From QED atoms to QCD hadrons

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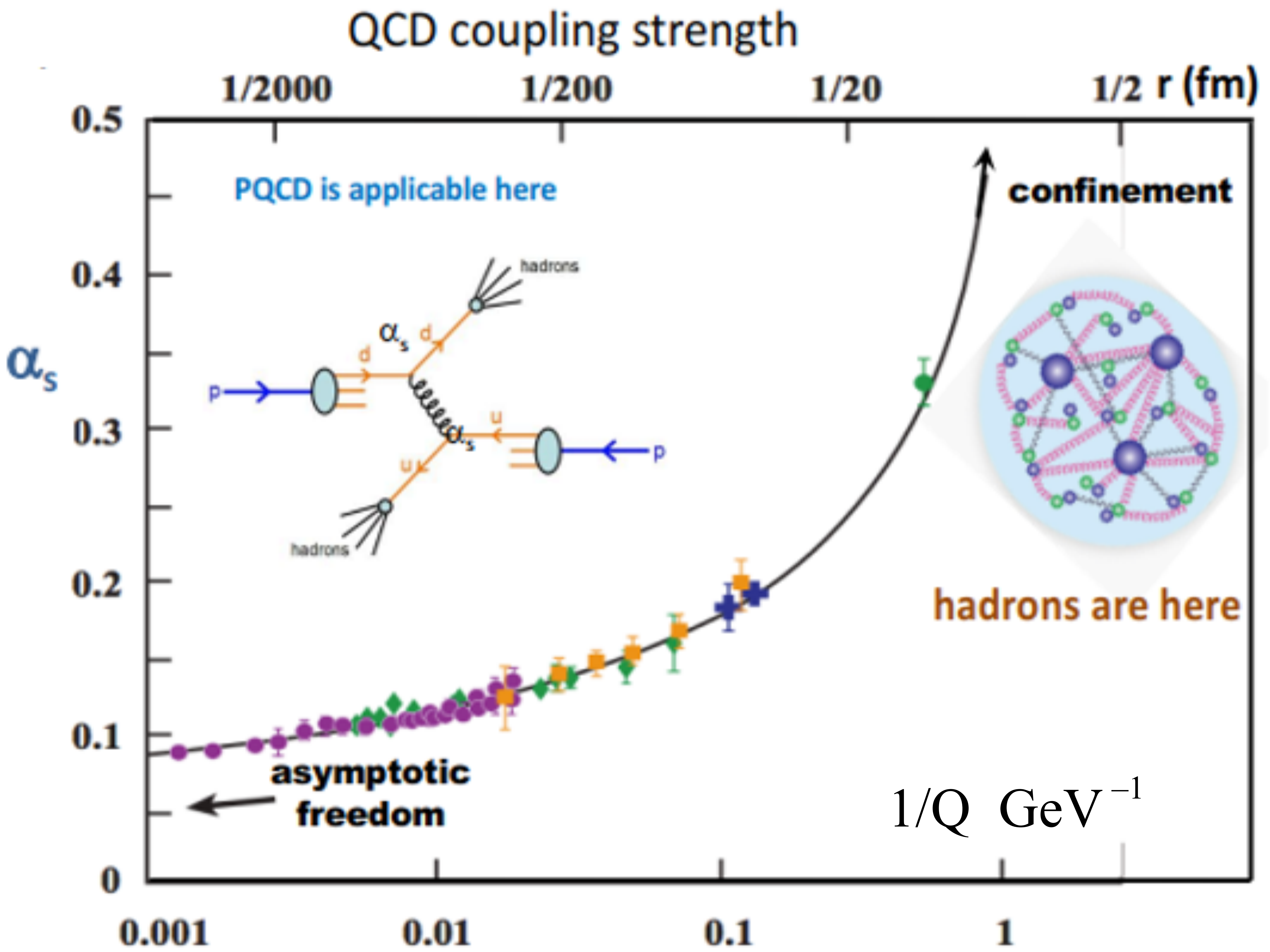
Bound state perturbation theory ??

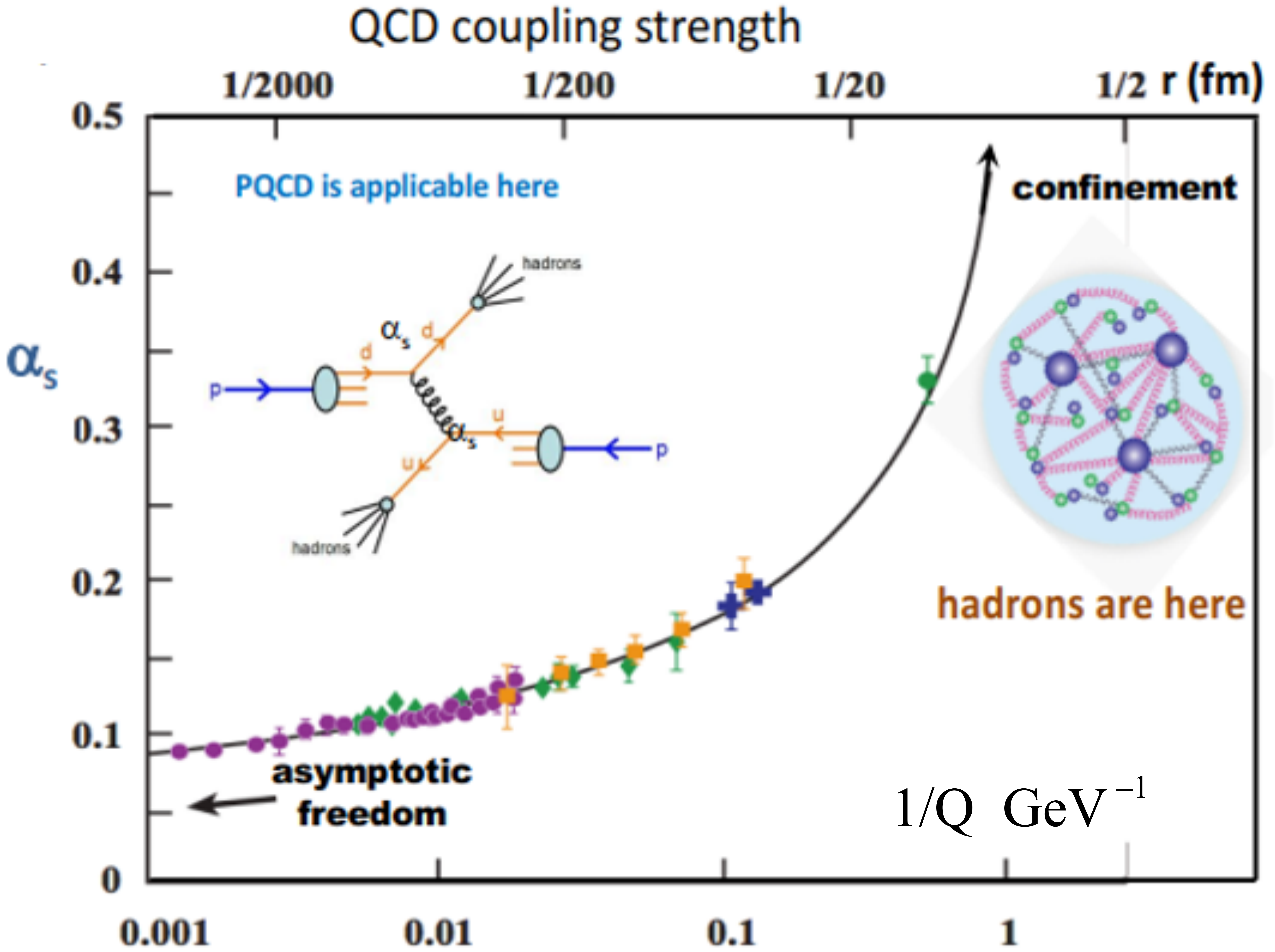
QED atoms: Yes, but tricky …

QCD hadrons: Non-perturbative!?

