Hadron Spectroscopy from lattice QCD simulations

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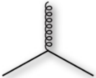
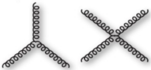
Quantum ChromoDynamics

$$L_{QCD} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \sum_{\alpha=1}^{N_f} \bar{\psi}_\alpha (i\gamma^\mu D_\mu - m_\alpha)\psi_\alpha = L_g[U] + L_q[\bar{\psi}, \psi, U]$$

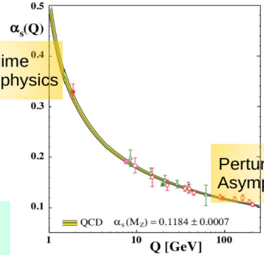
where $D_\mu = \partial_\mu - ig\sum_{i=1}^{8} \lambda^a A^a_\mu$, and $F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$



Quark-gluon vertex



Gluon-gluon vertices.
Not present in QED



Hadronic regime
Nonperturbative physics

Perturbative regime
Asymptotic freedom

Perturbative approaches fail in the hadronic regime.

Nonperturbative approaches required for first principles investigation: **Lattice QCD**

Disclaimer

- Neither an extenseive review, nor a pedagogical survey.
  A collection of selected topics by the speaker.

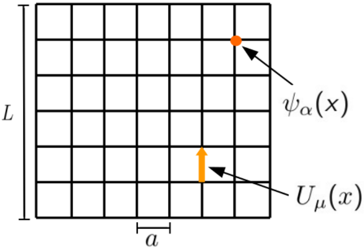
- Sincere apologies if any interesting work has not been included.

- Assume isospin symmetry ($m_u = m_d$) and only strong interactions.

Lattice QCD: theoretical aspects

LQCD : A non-perturbative, gauge invariant regulator for the QCD path integrals.

- Quark fields $\psi_\alpha(x)$ on lattice sites
- Gauge fields as parallel transporters U_μ
  Lives in the links. $U_\mu(x) = e^{igaA_\mu(x)}$
- $\bar{\psi}^i_\alpha(x)[U_\mu(x)]_{ij}\psi^j_\alpha(x + a\hat{\mu})$ is gauge invariant.
- Lattice spacing : UV cut off
- Lattice size : IR cut off



Employ Monte Carlo importance sampling methods on Euclidean metric for numerical studies.

Correlation functions

- Aim : to extract the physical states of QCD.
- Example case: mass of a pseudoscalar meson (pion)
  The simplest interpolating current: $\bar{\psi}\gamma_5\psi$
- Euclidean two point current-current correlation functions

$$\begin{aligned} C(t) &= \langle 0|[\bar{\psi}\gamma_5\psi](t)[\bar{\psi}\gamma_5\psi](0)|0\rangle \\ &= \langle 0|e^{Ht}[\bar{\psi}\gamma_5\psi](0)e^{-Ht}[\bar{\psi}\gamma_5\psi](0)|0\rangle \\ &= \sum_n e^{-E_nt}\langle 0|\bar{\psi}\gamma_5\psi(0)|n\rangle\langle n|\bar{\psi}\gamma_5\psi(0)|0\rangle \\ &= \sum_n |Z_n|^2 e^{-E_nt} \end{aligned}$$

Extraction of the mass spectrum

$C(t) = \sum_n |Z_n|^2 e^{-E_n t}$, which at large times, $C(t) \rightarrow |Z_0|^2 e^{-E_0 t}$

The operator can in principle couple with all the states that have its q. #s.
The strength of coupling Z_n determines the quality of signal.

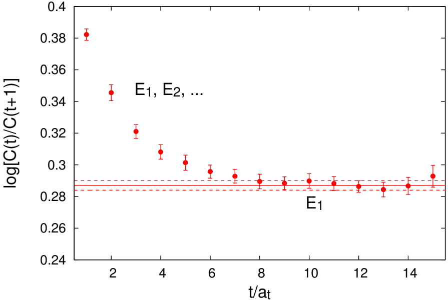
Effective mass defined as $m_{eff} = \frac{1}{dt} \log[\frac{C(t)}{C(t+dt)}]$
Mass extraction: Fit to $C(t)$ across multiple time slices.

Ground states: Single exponential fit forms

Excited states: Multi-exponential fit forms: Stability of fits!

Limited # time slices to extract excited state energies from multi-exponential fits.

Extraction of energy degenerate states is impossible this way.



Correlation matrices $C_{ji}(t)$ and GEVP

- Instead let us build a matrix of correlatiion functions:

$$C_{ji}(t) = \langle 0|\Phi_j(t)\bar{\Phi}_i(0)|0\rangle = \sum_n \frac{Z_i^{n*} Z_j^n}{2E_n} e^{-E_n(t)}$$

  where $\Phi_j(t)$ and $\bar{\Phi}_i(0)$ are the desired interpolating operators.
  $Z_j^n = \langle 0|\Phi_j|n\rangle$ are the operator-state overlaps.

- $C_{ji}(t)$ is Hermitian by construction. The eigensystem is automatically orthogonal.
  The eigenvalues representing the evolution of physical states.

- Solving the generalized eigenvalue problem for $C_{ji}(t)$. C. Michael (1985)

$$C_{ji}(t)v_j^{(n)}(t_0) = \lambda^{(n)}(t,t_0)C_{ji}(t_0)v_j^{(n)}(t_0)$$

- The m principal correlators given by eigenvalues behave as

$$\lambda_n(t,t_0) \sim e^{-E_n(t-t_0)}(1+\mathcal{O}(e^{-\partial E(t-t_0)})).$$

- Eigenvectors related to the operator state overlaps

$$Z_j^n = \langle 0|\Phi_j|n\rangle \propto v_j^{(n)}(t_0)$$

The interpolating operators: Example

Let us focus on the meson sector.

The simplest operators are local fermion bilinears:

$$\begin{aligned} 0^{-+} &\sim \bar{\psi}\gamma_5\psi \\ 1^{--} &\sim \bar{\psi}\gamma_i\psi \\ 0^{++} &\sim \bar{\psi}\psi \\ 1^{++} &\sim \bar{\psi}\gamma_5\gamma_i\psi \\ 1^{+-} &\sim \bar{\psi}\gamma_i\gamma_j\psi\epsilon_{ijk} \end{aligned}$$

No local fermion bilinear for $J^{PC} = 1^{-+}$, which is a quark model exotic q. #.

No higher spin local operators to extract orbital excitations.

Non-local operators:

Either involving displacements or using discrete derivatives.

Forward derivative: $\overrightarrow{D}_i\psi_x = \psi_{x+ai} - \psi_x$

Backward derivative: $\overleftarrow{D}_i\psi_x = \psi_x - \psi_{x-ai}$

Symmetric derivative: $\overleftrightarrow{D}_i = \overleftarrow{D}_i - \overrightarrow{D}_i$

A simple derivative operator: $\psi\overleftrightarrow{D}_i\psi$ $[J^{PC} = 1^{--}]$

Other possible operators:

Multi-meson operators, diquark-antidiquark operators, baryon-antibaryon-like operators, ...

Lattice systematics

Fermion related systematics

Unphysically heavy light quark masses: Chiral extrapolation
Tuning errors: strange, charm and bottom quark masses.
Discretization errors in heavy quark systems.

Non-zero lattice spacing

All calculations performed at finite non-zero lattice spacing.
Need for continuum extrapolation.

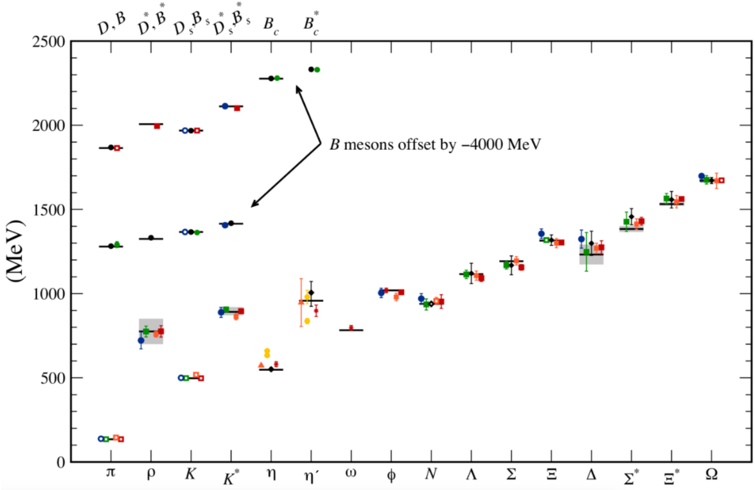
Finite volume

All calculations performed at finite physical lattice extent.
Need for infinte volume extrapolation.
Scattering and resonances: Need for multi-hadron operators, Quantization conditions, ...

Other systematics

Scale setting errors, effects from charm and bottom sea quenching, action specific uncertainties, mixed action effects, QED and strong isospin breaking effects, ...

Ground state spectrum from lattice QCD

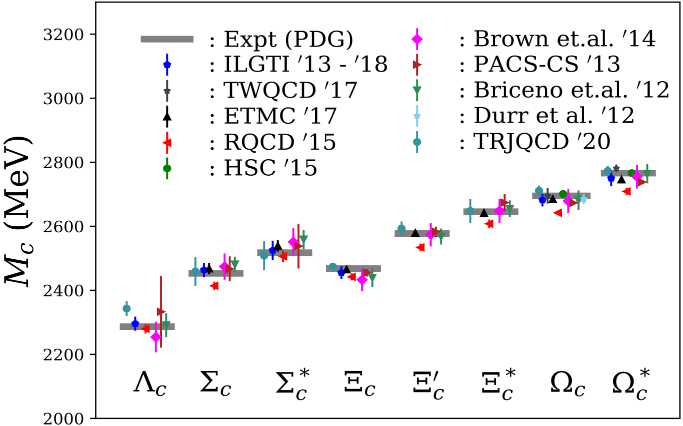


PDG review (2024) [Results from MILC, PACS-CS, BMW, QCDSF, ETMC]

$\eta - \eta'$ results from RBC-UKQCD, HSC, and Michael-Ottnad-Urbach

Heavy-light meson results from Fermilab-MILC, HPQCD, Mohler-Woloshyn

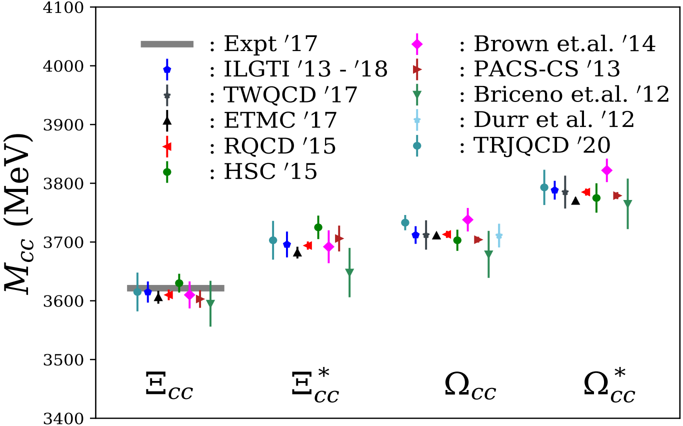
Singly charm baryons from lattice QCD



Early quenched lattice calculations : Lewis *et al.* '01; Mathur *et al.* '02; Flynn *et al.* '03

Dynamical (light quark) investigations : Liu *et al.* '10

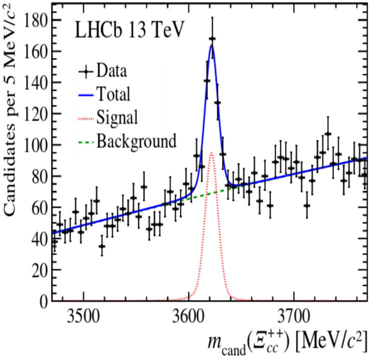
Doubly charm baryons

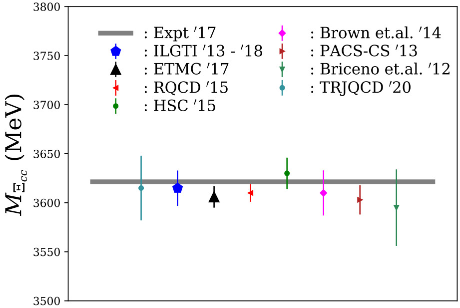


Another calculation of heavy baryon masses: QCDSF-UKQCD 1711.02485.
Heavy baryon mass splittings : BMW Science**347** 1452 '15

Early quenched lattice calculations : Lewis *et al.* '01; Mathur *et al.* '02; Flynn *et al.* '03
Dynamical (light quark) investigations : Liu *et al.* '10

The first doubly charm baryon : Ξ_{cc}





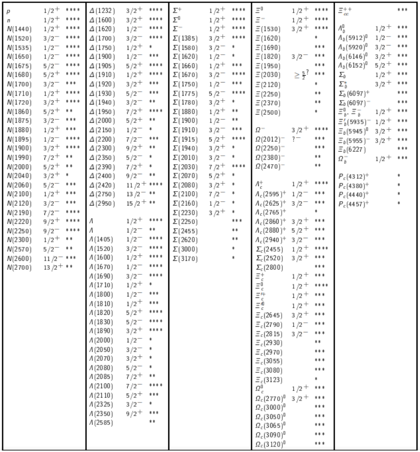
Ξ_{cc} isospin splitting (LQCD), 2.16(11)(17) MeV : BMW Science**347** 1452 '15
SELEX measurement (3519 MeV) : Mattson *et al.* PRL**89** 112001 '02

All lattice calculations disfavors SELEX peak to be a doubly charm baryon.

Family of strong interacting particles

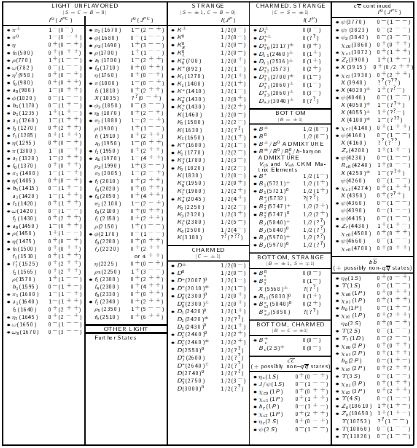
There is a big family of particles observed in nature, of which nucleon is just a member.

