Selected topics from Quantum Chromodynamics
Part I: Introduction

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Plan of the lectures

1. Introduction, the global symmetries of QCD, pions as Goldstone bosons and their interactions at low energy.
2. Electron structure and hadron structure, electromagnetic form factors, deep inelastic scattering and parton distributions
3. Diffractive and elastic scattering. Processes with rapidity gaps.

Quantum Electro Dynamics

QED Lagrangian

$$\mathcal{L} = \bar{\psi}(x)\Big(i\gamma^\mu D_\mu - m\Big)\psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)$$

$$D_\mu = \partial_\mu - ieA_\mu\,,\; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

- The construction principle: **gauge invariance**, a "symmetry" under local phase transformations.
- QED: $U(1)$ gauge group: $\psi(x) \to \exp(-ie\phi(x))\psi(x)$ and $A_\mu \to A_\mu - i\partial_\mu\phi(x)$.
- $D_\mu\psi(x) \to \exp(ie\phi(x))D_\mu\psi(x)$, and $F_{\mu\nu}(x) \to F_{\mu\nu}(x)$.
- We can add any number of fermions. Each of them can have its own charge e_f.
- Gauge invariance is a mathematical apparatus to construct **unitary**(=quantum) and **relativistically invariant** theories with **vector fields**. The latter are an ingredient extrapolated from experiment.
- The **massless** vector field A_μ has four components, but only two of them describe independent physical degrees of freedom (photons).
- In the quantum theory the charge α_{em} becomes scale dependent and increases at short distances. For practical purposes we are (almost) always in the domain of a weakly coupled system and may enjoy perturbation theory in $\alpha_{em} = e^2/(4\pi) \sim 1/137$.

Symmetries of Quantum Chromo Dynamics

QCD Lagrangian

$$\begin{aligned}
\mathcal{L} &= \sum_{f=1}^{n_f} \bar{\psi}_f(x)\Big(i\gamma^\mu D_\mu - m_f\Big)\psi_f(x) - \frac{1}{4}F^a_{\mu\nu}(x)F^{a\mu\nu}(x) \\
(D_\mu)_{ij} &= \partial_\mu\delta_{ij} - igA^a_\mu t^a_{ij} \\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g^2 f_{abc}A^a_\mu A^b_\nu\,,\ i,j = 1,\dots,N_c\,,\ a = 1,\dots,N_c^2-1
\end{aligned}$$

- The construction principle: **gauge invariance**, a “symmetry” under local phase transformations.
- QCD: $SU(N_c)$ gauge group, for $N_c = 3$ colors: $\psi_{f,i}(x) \to \Omega_{ij}(x)\psi_{f,j}(x)$.
- We want that the covariant derivative $D_\mu = \partial_\mu - ig\hat{A}_\mu$ (here $\hat{A}_\mu = A^a_\mu t^a_{ij}$) transforms as $D_\mu\psi \to \Omega(x)D_\mu\psi$.
- From that requirement one obtains gauge trf. of the gauge field, $\hat{A}_\mu \to \Omega\hat{A}_\mu\Omega^{-1} - \frac{i}{g}(\partial_\mu\Omega)\Omega^{-1} = \Omega(\hat{A}_\mu - \Omega^{-1}\partial_\mu\Omega))\Omega^{-1}$.
- Field strength tensor: $\hat{F}_{\mu\nu} = F^a_{\mu\nu}t^a \to \Omega\hat{F}_{\mu\nu}\Omega^{-1}$. Note, that $F^a_{\mu\nu}F^{a\mu\nu} \propto Tr[\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}]$.
- Here the gauge transformation is an element of the non-abelian group $SU(N_c)$. Nonabelian gauge invariance requires **that all quark flavors couple with the same coupling constant strength**.
- The theory without quarks is called a **Yang-Mills** theory. It shares with QCD the properties of **asymptotic freedom** (i.e. vanishing of $\alpha_S = g^2/(4\pi)$ at short distances) and **confinement of color charge** at large distances. These are related to the three- and four gluon couplings.

