis entropy and $n \geq 2$)
If

$$\rho = \rho^A \otimes \rho^B$$

then

$$\rho^A \rho^A = \frac{1}{k} \rho^A$$

$$\rho^A = \frac{1}{k} \mathrm{diag}[1, 1, ..., 1, 0, 0, ...]$$

Product state $\rho = \rho^A \otimes \rho^B$, diagonalize reduced density matrices,

$$\rho^A = \sum_a \alpha_a |a >< a| \quad , \quad \rho^B = \sum_b \beta_b |b >< b|$$

Leading contribution to the change of Tsalis entropy (for $n \geq 2$)

$$\delta S^{(n)} = \frac{n}{n-1} \sum_{a,a'} \alpha_a (\alpha_a^{n-1} - \alpha_{a'}^{n-1}) \Gamma_{aa'}$$

$$\Gamma_{aa'} = \sum_{b,b'} \left[\beta_b |\mathbf{T}_{aa'bb'}|^2 - \beta_b \beta_{b'} \mathbf{T}^*_{aa'bb} \mathbf{T}_{aa'b'b'} \right]$$

always positive for any $\mathbf{T}$, β's when

$$\rho^A = \frac{1}{k} \mathrm{diag}[1, 1, ..., 1, 0, 0, ...]$$

"projector state" "democratic ignorance"

C.Cheung et.al., arXiv:2304.13052v2 [hep-th]

Perturbation theory $\mathbf{T} = \sum_{k=1}^{\infty} \lambda^k \mathbf{T}^{(k)}$

For example, in the interaction picture,

$\mathbf{T}^{(k)} = \frac{1}{i}\frac{1}{k!}\mathcal{T}\left(-\frac{i}{\hbar}\int_{-\infty}^{\infty} dt H_I(t)\right)^k$

Time evolution $\rho \quad \rightarrow \quad U\rho U^\dagger = \rho + \delta\rho$

$$\delta\rho = i\lambda\left[\mathbf{T}^{(1)}, \rho\right] + i\frac{\lambda^2}{2}\left[\mathbf{T}^{(2)} + \mathbf{T}^{(2)^\dagger}, \rho\right] - \frac{\lambda^2}{2}\left[\mathbf{T}^{(1)}\left[\mathbf{T}^{(1)}, \rho\right]\right] + \ldots$$

Reduced density matrix $\rho^A \quad \rightarrow \quad \rho^A + \delta\rho^A$

$$\delta\rho^A = \mathrm{Tr}_B\left\{i\lambda\left[\mathbf{T}^{(1)}, \rho\right] + i\frac{\lambda^2}{2}\left[\mathbf{T}^{(2)} + \mathbf{T}^{(2)^\dagger}, \rho\right] - \frac{\lambda^2}{2}\left[\mathbf{T}^{(1)}\left[\mathbf{T}^{(1)}, \rho\right]\right] + \ldots\right\}$$

$$\delta\rho^A = \lambda\delta\rho^{A(1)} + \lambda^2\delta\rho^{A(2)} + \ldots$$

$$\mathrm{Tr}_A[\rho^A + \delta\rho^A] = 1\ ,\ 0 \leq \text{eigenvalues} \leq 1$$

The reduced density matrix has the properties

$$\rho^A = \sum_m \rho^A_m \; |m >< m| \quad , \quad \rho^A_m \geq 0 \quad , \quad \sum_m \rho^A_m = 1$$

$$m \in \ker \rho^A \text{ if } \rho^A_m = 0$$

First order in perturbation theory

$$\delta\rho^A_m = \text{eigenvalues of } \; < m|\mathrm{Tr}_B\left(i\lambda\left[\mathbf{T}^{(1)}, \rho\right]\right)|m' >$$

$$\rightarrow \quad \mathrm{Tr}\left(\mathbf{T}^{(1)}\left([\;\rho\;,\;|m >< m'| \otimes \mathcal{I}^B\;]\right)\right) = 0$$

must non-negative for any $\mathbf{T}^{(1)}$ whatsoever

$$\rightarrow \quad \boxed{\;[\;|m >< m'| \otimes \mathcal{I}^B\;,\;\rho\;] = 0 \quad , \quad \forall m, m' \in \ker \rho^A}$$

for any density matrix, independent of perturbation theory

Linear contribution to shift in entropy

$$\delta S = -\sum_m (\rho_m^A + \delta\rho_m^A)\ln(\rho_m^A + \delta\rho_m^A) + \sum_m \rho_m^A \ln \rho_m^A$$

$$= -\sum_{m \notin \ker \rho^A} \ln \rho_m^A \ \delta\rho_m^{A(1)} + \dots$$

$$= -\sum_{m \notin \ker \rho^A} \ln \rho_m^A < m|\mathrm{Tr}_B \left(\left[i\lambda \mathbf{T}^{(1)}, \rho\right]\right)|m > + \dots$$

$$= -\sum_{m \notin \ker \rho^A} \ln \rho_m^A \ i\lambda \mathrm{Tr}\left(\ \mathbf{T}^{(1)} \ \left[\ |m >< m| \otimes \mathcal{I}^B \ , \ \rho \ \right] \ \right) + \dots$$

is non-negative for any $\mathbf{T}^{(1)}$ whatsoever if $\left[|m >< m| \otimes \mathcal{I}^B, \rho\right] = 0$ $\forall m \notin \ker \rho^A$