Baryons



p	$1/2^+$	****
n	$1/2^+$	****
$N(1440)$	$1/2^+$	****
$N(1520)$	$3/2^-$	****
$N(1535)$	$1/2^-$	****
$N(1650)$	$1/2^-$	****
$N(1675)$	$5/2^-$	****
$N(1680)$	$5/2^+$	****
$N(1700)$	$3/2^-$	***
$N(1710)$	$1/2^+$	****
$N(1720)$	$3/2^+$	****
$N(1860)$	$5/2^+$	**
$N(1875)$	$3/2^-$	***
$N(1880)$	$1/2^+$	***
$N(1895)$	$1/2^-$	****
$N(1900)$	$3/2^+$	****
$N(1990)$	$7/2^+$	**
$N(2000)$	$5/2^+$	**
$N(2040)$	$3/2^+$	*
$N(2060)$	$5/2^-$	***
$N(2100)$	$1/2^+$	***
$N(2120)$	$3/2^-$	***
$N(2190)$	$7/2^-$	****
$N(2220)$	$9/2^+$	****
$N(2250)$	$9/2^-$	****
$N(2300)$	$1/2^+$	**
$N(2570)$	$5/2^-$	**
$N(2600)$	$11/2^-$	***
$N(2700)$	$13/2^+$	**
$\Delta(1232)$	$3/2^+$	****
$\Delta(1600)$	$3/2^+$	****
$\Delta(1620)$	$1/2^-$	****
$\Delta(1700)$	$3/2^-$	****
$\Delta(1750)$	$1/2^+$	*
$\Delta(1900)$	$1/2^-$	***
$\Delta(1905)$	$5/2^+$	****
$\Delta(1910)$	$1/2^+$	****
$\Delta(1920)$	$3/2^+$	***
$\Delta(1930)$	$5/2^-$	***
$\Delta(1940)$	$3/2^-$	**
$\Delta(1950)$	$7/2^+$	****
$\Delta(2000)$	$5/2^+$	**
$\Delta(2150)$	$1/2^-$	*
$\Delta(2200)$	$7/2^-$	***
$\Delta(2300)$	$9/2^+$	**
$\Delta(2350)$	$5/2^-$	*
$\Delta(2390)$	$7/2^+$	*
$\Delta(2400)$	$9/2^-$	**
$\Delta(2420)$	$11/2^+$	****
$\Delta(2750)$	$13/2^-$	**
$\Delta(2950)$	$15/2^+$	**
Λ	$1/2^+$	****
Λ	$1/2^-$	**
$\Lambda(1405)$	$1/2^-$	****
$\Lambda(1520)$	$3/2^-$	****
$\Lambda(1600)$	$1/2^+$	****
$\Lambda(1670)$	$1/2^-$	****
$\Lambda(1690)$	$3/2^-$	****
$\Lambda(1710)$	$1/2^+$	*
$\Lambda(1800)$	$1/2^-$	***
$\Lambda(1810)$	$1/2^+$	***
$\Lambda(1820)$	$5/2^+$	****
$\Lambda(1830)$	$5/2^-$	****
$\Lambda(1890)$	$3/2^+$	****
$\Lambda(2000)$	$1/2^-$	*
$\Lambda(2050)$	$3/2^-$	*
$\Lambda(2070)$	$3/2^+$	*
$\Lambda(2080)$	$5/2^-$	*
$\Lambda(2085)$	$7/2^+$	**
$\Lambda(2100)$	$7/2^-$	****
$\Lambda(2110)$	$5/2^+$	***
$\Lambda(2325)$	$3/2^-$	*
$\Lambda(2350)$	$9/2^+$	***
$\Lambda(2585)$		**
Σ^+	$1/2^+$	****
Σ^0	$1/2^+$	****
Σ^-	$1/2^+$	****
$\Sigma(1385)$	$3/2^+$	****
$\Sigma(1580)$	$3/2^-$	*
$\Sigma(1620)$	$1/2^-$	*
$\Sigma(1660)$	$1/2^+$	***
$\Sigma(1670)$	$3/2^-$	****
$\Sigma(1750)$	$1/2^-$	***
$\Sigma(1775)$	$5/2^-$	****
$\Sigma(1780)$	$3/2^+$	*
$\Sigma(1880)$	$1/2^+$	**
$\Sigma(1900)$	$1/2^-$	**
$\Sigma(1910)$	$3/2^-$	***
$\Sigma(1915)$	$5/2^+$	****
$\Sigma(1940)$	$3/2^+$	*
$\Sigma(2010)$	$3/2^-$	*
$\Sigma(2030)$	$7/2^+$	****
$\Sigma(2070)$	$5/2^+$	*
$\Sigma(2080)$	$3/2^+$	*
$\Sigma(2100)$	$7/2^-$	*
$\Sigma(2160)$	$1/2^-$	*
$\Sigma(2230)$	$3/2^+$	*
$\Sigma(2250)$		***
$\Sigma(2455)$		**
$\Sigma(2620)$		**
$\Sigma(3000)$		*
$\Sigma(3170)$		*
Ξ^0	$1/2^+$	****
Ξ^-	$1/2^+$	****
$\Xi(1530)$	$3/2^+$	****
$\Xi(1620)$		*
$\Xi(1690)$		***
$\Xi(1820)$	$3/2^-$	***
$\Xi(1950)$		***
$\Xi(2030)$	$\geq \frac{5}{2}^?$	***
$\Xi(2120)$		*
$\Xi(2250)$		**
$\Xi(2370)$		**
$\Xi(2500)$		*
Ω^-	$3/2^+$	****
$\Omega(2012)^-$	$?^-$	***
$\Omega(2250)^-$		***
$\Omega(2380)^-$		**
$\Omega(2470)^-$		**
Λ_c^+	$1/2^+$	****
$\Lambda_c(2595)^+$	$1/2^-$	***
$\Lambda_c(2625)^+$	$3/2^-$	***
$\Lambda_c(2765)^+$		*
$\Lambda_c(2860)^+$	$3/2^+$	***
$\Lambda_c(2880)^+$	$5/2^+$	***
$\Lambda_c(2940)^+$	$3/2^-$	***
$\Sigma_c(2455)$	$1/2^+$	****
$\Sigma_c(2520)$	$3/2^+$	***
$\Sigma_c(2800)$		***
Ξ_c^+	$1/2^+$	***
Ξ_c^0	$1/2^+$	****
$\Xi_c'^+$	$1/2^+$	***
$\Xi_c'^0$	$1/2^+$	***
$\Xi_c(2645)$	$3/2^+$	***
$\Xi_c(2790)$	$1/2^-$	***
$\Xi_c(2815)$	$3/2^-$	***
$\Xi_c(2930)$		**
$\Xi_c(2970)$		***
$\Xi_c(3055)$		***
$\Xi_c(3080)$		***
$\Xi_c(3123)$		*
Ω_c^0	$1/2^+$	***
$\Omega_c(2770)^0$	$3/2^+$	***
$\Omega_c(3000)^0$		***
$\Omega_c(3050)^0$		***
$\Omega_c(3065)^0$		***
$\Omega_c(3090)^0$		***
$\Omega_c(3120)^0$		***
Ξ_{cc}^{++}		***
Λ_b^0	$1/2^+$	***
$\Lambda_b(5912)^0$	$1/2^-$	***
$\Lambda_b(5920)^0$	$3/2^-$	***
$\Lambda_b(6146)^0$	$3/2^+$	***
$\Lambda_b(6152)^0$	$5/2^+$	***
Σ_b	$1/2^+$	***
Σ_b^*	$3/2^+$	***
$\Sigma_b(6097)^+$		***
$\Sigma_b(6097)^-$		***
Ξ_b^0, Ξ_b^-	$1/2^+$	***
$\Xi_b'(5935)^-$	$1/2^+$	***
$\Xi_b(5945)^0$	$3/2^+$	***
$\Xi_b(5955)^-$	$3/2^+$	***
$\Xi_b(6227)$		***
Ω_b^-	$1/2^+$	***
$P_c(4312)^+$		*
$P_c(4380)^+$		*
$P_c(4440)^+$		*
$P_c(4457)^+$		*

Mesons



LIGHT UNFLAVORED
$(S = C = B = 0)$

	$I^G(J^{PC})$
• $\pi^\pm$	$1^-(0^-)$
• π^0	$1^-(0^{-+})$
• η	$0^+(0^{-+})$
• $f_0(500)$	$0^+(0^{++})$
• $\rho(770)$	$1^+(1^{--})$
• $\omega(782)$	$0^-(1^{--})$
• $\eta'(958)$	$0^+(0^{-+})$
• $f_0(980)$	$0^+(0^{++})$
• $a_0(980)$	$1^-(0^{++})$
• $\phi(1020)$	$0^-(1^{--})$
• $h_1(1170)$	$0^-(1^{+-})$
• $b_1(1235)$	$1^+(1^{+-})$
• $a_1(1260)$	$1^-(1^{++})$
• $f_2(1270)$	$0^+(2^{++})$
• $f_1(1285)$	$0^+(1^{++})$
• $\eta(1295)$	$0^+(0^{-+})$
• $\pi(1300)$	$1^-(0^{-+})$
• $a_2(1320)$	$1^-(2^{++})$
• $f_0(1370)$	$0^+(0^{++})$
• $\pi_1(1400)$	$1^-(1^{-+})$
• $\eta(1405)$	$0^+(0^{-+})$
• $h_1(1415)$	$0^-(1^{+-})$
$a_1(1420)$	$1^-(1^{++})$
• $f_1(1420)$	$0^+(1^{++})$
• $\omega(1420)$	$0^-(1^{--})$
$f_2(1430)$	$0^+(2^{++})$
• $a_0(1450)$	$1^-(0^{++})$
• $\rho(1450)$	$1^+(1^{--})$
• $\eta(1475)$	$0^+(0^{-+})$
• $f_0(1500)$	$0^+(0^{++})$
$f_1(1510)$	$0^+(1^{++})$
• $f_2'(1525)$	$0^+(2^{++})$
$f_2(1565)$	$0^+(2^{++})$
$\rho(1570)$	$1^+(1^{--})$
$h_1(1595)$	$0^-(1^{+-})$
• $\pi_1(1600)$	$1^-(1^{-+})$
• $a_1(1640)$	$1^-(1^{++})$
$f_2(1640)$	$0^+(2^{++})$
• $\eta_2(1645)$	$0^+(2^{-+})$
• $\omega(1650)$	$0^-(1^{--})$
• $\omega_3(1670)$	$0^-(3^{--})$
• $\pi_2(1670)$	$1^-(2^{-+})$
• $\phi(1680)$	$0^-(1^{--})$
• $\rho_3(1690)$	$1^+(3^{--})$
• $\rho(1700)$	$1^+(1^{--})$
• $a_2(1700)$	$1^-(2^{++})$
• $f_0(1710)$	$0^+(0^{++})$
$\eta(1760)$	$0^+(0^{-+})$
• $\pi(1800)$	$1^-(0^{-+})$
$f_2(1810)$	$0^+(2^{++})$
$X(1835)$	$?^?(0^{-+})$
• $\phi_3(1850)$	$0^-(3^{--})$
• $\eta_2(1870)$	$0^+(2^{-+})$
• $\pi_2(1880)$	$1^-(2^{-+})$
$\rho(1900)$	$1^+(1^{--})$
$f_2(1910)$	$0^+(2^{++})$
$a_0(1950)$	$1^-(0^{++})$
• $f_2(1950)$	$0^+(2^{++})$
• $a_4(1970)$	$1^-(4^{++})$
$\rho_3(1990)$	$1^+(3^{--})$
$\pi_2(2005)$	$1^-(2^{-+})$
• $f_2(2010)$	$0^+(2^{++})$
$f_0(2020)$	$0^+(0^{++})$
• $f_4(2050)$	$0^+(4^{++})$
$\pi_2(2100)$	$1^-(2^{-+})$
$f_0(2100)$	$0^+(0^{++})$
$f_2(2150)$	$0^+(2^{++})$
$\rho(2150)$	$1^+(1^{--})$
• $\phi(2170)$	$0^-(1^{--})$
$f_0(2200)$	$0^+(0^{++})$
$f_J(2220)$	$0^+(2^{++}$ or $4^{++})$
$\eta(2225)$	$0^+(0^{-+})$
$\rho_3(2250)$	$1^+(3^{--})$
• $f_2(2300)$	$0^+(2^{++})$
$f_4(2300)$	$0^+(4^{++})$
$f_0(2330)$	$0^+(0^{++})$
• $f_2(2340)$	$0^+(2^{++})$
$\rho_5(2350)$	$1^+(5^{--})$
$f_6(2510)$	$0^+(6^{++})$

OTHER LIGHT

Further States

STRANGE
$(S = \pm1, C = B = 0)$

	$I(J^P)$
• $K^\pm$	$1/2(0^-)$
• K^0	$1/2(0^-)$
• K_S^0	$1/2(0^-)$
• K_L^0	$1/2(0^-)$
• $K_0^*(700)$	$1/2(0^+)$
• $K^*(892)$	$1/2(1^-)$
• $K_1(1270)$	$1/2(1^+)$
• $K_1(1400)$	$1/2(1^+)$
• $K^*(1410)$	$1/2(1^-)$
• $K_0^*(1430)$	$1/2(0^+)$
• $K_2^*(1430)$	$1/2(2^+)$
$K(1460)$	$1/2(0^-)$
$K_2(1580)$	$1/2(2^-)$
$K(1630)$	$1/2(?^?)$
$K_1(1650)$	$1/2(1^+)$
• $K^*(1680)$	$1/2(1^-)$
• $K_2(1770)$	$1/2(2^-)$
• $K_3^*(1780)$	$1/2(3^-)$
• $K_2(1820)$	$1/2(2^-)$
$K(1830)$	$1/2(0^-)$
$K_0^*(1950)$	$1/2(0^+)$
$K_2^*(1980)$	$1/2(2^+)$
• $K_4^*(2045)$	$1/2(4^+)$
$K_2(2250)$	$1/2(2^-)$
$K_3(2320)$	$1/2(3^+)$
$K_5^*(2380)$	$1/2(5^-)$
$K_4(2500)$	$1/2(4^-)$
$K(3100)$	$?^?(?^{??})$