The global symmetries of Quantum Chromo Dynamics

QCD Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \sum_{f=1}^{n_f} \bar{\psi}_f(x)\Big(i\gamma^\mu D_\mu - m_f\Big)\psi_f(x) - \frac{1}{4}F^a_{\mu\nu}(x)F^{a\mu\nu}(x) \\
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F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g^2 f_{abc} A^a_\mu A^b_\nu \,,\; i,j = 1,\ldots,N_c \,,\; a = 1,\ldots,N_c^2 - 1
\end{aligned}
$$

- Due to **confinement**, the degrees of freedom of the QCD Lagrangian are not the observable particles.
- The QCD Lagrangian in terms of quark and gluon degrees of freedom can be used for perturbation theory at short space/time distance scales or large momentum scales. Special types of observables can be calculated in terms of quarks and gluons.
- Highly involved numerical calculations (Lattice QCD) can give information on the spectrum and phase structure of QCD.
- At low energies strong interactions are often formulated in terms of **hadronic** degrees of freedom $\pi, K, \eta, N, \Delta, \Lambda \ldots$
- The formulation of QCD as a theory as a theory of hadrons becomes "rigorous" in the infrared limit. Low energy QCD is a theory of weakly interacting Goldstone bosons – the **pions**.
- Charge conjugation C, space reflection parity P and time reversal invariance T are exact.

The global symmetries of Quantum Chromo Dynamics

Quark part of QCD Lagrangian

$$\mathcal{L} \quad = \quad \sum_{f=1}^{n_f} \bar{\psi}_f(x)\Big(i\gamma^\mu D_\mu - m_f\Big)\psi_f(x)$$

- **Flavor conservation:** For each flavor we have a **global** phase symmetry $\psi_f \to \exp(i\theta_f)\psi_f$. **Conservation laws:** strangeness, baryon number, I_3 (3rd component of isospin), electric charge, charme, etc...
- **Approximate flavor symmetry:** If quark masses are equal, $\mathcal{L}$ is invariant under transformations of the type $\psi_f \to U_{f'f}\psi_f$ for a unitary transformation $U_{f'f}$. Existence of two very light quarks u and d and athird fairly light one, s accounts for the rather accurate **isospin** $SU(2)$ and approximate flavor $SU(3)$ invariance of the strong interactions.
- **Approximate chiral symmetry:** Ignoring masses of u, d, s quarks $\mathcal{L}$ is invariant under separate unitary rotations of right- and left handed quark fields.

$$\psi_{L,f} \to U^L_{f'f}\psi_{Lf}\;,\; \psi_{R,f} \to U^R_{f'f}\psi_{Rf}\;.$$

- This gives rise to a very large $U(3)\times U(3)$ symmetry. This symmetry is not visible in the spectrum and is **spontaneously broken** –or realized in the Nambu-Goldstone mode.
- a U(1) subgroup, the **axial** $U(1)$ is broken by quantum effects ("anomalous").

Chiral symmetry, right- and left-handed quarks

- right- and left handed fermion fields can be defined by the help of the matrix γ_5 which **anticommutes with all Dirac matrices**: $\gamma_\mu \gamma_5 + \gamma_5 \gamma_\mu = 0$ and fulfills $\gamma_5^2 = 1$.
- Projectors:

$$\hat{P}_R = \frac{1}{2}(1 + \gamma_5)\,,\ \hat{P}_L = \frac{1}{2}(1 - \gamma_5)\,,\ \hat{P}_R + \hat{P}_L = 1\,.$$

- Projectors on orthogonal subspaces: $\hat{P}_R \hat{P}_L = \hat{P}_L \hat{P}_R = 0,\ \hat{P}_R^2 = \hat{P}_L^2 = 1.$
- they can be used to decompose a fermion field into its right- and left handed parts:

$$\psi(x) = \hat{P}_R \psi(x) + \hat{P}_L \psi(x) = \psi_R(x) + \psi_L(x).$$

- conjugate quark field? $\bar{\psi}(x) = \psi^\dagger(x)\gamma_0$

$$\bar{\psi}_R = \overline{\hat{P}_R \psi} = (\hat{P}_R \psi)^\dagger \gamma_0 = \psi^\dagger \frac{1}{2}(1 + \gamma_5)\gamma_0 = \psi^\dagger \gamma_0 \frac{1}{2}(1 - \gamma_5) = \bar{\psi} \hat{P}_L\,.$$