Bound state derivations based on the QED action, (*c.f.* the *S*-matrix) are needed for textbooks and QCD.


QCD coupling strength
1/2000
1/200
1/20
1/2 r (fm)
0.5
0.4
0.3
0.2
0.1
0
α_s
PQCD is applicable here
hadrons
p
d
d
α_s
u
u
p
hadrons
confinement
hadrons are here
asymptotic
freedom
$1/Q\ \ \mathrm{GeV}^{-1}$
0.001
0.01
0.1
1

Strongly bound $q\bar{q}$ and qqq states??



$n^{2s+1}\ell_J$	J^{PC}	I $= 1$ $u\bar{d}$, $\bar{u}d$, $\frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$
1^1S_0	0^{-+}	$\pi(140)$
1^3S_1	1^{--}	$\rho(770)$
1^3P_0	0^{++}	$a_0(1450)$
1^1P_1	1^{+-}	$b_1(1235)$
1^3P_1	1^{++}	$a_1(1260)$
1^3P_2	2^{++}	$a_2(1320)$
1^3D_1	1^{--}	$\rho(1700)$
1^1D_2	2^{-+}	$\pi_2(1670)$
1^3D_3	3^{--}	$\rho_3(1690)$
1^3F_4	4^{++}	$a_4(1970)$
1^3G_5	5^{--}	$\rho_5(2350)$
2^1S_0	0^{-+}	$\pi(1300)$
2^3S_1	1^{--}	$\rho(1450)$
2^3P_1	1^{++}	$a_1(1640)$
2^3P_2	2^{++}	$a_2(1700)$
2^1D_2	2^{-+}	$\pi_2(1880)$
3^1S_0	0^{-+}	$\pi(1800)$

Particle Data Group

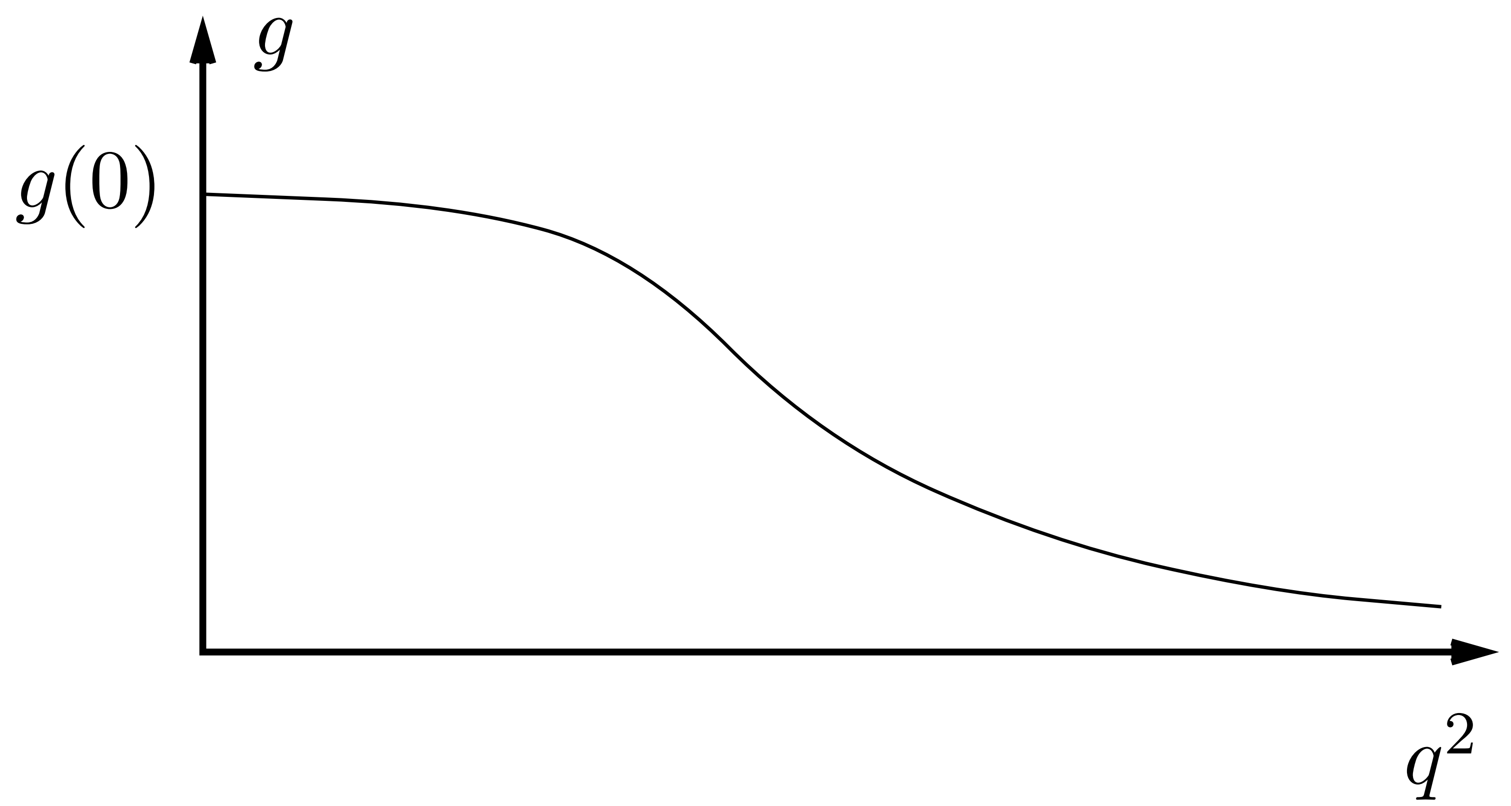
Gribov's perturbative confinement

There is a critical value of α_s :