CHARMED
$(C = \pm1)$

• $D^\pm$	$1/2(0^-)$
• D^0	$1/2(0^-)$
• $D^*(2007)^0$	$1/2(1^-)$
• $D^*(2010)^\pm$	$1/2(1^-)$
• $D_0^*(2300)^0$	$1/2(0^+)$
$D_0^*(2300)^\pm$	$1/2(0^+)$
• $D_1(2420)^0$	$1/2(1^+)$
$D_1(2420)^\pm$	$1/2(?^?)$
$D_1(2430)^0$	$1/2(1^+)$
• $D_2^*(2460)^0$	$1/2(2^+)$
• $D_2^*(2460)^\pm$	$1/2(2^+)$
$D(2550)^0$	$1/2(?^?)$
$D_J^*(2600)$	$1/2(?^?)$
$D^*(2640)^\pm$	$1/2(?^?)$
$D(2740)^0$	$1/2(?^?)$
$D_3^*(2750)$	$1/2(3^-)$
$D(3000)^0$	$1/2(?^?)$

CHARMED, STRANGE
$(C = S = \pm1)$

	$I(J^P)$
• $D_s^\pm$	$0(0^-)$
• $D_s^{*\pm}$	$0(?^?)$
• $D_{s0}^*(2317)^\pm$	$0(0^+)$
• $D_{s1}(2460)^\pm$	$0(1^+)$
• $D_{s1}(2536)^\pm$	$0(1^+)$
• $D_{s2}^*(2573)$	$0(2^+)$
• $D_{s1}^*(2700)^\pm$	$0(1^-)$
$D_{s1}^*(2860)^\pm$	$0(1^-)$
$D_{s3}^*(2860)^\pm$	$0(3^-)$
$D_{sJ}(3040)^\pm$	$0(?^?)$

BOTTOM
$(B = \pm1)$

• $B^\pm$	$1/2(0^-)$
• B^0	$1/2(0^-)$
• $B^\pm/B^0$ ADMIXTURE	
• $B^\pm/B^0/B_s^0/b$-baryon ADMIXTURE	
V_{cb} and V_{ub} CKM Matrix Elements	
• B^*	$1/2(1^-)$
• $B_1(5721)^+$	$1/2(1^+)$
• $B_1(5721)^0$	$1/2(1^+)$
$B_J^*(5732)$	$?(?^?)$
• $B_2^*(5747)^+$	$1/2(2^+)$
• $B_2^*(5747)^0$	$1/2(2^+)$
$B_J(5840)^+$	$1/2(?^?)$
$B_J(5840)^0$	$1/2(?^?)$
• $B_J(5970)^+$	$1/2(?^?)$
• $B_J(5970)^0$	$1/2(?^?)$

BOTTOM, STRANGE
$(B = \pm1, S = \mp1)$

• B_s^0	$0(0^-)$
• B_s^*	$0(1^-)$
$X(5568)^\pm$	$?(?^?)$
• $B_{s1}(5830)^0$	$0(1^+)$
• $B_{s2}^*(5840)^0$	$0(2^+)$
$B_{sJ}^*(5850)$	$?(?^?)$

BOTTOM, CHARMED
$(B = C = \pm1)$

• B_c^+	$0(0^-)$
$B_c(2S)^\pm$	$0(0^-)$

$c\overline{c}$
(+ possibly non-$q\overline{q}$ states)

• $\eta_c(1S)$	$0^+(0^{-+})$
• $J/\psi(1S)$	$0^-(1^{--})$
• $\chi_{c0}(1P)$	$0^+(0^{++})$
• $\chi_{c1}(1P)$	$0^+(1^{++})$
• $h_c(1P)$	$0^-(1^{+-})$
• $\chi_{c2}(1P)$	$0^+(2^{++})$
• $\eta_c(2S)$	$0^+(0^{-+})$
• $\psi(2S)$	$0^-(1^{--})$

$c\overline{c}$ continued

	$I^G(J^{PC})$
• $\psi(3770)$	$0^-(1^{--})$
• $\psi_2(3823)$	$0^-(2^{--})$
• $\psi_3(3842)$	$0^-(3^{--})$
$\chi_{c0}(3860)$	$0^+(0^{++})$
• $\chi_{c1}(3872)$	$0^+(1^{++})$
• $Z_c(3900)$	$1^+(1^{+-})$
• $X(3915)$	$0^+(0/2^{++})$
• $\chi_{c2}(3930)$	$0^+(2^{++})$
$X(3940)$	$?^?(?^{??})$
• $X(4020)^\pm$	$1^+(?^{?-})$
• $\psi(4040)$	$0^-(1^{--})$
$X(4050)^\pm$	$1^-(?^{?+})$
$X(4055)^\pm$	$1^+(?^{?-})$
$X(4100)^\pm$	$1^-(?^{??})$
• $\chi_{c1}(4140)$	$0^+(1^{++})$
• $\psi(4160)$	$0^-(1^{--})$
$X(4160)$	$?^?(?^{??})$
$Z_c(4200)$	$1^+(1^{+-})$
• $\psi(4230)$	$0^-(1^{--})$
$R_{c0}(4240)$	$1^+(0^{--})$
$X(4250)^\pm$	$1^-(?^{?+})$
$\psi(4260)$	$0^-(1^{--})$
• $\chi_{c1}(4274)$	$0^+(1^{++})$
$X(4350)$	$0^+(?^{?+})$
• $\psi(4360)$	$0^-(1^{--})$
$\psi(4390)$	$0^-(1^{--})$
• $\psi(4415)$	$0^-(1^{--})$
• $Z_c(4430)$	$1^+(1^{+-})$
$\chi_{c0}(4500)$	$0^+(0^{++})$
• $\psi(4660)$	$0^-(1^{--})$
$\chi_{c0}(4700)$	$0^+(0^{++})$

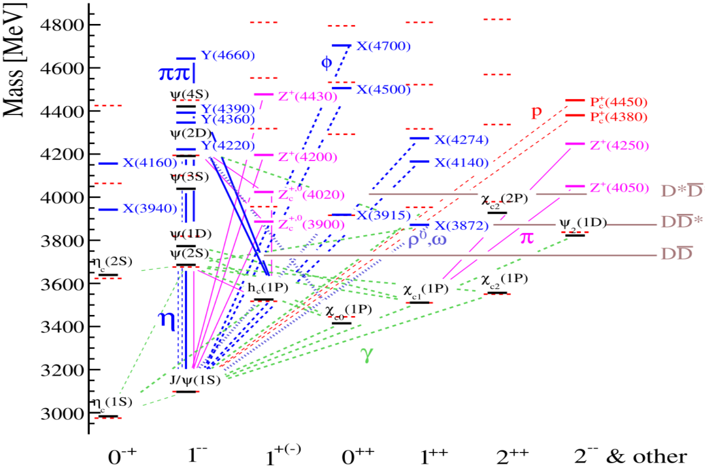
$b\overline{b}$
(+ possibly non-$q\overline{q}$ states)

• $\eta_b(1S)$	$0^+(0^{-+})$
• $\Upsilon(1S)$	$0^-(1^{--})$
• $\chi_{b0}(1P)$	$0^+(0^{++})$
• $\chi_{b1}(1P)$	$0^+(1^{++})$
• $h_b(1P)$	$0^-(1^{+-})$
• $\chi_{b2}(1P)$	$0^+(2^{++})$
$\eta_b(2S)$	$0^+(0^{-+})$
• $\Upsilon(2S)$	$0^-(1^{--})$
• $\Upsilon_2(1D)$	$0^-(2^{--})$
• $\chi_{b0}(2P)$	$0^+(0^{++})$
• $\chi_{b1}(2P)$	$0^+(1^{++})$
$h_b(2P)$	$0^-(1^{+-})$
• $\chi_{b2}(2P)$	$0^+(2^{++})$
• $\Upsilon(3S)$	$0^-(1^{--})$
• $\chi_{b1}(3P)$	$0^+(1^{++})$
• $\chi_{b2}(3P)$	$0^+(2^{++})$
• $\Upsilon(4S)$	$0^-(1^{--})$
• $Z_b(10610)$	$1^+(1^{+-})$
• $Z_b(10650)$	$1^+(1^{+-})$
$\Upsilon(10753)$	$?^?(1^{--})$
• $\Upsilon(10860)$	$0^-(1^{--})$
• $\Upsilon(11020)$	$0^-(1^{--})$

Summary tables taken from Particle Data Group website (2021): pdg.lbl.gov/

Going beyond 'stable' hadrons at the core of understanding strong interactions.

Charmonium



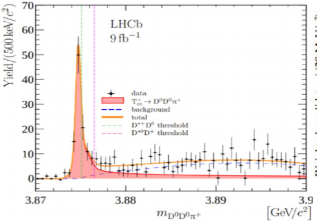
Rich energy spectrum. XYZ states.

$\bar{c}c$ picture works well for states below open charm threshold.

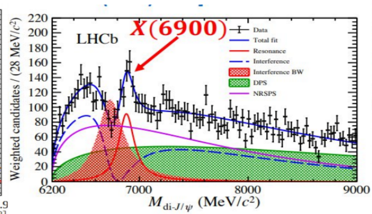
No single description for states above the open charm threshold.

Olsen *et al* 1708.04012

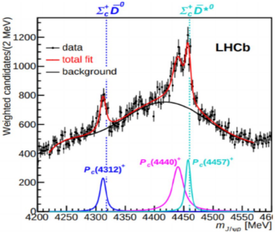
Beyond baryons and mesons in experiments



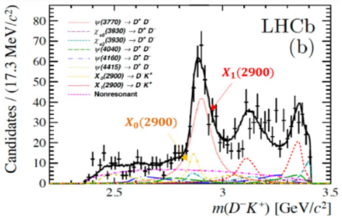
T_{cc} LHCb 2021



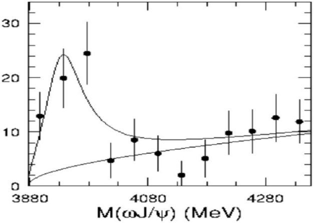
X(6900) LHCb 2020



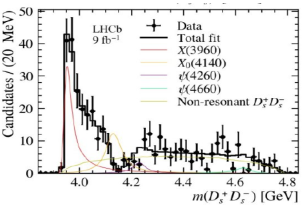
P_c LHCb 2019



X(2900) LHCb 2020



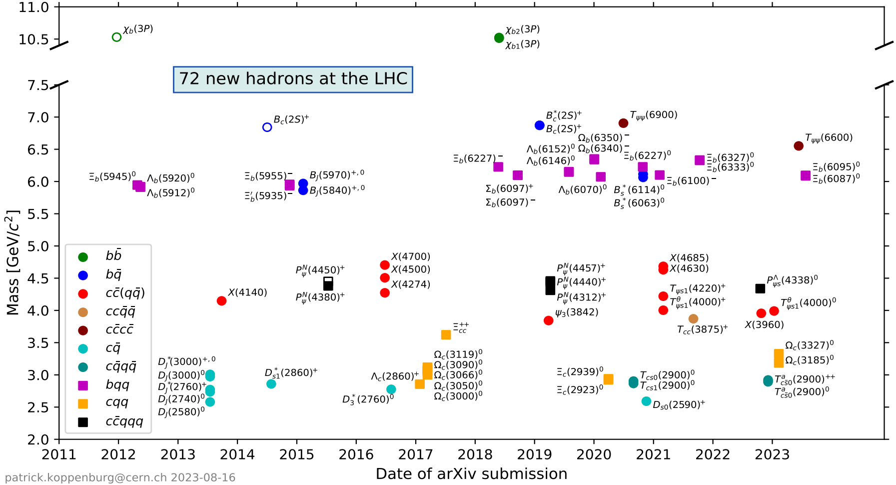
X(3915) Belle 2005



X(3960) LHCb 2021

See a recent talk by Liming Zhang here

Summary of LHCb discoveries

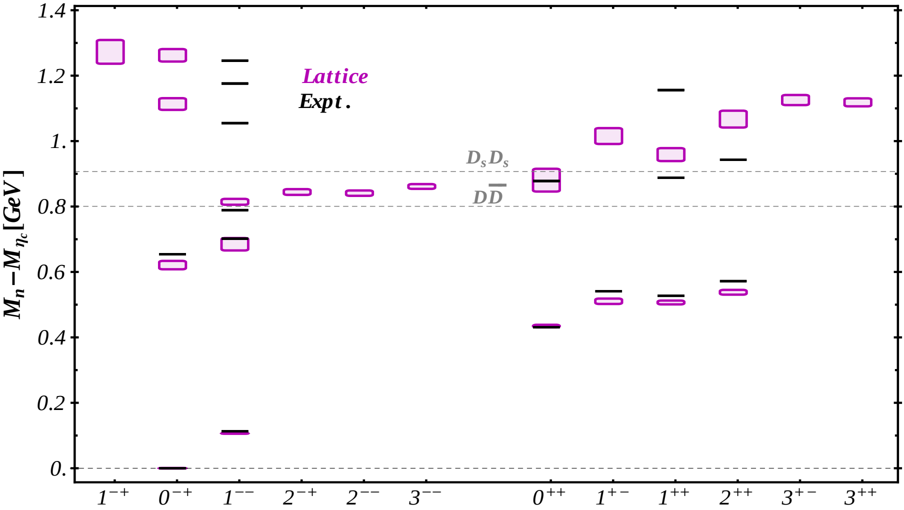


patrick.koppenburg@cern.ch 2023-08-16

https://www.nikhef.nl/∼pkoppenb/particles.html

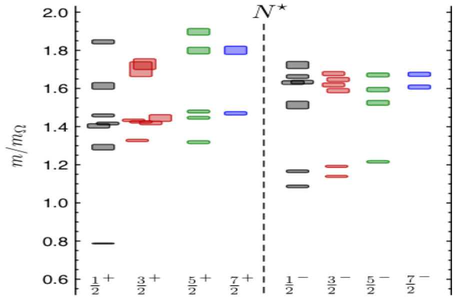
See a recent talk by Liming Zhang here

J^P identified excited charmonium spectrum from lattice

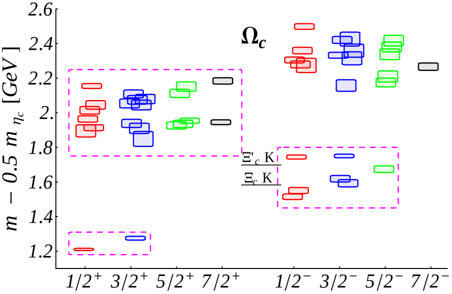


MP et al., 2018 PRD

Excited baryon spectroscopy from lattice : Example



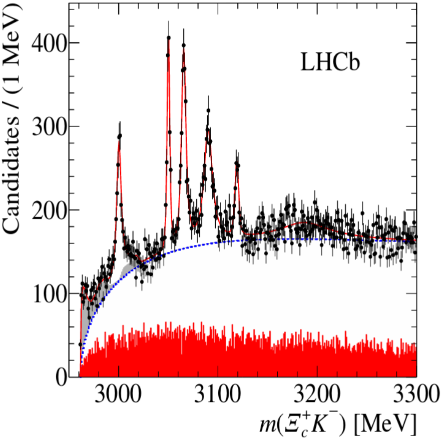
Edwards, *et al.* (HSC) PRD**84** 074508 ’11;
PRD**87** 054506 ’13



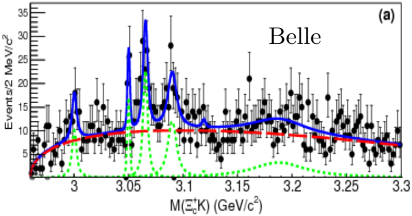
MP & Mathur (HSC) PRL**119** 042001 ’17

- Baryon interpolators : Basak *et al.* (LHPC) PRD**72** 074501, PRD**72** 094506 ’05
  Morningstar *et al.* PRD**88** 014511 ’13.
- Light baryons : Bulava *et al.* PRD**82** 014507, ’10
  Edwards *et al.* (HSC) PRD**84** 074508 ’11, PRD**87** 054506 ’13
- Heavy baryons : Meinel, PRD**85** 114510 ’12
  MP *et al.* (HSC) PRD**90** 074504 ’14, PRD**91** 094502 ’15
  MP & Mathur (HSC) PRL**119** 042001 ’17, 1508.07168.