Vector and scalar

$$\begin{aligned} \bar{\psi}\gamma_\mu\psi &= \bar{\psi}_R \gamma_\mu \psi_R + \bar{\psi}_L \gamma_\mu \psi_L \\ \bar{\psi}\psi &= \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R \end{aligned}$$

Chiral symmetry, right- and left-handed quarks

- Right and left handed spinors are equivalent to helicity for massless fermions.
- Recall Dirac four spinors, take limit $m \to 0$, or $E \gg m$:

$$u_\lambda(p) = \sqrt{\frac{E+m}{2E}} \begin{pmatrix} \chi_\lambda \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_\lambda \end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_\lambda \\ \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi_\lambda \end{pmatrix} .$$

with

$$\gamma_5 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \longrightarrow \hat{P}_{R,L} = \frac{1}{2} \begin{pmatrix} \mathbb{I} & \pm\mathbb{I} \\ \pm\mathbb{I} & \mathbb{I} \end{pmatrix}$$

- projects on right and left handed spinors

$$u_R(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_\uparrow \\ \chi_\uparrow \end{pmatrix} , \, u_L(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_\downarrow \\ -\chi_\downarrow \end{pmatrix}$$

- for **massless fermions** “chirality” is equivalent to helicity.

Invariance of action under continuous transformations of the field

- The relation between symmetries and conservation theorems is captured by **Noether's theorem**.
- continuous transformations of fields. Consider "infinitesimal" variations

$$\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x)\,.$$

- The action (and therefore Euler-Lagrange equations) are left invariant if e.g. the Lagrange density (Lagrangian) is invariant

$$\delta\mathcal{L} = \mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) - \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = 0$$

- in fact strict invariance of the Lagrangian is not necessary, it is enough that the Lagrangian changes by a total derivative

$$\delta\mathcal{L} = \partial_\mu f^\mu\,,$$

  the action will still be invariant up to a surface term, which is taken care of by boundary conditions.

Noether current and charge

- Let's for a moment assume we found a transformation under which the Lagrangian is invariant $\delta\mathcal{L} = 0$. Then:

$$0 = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta\phi = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta\phi = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)$$

- This means we have a conserved current!

Noether current:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\,,\ \partial_\mu j^\mu = 0 \leftrightarrow \frac{\partial j^0}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

- Together with the Noether current comes a **conserved charge**:

$$Q(t) = \int d^3\vec{x}\, j^0(t,\vec{x}) \rightarrow \frac{dQ(t)}{dt} = -\int d^3\vec{x}\,\vec{\nabla}\vec{j}(t,\vec{x}) = 0$$

Consequences of symmetries, Wigner-Weyl realization

- The interesting point here is that global symmetries can manifest themselves in phenomena in two very different manners.
- From Quantum Mechanics we are familiar with the following situation, say our Hamiltonian is invariant under some symmetry represented by a unitary transformation U:

$$U\hat{H}U^\dagger = \hat{H}\,.$$

- we have eigenstates $|A\rangle$ with energies E_A: $\hat{H}|A\rangle = E_A|A\rangle$.
- If we apply our symmetry U to the eigenstates $|A\rangle$, the transformed states $|B\rangle = U|A\rangle$ are also eigenstates and are **degenerate** with $|A\rangle$. Proof:

$$E_A = \langle A|H|A\rangle = \langle A|U^\dagger U\hat{H}U^\dagger U|A\rangle = \langle B|\hat{H}|B\rangle = E_B\,.$$

- we deal with a **multiplet** of degenerate states which transform into each other under the symmetry transformation.
- An example from hadronic physics would be the **isospin symmetry**. Isospin doublets $(p, n), (K^+, K^0)$, triplets π^+, π^0, π^-, ρ^+, ρ^0, ρ^-, quadruplets $\Delta^{++}, \Delta^+, \Delta^0, \Delta^-$.
- This situation is called the **Wigner-Weyl** realization (or mode) of the symmetry.
- It is common for systems with a finite number of degrees of freedom. The hidden assumption is the existence of a unique ground state.