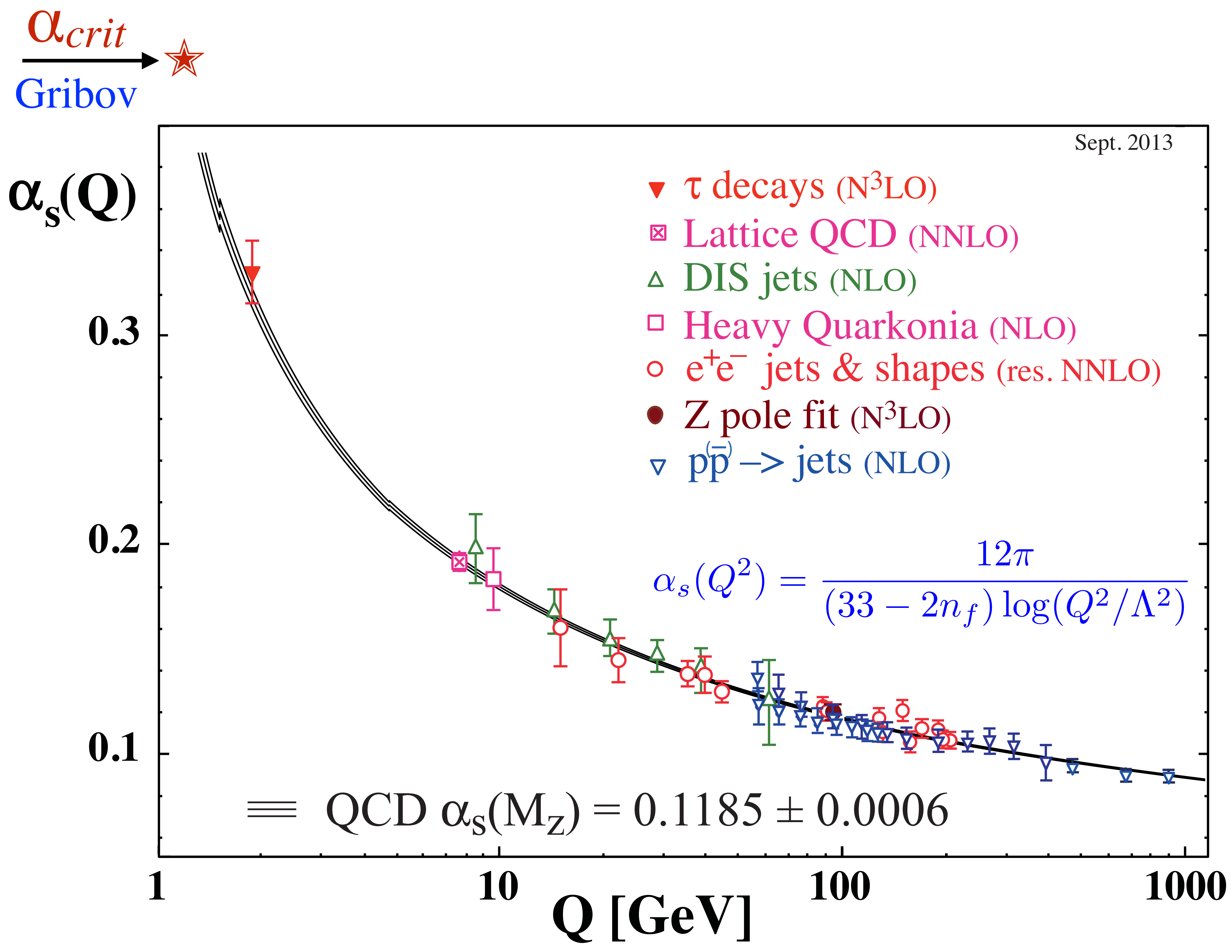
$$\alpha_s^{crit} = \pi C_F^{-1}\left(1 - \sqrt{2/3}\right) \simeq 0.43$$

At α_s^{crit} the structure of the vacuum changes and the coupling freezes.

D&K: "This number … is numerically small. Gribov's ideas … offer an intriguing possibility to address all the diversity and complexity of the hadron world from within the field theory with a reasonably small effective interaction strength."



Gribov, hep-ph/9807224 and hep-ph/9902279
Dokshitzer and Kharzeev, hep-ph/0404216



Positronium in QED (1)

$$|e^-e^+, \boldsymbol{P} = t = 0\rangle \equiv \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\, \bar{\psi}(\boldsymbol{x}_1)\,\Gamma\, F(\boldsymbol{x}_1 - \boldsymbol{x}_2)\psi(\boldsymbol{x}_2)\,|0\rangle$$

$\Gamma_{Para} = \gamma_5 \qquad J^{PC} = 0^{-+}$

$\Gamma^{\lambda}_{Ortho} = \boldsymbol{e}_\lambda \cdot \gamma^0 \boldsymbol{\gamma} \qquad J^{PC} = 1^{--}$

$H_{QED}\,|e^-e^+\rangle = (2m + E_B)\,|e^-e^+\rangle$

$\alpha \ll 1$ **NR approximation**

$$\Big(-\frac{\nabla^2}{m_e} - \frac{\alpha}{r} \Big) F(r) = E_B F(r)$$

$$F(r) = N \exp(-\alpha\, m\, r/2)$$

$$E_B = -\alpha^2 m/4$$

Analytic, non-perturbative solution for e^-e^+ eigenstates in the classical potential.

May serve as the starting point for a formally exact perturbative expansion.

Caswell and Lepage (1978)

Positronium in QED (2)

NRQED is used to calculate the perturbative expansion Caswell and Lepage (1986)

Positronium hyperfine splitting $\Delta E = M(^3S_1) - M(^1S_0)$ is given by a power expansion in α

$$\frac{\Delta E}{m_e} = \frac{7}{12}\alpha^4 - \left(\frac{8}{9} + \frac{\ln 2}{2}\right)\frac{\alpha^5}{\pi} - \frac{5}{24}\alpha^6 \ln\alpha + \left[\frac{1367}{648} - \frac{5197}{3456}\pi^2 + \left(\frac{221}{144}\pi^2 + \frac{1}{2}\right)\ln 2 - \frac{53}{32}\zeta(3)\right]\frac{\alpha^6}{\pi^2}$$

$$-\frac{7\alpha^7}{8\pi}\ln^2\alpha + \left(\frac{17}{3}\ln 2 - \frac{217}{90}\right)\frac{\alpha^7}{\pi}\ln\alpha + \mathcal{O}(\alpha^7)$$

Agrees with experiment

Adkins, Cassidy, and Pérez-Ríos (2022)

QED calculations for atoms are done in the rest frame only

Poincaré covariance is challenging for bound states (spatially extended)

In IF: Does the potential $-\alpha/|\boldsymbol{x}|$ remain instantaneous when $|\boldsymbol{P}| \gg M$? [Yes]

Do other Fock states than $|e^-e^+\rangle$ contribute when $|\boldsymbol{P}| \gg M$? [Yes]

Instantaneous potential (QED in IF)

No physical particle can move faster than light.

Gauge theory Lagrangians lack $\partial_t A^0$ and $\nabla \cdot \boldsymbol{A}$ terms.

$\Longrightarrow$ A^0 and A_L do not propagate in space-time. They are gauge dependent:

$\nabla \cdot \boldsymbol{A}(t,\boldsymbol{x}) = 0$ Coulomb gauge

$A^0(t,\boldsymbol{x}) = 0$ Temporal gauge

Gauge condition for all $\boldsymbol{x}$ at the same t induces an instantaneous potential

Consider here temporal gauge: quantization without constraints, $\boldsymbol{E} = -\partial_t \boldsymbol{A}$

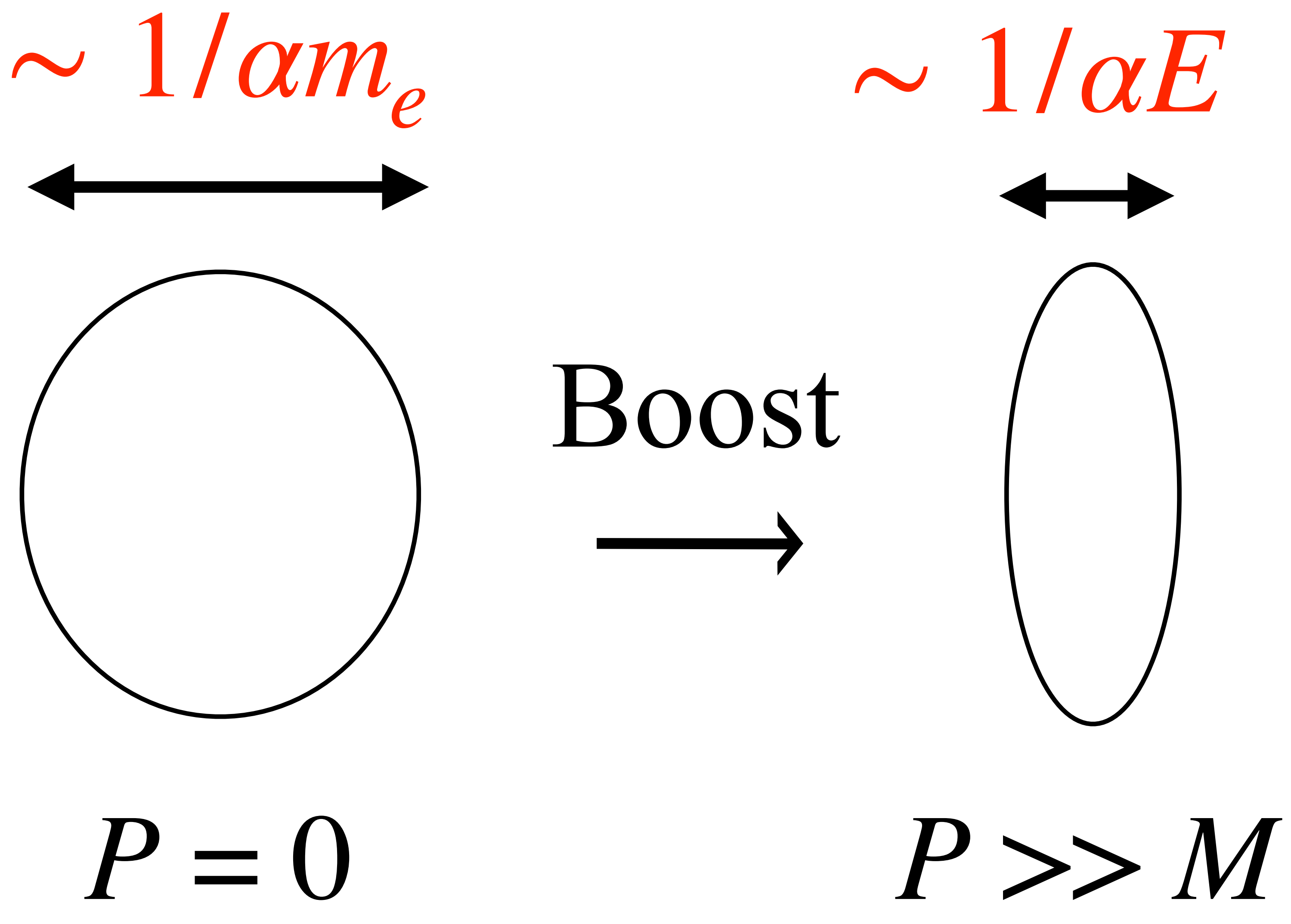
Invariance of physical states under t-independent gauge transformations requires:

