Inclusive Precision QCD in the High-Energy Domain

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Breaking the Ground

James Daniel Bjorken

1934-2024 © SLAC

- Bjorken scaling
- implied Feynman's naive parton model
- unpolarized and polarized Bjorken sum rule
- Bjorken-Paschos parton model
- The Bjorken Limit: essential for deciphering the sub-structure of nucleons: SLAC-MIT experiments
- How do quarks interact? What is the strong force ?

The Beginning

1966: SU_{3c}

Y. Nambu, Preludes in Theoretical Physics: In Honor of V.F. Weisskopf

1972/73: SU_{3c}

H. Fritzsch, M. Gell-Mann and H. Leutwyler, Phys. Lett. B 47 (1973) 365.

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \sum_{k=1}^{N_F} \overline{\psi}_k (i\gamma^\mu D_\mu - m_k)\psi_k + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FP}}$$

Yang-Mills:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \quad D_\mu = \partial_\mu - igT^a A^a_\mu$$

- Renormalizability, 't Hooft (1971)
- Anomaly Freedom of the SM, Bouchiat, Iliopoulos, Meyer (1972)
- Asymptotic freedom: $\beta(g(N_F)) < 1$ allowed perturbation theory to higher orders, Gross, Wilczek, Politzer (1973).
- **Precision analyzes of the strong sector are possible.**

Testing of a Theory

- Only simple constructions work.
- The drop of an apple lead to Newton's law (after Tycho Brahe & Kepler).
- Do not test the Bohr-Sommerfeld atom model beginning with Xenon, but start with hydrogen.
- Do not test QCD in low energy nuclear reactions first [or even in color-neutral environments], but at high scales under inclusive conditions, etc.

- The extraction of fundamental quantities need different clear and simple experiments.
- More involved situations can be described as well, but there the basic key quantities are harder to isolate unambiguously.

What do we want to know ?

Fundamental quantities of the Standard Model

- $g_s = \sqrt{4\pi\alpha_s}$
- m_c, m_b, m_t
- The twist–2 parton densities: $q_i, \overline{q}_i, g$, $\Delta q_i, \Delta\overline{q}_i, \Delta g$.

- as precisely as possible.

Because of confinement: high virtualities Q^2 are needed to probe these quantities.

- Deep-inelastic scattering (fixed target, HERA, EIC)
- hard scattering processes at e^+e^- (LEP) and hadron colliders (LHC)

The QCD β function

- 1973 : One loop: Gross, Wilczek; Politzer
- 1974 : Two loops: Caswell; Jones
- 1980 : Three loops: Tarasov, Vladimirov, Zharkov (conf.: Larin, Vermaseren, 1993; Chetyrkin, Misiak, Münz, 1997)
- 1997 : Four loops: Larin, van Ritbergen, Vermaseren (conf.: Czakon, 2005)
- 2016/17 : Five loops: Baikov, Chetyrkin, Kühn; Herzog, Ruijl, Ueda, Vermaseren, Vogt; Luthe, Maier, Marquard, Schröder

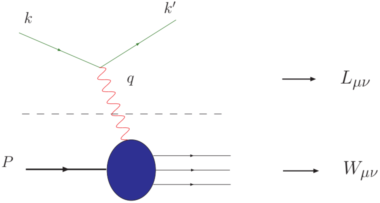
$$\frac{da_s}{d\ln(\mu^2)} = -\sum_{k=0}^{\infty} \beta_k a_s^{k+2}, \qquad a_s = \frac{\alpha_s}{4\pi}$$

$$\beta_0 = 11 - \frac{2}{3} N_F$$

$$\beta(\alpha_s, N_F = 5) = \beta_0[1 + 0.40135\alpha_s + 0.14943\alpha_s^2 + 0.31722\alpha_s^3 + 0.08092\alpha_s^4]$$

Also: other zero-scale anomalous dimensions.

Unpolarized Deep–Inelastic Scattering (DIS):



$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q} \quad \text{Bjorken–x}$$

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)\ .$$

Structure Functions: $F_{2,L}$ contain light and heavy quark contributions.
At 3-Loop order also graphs with two heavy quarks of different mass contribute.
$\Longrightarrow$ Single and 2-mass contributions: c and b quarks in one graph.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \, \delta(x - yz) f(y) g(z) \; .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx \, x^{N-1} f(x) \; .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)\ .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j \mid O_i \mid j\rangle\ .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO [Moch, Vermaseren, Vogt, 2005; JB, Marquard, Schneider, Schönwald, 2022].