Consequences of symmetries, Nambu-Goldstone realization

- The situation is more complicated in Quantum Field Theory. The Wigner-Weyl mode is certainly a possibility and as we have seen of practical importance. QFT is a framework with infinitely many degrees of freedom. It turns out that global symmetries do **not necessary imply a degenerate set of equal mass particles**.
- We can have a situation, where the **ground state is not invariant under the symmetry transformation**. This is "spontaneous symmetry breaking", or a realization of the symmetry in the **Nambu-Goldstone** mode.
- Everything we said about the existence of Noether currents and charges remains true. Let us try to indicate what changes:
- In QFT the **fields** themselves become **operators**. Particle states can be written as fields acting on the ground state ("vacuum", $|0\rangle$).

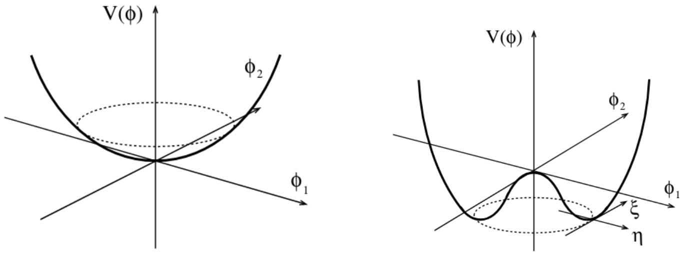
$$|A\rangle = \hat{\Phi}_A|0\rangle \, , \, |B\rangle = \hat{\Phi}_B|0\rangle \ .$$

- Symmetry transformation, say rotate Φ_A into Φ_B:

$$\begin{aligned} \hat{\Phi}'_A &= U\hat{\Phi}_A U^\dagger = \Phi_B \\ U|A\rangle &= U\hat{\Phi}_A|0\rangle = U\hat{\Phi}_A U^\dagger U|0\rangle = \hat{\Phi}_B U|0\rangle \neq \hat{\Phi}_B|0\rangle \, . \end{aligned}$$

- For the symmetry transformation to rotate between the states $|A\rangle, |B\rangle$, we would have to have that the **vacuum is invariant** $U|0\rangle = |0\rangle$, or $(1+\epsilon^a Q^a)|0\rangle = |0\rangle \rightarrow \hat{Q}^a|0\rangle = 0$.
- Spontaneous symmetry breaking: Noether charges do not "annihilate the vacuum". For each $\hat{Q}^a|0\rangle \neq 0$ there exists a massless boson (Nambu-Goldstone boson).

Spontaneous symmetry breaking for a complex scalar field



- complex scalar field with Lagrangian:

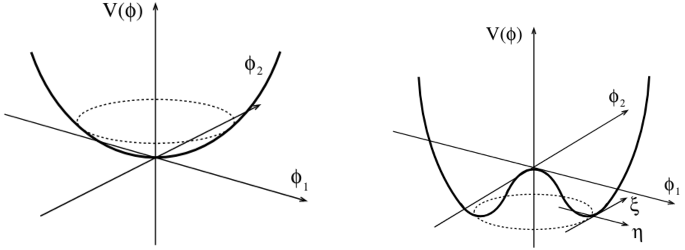
$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi)\,,\; V(\phi) = \mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2 \;= \lambda \left(\phi^* \phi + \frac{\mu^2}{2\lambda} \right)^2 + \mathrm{const}\,.$$

- For $\mu^2 > 0$ we have the potential on the LHS. μ is the mass of the particle, and in the ground state we have $\phi = 0$, or $\langle 0 | \phi | 0 \rangle = 0$.
- Again, the situation completely changes, if we allow $\mu^2 < 0$. Then we **cannot interpret the quadratic term as a mass term**. Minima of potential:

$$|\phi|^2_{\mathrm{min}} = -\frac{\mu^2}{2\lambda}\,.$$

- Now we have **infinitely many** degenerate ground states with $\phi_{\mathrm{min}} = |\phi|_{\mathrm{min}} e^{i\alpha}$.