$\left(\nabla \cdot \boldsymbol{E} - e\psi^\dagger\psi\right)|phys\rangle = 0$ Determines E_L from instantaneous electron positions:

$$\boldsymbol{E}_L(t,\boldsymbol{x})\,|phys\rangle = -\nabla_x \int d\boldsymbol{y}\, \frac{e}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\, \psi^\dagger\psi(t,\boldsymbol{y})\,|phys\rangle$$

Positronium in motion (IF)

Lorentz contraction



The Coulomb potential $-\dfrac{\alpha}{z}\dfrac{E}{M}$ grows with P,

wheras excitation energies decrease with P:

$$\sqrt{P^2+(2m+E_B)^2}-\sqrt{P^2+4m^2}\simeq\frac{2m\,E_B}{P}$$

The Poincaré covariance of atoms is realised dynamically in the IF.

QED: $|e^-e^+\gamma\rangle$ Fock state contributes to E_B at leading $\mathcal{O}(\alpha^2)$ for $P > 0$.

It subtracts the large Coulomb energy, ensuring $E=\sqrt{M^2+P^2}$

M. Järvinen, Phys. Rev. D71 (2005) 085006 [hep-ph/0411208]

Fock expansion in A^0=0 gauge (QED)

$$H(t)\,|M,\boldsymbol{P},t\rangle = \sqrt{M^2+\boldsymbol{P}^2}\,|M,\boldsymbol{P},t\rangle$$

M: Rest mass $\boldsymbol{P}$: CM momentum

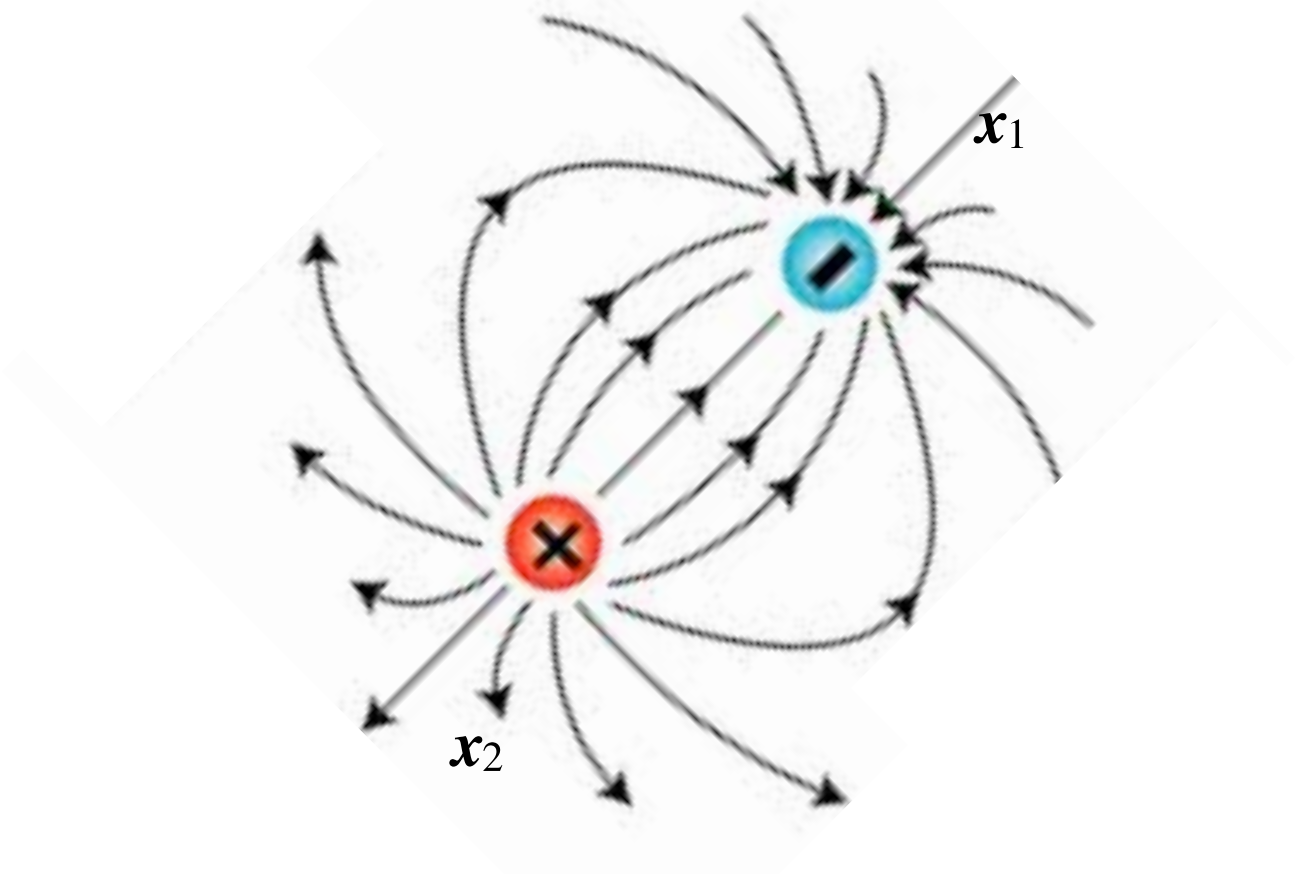
$$|M,\boldsymbol{P},t\rangle = \sum_j \Phi_j\,|\{e^-\},\{e^+\},\{\gamma\}\rangle_j$$

Fock expansion in terms of e^-, e^+, γ in temporal gauge ($A^0 = 0$)

Recall: $\big(\nabla\cdot\boldsymbol{E} - e\psi^\dagger\psi\big)\,|phys\rangle = 0$