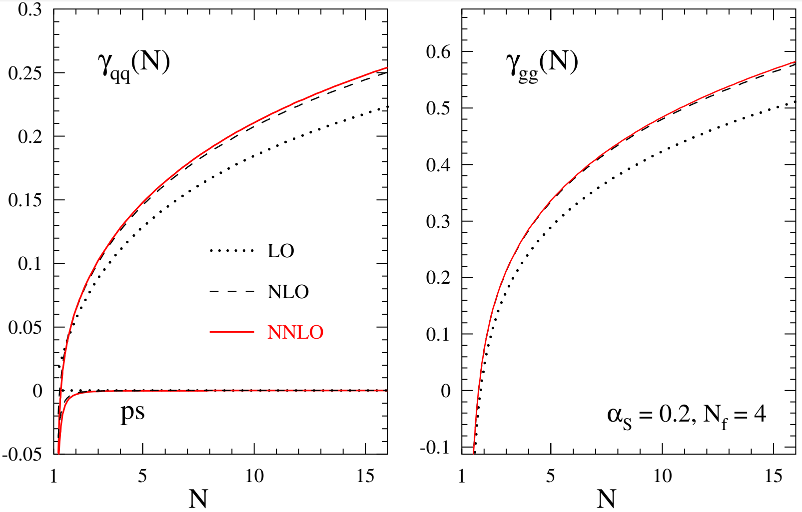
For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

The anomalous dimensions

- 1973 : One loop: Gross, Wilczek; Georgi, Politzer
- 1975 : One loop: Sasaki; Ahmed, Ross [pol.]
- 1977 : One loop: partonic splitting functions Altarelli, Parisi
- 1977/92 : Two loops: Floratos, Ross, Sachrajda, Gonzalez-Arroyo, Lopez, Yndurain, Curci, Furmanski, Petronzio, Kounnas, Floratos, Lacaze, Hamberg, van Neerven
- 1995 : Two loops: Mertig, van Neerven; Vogelsang [pol.]
- 2004 : Three loops: Moch, Vermaseren, Vogt; Ablinger et al. (2014, 2017); Anastasiou et al. (2015); Mistlberger (2018); Duhr et al. (2020); Luo et al. (2019); Ebert et al. (2020), JB, Marquard, Schneider, Schönwald (2021); Baranowski et al. (2022); Gehrmann et al. (2023)
- 2014 : Three loops: Moch, Vermaseren, Vogt; Behring et al. (2019); JB, Marquard, Schneider, Schönwald (2021) [pol.]
- 2006(16): Four loops: [Moments] Baikov, Chetyrkin, Kühn; Velizhanin; Davies, Falcioni, Herzog, Moch, Ruijl, Ueda, Vermaseren, Vogt
- > 1994 : Large N_F expansion to all orders Gracey et al.

Different methods: Forward Compton amplitude; on-shell massive OMEs; off-shell massless OMEs.

