NNLO QCD Calculation
of $\bar{B} \rightarrow X_s \gamma$

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“Flavour in the era of the LHC” workshop,
3rd meeting, 15-17 May, 2006, CERN
Next 20 Minutes

• Introduction and Motivation
• Status of NNLO QCD Calculation
• Conclusions and Outlook
Introduction

- Inclusive radiative $b \rightarrow s\gamma$ decay offers important precision tests of flavour sector in and beyond SM.
- Strong constraints on NP crucially depend on theoretical uncertainty of SM prediction.

\[ M_H \text{ [GeV]} \]

\[ \tan \beta \]

\[ \bar{B} \rightarrow X_{s\gamma} \]

\[ \text{LEP2} \]

\[ R_b \]

\[ B \rightarrow \tau \nu \]

\[ B \rightarrow X_{\tau \nu} \]

Gambino & Misiak '01
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### General Structure

<table>
<thead>
<tr>
<th>Perturbative Contributions</th>
<th>Non-Perturbative Contributions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{B}(\bar{B} \to X_s \gamma)_{\text{SM}} \geq 1.6 \text{ GeV}$</td>
<td>$\mathcal{B}(\bar{B} \to X_c e\bar{\nu})_{\text{exp}}$</td>
</tr>
<tr>
<td>$\frac{\Gamma(b \to s\gamma)}{\Gamma(b \to c e\bar{\nu})}_{\text{LO}}$</td>
<td>$f \left( \frac{\alpha_s(M_W)}{\alpha_s(m_b)} \right)$</td>
</tr>
</tbody>
</table>

$$
\times \left\{ 1 + \mathcal{O}(\alpha_s) + \mathcal{O}(\alpha) + \mathcal{O}(\alpha_s^2) + \mathcal{O} \left( \frac{\Lambda^2}{m_b^2} \right) + \mathcal{O} \left( \frac{\Lambda^2}{m_c^2} \right) + \mathcal{O} \left( \frac{\alpha_s \Lambda}{m_b} \right) \right\}
$$

- **NLO QCD**
- **NLO EW**
- **NNLO QCD**
- **LO QCD + NLO $m_b$**
- **LO QCD + NLO $m_c$**
- **NLO QCD + LO $m_b$**

- perturbative
- non-perturbative

$\approx 25\%$
$\approx 7\%$
$\approx 4\%$
$\approx 1\%$
$\approx 3\%$
$\approx 5\%$
Effective Theory

\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QCD} \times \text{QED}} + \frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{8} C_i(\mu) Q_i + \ldots \]

\[ Q_{1,2} = (\bar{s} \Gamma_i c)(\bar{c} \Gamma'_i b) \]

\[ |C_i(m_b)| \sim 1 \]

\[ Q_{3-6} = (\bar{s} \Gamma_i b) \sum_q (\bar{q} \Gamma'_i q) \]

\[ |C_i(m_b)| < 0.07 \]

\[ Q_7 = \frac{e m_b}{16\pi^2} (\bar{s}_L \sigma^{\mu\nu} T^a b_R) F_{\mu\nu} \]

\[ C_7(m_b) \sim -0.3 \]

\[ Q_8 = \frac{g m_b}{16\pi^2} (\bar{s}_L \sigma^{\mu\nu} T^a b_R) G_{\mu\nu}^a \]

\[ C_8(m_b) \sim -0.15 \]

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- \( |C_i(m_b)| < 0.07 \)
- \( Q_{3-6} = (\bar{s}\Gamma_i b) \sum_q (\bar{q}\Gamma'_i q) \)
- \( C_7(m_b) \simeq -0.3 \)
- \( C_8(m_b) \simeq -0.15 \)

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Experiment vs. Theory

\[
\mathcal{B}_{\exp}^{E\gamma > 1.6\text{ GeV}} = \left(3.55 \pm 0.24^{+9}_{-10} \pm 3\right) \times 10^{-4}
\]

\[
\mathcal{B}_{\text{SM}}^{E\gamma > 1.6\text{ GeV}} = \left(3.33 \pm 0.29\right) \times 10^{-4} \quad \text{(NLO)}
\]

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Charm Quark Mass

\[ m_c / m_b = 0.29 \pm 0.02 \] (on-shell)

\[ m_c / m_b = 0.22 \pm 0.04 \] (\( \overline{\text{MS}} \))
• unrenormalized quark self-energy \( \Sigma(q) \) is gauge-dependent

• consider gauge-independent mass insertion

\[
\delta m_c^{\text{eff}} = \Sigma(q = m_c) + \delta m_c
\]

• including \( \delta m_c^{\text{eff}} \) insertions reduces uncertainty due to \( m_c \) from \( \pm 6\% \) to \( \pm 3\% \)

Asatrian et al. '05

\[ B_{SM} \]

\[ \frac{B_{\gamma > m_b/20}}{10^{-4}} \]

\[ \mu_b [\text{GeV}] \]

\[ 2, 4, 6, 8, 10 \]

\[ 3.0, 3.2, 3.4, 3.6, 3.8, 4.0 \]

NLO

NLO + \( \delta m_c^{\text{eff}} \)
NNLO Error Estimate

- no loop calculation needed since RG gives us answer
  \[ \delta A_{NNLO} \propto \gamma_m^{(0)} \frac{dA_{NLO}}{d \ln m_c} \ln \frac{m_b}{m_c} \]

- NNLO corrections expected to drastically reduce theoretical due to definition of \( m_c \)

- to determine shift of central value of \( B_{SM} \) going beyond NLO unavoidable

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NNLO Matching Conditions

- Analytic expressions of 2-loop $O(\alpha_s^2)$ corrections to $C_{1-6}(M_W)$ known
  
  Bobeth et al. '00, Buras et al. '06

- Asymptotic expansions of 3-loop $O(\alpha_s^2)$ corrections to $C_{7,8}(M_W)$ available
  
  Misiak & Steinhauser '04
NNLO Anomalous Dimensions

- 3-loop $O(\alpha_s^3)$ mixing in $Q_{1-6}$ sector calculated
  Gorbahn & UH ‘04

- 3-loop $O(\alpha_s^3)$ mixing in $Q_{7,8}$ sector found
  Gorbahn et al. ‘05

- 4-loop $O(\alpha_s^3)$ mixing of $Q_{1-6}$ into $Q_7$ already computed and $Q_8$ almost finished
  Czakon et al.