Electrons in every Fock state interact through their instantaneous electric field:

$$\boldsymbol{E}_L(t,\boldsymbol{x})\,|phys\rangle = -\nabla_x\int d\boldsymbol{y}\,\frac{e}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,\psi^\dagger\psi(t,\boldsymbol{y})\,|phys\rangle$$



$$H(t) = \int d\boldsymbol{x}\big[H_0 + H_V + H_{int}\big] \qquad H_V = \int d\boldsymbol{x}\,\tfrac{1}{2}\boldsymbol{E}_L^2 \qquad H_{int} = -e\int d\boldsymbol{x}\,\psi^\dagger\boldsymbol{\alpha}\cdot\boldsymbol{A}\,\psi$$

Perturbative bound state expansion (QED)

Start from the $|e^-e^+\rangle$ Fock state of Positronium with momentum $\boldsymbol{P}$

$$|e^-e^+;\boldsymbol{P},t=0\rangle \equiv \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\,\bar{\psi}(\boldsymbol{x}_1)\,e^{i\boldsymbol{P}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2}\,\Phi^{(P)}(\boldsymbol{x}_1-\boldsymbol{x}_2)\,\psi(\boldsymbol{x}_2)\,|0\rangle$$

Caswell and Lepage (1978)

Operating with $H(t)$ creates an $|e^-e^+\gamma\rangle$ Fock component at $\mathcal{O}(e)$

Operating with $H(t)$ on $c_{e^+e^-}|e^-e^+\rangle + c_{e^+e^-\gamma}|e^-e^+\gamma\rangle$ creates $|e^-e^+\gamma\gamma\rangle$ and $|e^-e^+e^-e^+\rangle$ components at $\mathcal{O}(e^2)$, *etc.*

Form eigenstates of $H(t)$ at $\mathcal{O}(e^n)$ by including a sufficient number of Fock states.

This defines a perturbative expansion for bound states of any CM momentum $\boldsymbol{P}$

A systematic PQED approach to atoms allows to consider an extension to hadrons.

Quarkonia

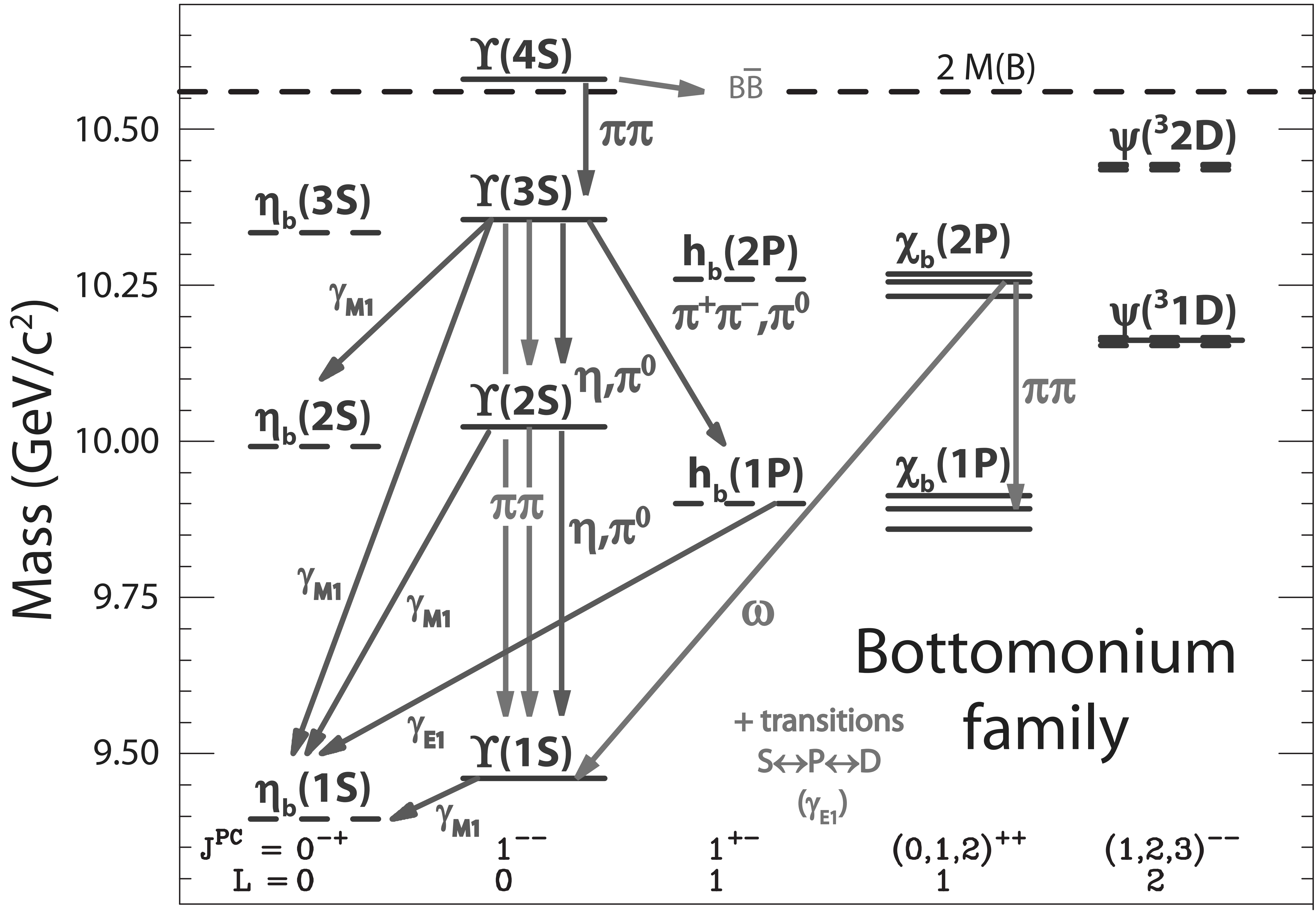
The $c\bar{c}$ and $b\bar{b}$ states are described by the Schrödinger eq. with the "Cornell potential"