Spontaneous symmetry breaking for a complex scalar field



- We can parametrize $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. Let's choose for the ground state the mimimum for $\alpha = 0$ (ϕ_1-direction).
- Now the minimum is at $\phi_1 = \sqrt{-\mu^2/\lambda} = v$. Then we again shift the fields

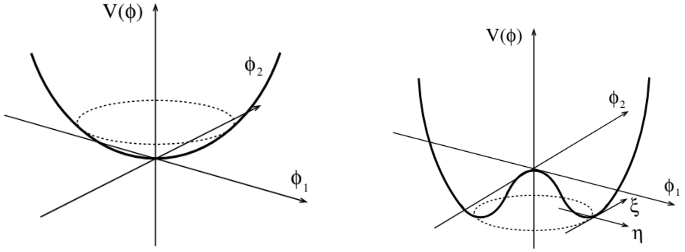
$$\phi_1 = v + \eta(x), \qquad \phi_2 = \xi(x)$$

- The Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \eta \partial^\mu \eta + \frac{1}{2}\partial_\mu \xi \partial^\mu \xi - \frac{1}{2}(-2\mu^2)\eta^2 - \frac{\lambda}{4}(\eta^2 + \xi^2)^2 - \lambda v(\eta^2 + \xi^2)\eta$$

- η is a massive particle, $m_\eta = -2\mu^2$.
- We have a **massless mode**. ξ is a Nambu-Goldstone boson.

Spontaneous symmetry breaking for a complex scalar field



- Despite the spontaneous breaking of the symmetry, the Noether current is still conserved.

$$j_\mu = -i\left(\phi^* \partial_\mu \phi - \partial_\mu \phi^* \phi\right) = v\partial_\mu \xi + \eta \partial_\mu \xi - \xi \partial_\mu \eta$$

- The Noether current has a nonzero matrix element between a 1-Goldstone boson state and the vacuum

$$\langle \xi(p)| j_\mu |0\rangle = v\, p_\mu \, \exp(-ip \cdot x)$$

- It means that the Noether charge does not annihilate the vacuum. It creates a zero-momentum Goldstone boson.

$$\hat{Q}|0\rangle \propto \delta^{(3)}(\vec{p})|\xi(p)\rangle$$

Exponential parametrization

- a subtlety of field theory is that we have a freedom to redefine fields, while maintaining all observables (on-shell amplitudes). It is a certain freedom in the choice of "coordinates" in the field space.
- For the case at hand, there is a more convenient parametrization of $\phi(x)$ which highlights another property of Goldstone boson physics.

$$\phi = \frac{1}{\sqrt{2}} (v + \eta(x)) \exp \left[i \frac{\theta}{v} \right]$$

- Now, the field $\theta(x)$ will be the Goldstone degree of freedom, while $\eta(x)$ still is a massive mode.

$$\mathcal{L} = \frac{1}{2} \left(1 + \frac{\eta}{v} \right)^2 \partial_\mu \theta \partial^\mu \theta + \frac{1}{2} \partial_\mu \eta \partial^\mu \eta - \frac{1}{2} (-2\mu^2) \eta^2 + a\eta^3 + b\eta^4$$

- **The Goldstone particle does not appear in the potential terms**. It only has derivative interactions with the η-field. $\propto \eta \partial_\mu \theta \partial^\mu \theta$ and $\eta^2 \partial_\mu \theta \partial^\mu \theta$.
- **Goldstone bosons are weakly coupled at low momenta**. A new possibility for a perturbative expansion emerges.