Anomalous dimensions

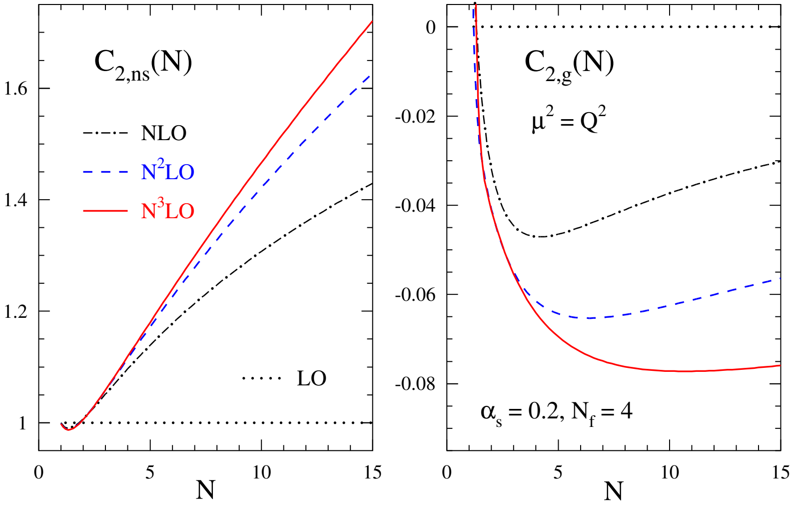


from: Moch, Vermaseren, Vogt, 2004

The massless DIS Wilson coefficients

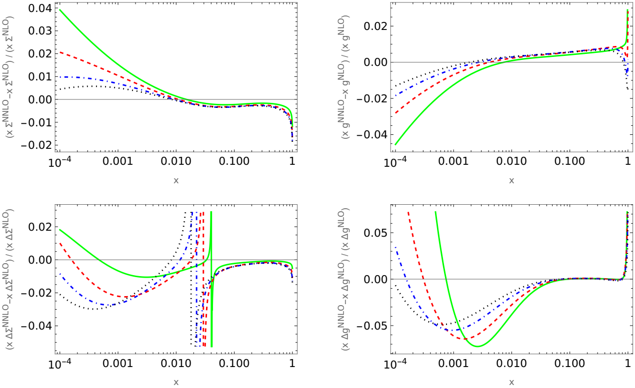
- 1979/80: One loop: Bardeen, Buras, Duke, Muta + various other authors. Consensus on correct results are reached in: Furmanski, Petronzio
- 1989 : One loop: Bodwin, Qiu [pol.]
- 1987/92: Two loops: Kazakov, Kotikov; Sanchez-Guillen et al., van Neerven, Zijlstra; Moch, Vermaseren (1999)
- 1993 : Two loops: van Neerven, Zijlstra [pol.]
- 2005/22: Three loops: Moch, Vermaseren, Vogt; JB, Marquard, Schneider, Schönwald
- 2022 : Three loops: JB, Marquard, Schneider, Schönwald [pol.]
- 2023 : Four loops: NS $O(N_F^2)$ Basdew-Sharma, Pelloni, Herzog, Vogt

Wilson coefficients



from: Moch, Vermaseren, Vogt, 2005

The unpolarized and polarized NNLO evolution



$Q^2 = 10, 10^2, 10^3, 10^4$ GeV^2 dotted, dash-dotted, dashed, full lines.

At the 1% level of DIS structure functions N^{2}LO is not yet enough at the theory side. High luminosity measurements at EIC and perhaps LHeC.

Why are Heavy Flavor Contributions important ?

- They form a significant contribution to F_2, F_L and g_1 particularly at small x and high Q^2.
- Concise 3-loop corrections are needed to determine $\alpha_s(M_Z)$, m_c and perhaps m_b.
- The accuracy of measurements at the LHC reaches a level of precision requiring 3-loop VFNS matching, including 2–mass corrections.

Heavy flavor Wilson coefficients

- 1976/80 : One loop: Witten; Babcock, Sivers, Wolfram; Shifman, Vainshtein, Zakharov; Leveille, Weiler; Glück, Hoffmann, Reya
- 1981/90 : One loop: Watson; Glück, Reya, Vogelsang [pol.]
- 1992 : Two loops: Laenen, Riemersma, Smith, van Neerven
- 1995 : Two loops: Buza, Matiounine, Smith, van Neerven; Hekhorn, Stratmann (2018) [pol.]
- 2010/now: Three loops: Ablinger, Behring, JB, De Freitas, Hasselhuhn, von Manteuffel, Raab, Round, Saragnese, Schneider, Schönwald, Wißbrock [unpol + pol.] + two-mass case.
- 2024: all finished unpolarized and polarized, except the 2-mass contribution to $(\Delta)A_{Qg}^{(3)}$ coming soon