$$V(r) = V'r - \frac{4}{3}\frac{\alpha_s}{r}$$

$$V' \simeq 0.18\ \mathrm{GeV}^2, \quad \alpha_s = \alpha_s(m_Q^2)$$

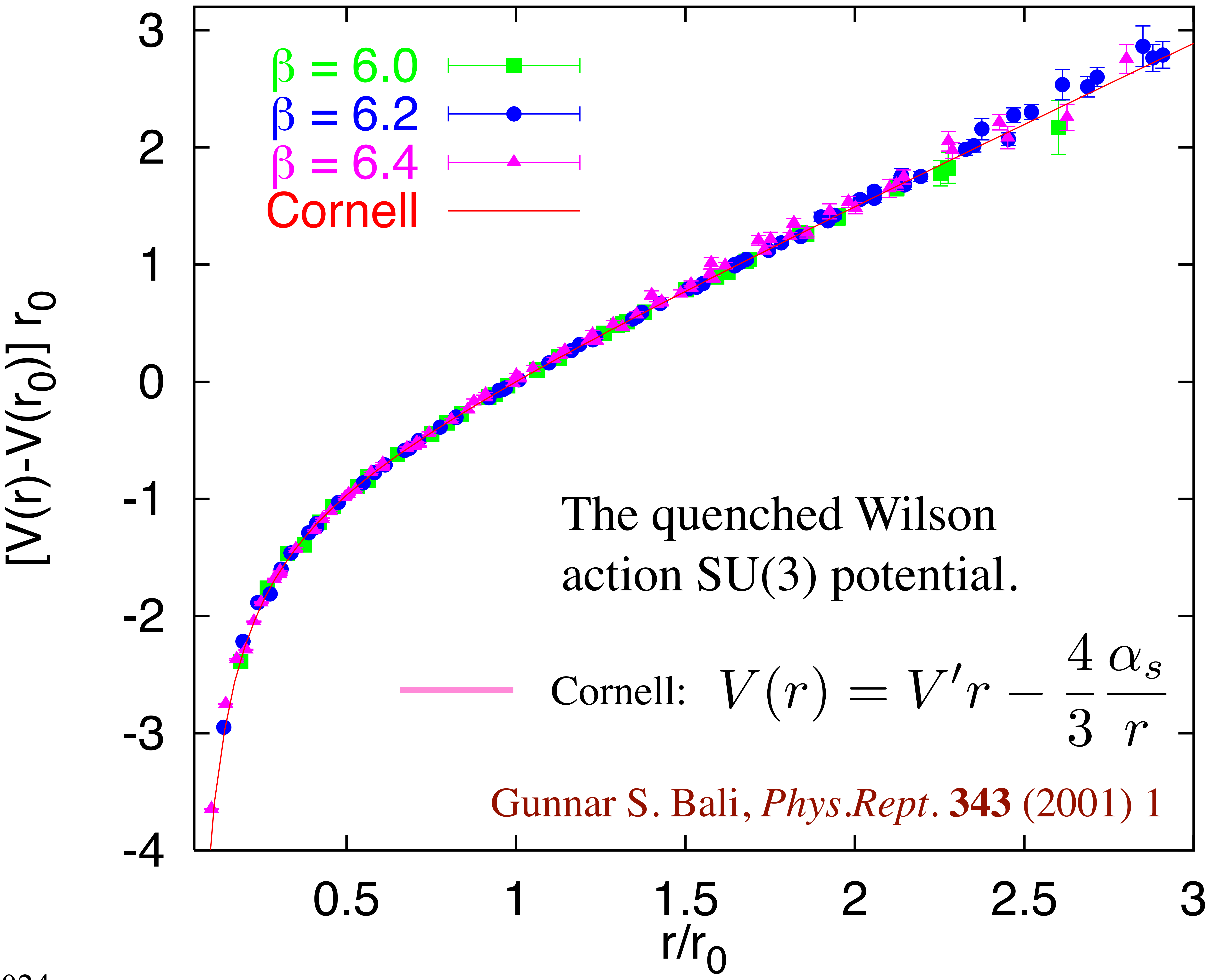
$V'r$ is treated as a classical potential.
It involves the confinement scale V'.

Transitions are calculated perturbatively in α_s, as in QED.



E. Eichten et al, Rev. Mod. Phys. **80** (2008) 1161

Lattice QCD confirmed the Cornell potential



Transitions are determined perturbatively

$\eta_c \to$ hadrons is calculated perturbatively, as $\eta_c \to gg$:

$$\frac{\Gamma(\eta_c \to gg)}{\Gamma(\eta_c \to \gamma\gamma)} = \frac{9[\alpha_S(m_c^2)]^2}{8\alpha^2}\left[1 + 8.2\frac{\alpha_S(m_c^2)}{\pi}\right]$$

No V´ dependence: The confining interaction does not create gluons

Consistent with the dominance of $q\bar{q}$ and qqq constituents
also for strongly bound states.

The inclusion of Λ_{QCD}

$\mathscr{L}_{QCD}$ does not have the confinement scale Λ_{QCD} .

Neither do the field equations of motion, nor the E_L constraint of $A^0 = 0$ gauge:

$$\left(\nabla \cdot \boldsymbol{E}_L^a + g f_{abc} \boldsymbol{A}_b \cdot \boldsymbol{E}_c - g \psi^\dagger T^a \psi\right) |phys\rangle = 0$$

The confinement scale must be introduced without changing $\mathscr{L}_{QCD}$:

Steven Weinberg: Quantum Theory of Fields (Vol. I)

"... Quantum field theory is the way it is because ... it is the only way to reconcile the principles of quantum mechanics ... with those of special relativity."

Λ_{QCD} may be introduced through a boundary condition: $E_L^a(|\boldsymbol{x}| \to \infty) \neq 0$.

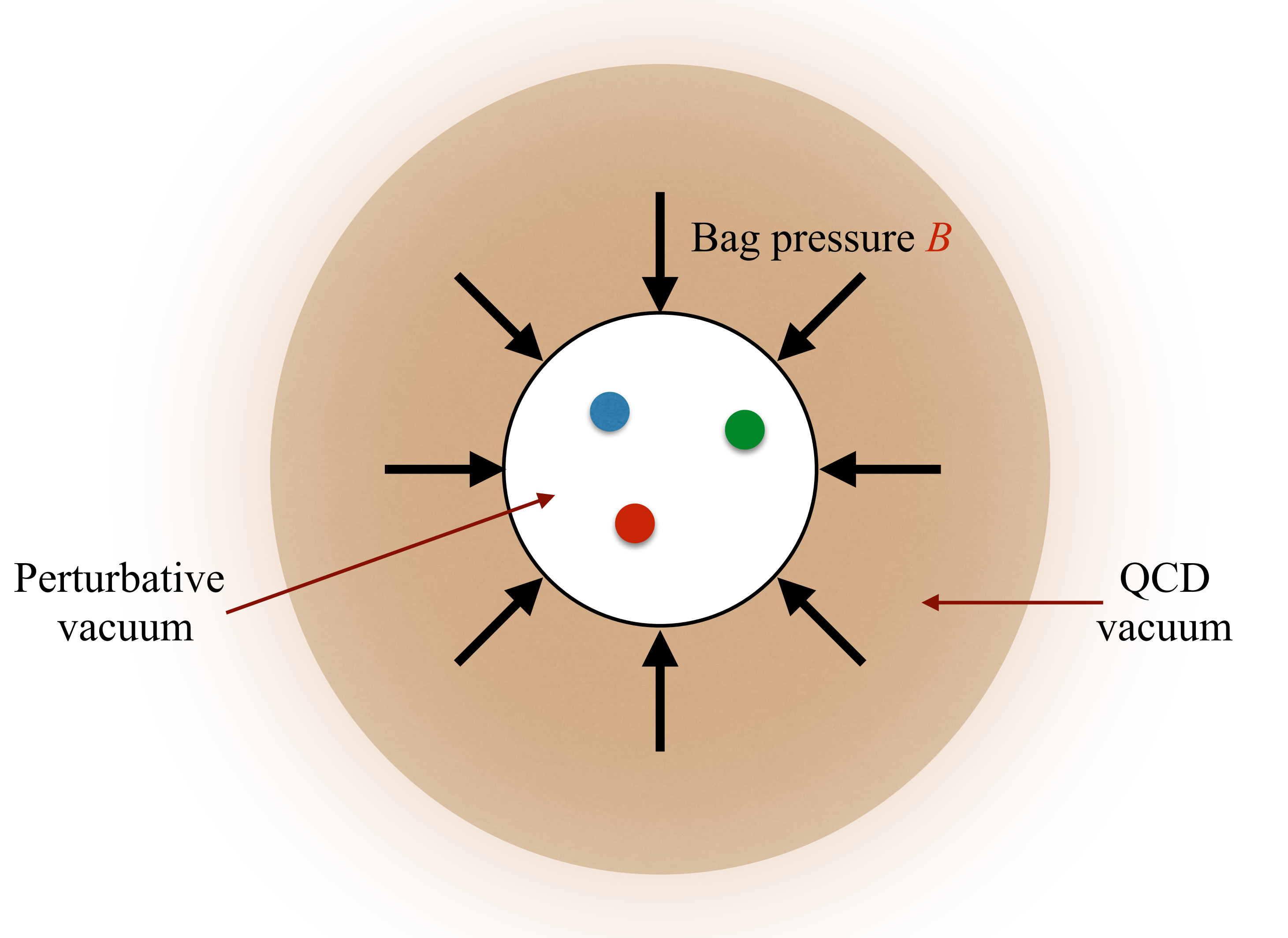
This implies a non-vanishing gluon field energy in the vacuum.

