QCD and Feynman Diagrams

Lecture 1

Sebastian Sapeta

IFJ PAN

14th Trans-European School of High Energy Physics, Bezmiechowa Górna, 14-22 July, 2023

The theory of QCD

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$$[T^a, T^b] = i f^{abc} T^c \,.$$

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Field-strength tensor

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$$\left(T^a_{ij}\right)_F = \frac{1}{2}\lambda^a_{ij} \qquad \text{where } \lambda^a \text{ are the Gell-Mann matrices}$$

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- gluons are in the adjoint, octet representation

$$(T^a_{bc})_A = -if^{abc}$$

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Useful relations:

$$
\begin{aligned}
\mathrm{Tr}\,(T^a)_F\,(T^b)_F &= T_R\delta^{ab}\,, \qquad T_R = \frac{1}{2} \text{ (by convention)} \\
\sum_a (T^a_{ij})_F\,(T^a_{jk})_F &= C_F\,\delta_{ik}\,, \qquad C_F = \frac{N^2-1}{2N}\,, \\
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where C_F and C_A are the Casimirs of the fundamental and the adjoint representation, respectively ($T^a T^a$ is an invariant of the algebra).

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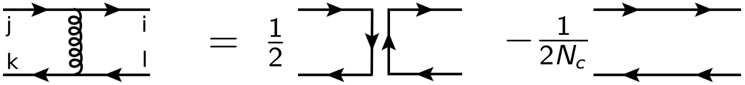
And also

$$
(T^a_{ij})_F\,(T^a_{kl})_F = \frac{1}{2}\left[\delta_{jk}\delta_{il} - \frac{1}{2N}\delta_{ij}\delta_{kl}\right] \quad \text{(Fierz identity)}
$$

Fierz identity and the large N_c limit

$$\left(T^a_{ij}\right)_F \left(T^a_{kl}\right)_F = \frac{1}{2}\left[\delta_{jk}\delta_{il} - \frac{1}{2N_c}\delta_{ij}\delta_{kl}\right]$$



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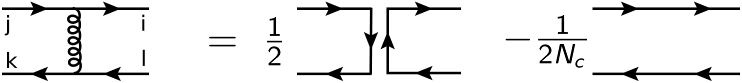
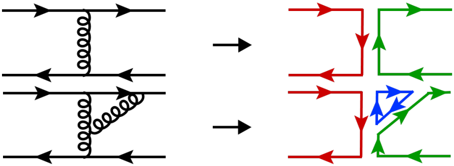


If $N_c \gg 1$, we can replace a gluon with the $q\bar{q}$ pair:



QCD Lagrangian: local gauge symmetry

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Redefinition of the quark fields by the SU(3) group element

$$U(x) = \exp\left(i\theta_a(x) T^a \right) ,$$

independently at each phase space point, does not change the physical content of the theory.

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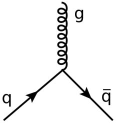
SU(3) transformation:

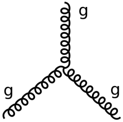
$$\begin{aligned}
q_i(x) &\mapsto q'_i(x) = U(x)_{ij} q_j(x) \\
D_\mu q(x) &\mapsto D'_\mu(x) = U(x)_{ij} D_\mu q_j(x) \\
A^\mu &\mapsto U(x) A^\mu U(x)^{-1} + \frac{i}{g_s} \left[\partial^\mu U(x)\right] U(x)^{-1} \\
F_{\mu\nu} &\mapsto U(x) F_{\mu\nu} U(x)^{-1}
\end{aligned}$$

QCD Lagrangian: the interactions

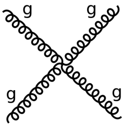
$$\mathcal{L}_{\text{classical}} = \sum_{\text{flavours}} \bar{q}_i \Big(i\gamma_\mu \left(\partial^\mu + i g_s A^\mu\right) - m \Big)_{ij} q_j$$
$$- \frac{1}{4} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f^{abc} A^b_\mu A^c_\nu\right)\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a - g_s f^{ade} A^\mu_d A^\nu_e\right)$$



$$-g_s\, \bar{q}_i \gamma_\mu A^\mu_{ij} q_j$$



$$\frac{g_s}{4} f^{abc} A^b_\mu A^c_\nu \left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right)$$



$$-\frac{g_s g_s}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\,\mu} A^{e\,\nu}$$

Gauge-fixing and ghost terms

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The gauge-fixing and ghost part:

$$\begin{aligned} \mathcal{L}_{\text{gauge-fixing}} &= -\frac{1}{2\xi}\left(\partial_\mu A^{a\mu}\right)\left(\partial_\nu A^{a\nu}\right) \\ \mathcal{L}_{\text{ghost}} &= \partial_\mu \eta^{a\dagger}\left(\partial^\mu \delta^{ab} + g_s f_{abc} A^{c\mu}\right)\eta_b \end{aligned}$$

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- The gauge-fixing term is needed because of a degeneracy of sets of gluon field configurations that enter the path-integral formulation of QCD and which are equivalent under gauge transformation.