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
\color{red}{2014}\ L_{2,q}^{\mathsf{NS}}(n_f) &= a_s^2\left[A_{qq,Q}^{\mathsf{NS},(2)}(n_f)+\hat{C}_{2,q}^{\mathsf{NS},(2)}(n_f)\right]\\
&+ a_s^3\left[A_{qq,Q}^{\mathsf{NS},(3)}(n_f)+A_{qq,Q}^{\mathsf{NS},(2)}(n_f)C_{2,q}^{\mathsf{NS},(1)}(n_f)+\hat{C}_{2,q}^{\mathsf{NS},(3)}(n_f)\right]\\
\color{red}{2010}\ \tilde{L}_{2,q}^{\mathsf{PS}}(n_f) &= a_s^3\left[\ \tilde{A}_{qq,Q}^{\mathsf{PS},(3)}(n_f)+A_{gq,Q}^{(2)}(n_f)\ \tilde{C}_{2,g}^{(1)}(n_f+1)+\hat{\tilde{C}}{}_{2,q}^{\mathsf{PS},(3)}(n_f)\right]\\
\color{red}{2010}\ \tilde{L}_{2,g}^{\mathsf{S}}(n_f) &= a_s^2A_{gg,Q}^{(1)}(n_f)\tilde{C}_{2,g}^{(1)}(n_f+1)\\
&+ a_s^3\Big[\ \tilde{A}_{qg,Q}^{(3)}(n_f)+A_{gg,Q}^{(1)}(n_f)\ \tilde{C}_{2,g}^{(2)}(n_f+1)+A_{gg,Q}^{(2)}(n_f)\ \tilde{C}_{2,g}^{(1)}(n_f+1)\\
&\qquad + A_{Qg}^{(1)}(n_f)\ \tilde{C}_{2,q}^{\mathsf{PS},(2)}(n_f+1)+\hat{\tilde{C}}{}_{2,g}^{(3)}(n_f)\Big]\\
\color{red}{2014}\ H_{2,q}^{\mathsf{PS}}(n_f) &= a_s^2\left[\ A_{Qq}^{\mathsf{PS},(2)}(n_f)+\ \tilde{C}_{2,q}^{\mathsf{PS},(2)}(n_f+1)\right]\\
&+ a_s^3\Big[\ A_{Qq}^{\mathsf{PS},(3)}(n_f)+\ \tilde{C}_{2,q}^{\mathsf{PS},(3)}(n_f+1)+A_{gq,Q}^{(2)}(n_f)\ \tilde{C}_{2,g}^{(1)}(n_f+1)\\
&\qquad +A_{Qq}^{\mathsf{PS},(2)}(n_f)\ C_{2,q}^{\mathsf{NS},(1)}(n_f+1)\Big]\\
\color{red}{2024}\ H_{2,g}^{\mathsf{S}}(n_f) &= a_s\left[\ A_{Qg}^{(1)}(n_f)+\ \tilde{C}_{2,g}^{(1)}(n_f+1)\right]\\
&+ a_s^2\Big[\ A_{Qg}^{(2)}(n_f)+\ A_{Qg}^{(1)}(n_f)\ C_{2,q}^{\mathsf{NS},(1)}(n_f+1)+\ A_{gg,Q}^{(1)}(n_f)\ \tilde{C}_{2,g}^{(1)}(n_f+1)\\
&\qquad + \tilde{C}_{2,g}^{(2)}(n_f+1)\Big]\\
&+ a_s^3\Big[\ A_{Qg}^{(3)}(n_f)+\ A_{Qg}^{(2)}(n_f)\ C_{2,q}^{\mathsf{NS},(1)}(n_f+1)+\ A_{gg,Q}^{(2)}(n_f)\ \tilde{C}_{2,g}^{(1)}(n_f+1)\\
&\qquad + A_{Qg}^{(1)}(n_f)\left[C_{2,q}^{\mathsf{NS},(2)}(n_f+1)+\ \tilde{C}_{2,q}^{\mathsf{PS},(2)}(n_f+1)\right]+\ A_{gg,Q}^{(1)}(n_f)\ \tilde{C}_{2,g}^{(2)}(n_f+1)\\
&\qquad + \tilde{C}_{2,g}^{(3)}(n_f+1)\Big]\ .
\end{aligned}
$$

All unpolarized logarithmic corrections are known since 2010 [polarized: since 2021].

Variable Flavor Number Scheme

$$
\begin{aligned}
f_k(n_f+1,\mu^2) + f_{\bar{k}}(n_f+1,\mu^2) &= \boxed{A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right)} \otimes \left[f_k(n_f,\mu^2) + f_{\bar{k}}(n_f,\mu^2)\right] \\
&+ \boxed{\tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)} \otimes \Sigma(n_f,\mu^2) + \boxed{\tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)} \otimes G(n_f,\mu^2) \\
f_{Q+\bar{Q}}(n_f+1,\mu^2) &= \boxed{\tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)} \otimes \Sigma(n_f,\mu^2) + \boxed{\tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)} \otimes G(n_f,\mu^2)\ . \\
G(n_f+1,\mu^2) &= \boxed{A_{gq,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)} \otimes \Sigma(n_f,\mu^2) + \boxed{A_{gg,Q}^{S}\left(n_f,\frac{\mu^2}{m^2}\right)} \otimes G(n_f,\mu^2)\ . \\
\Sigma(n_f+1,\mu^2) &= \sum_{k=1}^{n_f+1}\left[f_k(n_f+1,\mu^2) + f_{\bar{k}}(n_f+1,\mu^2)\right] \\
&= \left[\boxed{A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right)} + n_f \boxed{\tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)} + \boxed{\tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)}\right] \\
&\otimes\Sigma(n_f,\mu^2) \\
&+ \left[n_f \boxed{\tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)} + \boxed{\tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)}\right] \otimes G(n_f,\mu^2)
\end{aligned}
$$