A non-vanishing field energy of the vacuum

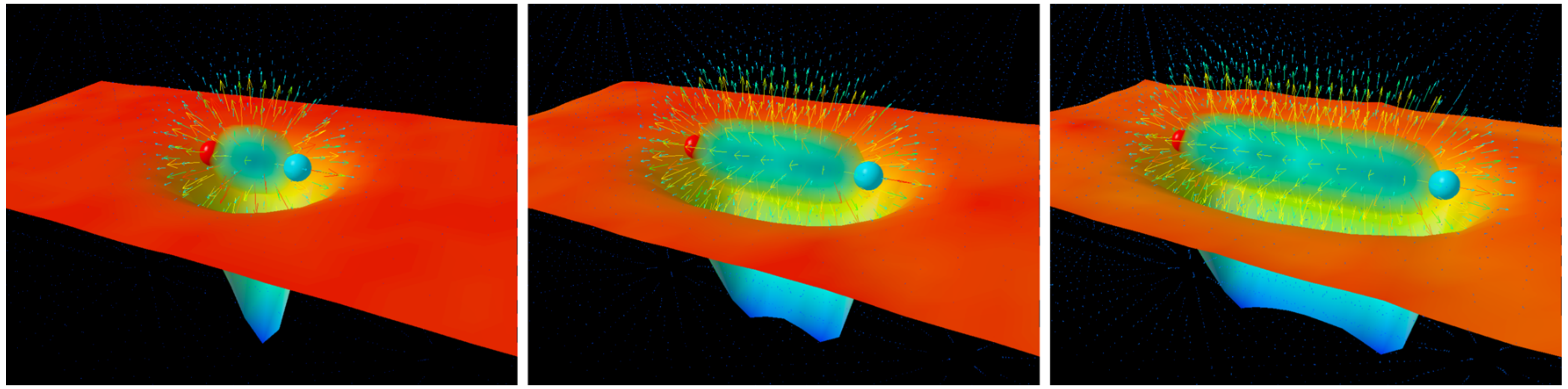
Recall the "Bag model": $\mathscr{L}_{bag} = (\mathscr{L}_{QCD} - B)\,\theta(bag)$

A. Chodos, et al., Phys. Rev. **D9** (1974) 3471

However, this modified $\mathscr{L}_{QCD}$.



Lattice QCD supports the physical picture of the bag model.



F. Gross, et al., Eur.Phys.J.C 83 (2023) 1125 [2212.11107]

Long-range effects from $E_L^a(|\boldsymbol{x}| \to \infty) \neq 0$?

QED: $\boldsymbol{E}_L(t,\boldsymbol{x})\,|phys\rangle = -\nabla_x \int d\boldsymbol{y}\, \frac{e}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\, \psi^\dagger\psi(t,\boldsymbol{y})\,|phys\rangle$ $\to 0$ for $|\boldsymbol{x}| \to \infty$

This gives the $-\alpha/r$ potentials of the e^- and e^+ in $\bar\psi(\boldsymbol{x}_1)\psi(\boldsymbol{x}_2)\,|0\rangle$
An external observer sees a dipole field, which must vanish for $|\boldsymbol{x}| \to \infty$

QCD: Color singlet states give $E_L^a(\boldsymbol{x}) = 0$ at all $\boldsymbol{x}$ (after summing over quark colors)

In $\bar\psi^A(\boldsymbol{x}_1)\psi^A(\boldsymbol{x}_2)\,|0\rangle$ the $q^A(\boldsymbol{x}_1)$ quark sees the $E_L^a(\boldsymbol{x})$ field of the $\bar q^A(\boldsymbol{x}_2)$ anti-quark

The $E_L^a(\boldsymbol{x})$ field of $\bar q^A(\boldsymbol{x}_2)$ need not, separately for each color A, vanish as $|\boldsymbol{x}| \to \infty$.

Consider a homogenous solution $\nabla \cdot \boldsymbol{E}_L^a(\boldsymbol{x})\,|phys\rangle = 0$ in solving for $\boldsymbol{E}_L^a(\boldsymbol{x})$ from

$$\nabla \cdot \boldsymbol{E}_L^a(\boldsymbol{x})\,|phys\rangle = g\big[-f_{abc}\boldsymbol{A}_b \cdot \boldsymbol{E}_c + \psi^\dagger T^a \psi(\boldsymbol{x})\big]\,|phys\rangle$$

Λ_{QCD} from a boundary condition

Poincaré symmetry restricts the form of the homogeneous solution:

$$\boldsymbol{E}_L^a(\boldsymbol{x})\,|phys\rangle = \nabla_x \int d\boldsymbol{y}\left[\kappa\, \boldsymbol{x}\cdot\boldsymbol{y} + \frac{g}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\right]\underbrace{\left[f_{abc}\boldsymbol{A}_b\cdot\boldsymbol{E}_c(\boldsymbol{y}) - \psi^\dagger T^a\psi(\boldsymbol{y})\right]}_{\mathscr{E}_a(\boldsymbol{y})}|phys\rangle$$

The contribution to the Hamiltonian is

$$\mathscr{H}_V \equiv \frac{1}{2}\int d\boldsymbol{x}\sum_a \boldsymbol{E}_L^a\cdot\boldsymbol{E}_L^a = \int d\boldsymbol{y}d\boldsymbol{z}\left\{\boldsymbol{y}\cdot\boldsymbol{z}\left[\tfrac{1}{2}\kappa^2\int d\boldsymbol{x} + g\kappa\right] + \frac{1}{2}\frac{\alpha_s}{|\boldsymbol{y}-\boldsymbol{z}|}\right\}\mathscr{E}_a(\boldsymbol{y})\mathscr{E}_a(\boldsymbol{z})$$

The term $\propto \kappa^2$ gives an $\boldsymbol{x}$-independent field energy density: $H_V \propto \int d\boldsymbol{x}$

For each state $|phys\rangle$, κ is determined such that the energy density is universal.

This leaves one physical scale Λ_{QCD}, given by the energy density of the vacuum

$q\bar{q}$ Fock state potential

$$|q(\boldsymbol{x}_1)\bar{q}(\boldsymbol{x}_2)\rangle \equiv \sum_A \bar{\psi}^A(\boldsymbol{x}_1)\psi^A(\boldsymbol{x}_2)|0\rangle$$

globally color singlet $q\bar{q}$ state

$$\mathscr{H}_V \equiv \frac{1}{2}\int d\boldsymbol{x}\sum_a \boldsymbol{E}_L^a \cdot \boldsymbol{E}_L^a$$

$$\mathscr{H}_V|q\bar{q}\rangle = V_{q\bar{q}}|q\bar{q}\rangle$$

Eigenstate of $\mathscr{H}_V$

$$V_{q\bar{q}}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| - C_F\frac{\alpha_s}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}$$

Cornell potential

This potential is valid also for relativistic $q\bar{q}$ Fock states, in any frame

The linear, confining part $\propto \Lambda^2$ is of $\mathcal{O}(\alpha_s^0)$.