Symmetries of massless QCD

Quark part of massless QCD Lagrangian

$$\mathcal{L} = \sum_{f=1}^{n_f} \bar{\psi}_f(x) i\gamma^\mu D_\mu \psi_f(x) = \sum_{f=1}^{n_f} \left(\bar{\psi}_{R,f}(x) i\gamma^\mu D_\mu \psi_{R,f}(x) + \bar{\psi}_{L,f}(x) i\gamma^\mu D_\mu \psi_{L,f}(x) \right)$$

- We can perform **independent** $SU(n_f)$ flavor rotations on R and L fields! Huge $SU(n_f) \times SU(n_f)$ symmetry.
- a global symmetry comes with conserved Noether currents and charges. They are:

$$V_\mu^a = \bar{\psi}\gamma_\mu \frac{\lambda^a}{2}\psi = \bar{\psi}_R\gamma_\mu \frac{\lambda^a}{2}\psi_R + \bar{\psi}_L\gamma_\mu \frac{\lambda^a}{2}\psi_L$$

$$A_\mu^a = \bar{\psi}\gamma_\mu\gamma_5 \frac{\lambda^a}{2}\psi = \bar{\psi}_R\gamma_\mu \frac{\lambda^a}{2}\psi_R - \bar{\psi}_L\gamma_\mu \frac{\lambda^a}{2}\psi_L$$

- Here $\lambda^a/2$ are the $n_f^2 - 1$ generators of the $SU(n_f)$ flavor group. In practice $n_f = 2$ or $n_f = 3$.
- The **vector charges** generate the isospin symmetry for $n_f = 2$ and the $SU(3)$ flavor symmetry for $n_f = 3$, which we studied in the quark model.
- The **axial charges** are **spontaneously broken**! Each broken charge gives rise to a **Goldstone boson**. These are 3 GBs for $n_f = 2$ massless flavors – π^+, π^-, π^0, and 8 GBs for 3 massless flavours (3 pions, 4 Kaons and the η (stricly speaking the η_8)).

How do we know that axial charges are spontaneously broken?

- Could it be that the chiral symmetry is realized in the Wigner-Weyl mode?
- Then we should see the $SU_L(2) \times SU_R(2)$ symmetry **in the spectrum**.
- in WW-mode, the axial charges have to annihilate the vacuum $\hat{Q}_5^a|0\rangle = 0$. We could use the standard argument from QM, that the charge has to commute with the Hamiltonian. Say, we have an eigenstate $\hat{H}|P\rangle = E_P|P\rangle$. Then

$$\hat{H}\hat{Q}_5^a|P\rangle = \hat{Q}_5^a\hat{H}|P\rangle = E_P\hat{Q}_5^a|P\rangle\,..$$

- The state $\hat{Q}_5^a|P\rangle$ is degenerate with $|P\rangle$, but has **opposite parity**. "Parity doubling".
- There is no indication of parity doubling in the meson or baryon spectrum of QCD. E.g. The Nucleon, with $J^\pi = \frac{1}{2}^+$ has $M \sim 1$ GeV, but the closest $J^\pi = \frac{1}{2}^-$ resonance is at about $M \sim 1.6$ GeV. Similarly, there is a large splitting between the octet of pseudoscalar 0^- mesons and scalar 0^+ mesons.

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- Note that in QCD, the chiral symmetry is **broken explicitly** by the quark mass terms.

Constructing a Lagrangian for the Goldstone bosons

- We have learned, that spontaneous symmetry breaking is associated with a **vacuum expectation value** (VEV) for some field. What is the VEV in QCD?
- **Chiral condensate:**

$$\hat{\Sigma}^{ij} = \langle 0 | q_L^i \bar{q}_R^j | 0 \rangle = \delta^{ij} v .$$

- It transforms under the chiral group $SU_L(n_f) \times SU_R(n_f)$

$$\hat{\Sigma} \to g_L \hat{\Sigma} g_R^\dagger .$$

- It is invariant under a "diagonal subgroup is we rotate by $g_L = g_R = g$: $\hat{\Sigma} = g \hat{\Sigma} g^\dagger$.
- We can easily construct a model("Sigma-model") with the same symmetry breaking pattern $SU_L(n_f) \times SU_R(n_f) \to SU(n_f)$.

$$\mathcal{L} = \frac{1}{4} Tr \left[\partial_\mu \hat{\Sigma}^\dagger(x) \partial^\mu \hat{\Sigma}(x) \right] - \frac{\lambda}{4} \left(\frac{1}{2} Tr \left[\hat{\Sigma}^\dagger(x) \hat{\Sigma}(x) \right] - v^2 \right)^2 .$$