$A_{qq,Q}^{(3),PS}$, $A_{qg,Q}^{(3)}$: 2010, $A_{qq}^{(3),NS}$, $A_{Qq}^{(3),PS}$, $A_{gq}^{(3)}$: 2014, $A_{gg}^{(3)}$: 2022, $A_{Qg}^{(3)}$: 2024

There are generalizations necessary in the 2-mass case.

J. Ablinger et al., Nucl.Phys. B921 (2017) 585 [arXiv:1705.07030]

Computer-algebraic framework: Principal computation steps

- QGRAF, Nogueira, 1993 Diagram generation
- FORM, Vermaseren, 2001; Tentyukov, Vermaseren, 2010 Lorentz algebra
- Color, van Ritbergen, Schellekens and Vermaseren, 1999 Color algebra
- Reduze 2 Studerus, von Manteuffel, 2009/12, Crusher, Marquard, Seidel IBPs
- Method of arbitrary high moments, JB, Schneider, 2017 Computing large numbers of Mellin moments
- Guess, Kauers et al. 2009/2015; JB, Kauers, Schneider, 2009 Computing the recurrences
- Sigma, EvaluateMultiSums, SolveCoupledSystems, Schneider, 2007/14 Solving the recurrences
- OreSys, Zürcher, 1994; Gerhold, 2002; Bostan et al., 2013 Decoupling differential and difference equations
- Diffeq, Ablinger et al, 2015, JB, Marquard, Rana, Schneider, 2018 Solving differential equations
- HarmoncisSums, Ablinger and Ablinger et al. 2010-2019 Simplifying nested sums and iterated integrals to basic building blocks, performing series and asymptotic expansions, Almkvist-Zeilberger algorithm etc.

How to integrate analytically ?

In the massive case the simple view of just harmonic sums, like widely in the massless case, fails.

Anti-Differentiation:

- Risch-like algorithms: define the proper function space, in which your problem can be solved
- obtain the basis of this function space over which the integrals are expressed
- shuffle algebras reduce the original integrals

Iterative integrals over $\mathfrak{A}$:

$$G_{a,\vec{b}}(x) = \int_0^x dy f_b(y) G_{\vec{b}}(y)$$

Alphabet:

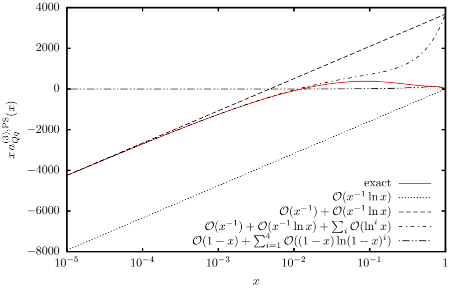
$$\mathfrak{A} = \left\{ \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, ..., g(x), ... \right\}$$

Here $g(x)$ can be a higher transcendental function, which is no iterative integral.

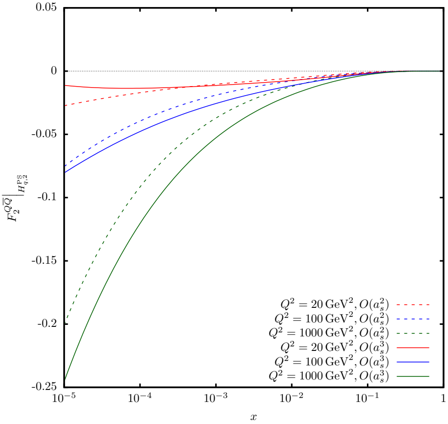
For more details on the different methods see:

JB and C. Schneider, Int. J. Mod. Phys. **A 33** (2018) 17, 1830015.