LHCb discovery of excited Ω_c^0 baryons



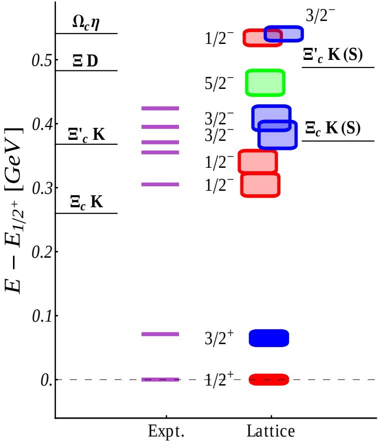
Resonance	Energy	Width	Q.no.
Ω_c^0	2695(2)	-	$1/2^+$
$\Omega_c^0(2770)$	2766(2)	-	$3/2^+$
$\Omega_c^0(3000)$	3000(1)	4.5(1)	?
$\Omega_c^0(3050)$	3050(1)	1(-)	?
$\Omega_c^0(3066)$	3066(1)	3.5(-)	?
$\Omega_c^0(3090)$	3090(1)	8.7(1)	?
$\Omega_c^0(3119)$	3119(1)	1(1)	?



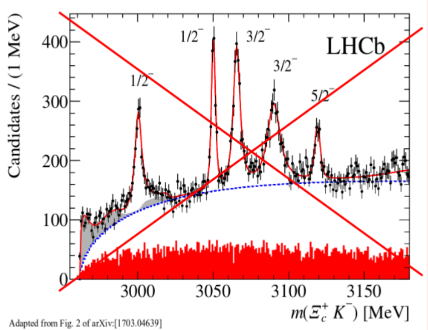
Aaij *et al.* (LHCb) PRL**118** 182001 '17

Confirmation by Belle : Yelton *et al.* (Belle) PRD**97** 051102 '18

Quantum number assignment and falsification



Excluded by 3.6σ



Adapted from Fig. 2 of arXiv:[1703.04639]

to be revisited!

MP and Mathur 2017 PRL and other pheno predictions.

LHCb PRD 104, L091102 (2021)

On the lattice, strong decays are ignored, and there remain various unattended systematics.

Excited state spectroscopy from lattice: Single hadron approach

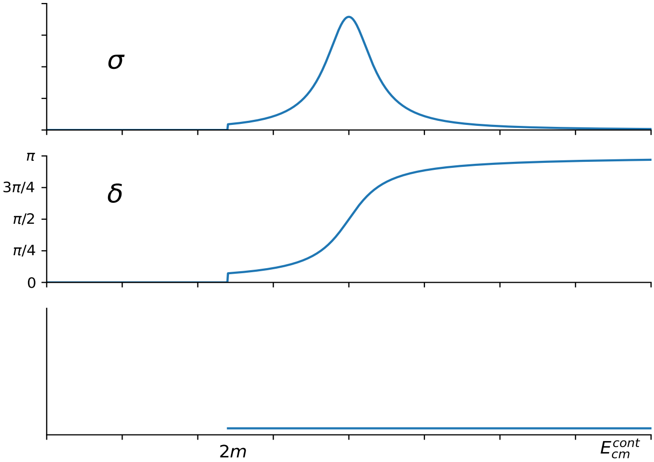
- Large basis of carefully constructed hadron interpolators
  Mesons : Liao & Manke hep-lat/0210030 '02; Thomas (HSC) PRD**85** 014507 '12
  Baryons : Basak *et al.* (LHPC) PRD**72** 074501, PRD**72** 094506 '05
  Morningstar *et al.* PRD**88** 014511 '13.
- Matrix of correlation functions & Variational study
  Dudek *et al.* PRD**77** 034501 '08, Michael NPB**259** 58 (1985)
- Established and practised by many groups

  Relatively old summary. Many more in the recent years

  Light mesons : Dudek *et al.* (HSC) PRL**103** 262001 '09, PRD**82** 034508 '10
  Dudek *et al.* (HSC) PRD**85** 014507 '12
  Light baryons : Bulava *et al.* (HSC) PRD**82** 014507, '10
  Edwards *et al.* (HSC) PRD**84** 074508 '11, PRD**87** 054506 '13
  Heavy mesons : Liu *et al.* (HSC) JHEP**1207** 126; Moir *et al.* JHEP**1305** 021
  Cheung *et al.* JHEP**1612** 089; Mohler *et al.* PRD**87** 034501 '13
  Bali *et al.* PRD**84** 094506 '11; Wurtz *et al.* PRD**92** 054504 '15
  Heavy baryons : Meinel, PRD**85** 114510 '12
  MP *et al.* (HSC) PRD**90** 074504 '14, PRD**91** 094502 '15
  MP & Mathur (HSC) PRL**119** 042001 '17, 1508.07168.
- Single hadron approach. Naive expectation : correct up to $\mathcal{O}(\Gamma)$

The challenge on lattice: Resonances in the infinite volume continuum

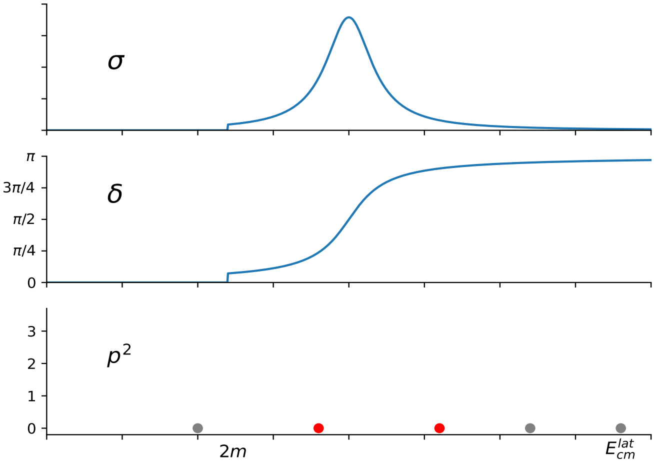
Scattering cross sections, phase shifts, branch cuts, Riemann sheets.



Schematic picture for illustration. Should not be taken quantitatively.

Resonances on the lattice (elastic) : ??

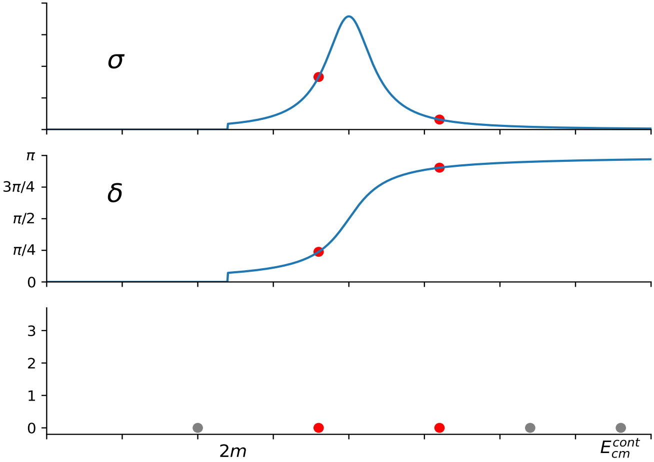
Discrete spectrum: No branch cuts, no Riemann sheets, no resonances!



Maiani-Testa no-go theorem [1990]

Resonances on the lattice (elastic) : Lüscher (1991)

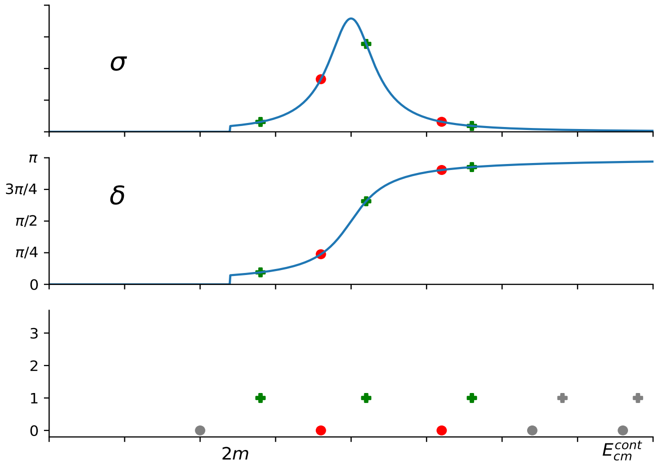
Infinite volume scattering amplitudes ⇔ Finite volume spectrum



Lüscher [1991]

Resonances on the lattice (elastic) : Lüscher (1991)

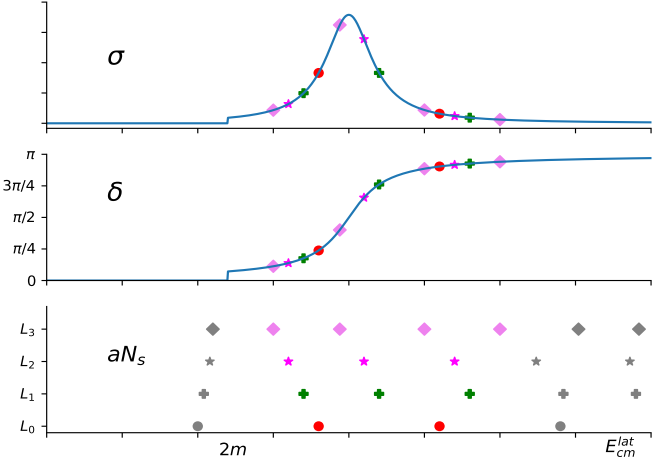
Infinite volume scattering amplitudes $\Leftrightarrow$ Finite volume spectrum



Different inertial frames can be utilized to extract more information

Resonances on the lattice (elastic) : Lüscher (1991)

Infinite volume scattering amplitudes ⇔ Finite volume spectrum



Multiple physical volumes can also be utilized to extract more information.

For generalizations of Lüscher framework, *c.f.* Briceño, Hansen 2014-15

Finite volume spectrum and infinite volume physics

- On a finite volume Euclidean lattice : Discrete energy spectrum
  Cannot constrain infinite volume scattering amplitude away from threshold.

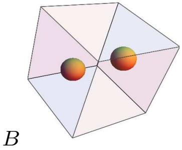
Maiani-Testa 1990

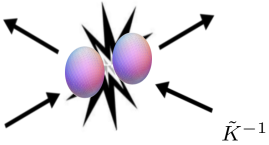
- Non-interacting two-hadron levels are given by
  $$E(L) = \sqrt{m_1^2 + \boldsymbol{k}_1^2} + \sqrt{m_2^2 + \boldsymbol{k}_2^2} \text{ where } \boldsymbol{k}_{1,2} = \tfrac{2\pi}{L}(n_x, n_y, n_z).$$

- Switching on the interaction: $\boldsymbol{k}_{1,2} \neq \frac{2\pi}{L}(n_x, n_y, n_z)$. *e.g.* in 1D $\boldsymbol{k}_{1,2} = \frac{2\pi}{L}n + \frac{2}{L}\delta(k)$.

- Lüscher's formalsim: **finite volume level shifts $\Leftrightarrow$ infinite volume phase shifts.**

Lüscher 1991





- Generalizations of Lüscher's formalism: *c.f.* Briceño 2014
  Quite complex problem: inelastic resonances ($R \to H_1H_2, H_3H_4$)
  Quantization condition is a determinant equation: $Det(B(L,k^2) - \tilde{K}^{-1}(k^2)) = 0$
  becomes an underconstrained problem with only few energy levels at hand.

Extensions and other methods

- Extensions within and beyond elastic scattering :
  - different inertial frames, boundary conditions
  - multiple scattering channels
  - particles with different identities   Briceño 1411.6944; Hansen 1511.04737
  - 2-particle scattering in finite volume code: https://github.com/cjmorningstar10/TwoHadronsInBox
  - 3-particle scattering : Hansen, Sharpe, Lopez, Mai, Döring, Rusetsky, ...
- HALQCD method :
  - Determine the potential between scattering particles
  - Extract resonance information solving Schrödinger equation.

  Ishii *et al.* PRL**99** 022001 ’07; PLB**712** 437 ’12
- finite volume Hamiltonian EFT / Quantization condition in plane wave basis :
  - Constrain free parameters of the Hamiltonian based on lattice spectrum
  - Solve for EVP to extract resonance information.