- We can parametrize

$$\hat{\Sigma}(x) = (v + s(x)) U(x) , \text{ with } U^\dagger U = 1 .$$

- The Lagrangian becomes

$$\mathcal{L} = \frac{v^2}{4} (1 + \frac{s}{v})^2 Tr \left[\partial_\mu U^\dagger \partial^\mu U \right] + \frac{1}{2} \partial_\mu s \partial^\mu s - \frac{1}{2} M^2 s^2 + \dots$$

Effective Lagrangian for pions

- The lowest order Lagrangian for Goldstone bosons is universal, and will look the same for every theory with the same breaking pattern. Let us concentrate on the case $SU_L(2) \times SU_R(2) \to SU_{\mathrm{Isospin}}(2)$ in massless QCD:

$$\mathcal{L}^{(2)} = \frac{F^2}{4} Tr\left[\partial_\mu U^\dagger \partial^\mu U\right] , \; U(x) = \exp\left[\frac{i\vec{\pi}(x)\cdot\vec{\tau}}{F}\right] .$$

- F is a constant of dimension Mass, called the pion decay constant.
- explicit chiral symmetry breaking from quark masses:

$$\delta\mathcal{L} = m_u \bar{u}_R u_L + m_d \bar{d}_R d_L + h.c.$$

  it can be included into the effective Lagrangian as

$$\delta\mathcal{L}^{(2)} = \frac{1}{4} F^2 B Tr\left[\hat{M}\left(U + U^\dagger\right)\right] , \; \hat{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} .$$

- We can expand to second order in the pion fields:

$$\mathcal{L}^{(2)} \to (m_u + m_d)F^2 B + \frac{1}{2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - \frac{1}{2}(m_u + m_d)B\vec{\pi}\cdot\vec{\pi}$$

- pion mass term: $m_\pi^2 = (m_u + m_d)B$.
- constant term shifts the vacuum energy by

$$\Delta E_{vac} \quad = \quad -(m_u + m_d)F^2 B \times \mathrm{Volume} = \langle 0|\hat{H}_0|0\rangle = \langle 0|m_u \bar{u}u + m_d \bar{d}d|0\rangle \times \mathrm{Volume}$$

- Therefore: $\langle 0|m_u \bar{u}u + m_d \bar{d}d|0\rangle = -(m_u + m_d)F^2 B$

Chiral perturbation theory

- Gell-Mann–Oakes–Renner relation: (in the limit $\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle$.)

$$m_\pi^2 = \frac{1}{F^2}\,(m_u + m_d)\,|\langle 0|\bar{u}u|0\rangle|\,.$$

- pions indeed are massless in the limit $m_u, m_d \to 0$.
- The chiral condensate can be determined e.g. form Lattice QCD $\langle 0|\bar{u}u|0\rangle \sim -(245\,\mathrm{MeV})^3$
- to second order in derivatives, we also have interactions between pions. Let us extract the four-point vertices:

$$\mathcal{L}^{(2)} \to \frac{1}{6F^2}\vec{\pi}^2\,\vec{\pi}\partial_\mu\partial^\mu\cdot\vec{\pi} + \frac{1}{2F^2}(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 + \frac{m_\pi^2}{24F^2}(\vec{\pi}\cdot\vec{\pi})^2\,.$$

- **effective field theory** with expansion parameter $\epsilon = p^2/(4\pi F)^2, m_\pi^2/(4\pi F)^2$.
  $F \sim 92.2$ MeV, so that $4\pi F \sim 1$ GeV.
- From here we can obtain $\pi\pi$ scattering amplitudes.
  The $\pi\pi$ system can have isospin $I = 0, 1, 2$.
  By Bose symmetry the permissible angular momenta are for $I = 0, 2 : \ell = 0, 2, \ldots,$
  $I = 1, \ell = 1$.
- scattering amplitudes are expanded in cm-momentum k:

$$T_\ell^I = k^{2\ell}\left(a_\ell^I + 2b_\ell^I\frac{k^2}{m_\pi^2} + \ldots\right)\,.$$