$H_{q,2}^{\mathrm{PS}}$



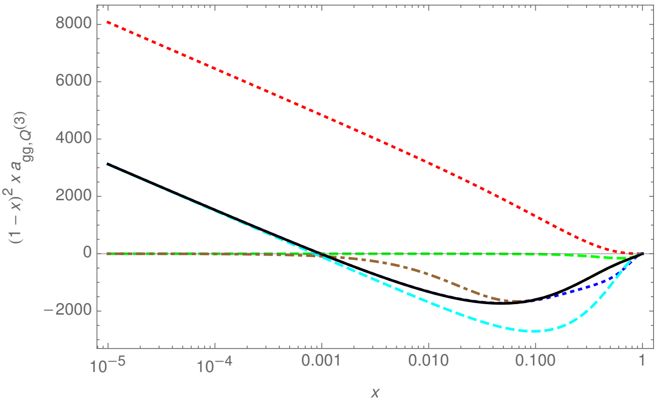
$a_{Qq}^{(3),\mathrm{PS}}$



Contribution to $F_2(x, Q^2)$

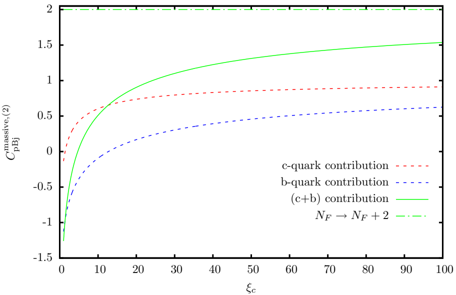
The leading small x approximation corresponding to High-energy factorization and small x heavy flavor production S. Catani, M. Ciafaloni, F. Hautmann, Nucl.Phys. B366 (1991) 135 departs from the physical result everywhere except for $x = 1$ (dotted line).

$a_{gg,Q}^{(3)}$

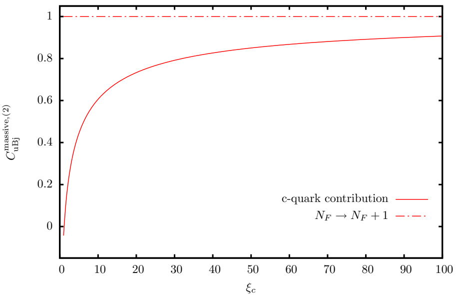


The non–N_F terms of $a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Full line (black): complete result; upper dotted line (red): term $\propto \ln(x)/x$, BFKL limit; lower dashed line (cyan): small x terms $\propto 1/x$; lower dotted line (blue): small x terms including all $\ln(x)$ terms up to the constant term; upper dashed line (green): large x contribution up to the constant term; dash-dotted line (brown): complete large x contribution.

$O(\alpha_s^2)$ Complete NS corrections [including power corrections]



pol BJ sum rule

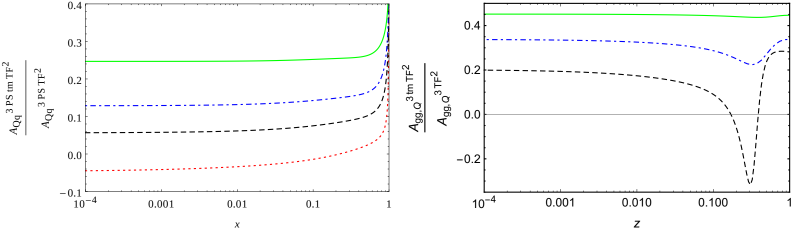


unp. BJ sum rule

Note the negative corrections at low Q^2!
Already for charm it takes quite a while to become massless.

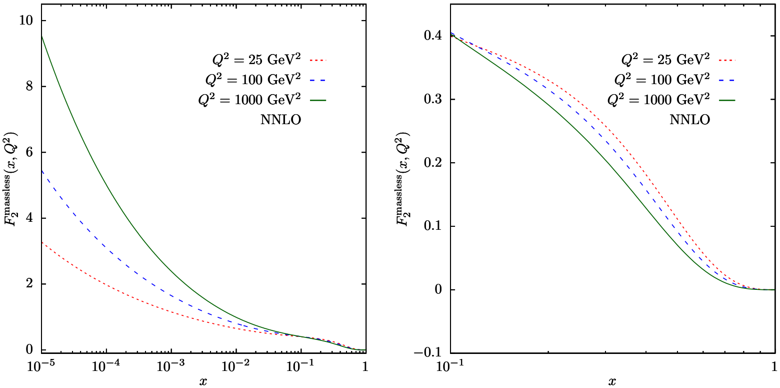
JB, G. Falcioni, A. De Freitas, Nucl. Phys. B910 (2016) 568.