  Hall *et al.* PRD**87** 094510 ’12

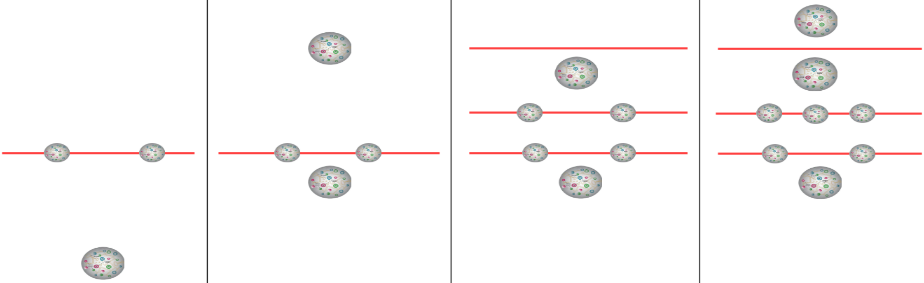
  Meng & Epelbaum JHEP**10** (2021) 051

  Mai & Döring Eur.Phys.J.A**53** (2017) 12, 240
- Optical potential :

  Agadjanov *et al.* JHEP**06** 043 ’16 [HSI]

  Hammer, Pang, Rusetsky, JHEP**1709** 109

Complexity in Hadron spectroscopy



Straightforward

Deeply bound;
Strong decay stable;

$\pi, K, D, p, n, \Lambda, \Xi_{cc}, \ldots$

Exponential volume corrections
$[E_\infty - E_L \propto e^{-mL}]$

Relatively Easy

Shallow bound states;
Elastic resonances;
Only two body decays.

Deuteron,
$\Delta, \rho, D^*, D^*_{s0}, D_{s1}, \ldots$

Difficult

Inelastic resonances;
Multiple two-body final states;
Most hadrons.

Additionally three-body decays.

Quite complex

Inelastic resonances;
Multiple final state configs;
Most hadrons.

XYZTPs, glueballs, nuclei, ...

Power law volume corrections $\Delta E \propto \frac{a_0}{L^3} + O(\frac{1}{L^4})]$
Need a rigorous finite-volume amplitude analysis.

Scattering amplitude parametrization

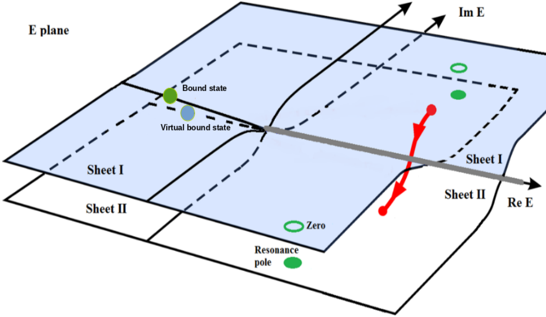
- Scattering amplitude: $S = 1 + i\frac{4k}{E_{cm}}t$
- For an elastic scattering, and assuming only S-wave,

$$t^{-1} = \frac{2\tilde{K}^{-1}}{E_{cm}} - i\frac{2k}{E_{cm}}, \quad \text{with} \quad \tilde{K}^{-1} = k.cot\delta(k)$$

(virtual/bound) state constraint below threshold: $k.cot\delta(k) = (+/-)\sqrt{-k^2}$

- Lüscher's prescription: $k.cot\delta(k) = B(L, k^2)$: a known mathematical function.
  k^2 is determined from each extracted finite volume energy splittings.
- Parametrize $k.cot\delta(k)$ as different functions of k.
  Effective Range Expansion (ERE): $k.cot\delta(k) = a_0^{-1} + 0.5r_0k^2 + \beta_i k^{2i+4}$.
  The best fits determined to represent the energy dependence.
- For multichannel processes, $\tilde{K}^{-1}(k^2)$ and $B(L, k^2)$ become matrices,
  the Quantization conditions become a matrix equarion, each energy level gives a constraint, and each $\tilde{K}^{-1}$-matrix element* needs to be parametrized.

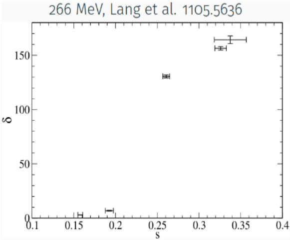
Virtual/bound states

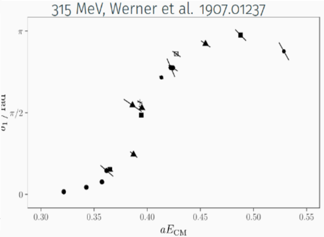


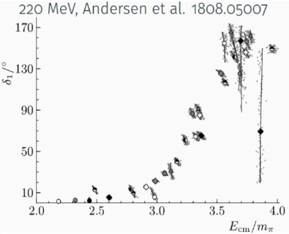
- $T \;\propto\; (p cot \delta_0 - ip)^{-1}$. Bound state is a pole in T with $p = i|p|$. Virtual bound state is a pole in T with $p = -i|p|$.

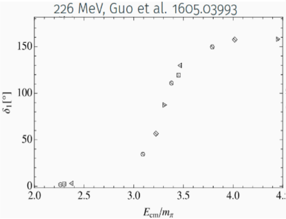
- An example for virtual bound state: spin-singlet dineutron.

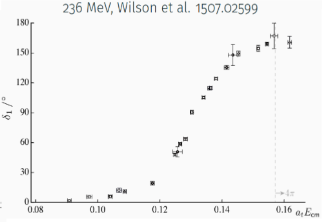
Vanilla resonance: $\rho \to \pi\pi$

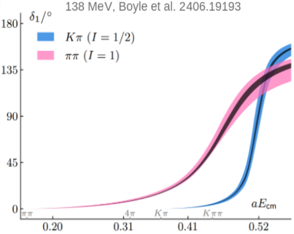

266 MeV, Lang et al. 1105.5636


315 MeV, Werner et al. 1907.01237


220 MeV, Andersen et al. 1808.05007


226 MeV, Guo et al. 1605.03993

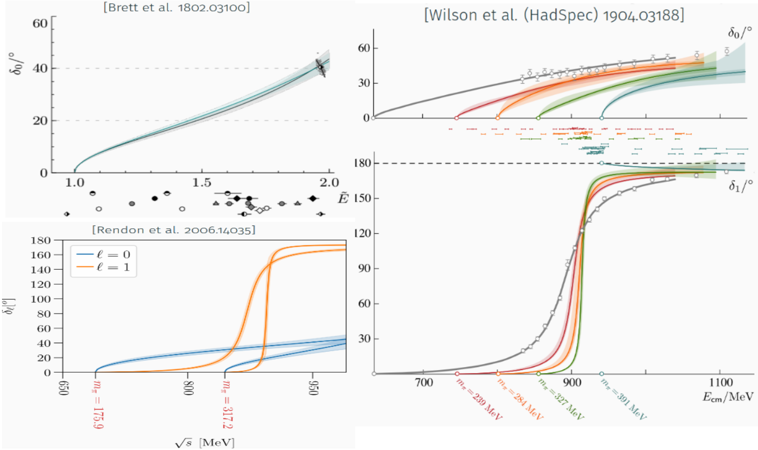

236 MeV, Wilson et al. 1507.02599


138 MeV, Boyle et al. 2406.19193

Incomplete list of lattice calculations

See also the talk by Ben Hörz in Lattice 2021

Another vanilla resonance: $K^* \to K\pi$



[Brett et al. 1802.03100]

[Rendon et al. 2006.14035]

[Wilson et al. (HadSpec) 1904.03188]

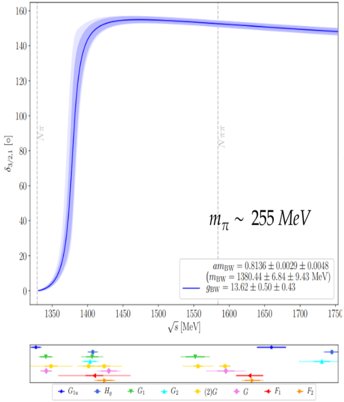
K^* and κ resonances

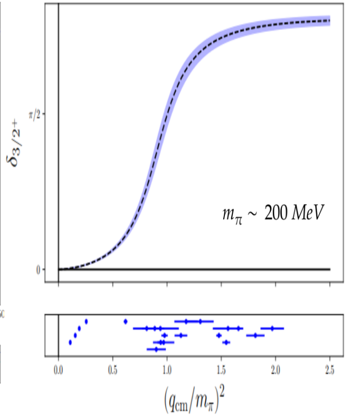
$K\pi$ atoms at DIRAC experiment 1605.06103

Incomplete list of lattice calculations

See talk by Ben Hörz at Lattice 2021

Baryon-meson scattering: $\Delta \to N\pi$ in $I = 3/2$





Silvi *et al*, PRD 2021

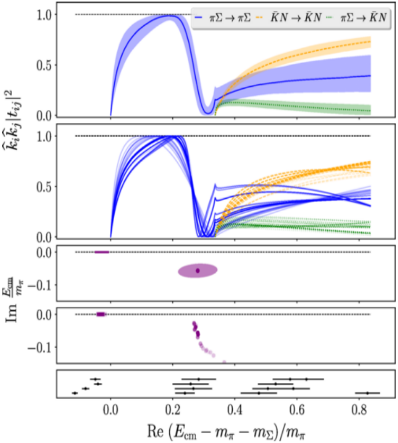
Bulava *et al*, NPB 2022

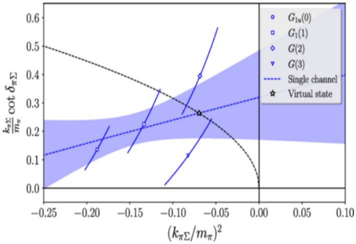
Lightest known baryon resonance

$N\pi$ scattering in $I = 1/2$ channel $(P = \pm)$: Lang&Verduci PRD 2013, Lang MP *et al*, PRD 2017

Liu *et al*, PRD 2017 [HEFT formulation]

$\Lambda(1405)$ and $\pi\Sigma - \bar{K}N$ coupled channel scattering





$m_\pi \sim 200\ MeV$
$m_K \sim 487\ MeV$

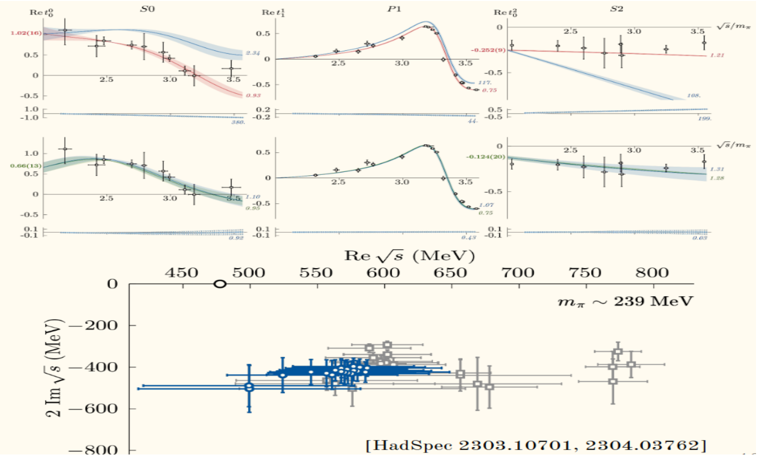
Bulava *et al*, PRD PRL 2024

Strange baryon excitation lower than the first radial excitation of nucleon

A previous lattice calculation Menadue *et al*, PRL 2012

A pedagogical survey Mai EPJS 2021

Lightest meson resonance σ



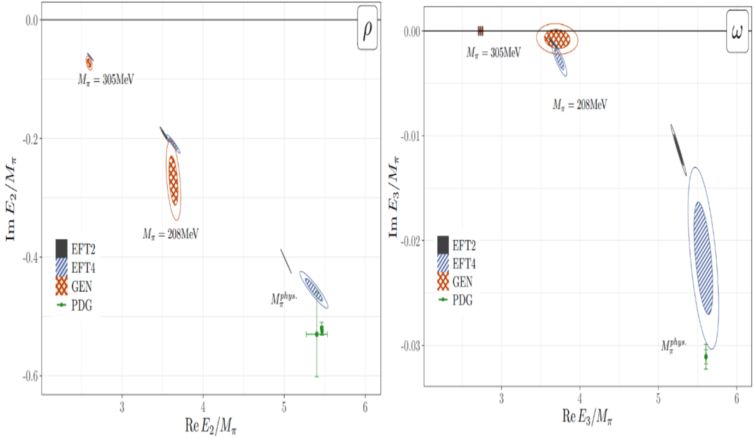
[HadSpec 2303.10701, 2304.03762]

Rodas, Dudek and Edwards PRD 2023, 2024

Several lattice calculations Briceño *et al*, PRL 2017, Guo, Mai *et al*, PRD 2018, 2019.

A rich history in literature from nonlattice perspective, *e.g.* Caprini, Colamgelo, Leutwyler, Pelaez, ...

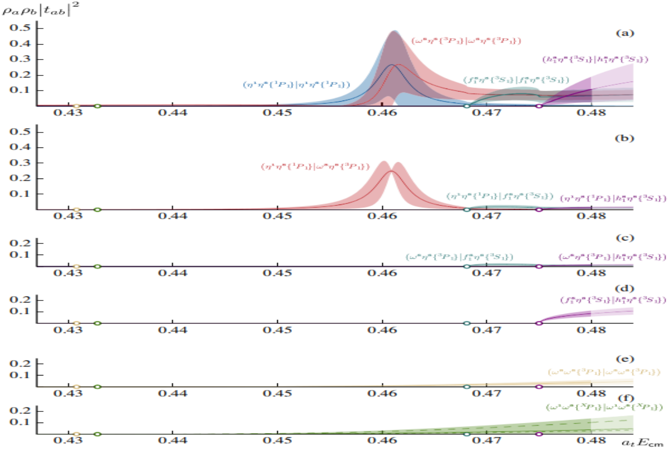
Isoscalar vector meson ω and $\pi\pi\pi$ scattering



First lattice study incorporating three body dynamics in this channel Yan *et al*, 2024.16659

See also Mai *et al*, PRL 2021 for a study of $a_1(1260)$ resonance from three body dynamics.

Exotic 1^{-+} hybrid meson resonance



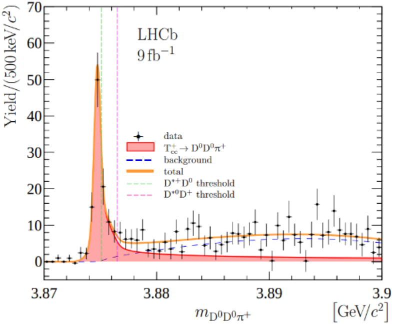
First lattice study addressing exotic quantum numbers $J^{PC} = 1^{-+}$

Woss *et al*, PRD 2021

$m_\pi \sim m_K \sim 700$ MeV and includes several two-body decay channels, which are otherwise unstable in the real world.