Pion decay constant

- By the Noether method, we can find that the axial current in our effective theory is given by

$$\vec{j}^A_\mu = -F\partial_\mu\vec{\pi}$$

- Then, as we studied in our example, the axial current has a matrix element between the (strong-interaction) vacuum and the Goldstone boson.

$$\langle 0|j^{A,i}_\mu|\pi^j(p)\rangle = \delta_{ij}\, p_\mu\, F$$

- We are lucky, that this matrix element is just related to the decay width of the charge pion $\pi^+ \to \mu^+\nu_\mu$, from where it is determined to be $F \sim 92.4\,\mathrm{MeV}$.

Chiral perturbation theory

from Donnelly et al. Foundations of Nuclear and Particle Physics, CUP

	Experimental	Lowest Order[3]	First Two Orders[3]
a_0^0	0.220 ± 0.005	0.16	0.20
b_0^0	0.250 ± 0.030	0.18	0.26
a_0^2	-0.044 ± 0.001	-0.045	-0.041
b_0^2	-0.082 ± 0.008	-0.089	-0.070
a_1^1	0.038 ± 0.002	0.030	0.036
b_1^1		0	0.043
a_2^0	$(17 \pm 3) \times 10^{-4}$	0	20×10^{-4}
a_2^2	$(1.3 \pm 3) \times 10^{-4}$	0	3.5×10^{-4}

- Weinberg's famous results:

$$a_0^0 = \frac{7m_\pi^2}{32\pi F^2}\,,\ a_0^2 = -\frac{m_\pi^2}{16\pi F^2}\,,\ a_1^1 = \frac{m_\pi^2}{24\pi F^2}\,,\ b_0^0 = \frac{m_\pi^2}{4\pi F^2}\,,\ b_0^2 = -\frac{m_\pi^2}{8\pi F^2}$$

- As F is fixed from $\pi \to \mu\nu$ decays these are absolute predictions!
- experimental information comes e.g. form $K \to 3\pi$ decays, or the lifetime of $\pi^+\pi^-$ atoms.
- the higher order corrections require the **Lagrangian** at higher order! A limitation to the formalism are possible **resonances**.

Relations between quark masses

- Just from symmetry considerations, we cannot say anything about the values of the light quark masses. We can however extract information on their **ratios**.
- Extending to $SU(3)$ flavor symmetry, one can obtain:

$$\begin{aligned} M_{\pi^\pm}^2 &= 2\hat{m}B, \quad M_{\pi^0}^2 = 2\hat{m}B - \varepsilon \\ M_{K^\pm}^2 &= (m_u + m_s)B, \quad M_{K^0}^2 = (m_d + m_s)B \\ M_\eta^2 &= \frac{2}{3}(\hat{m} + 2m_s)B + \varepsilon. \end{aligned}$$

  with

$$\hat{m} = \frac{1}{2}(m_u + m_d), \quad \varepsilon = \frac{B}{4}\frac{(m_u - m_d)^2}{m_s - \hat{m}}.$$

- Then, we can calculate from the measured meson masses:

$$\begin{aligned} \frac{m_d - m_u}{m_d + m_u} &= \frac{M_{K^0}^2 - M_{\pi^0}^2 + (M_{\pi^\pm}^2 - M_{K^\pm}^2)}{M_{\pi^0}^2} = 0.29\,. \\ \frac{m_s - \hat{m}}{2\hat{m}} &= \frac{M_{K^0}^2 - M_{\pi^0}^2}{M_{\pi^0}^2} = 12.6 \end{aligned}$$

- Light quark mass differences are not small! Remeber that in the constituent quark model, we argued that u and d constituent quarks have the same mass ~ 300 MeV.
- PDG: $m_u \sim 2.16\,\mathrm{MeV}$, $m_d \sim 4.67\,\mathrm{MeV}$, $m_s \sim 93.4\,\mathrm{MeV}$. Isospin emerges, because $m_u, m_d \ll \Lambda_{\mathrm{QCD}} \sim 200\,\mathrm{MeV}$.