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Results of 4-Loop Mixing

- numerical effect of $O(\alpha_s^3)$ mixing of $Q_{1-6}$ into $Q_7$ on $B_{SM}$ is $-2.4\%$

- impact of remaining $O(\alpha_s^3)$ mixing of $Q_{1-6}$ into $Q_8$ expected to be $\approx 10$ times smaller

$\gamma^{(2)}_{1-6,7} = \begin{pmatrix}
\frac{150994745}{1062882} + \frac{1272596}{6561} \zeta_3 \\
\frac{138336202}{177147} - \frac{2713672}{2187} \zeta_3 \\
-\frac{58397866}{177147} + \frac{3236560}{2187} \zeta_3 \\
-\frac{5108749081}{2125764} + \frac{2007886}{6561} \zeta_3 \\
\frac{5824017302}{177147} + \frac{112180720}{2187} \zeta_3 \\
\frac{3603565835}{531441} + \frac{15361912}{6561} \zeta_3 
\end{pmatrix}$
NNLO Matrix Elements

- Large $\beta_0$ limit of 2- and 3-loop $O(\alpha_s^2)$ matrix elements of $Q_{7,8}$ and $Q_{1,2}$ worked out
  Bieri et al. '03

- Analytic results for 2-loop $O(\alpha_s^2)$ corrections to self-interference of $Q_7$ derived
  Blokland et al. '05, Asatrian et al. '06; Melnikov & Mitov '05, Asatrian et al., Ferroglia & Gambino

- Interpolation in $m_c$ of 3-loop $O(\alpha_s^2)$ matrix elements of $Q_{1,2}$ very far advanced
  Misiak & Steinhauser

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Individual NNLO Contributions

\[ C_i = \sum_{n=0}^{2} \left( \frac{\alpha_s}{4\pi} \right)^n C_i^{(n)} \]

\[ \mathcal{B}_{\text{NNLO}}^{E_{\gamma} > 1.6 \text{ GeV}} \equiv \mathcal{B}_{\text{NLO}}^{E_{\gamma} > 1.6 \text{ GeV}} \left( 1 + \delta_1 + \delta_2(r) + \delta_3(r) \right) \]

\[ r \equiv \frac{m_c(m_c)}{m_{b1S}} \]

\[ \delta_1 \propto C_i^{(0)} C_j^{(2)} , C_i^{(1)} C_j^{(1)} \]

\[ \delta_2(r) \propto C_i^{(0)} C_j^{(0)} \]

\[ \delta_3(r) \propto C_i^{(0)} C_j^{(1)} \]

\[ \delta_2(r) = A n_f + B = -\frac{3}{2} \left( 11 - \frac{2}{3} n_f \right) A + \left( \frac{33}{2} A + B \right) \equiv \delta_2^{\beta_0}(r) + \delta_2^{\text{non-}\beta_0}(r) \]

- \( \delta_2^{\beta_0}(r) \) known for arbitrary \( r \)
- \( \delta_2^{\text{non-}\beta_0}(r) \) has been computed so far only in limit \( r \gg 1/2 \)
Interpolation in Charm Quark Mass

- Interpolate beyond large $\beta_0$ correction $\delta_2^{\text{non-}\beta_0}(r)$ by

$$a \frac{d\beta_{\text{NLO}}}{d\ln r} + b \beta_{\text{NLO}} + c \delta_2^{\beta_0}(r) + d$$

- Coefficients $a, b, c$ and $d$ are determined from behaviour at large $r$ and requiring that correction vanishes at $r = 0$

- Lower curves correspond to $\delta_2^{\text{non-}\beta_0}(0) = 0$ upper ones to $\delta_1 + \delta_2^{\text{non-}\beta_0}(0) + \delta_3(0) = 0$

Misiak et al.
Summary and Outlook

• current experimental world average already slightly more accurate than SM prediction at NLO

• completion of NNLO enterprise will reverse situation and drastically increase sensitivity of $\bar{B} \rightarrow X_s \gamma$ to NP

• to resolve all doubts in interpolation in charm quark mass explicit calculation of 3-loop matrix elements for $m_c = 0$ should be attempted

• mandatory to perform a study of non-perturbative effects that scale like $\mathcal{O}(\alpha_s \Lambda / m_b)$ in limit of vanishing charm quark mass
Credits

Michal Czakon,
Paolo Gambino,
Martin Gorbahn,
Mikolaj Misiak &
Matthias Steinhauser
2-loop $O(\alpha_s^2)$ corrections to soft function $\tilde{s}(L, \mu)$ available now

Becher & Neubert '05

2-loop $O(\alpha_s^2)$ corrections to jet function $\tilde{j}(L, \mu)$ calculated recently

Becher & Neubert '06

analysis of non-perturbative effects in matrix elements of $Q_{1,2}$ at $O(\alpha_s)$

open issue

≈80%  ≈20%

done  to do