Find a broad resonance potentially related to $\pi_1(1600)$.

Doubly heavy tetraquarks: T_{cc}^+



LHCb: 2109.01038, 2109.01056

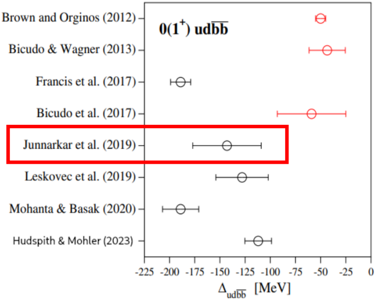
$$\delta m \equiv m_{T_{cc}^+} - (m_{D^{*+}} + m_{D^0})$$

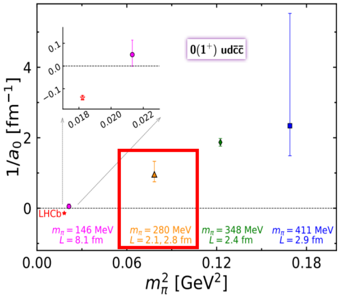
$$\begin{aligned} \delta m_{\rm pole} &= -360 \pm 40\,^{+4}_{-0}\ \mathrm{keV}/c^2\,, \\ \Gamma_{\rm pole} &= 48 \pm 2\,^{+0}_{-14}\,\mathrm{keV}\,. \end{aligned}$$

- The doubly charmed tetraquark T_{cc}^+, $I = 0$ and favours $J^P = 1^+$. Nature Phys., Nature Comm. 2022
  Striking similarities with the longest known heavy exotic, X(3872).
- No features observed in $D^0D^+\pi^+$: possibly not $I = 1$.
- Many more exotic tetraquark candidates discovered recently, T_{cs}, $T_{c\bar{s}}$, $X(6900)$.
  Prospects also for T_{bc} in the near future. See talk by Ivan Polyakov at Hadron 2023
- Doubly heavy tetraquarks: theory proposals date back to 1980s.

c.f. Ader&Richard PRD25(1982)2370

Doubly heavy tetraquarks using lattice QCD, T_{bb} and T_{cc}: $I(J^P) = 0(1^+)$





- Deeper binding in doubly bottom tetraquarks $\mathcal{O}(100 MeV)$.

  Fig: Hudspith&Mohler 2023

  Red box: ILGTI work on T_{QQ} tetraquarks: Junnarkar, Mathur, MP PRD 2019

- Shallow bound state in doubly charm tetraquarks $\mathcal{O}(100 keV)$.

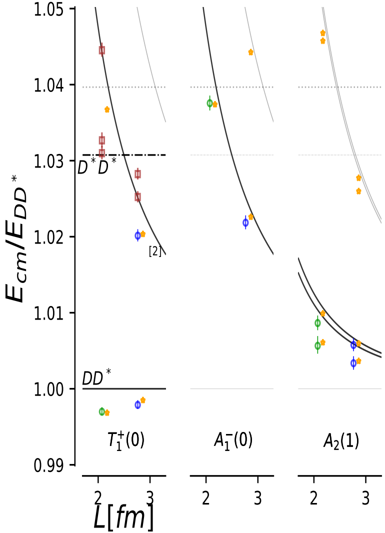
  Fig: Lyu *et al.*PRL 2023

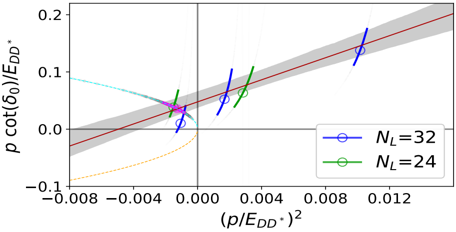
  Red box: T_{cc} (RQCD) [PRL 2022] and its quark mass dependence [2402.14715].

- Several recent calculations in the bottom-charm tetraquark sector.
  A summary of different lattice investigations →

  see review by Pedro Bicudo, 2212.07793

DD^* scattering in $l = 0, 1$ @ $m_c^{(h)}$ with an ERE: T_{cc}^+





MP, Prelovsek PRL 2022

$m_\pi \sim 280$ MeV

- Fit quality:
  $$\chi^2/d.o.f. = 3.7/5.$$
- Fit parameters:
  $a_0^{(1)} = 1.04(0.29)$ fm & $r_0^{(1)} = 0.96(^{+0.18}_{-0.20})$ fm
  $a_1^{(0)} = 0.076(^{+0.008}_{-0.009})$ fm^3 & $r_1^{(0)} = 6.9(2.1)$ fm^{-1}
- Binding energy:
  $$\delta m_{T_{cc}} = -9.9(^{+3.6}_{-7.2}) \text{ MeV}.$$
- First evaluation of the DD^* amplitude in T_{cc} channel.

+/g refers to positive parity, -/u refers to negative parity.

Pion exchange interactions/left-hand cut: ERE and QC

- A two fold problem: (Unphysical pion masses used in lattice)

  $m_\pi > m_{D^*} - m_D \quad \Rightarrow \quad D^* \to D\pi$ is kinematicaly forbidden.

  $2 \to 2$ Generalized LQC: does not subthreshold lhc effects.

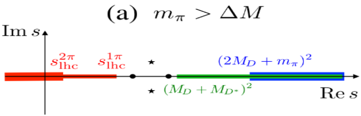
  Raposo&Hansen 2311.18793, Dawid *et al* 2303.04394, Hansen *et al* 2401.06609

  ERE convergence fails at the nearest singularity.

  Left-hand cut in the DD^* system close below the DD^* threshold.

  Du *et al* 2303.09441[PRL]

- Unphysical pion masses ($m_\pi > \Delta M = M_{D^*} - M_D$, stable D^* meson):



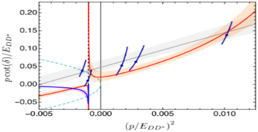


Figure taken from Du *et al* 2303.09441[PRL]

Long range pion exchange interactions: the origin of left-hand singularity and cut.
Fits with a potential that incorporates the one pion exchange:

Virtual bound states ⇒ Virtual resonances

One-pion exchange interaction/left-hand cut

- OPE from the lowest order NR Lagrangian

$$\mathcal{L} = \frac{g_c}{2f_\pi}\boldsymbol{D}^{*\dagger}\cdot\boldsymbol{\nabla}\pi^a\tau^a D + h.c. \,. \quad \Rightarrow \quad V_\pi(\boldsymbol{p},\boldsymbol{p}') = 3\left(\frac{g_c}{2f_\pi}\right)^2 \frac{(\boldsymbol{\epsilon}\cdot\boldsymbol{q})(\boldsymbol{q}\cdot\boldsymbol{\epsilon}'^*)}{u-m_\pi^2}$$

Fleming *et al.* hep-ph/0703168, Hu&Mehen hep-ph/0511321

- Upon S-wave projection, we have

$$V_\pi^S(p,p) = \frac{g_c^2}{4f_\pi^2}\left[\frac{m_\pi^2-q_0^2}{4p^2}\ln\left(1+\frac{4p^2}{m_\pi^2-q_0^2}\right)-1\right]$$

Logarithmic function branch cut → infinite set of Riemann sheets

- With the finite branch point at

$$\boldsymbol{p_{\mathrm{lhc}}^2 = \frac{1}{4}(q_0^2 - m_\pi^2) < 0} \text{ for all lattice setups.}$$

with $q_0 \simeq m_{D^*} - m_D$, where the $D^{(*)}$-meson recoil terms are ignored.

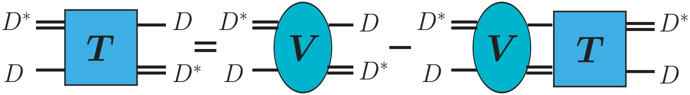
Du *et al.* 2303.09441[PRL]

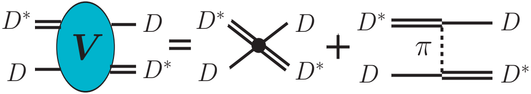
- Consequences:

  Complex phase shifts below the lhc.

  Modified near-threshold energy dependence.

Solving Lippmann-Schwinger Equation for the DD^* amplitude





- The potential: a sum of short range and long range interactions

$$V(\boldsymbol{p},\boldsymbol{p}') = V_{\rm CT}(p,p') + V_\pi^S(p,p') \quad \text{with} \quad V_{\rm CT}(p,p') = 2c_0 + 2c_2(p^2 + p'^2) + \mathcal{O}(p^4, p'^4)$$

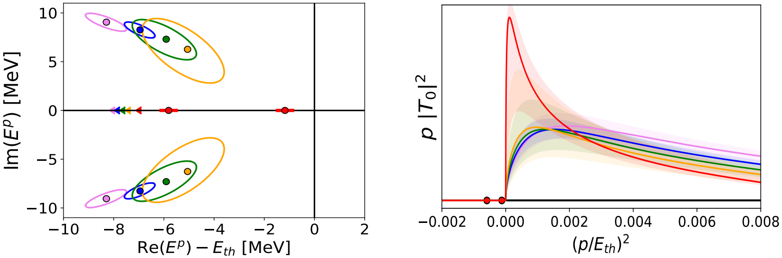
- The scattering amplitude $\qquad T^{-1} \propto p\cot\delta_0 - ip$

- The pion decay constant f_π and $DD^*\pi$ coupling g_c
  at $m_\pi \sim 280$ MeV following the 1-loop χPT.

Du *et al* 2303.09441[PRL]

Collins, Nefediev, MP, Prelovsek 2402:14715

Pole positions and scattering rate [EFT]



- Subthreshold resonance pole pair moving towards the real axis with increasing m_c.

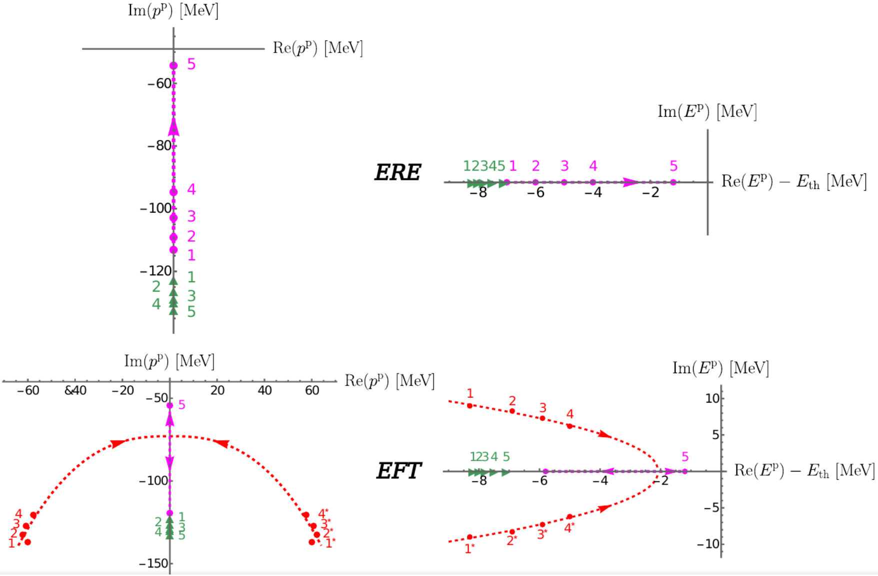
- Collide on the real axis below threshold and turn back-to-back.
  At the heaviest m_c: virtual bound poles [in Red]

- With increasing m_c, subthreshold resonance poles evolves to become
  a pair of virtual bound poles.

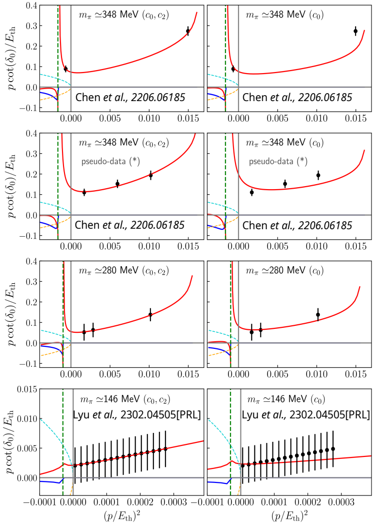
- Enhancement in the DD^* scattering rate $(p|T_0|^2)$.

Collins, Nefediev, MP, Prelovsek 2402:14715

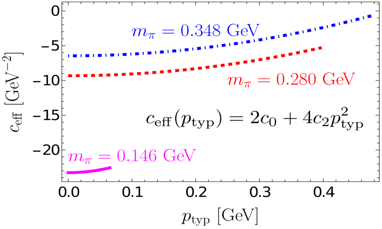
Pole trajectory of T_{cc}^+: ERE Vs EFT



m_π dependence of the T_{cc} pole [EFT]



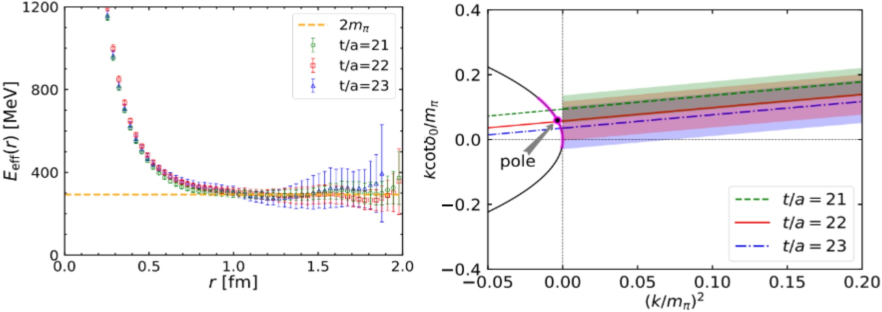
Collins, Nefediev, MP, Prelovsek 2402:14715



- Qualitative study of m_π dependence using $V_{\rm CT}(p, p') = 2c_0 + 2c_2(p^2 + p'^2)$
- Two parameter fit (c_0, c_2) [left] and a single parameter fit (c_0, with $c_2 = 0$) [right].
- Resonance poles at m_π ∼348 and ∼280 MeV. Shallow virtual bound poles at m_π =146 MeV.
- Stronger attraction for lighter m_π. [$c_{\rm eff}$] stronger binding in T_{cc} for lighter pions.
- m_π =146 MeV: HALQCD procedure.