The universal vacuum energy density is $$E_\Lambda = \frac{\Lambda^4}{2g^2 C_F}$$

Baryon Fock state potential

$$|\, q(\boldsymbol{x}_1)q(\boldsymbol{x}_2)q(\boldsymbol{x}_3)\rangle \equiv \sum_{A,B,C} \epsilon_{ABC}\psi_A^\dagger(\boldsymbol{x}_1)\psi_B^\dagger(\boldsymbol{x}_2)\psi_C^\dagger(\boldsymbol{x}_3)\,|\,0\rangle$$

Baryon Fock state

$$V_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = \Lambda^2\, d_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) - \frac{2}{3}\alpha_s\left(\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} + \frac{1}{|\boldsymbol{x}_2-\boldsymbol{x}_3|} + \frac{1}{|\boldsymbol{x}_3-\boldsymbol{x}_1|}\right)$$

Confining potential

$$d_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) \equiv \frac{1}{\sqrt{2}}\sqrt{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2 + (\boldsymbol{x}_2-\boldsymbol{x}_3)^2 + (\boldsymbol{x}_3-\boldsymbol{x}_1)^2}$$

When two of the quarks coincide the potential reduces to the $q\bar{q}$ potential:

$$V_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_2) = \Lambda^2\,|\boldsymbol{x}_1-\boldsymbol{x}_2| - \frac{4}{3}\frac{\alpha_s}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} = V_{q\bar{q}}(\boldsymbol{x}_1,\boldsymbol{x}_2)$$

Analogous potentials are obtained for any globally color singlet quark and gluon Fock state, such as $q\bar{q}g$ and gg.

$q\bar{q}$ bound states at $\mathcal{O}(\alpha_s^0)$

$$|q\bar{q}, \boldsymbol{P} = t = 0\rangle \equiv \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\, \bar{\psi}_\alpha^A(\boldsymbol{x}_1)\delta^{AB}\Phi_{\alpha\beta}(\boldsymbol{x}_1 - \boldsymbol{x}_2)\psi_\beta^B(\boldsymbol{x}_2)\,|0\rangle$$

The bound state condition $H\,|q\bar{q}\rangle = M\,|q\bar{q}\rangle$ gives

$$\big[i\gamma^0\boldsymbol{\gamma}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0\big]\Phi(\boldsymbol{x}) + \Phi(\boldsymbol{x})\big[i\gamma^0\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{\nabla}} - m\gamma^0\big] = \big[M - V(|\boldsymbol{x}|)\big]\Phi(\boldsymbol{x})$$

where $\boldsymbol{x} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$ and $V(\boldsymbol{x}) = \Lambda^2\,|\boldsymbol{x}\,|$

In the non-relativistic limit ($m \gg \Lambda$) this reduces to the Schrödinger equation.

$\Rightarrow$ The quarkonium phenomenology with the Cornell potential.

Example: $-\eta_P = \eta_C = (-1)^j$ states at $\mathcal{O}(\alpha_s^0)$

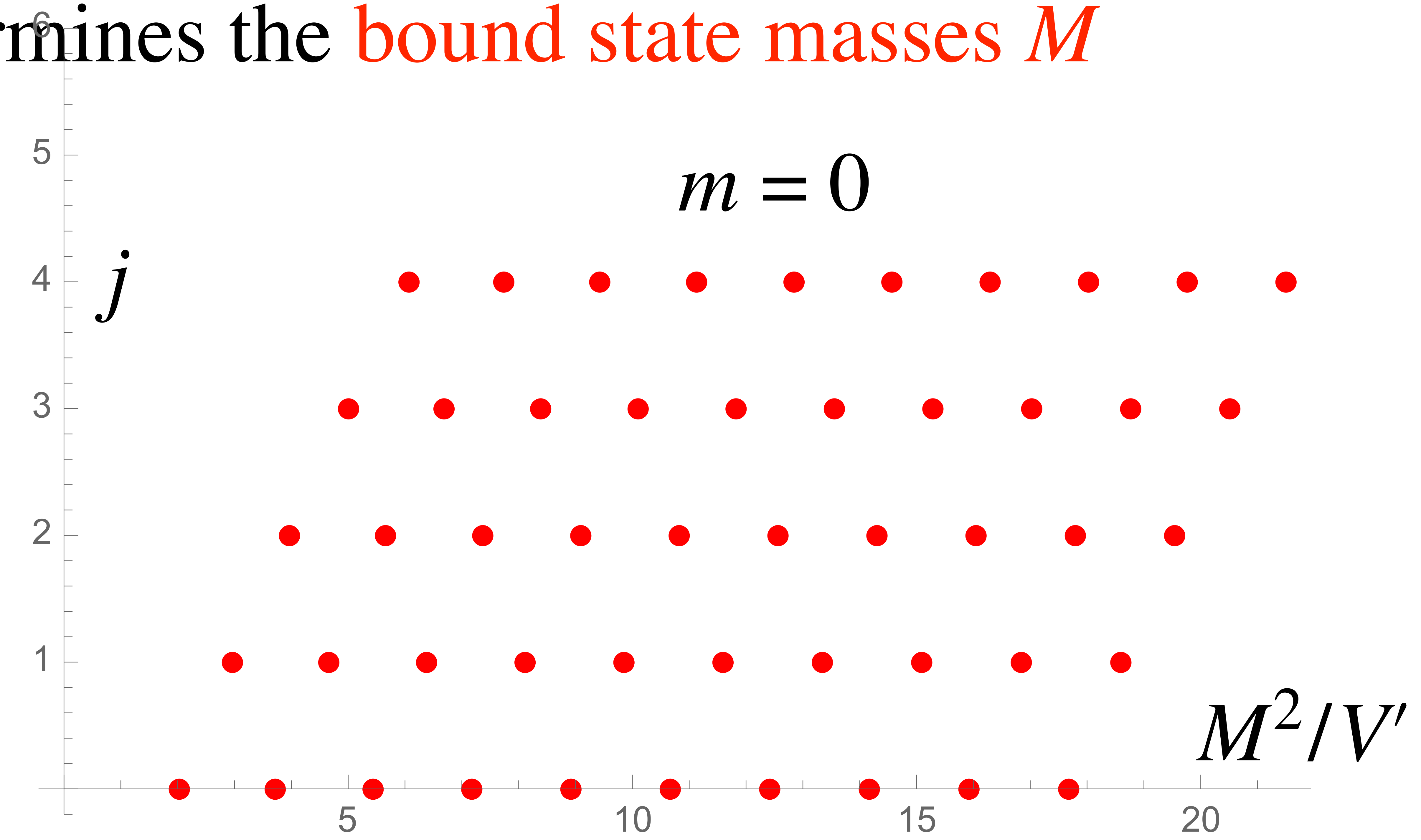
$$\Phi_{-+}(\boldsymbol{x}) = \Big[\frac{2}{M-V}(i\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}} + m\gamma^0) + 1\Big]\gamma_5\, F_1(r)Y_{j\lambda}(\hat{\boldsymbol{x}})$$

$$F_1'' + \Big(\frac{2}{r} + \frac{V'}{M-V}\Big)F_1' + \Big[\tfrac{1}{4}(M-V)^2 - m^2 - \frac{j(j+1)}{r^2}\Big]F_1 = 0$$ Radial equation

Regularity of the wave function determines the bound state masses M

Linear Regge trajectories with daughters:

Spectrum similar to dual models



Frame dependence of EM form factors

In a perturbative expansion each order in α_s, including $\mathcal{O}(\alpha_s^0)$,
must have exact Poincaré covariance. Boost covariance is dynamical in IF.

Check with electromagnetic form factor for any states A, B:

$$F_{AB}^{\mu}(y) = \langle B, \boldsymbol{P}_B | j^{\mu}(y) \, | A, \boldsymbol{P}_A \rangle = e^{i(P_B - P_A)\cdot y} \langle B, \boldsymbol{P}_B | j^{\mu}(0) \, | A, \boldsymbol{P}_A \rangle$$

$\partial_\mu F_{AB}^{\mu} = 0$ Gauge invariance OK

F_{AB}^{μ} Transforms as a 4-vector under boosts: Poincaré invariance OK

The three main points

1. Systematic perturbative methods bound states, based on the action, should be developed for bound states.

   Cf. the derivation of the perturbative S-matrix in the Interaction Picture.

2. The Poincaré covariance of bound states merits attention.

3. The confinement scale Λ_{QCD} must be introduced without changing $\mathscr{L}_{QCD}$.

PH: 2101.06721, 2304.11903