3-Loop 2 Mass contributions: PS and gg



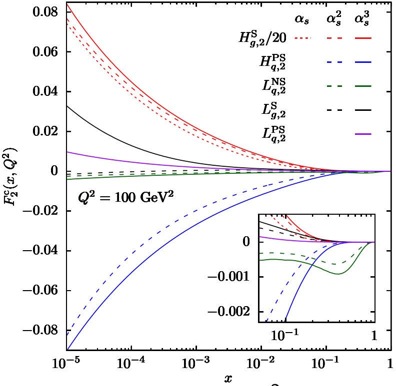
The 2-mass contributions are a significant part of the the T_F^2 terms (also in the other channels).

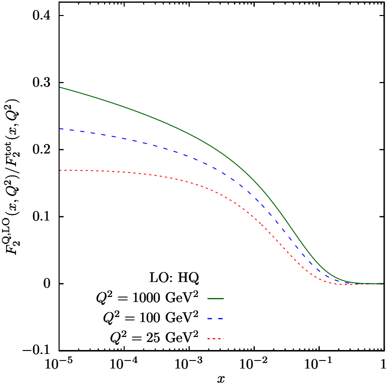
The massless contributions to F_2



$N_F = 3$ massless quarks @ NNLO.

Single-mass contributions to $F_2^{c,b}$

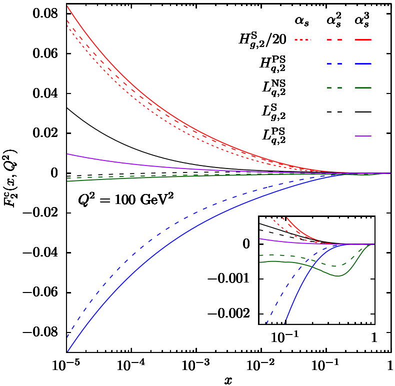


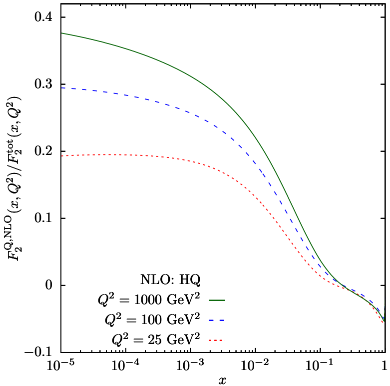


charm for $Q^2 = 100$ GeV^2. c and b single mass

Allows to strongly reduce the current theory error on m_c.

Single-mass contributions to $F_2^{c,b}$





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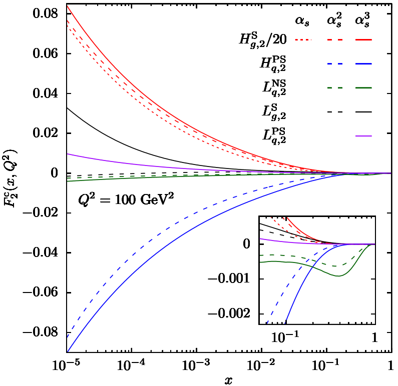

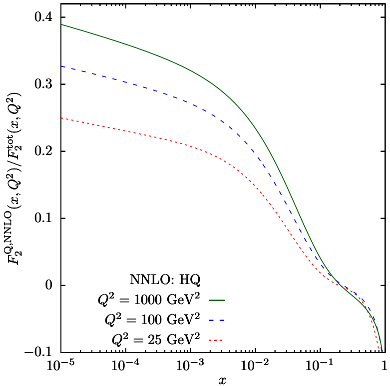

charm for $Q^2 = 100$ GeV^2.

c and b single mass

Allows to strongly reduce the current theory error on m_c.

Conclusions

- After 50 years we know a lot more on strong interactions at very small distances.
- QCD is very well established and reached a very good level of precision.
- We have learnt much more than believed on the structure of Feynman integrals and what QFTs are at the quantitative side.
- Theoretical physics and mathematics inspired each other.
- This fantastic development started in 1972/73.
- **As for the Future:** Fcc_ee and Fcc_pp are needed to probe and explore even higher scales and finer structures.
- One shall start to construct and build these machines as soon as possible.
- Associated theory developments have to proceed with even more and new efforts, along with the development of further computeralgebraic and mathematical technologies.