HALQCD approach @ near physical m_π

- DD^* s-wave scattering amplitudes from the lattice extracted DD^* potential.



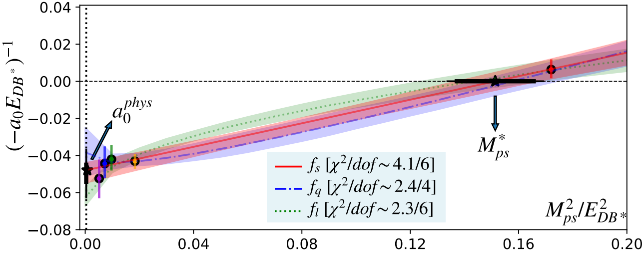
$$E_{\rm eff}(r) = -\frac{\ln[V(r)r^2/a_3]}{r}$$

$$V^B_{\rm fit}(r;m_\pi) = \sum_{i=1,2} a_i \; e^{-(r/b_i)^2} + a_3 \; \left(1 - e^{-(r/b_3)^2}\right)^n V^n_\pi(r)$$

Lyu *et al* arXiv:2302.04505

Long distance potential dominated by two pion exchange, not OPE.
Phase shifts extracted from long distance behaviour.
Shallow virtual bound state turning to a real bound state at physical m_π

T_{bc} $(I)J^P = (0)1^+$ bound state



MP *et al* 2307.14128, Archana Radhakrishnan's talk on Friday.

- Light quark mass ($m_{u/d}$ or M_{ps}) dependence indicates a real bound state at physical pion mass.
- DB^* scattering length[1] and binding energy (w.r.t. E_{DB^*}) in the continuum limit

$$a_0^{phys} = 0.57(^{+4}_{-5})(17) \text{ fm} \quad \text{and} \quad \delta m_{T_{bc}} = -43(^{+6}_{-7})(^{+14}_{-24}) \text{ MeV}$$

- A more recent lattice investigation also suggesting attractive interactions.

Alexandrou *et al* 2312.02925

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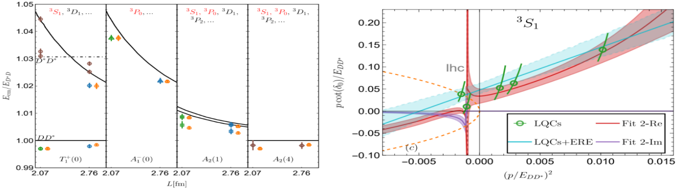
[1] Note the sign convention used: [$kcot\delta_0 \sim -1/a_0$]

Work around to LQC: A plane-wave approach and modified LQC

- An effective field theory incorporating OPE with a plane wave basis expansion.



Lu Meng *et al* arXiv:2312.01930

Virtual bound states ⇒ Virtual resonances [m_π ∼280 MeV]

- Modified 3-particle (Lüscher) Quantization Condition:

Hansen, Romero-Lopez, Sharpe, 2401.06609, Raposo, Hansen, 2311.18793

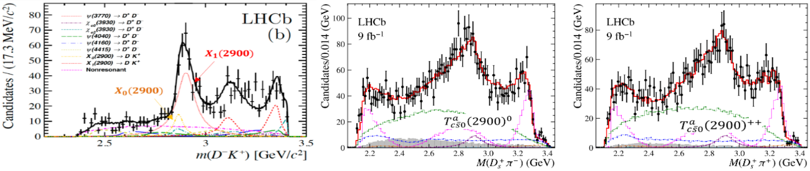
See a recent talk by Romero-Lopez here

A rigorous procedure, but demands multiple lattice inputs.
- $D\pi$ finite volume spectrum up to the $D\pi\pi$ threshold.
- Isovector DD finite volume spectrum up to the $DD\pi$ threshold.
- Isoscalar $DD\pi$ finite volume spectrum up to the $DD\pi\pi$ threshold.

Excited charmed-light and charmed-strange mesons

- Scalar D_0^* a broad feature in the $D\pi$ amplitudes, whereas a narrow D_{s0}^* below the DK threshold.

- Recent [LHCb] discoveries of T_{cs} [$X_1(2900)$, $X_0(2900)$], $T_{c\bar{s}0}(2900)^{0/++}$.



See a recent talk by Liming Zhang here

- A new framework of four quark systems with a charm quark and remaining light/strange quarks [$cs\bar{u}\bar{d}$, $cu\bar{s}\bar{d}$, $cd\bar{s}\bar{u}$].

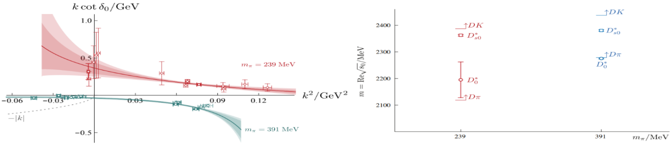
LHCb discoveries

- A handful of lattice calculations (not explicitly exotic channels):

Mohler *et al* 1308.3175 (PRL), Lang *et al* 1403.8103, Bali *et al* 1706.01247, Gayer *et al* 2102.04973, Mohler *et al* 1208.4059, Moir *et al* 1607.07093, Gregory *et al* 2106.15391, Yan *et al* 2312.01078

Recent lattice investigations

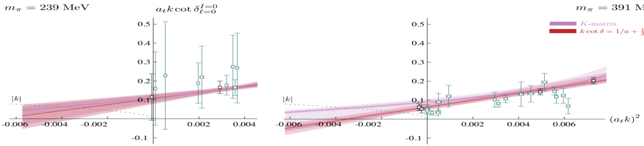
- Scalar charmed mesons and the $D\pi$ amplitudes,



Gayer *et al* 2102.04973

D_0^* pole real part consistently below that for D_{s0}^* for either m_π.

- Isoscalar $D\bar{K}$ scattering in s-wave (explicitly flavor exotic channel "$cs\bar{q}_1\bar{q}_2$"):



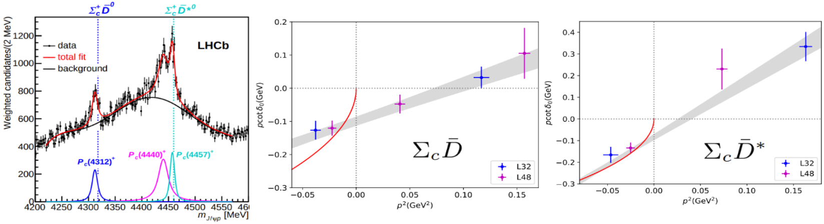
Cheung *et al* 2008.06432

Weak attraction indicating presence of a virtual state.

Pentaquarks, P_c in $J/\psi p$ final states

- Narrow pentaquark structures $P_c(4312)^+$, $P_c(4440)^+$, and $P_c(4457)^+$ in $J/\psi p$ final states. Features close below the $\Sigma_c\bar{D}$ and $\Sigma_c\bar{D}^*$

LHCb 1904.03947 (PRL)



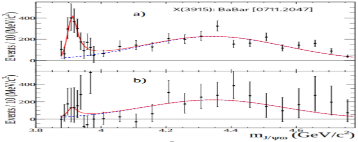
- Indications for shallow bound states in $\Sigma_c\bar{D}$ and $\Sigma_c\bar{D}^*$ from lattice. Coupling to $J/\psi p$ omitted in the analysis. $m_\pi \sim 294$ MeV.

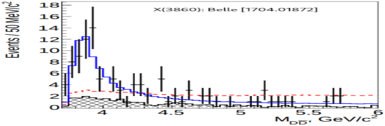
Xing *et al* 2210.08555

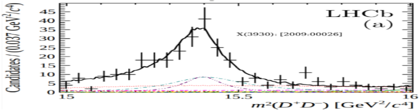
- Evidence for $P_{cs}(4459)^0$ $(\bar{c}csud)$. No lattice investigation yet.

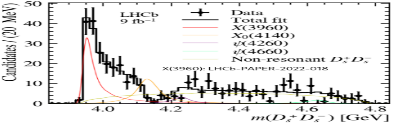
LHCb Science Bulletin 2021

Scalar charmonium-like states









- Several likely related features, $X(3915)$, $X(3930)$, $X(3960)$.
  Proximity to the $\bar{D}_s D_s$ threshold: Possible hidden strange content [$\mathbf{cs\bar{c}\bar{s}}$]
  $\Rightarrow$ narrow width from $\bar{D}D$

- Several phenomenological studies supporting this:

  Lebed Polosa 1602.08421, Chen *et al* 1706.09731, Bayar *et al* 2207.08490

- Another feature named as X(3860) observed by Belle. No evidence from LHCb.

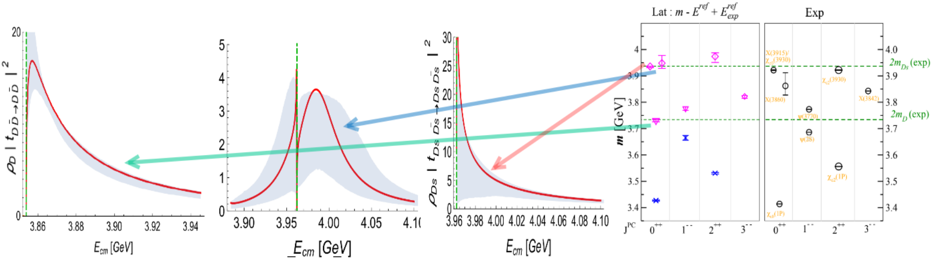
- Yet unknown $\bar{D}D$ bound state, predicted by models.

  Gamermann *et al* 0612179, Hidalgo-Duque *et al* 1305.4487, Baru *et al* 1605.09649

- Such a $\bar{D}D$ bound state is supported by re-analysis of the exp. data.

  Danilkin *et al* 2111.15033, Ji *et al* 2212.00631.

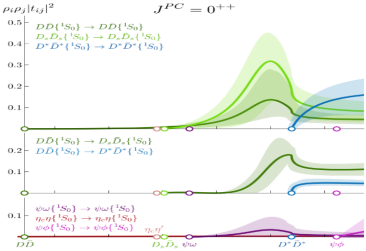
Charmonium-like resonances and bound states on the lattice

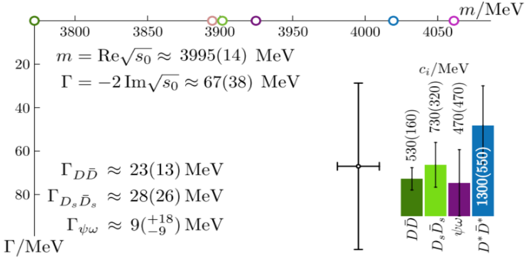


- ✿ First extraction of coupled $\bar{D}D$-$\bar{D}_sD_s$ scattering amplitude.
  $[\mathbf{\bar{c}c}, \mathbf{\bar{c}c\bar{q}q}; \mathbf{q} \to \mathbf{u}, \mathbf{d}, \mathbf{s}, \text{and } \mathbf{I} = \mathbf{0}]$.
- ✿ Lattice QCD ensembles : CLS Consortium
  $m_\pi \sim$ 280 MeV, $m_K \sim$ 467 MeV, $m_D \sim$ 1927 MeV, $a \sim 0.086$ fm
- ✿ In addition to conventional charmonium states, we observe candidates for three excited scalar charmonium states
  - ⇒ a yet unobserved shallow $\bar{D}D$ bound state.
  - ⇒ a $\bar{D}D$ resonance possibly related to X(3860).
  - ⇒ a narrow resonance just below and with large coupling to $\bar{D}_sD_s$ threshold.
    possibly related to X(3960) / X(3930) / X(3915).
- ✿ Our (RQCD) recent publications on charmonium:

Collins, Mohler, MP, Piemonte, Prelovsek 2111.02934, 2011.02541, 1905.03506.

Recent lattice investigation by HSC

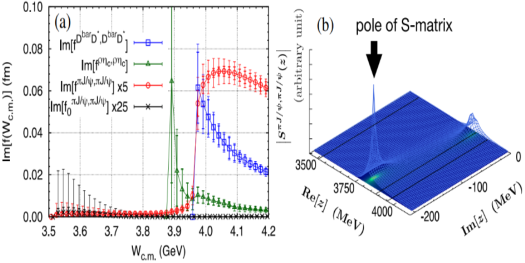


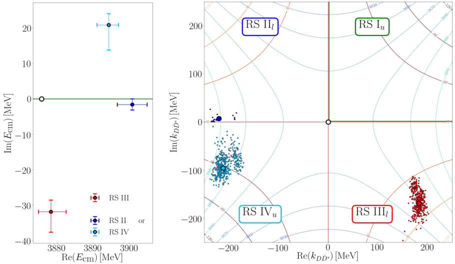


HSC 2309.14070, 2309.14071.

- Two-hadron channels considered: $\eta_c\eta$, $\eta_c\eta'$, $\bar{D}D$, $\bar{D}_sD_s$, $\psi\omega$, $\psi\phi$, $\bar{D}^*D^*$, $\chi_{c1}\eta$.
- Anisotropic lattice QCD ensembles : Hadron Spectrum Collaboration
  $m_\pi \sim$ 391 MeV, $m_K \sim$ 540 MeV, $m_D \sim$ 1852 MeV, $a_s \sim 0.12$ fm
- In addition to conventional charmonium states, only a single scalar resonance below 4 GeV
  $\Rightarrow$ with large coupling to all open charm channels.
  relation to X(3960) / X(3930) / X(3915) / χ_{c0}(3860) features ?
- Results in conflict with several other theoretical and experimental studies.
  Resolution: quark mass dependence ?

Charged charmonium-like states from lattice $[Z_c(3900)^+]$





HALQCD 1602.03465 (PRL).