To linear order

$\delta S \geq 0$ for any $\mathbf{T}^{(1)}$ requires $\left[\ \rho^A \otimes \mathcal{I}^B \ , \ \rho \ \right] \ = \ 0 \ , \ \ \rho^A = \mathrm{Tr}_B[\rho]$

To linear order

$$\delta S \geq 0 \text{ for any } \mathbf{T}^{(1)} \text{ requires } \left[\rho^A \otimes \mathcal{I}^B \ , \ \rho \right] \ = \ 0$$

Includes any product states $\rho = \rho^A \otimes \rho^B$ and it is a special case of separable state which is defined as any state of the form

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B \quad , \quad 0 < p_i < 1, \ \sum p_i = 1$$

A separable state can be assembled using classical processes alone.

$$\rho^A \otimes \mathcal{I}^B |m, \tilde{m} >= \rho_m^A |m, \tilde{m} >$$

$$\rho|m, \tilde{m} >= \rho_{m,\tilde{m}} |m\tilde{m} >$$

$$\rho = \sum_{m,\tilde{m}} \rho_{m,\tilde{m}} |m, \tilde{m} >< m, \tilde{m}|$$

Second order

$$\delta S \;=\; \lambda^2 \ln \frac{1}{\lambda^2} \sum_{m \in \ker \rho^A} \delta\rho_m^{A(2)} + \dots$$

Since we must have $0 \le \rho_m^A + \lambda\delta\rho_m^{A(1)} + \lambda^2\delta\rho_m^{A(2)} + \dots\;,\ \forall m$
$m \in \ker \rho^A$: Since $\rho_m^A = 0$ and we have already shown that $\delta\rho_m^{A(1)} = 0$ for $m \in \ker\rho^A$ it must be that $\delta\rho_{m\in\ker\rho^A}^{A(2)} \ge 0$.

Conclusion: von Neumann entropy always increases if

$$\ker\rho^A \ne \emptyset \quad \text{and} \quad \left[\, \rho^A \otimes \mathcal{I}^B \;,\; \rho \,\right] \;=\; 0$$

and $\exists \delta\rho_{n\in\ker\rho^A}^{A(2)} \ne 0$**.**
This includes any pure state, any product state and more general separable states.

Example: entropy generated by scattering

Initial pure state: $\rho_{m,\tilde{m}} = \delta^1_m \delta^1_{\tilde{m}}$

$$\delta S = \lambda^2 \ln\left[\frac{1}{\lambda^2}\right] \sum_{m\neq 1,\tilde{m}\neq 1} \left| < m,\tilde{m}|\mathbf{T}^{(1)}|1,1 > \right|^2 + \dots$$

which is proportional to the total transition probability to states other than the initial state with the only constraint that the states of both subsystems must change.

Scattering "area law":

$$\frac{\delta S}{\text{unit time} \times \text{ unit beam flux}} = \lambda^2 \ln\left[\frac{1}{\lambda^2}\right] \times \text{ total cross} - \text{section}$$

S. Seki, I.Y. Park, S.-J. Sin, Phys.Lett. **B 743** (2015) 147-153.
G. Grignani, G. W. S., Phys. Lett. B **772**, 699-702 (2017).

What if the $\lambda^2 \ln \frac{1}{\lambda^2}$ contribution vanishes $\delta\rho_m^{A(2)} = 0 \;\; \forall m \in \ker \rho^A$

Then, the leading contribution is

$$\delta S = \tag{1}$$

$$\frac{\lambda^2}{2} \sum_{\substack{m \\ m'}} \frac{\ln \frac{\rho_m^A}{\rho_{m'}^A}}{\rho_m^A - \rho_{m'}^A} \Bigg\{ \sum_{\substack{\tilde{m} \\ \tilde{m}', \tilde{m}''}} [\rho_{m,\tilde{m}} - \rho_{m',\tilde{m}'}][\rho_{m,\tilde{m}''} - \rho_{m',\tilde{m}''}] \mathbf{T}^{(1)}_{\substack{m\tilde{m} \\ m'\tilde{m}'}} \mathbf{T}^{(1)}_{\substack{m'\tilde{m}' \\ m\tilde{m}}}$$

$$- \sum_{\substack{\tilde{m} \\ \tilde{m}'}} [\rho_{m',\tilde{m}} - \rho_{m,\tilde{m}}][\rho_{m',\tilde{m}'} - \rho_{m,\tilde{m}'}] \mathbf{T}^{(1)}_{\substack{m\tilde{m} \\ m'\tilde{m}}} \mathbf{T}^{(1)}_{\substack{m'\tilde{m}' \\ m\tilde{m}'}} \Bigg\}$$

If $\rho = \rho^A \times \rho^B$, same conclusion as

C.Cheung et.al., arXiv:2304.13052v2 [hep-th] Phys.Rev.D 108 (2023) 4, 045013

$$\rho^A = \frac{1}{k} \mathrm{diag}[1, 1, ..., 1, 0, 0, ...]$$

Thank you!