Sadl *et al*, 2406.09842

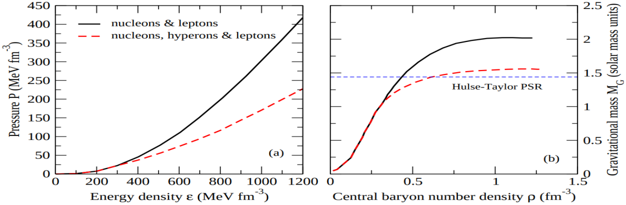
- Lattice calculations from two different fronts:
  Calculations based on Lüscher's formalism and using HALQCD approach
- HALQCD work: Coupled $J/\psi\pi$-$\rho\eta_c$-$\bar{D}D^*$ scattering. $m_\pi \sim$ 400-700 MeV, $a \sim 0.09$ fm
  Strong coupling between $\bar{D}D^*$ and other two channels.
  $Z_c(3900)$ not a usual resonance, but a threshold cusp
- Lüscher's formalism: no robust supporting/exluding remarks for such a near threshold state.
  Prelovsek *et al* 1405.7623, Chen *et al* 1403.1318, 1503.02371, CLQCD 1907.03371
- More recent lattice study of isovector $1^{-\pm}$ systems in coupled channel scenario.
  Lüscher based QC.
  Sadl, MP, *et al*, 2406.09842

Baryon-baryon interactions: Other prospects

- Hyperon formation ⇐ Large nuclear densities in astrophysical objects

Bazavov *et al*, 1404.6511 PRL, 1404.4043 PLB

Chatterjee and Vidaña 1510.06306 EPJA, Vidaña *et al* 1706.09701 PLB



- A handful of experimental efforts using large nuclei reactions.
  Inputs on LECs to EFTs ⇒ nuclear many body calculations.

Epelbaum 2005, INT-NFPNP 2022, $0\nu\beta\beta$ PSWR 2022

- Heavy dibaryons: Relatively free of the light quark chiral dynamics.

- Heavy dibaryons: no near three or four particle thresholds.
  Simple model studies ($\Omega\Omega$ scattering): widely different inferences.

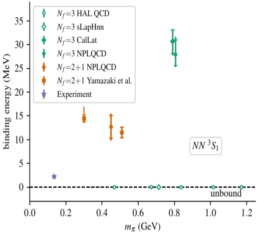
Richard *et al* 2005.06894 PRL, Liu *et al* 2107.04957 CPL, Huang *et al* 2011.00513 EPJC

Baryon-baryon interactions from lattice QCD

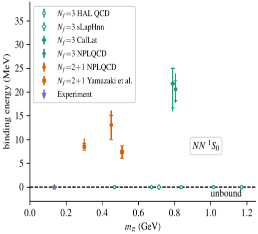
- A handful of lattice QCD efforts on baryon-baryon scattering typically at $m_\pi > m_\pi^{phys}$.

  see works by NPLQCD, HALQCD*, Mainz, CalLat, and others in the past decade.

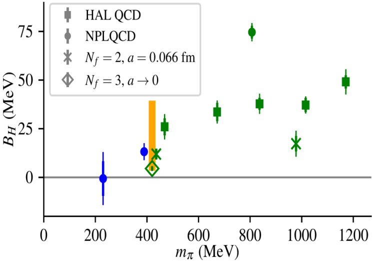
- Focus on light and strange six quark systems: Deuteron, dineutron, H-dibaryon, ... Discretization effects could be crucial.



deuteron

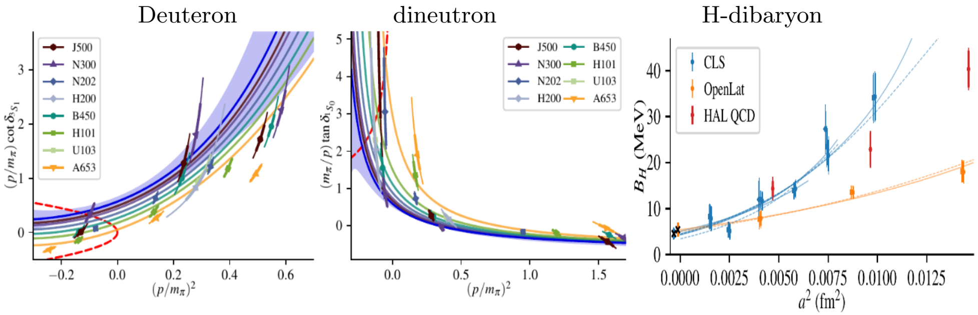


dineutron



Talk by Green @ Santa Fe Workshop 2023, Briceño *et al* Chapter 16 of 2202.01105 FBS

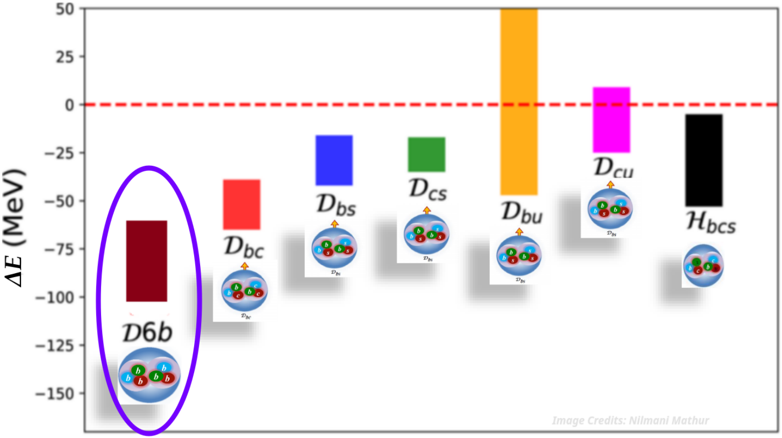
Baryon-baryon interactions from lattice QCD



Talk by Green @ Santa Fe Workshop 2023
Green @ Liverpool Lattice 2024

- Discretization effects could be crucial.
- Results at SU(3) point:
  HALQCD @ $m_\pi \sim 840$ MeV and other points @ $m_\pi \sim 420$ MeV.
- Deutron and dineutron potentially a virtual bound pole at $m_\pi \sim 420$ MeV.
  H-dibaryon is a shallow bound state.
- @ the largest lattice spacing Deuteron is nearly a bound state.

Baryon-baryon interactions in heavy sector

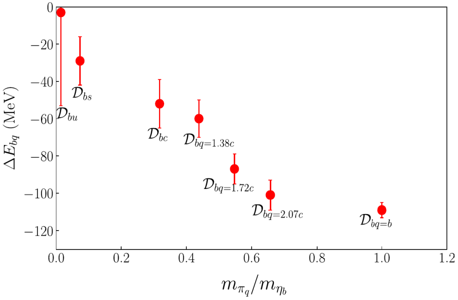


Mathur, MP, Chakraborty 2205.02862 PRL

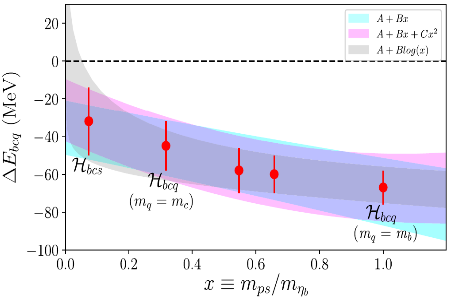
Junnarkar and Mathur 1906.06054 PRL & 2206.02942 PRD

Ellipsed: Not limited to just a finite volume spectrum extraction.
Involved scattering analysis with a zero-range approximation.

Light quark mass dependence



Junnarkar and Mathur 1906.06054 PRL,

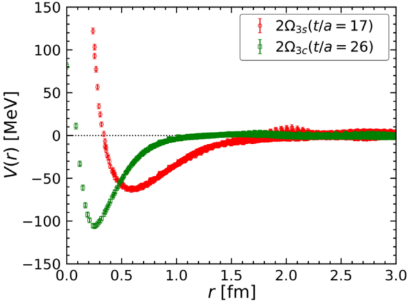


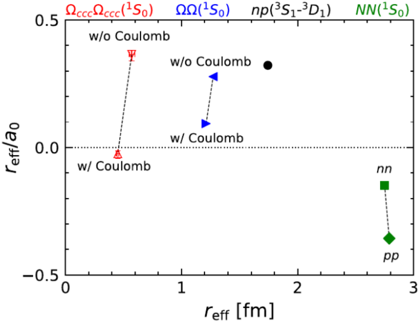
Junnarkar and Mathur 2206.02942 PRD

Heavier the quark masses, stronger the binding.
Different pattern of binding compared to T_{QQ}

MP, Prelovsek 2202.10110 PRL, Collins, MP, *et al.*, 2402.14715 PRD

Other calculations [$\mathcal{D}_{6c}$ & $\mathcal{D}_{6s}$]

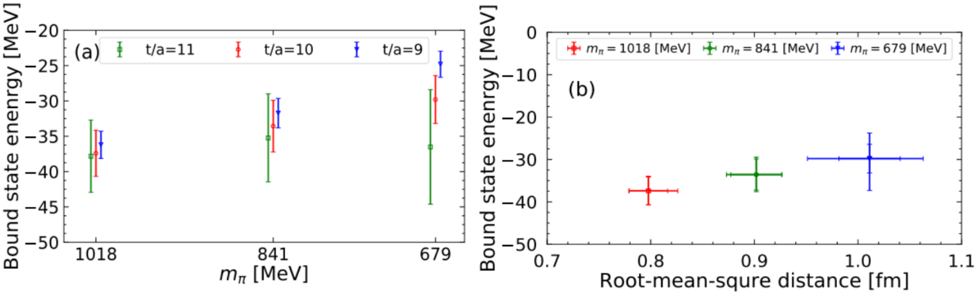




- S-wave $\Omega\Omega$ scattering using HALQCD procedure.
  $m_\pi \sim 146$MeV, $L \sim 8.1$ fm
- System close to the point where scattering length diverges.

HALQCD 2102.00181 PRL

$\Delta\Delta$ scattering and $d^*(2380)$ from lattice



- $\Delta\Delta$ scattering on the lattice.

Gongyo *et al*, 2008.00856 PLB

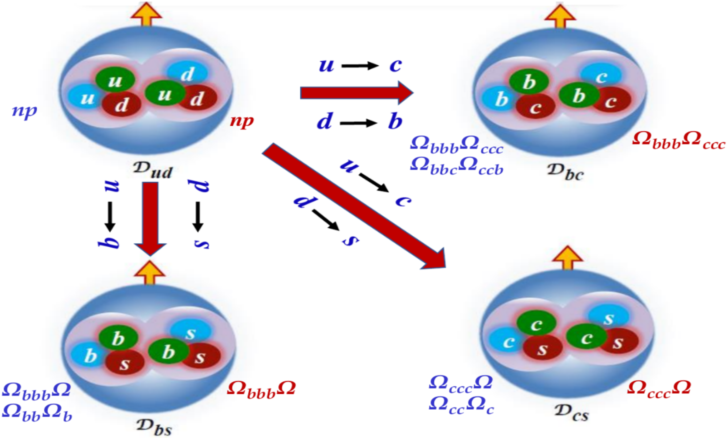
- Results at SU(3) point: HALQCD @ $m_\pi \sim 680$, 840 and 1018 MeV. Stable Δ baryons.
- Lattice spacing $a \sim 0.121$ fm and lattice size $L \sim 3.87$ fm. $d*$ as a quasi-bound state.
- Coarse lattice spacing used.

Summary

- We have a handful of hadrons, with a large set of them still demanding an understanding based on first principles. The list is proliferating with those several experimental efforts across globe.
- Lattice QCD, being a suitable nonperturbative framework, has been used to study several of these hadrons.
- Made a ‘very’ brief outline of how hadron masses are extracted and how resonances are studied in a finite volume.
- Presented a selected list of lattice investigations, particularly addressing shallow bound states, near threshold poles and conventional resonances.
- Many hadronic states remain unaddressed and several remaining challenges even before addressing lattice systematics. Formalisms accounting three body dynamics. New ideas to access highly excited states. ...
- Quark mass dependence as a probe to understand the nature of resonances.
  Heavy hadron sector serving as an excellent test bed.
- Lattice systematics. *e.g. H-dibaryon studies.* Need for huge computation resources.

Thank you

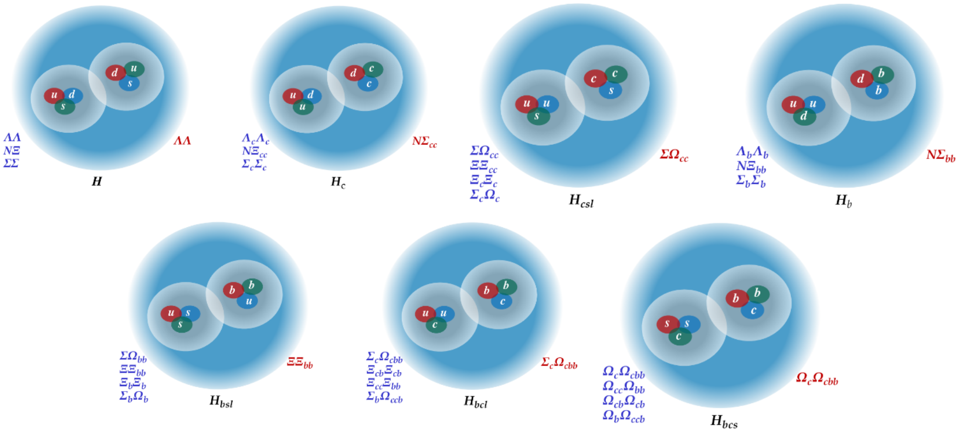
Deuteron-like Heavy dibaryons



Elastic thresholds in red text

Junnarkar and Mathur 1906.06054 PRL

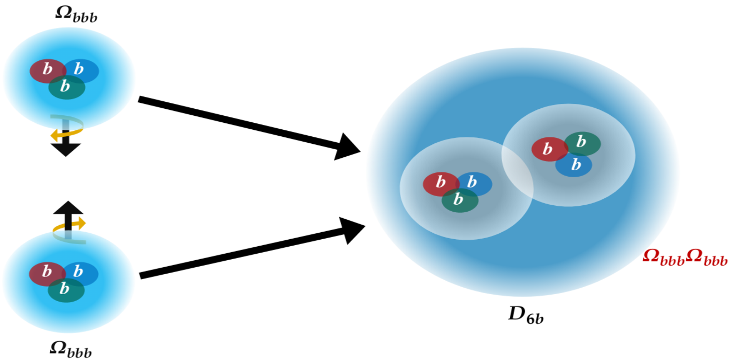
Triply flavored heavy dibaryons



Elastic thresholds in red text

Junnarkar and Mathur 2206.02942 PRD

Single flavored heavy dibaryons ($\mathcal{D}_{6q}$)



Heavy spin 0 single flavored partner of $d^*(2380)$?? Dyson and Xuong PRL 13 815 (1964)

Leading m_l dependence could arise from pair produced 2π exchanges.

Calculations at m_Q: Relatively cheap calculations with clean signals.

Mathur, MP and Chakraborty 2205.02862 PRL