  $\hookrightarrow$ This degeneracy makes it impossible to write a gluon propagator. Adding gauge-fixing term to the Lagrangian solves the problem.

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  ↪ Choice of the parameter ξ fixes the gauge.
- On top of that, non-abelian gauge theory needs unphysical degrees of freedom, called ghosts, η, which are complex scalar fields obeying Fermi statistics.

Ways to solve QCD

When coupling is small $g_s \ll 1$:

- Perturbative expansion

$$\sigma = \underbrace{\sigma^{(1)} g_s^2}_{\text{leading order (LO)}} + \underbrace{\sigma^{(2)} g_s^4}_{\text{next-to-leading order (NLO)}} + \underbrace{\sigma^{(3)} g_s^6}_{\text{NNLO}} + \ldots$$

+ provides very precise results at high energies
– relies on $\sigma^{(i)}$ being all of the same order: not always true!
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In principle, for any value of g_s:

- **Lattice QCD**

  Put quarks and gluons on 4D-lattice and compute which configurations are most likely.

  + excellent at calculating static properties like hadron masses
  – only limited lattice sizes (hence large spacings) can be used in practice because of very high computational costs
  – unable to address questions in collider physics because of missing analytic continuation from the imaginary to the real time

Feynman rules

$$= \quad \delta^{ab}\left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2+i\epsilon}\right]\frac{i}{p^2+i\epsilon}$$

$$= \quad \delta^{ij}\frac{i(\not p + m)_{\mu\nu}}{p^2-m^2+i\epsilon}$$

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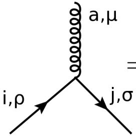

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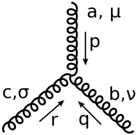
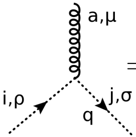
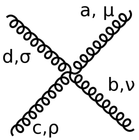


$$= \quad -g_s f^{abc}\Big[(p-q)^\sigma g^{\mu\nu} + (q-r)^\mu g^{\nu\sigma} + (r-p)^\nu g^{\sigma\mu}\Big]$$



$$= \quad g_s f^{aij} q^\mu$$



$$\begin{aligned} = &-ig_s^2 f^{xac} f^{xbd}\left[g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\gamma}\right] \\ &-ig_s^2 f^{xad} f^{xbc}\left[g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}\right] \\ &-ig_s^2 f^{xab} f^{xcd}\left[g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\mu\rho}\right] \end{aligned}$$

Types of gauges

Covariant gauges: depend on a parameter ξ

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Choices of ξ correspond to various gauges in this class:

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Axial gauges: depend on an arbitrary vector n_μ

$$= \quad \delta^{ab}\left[-g_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{k\cdot n} - \frac{n^2}{(k\cdot n)^2}k_\mu k_\nu\right]\frac{i}{p^2+i\epsilon}$$

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- big advantage: ghost contributions disappear
  $\hookrightarrow$ Faddeev-Popov determinant is A^a_μ-independent
- **Light-cone gauge**: a special case of axial-gauge with $n^2 = 0$
  $\hookrightarrow$ subtleties related to $k\cdot n$ singularities

Meaning of interactions

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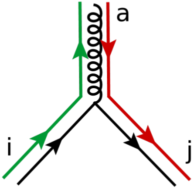

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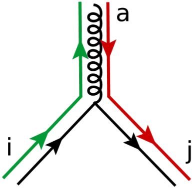

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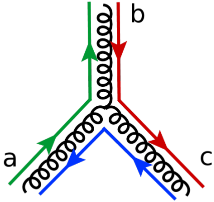
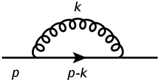


$$-g_s f^{abc}\left[(p-q)^\nu g^{\lambda\mu} + (p-q)^\nu g^{\lambda\mu} + (p-q)^\nu g^{\lambda\mu}\right]$$
$$= -g_s f^{\text{(green-antiblue) (antigreen-red) (antiblue-red)}}$$
$$\times\left[(p-q)^\nu g^{\lambda\mu} + (p-q)^\nu g^{\lambda\mu} + (p-q)^\nu g^{\lambda\mu}\right]$$

Renormalization

Let's calculate the quark self-energy graph in 4 dimensions

$\int d^4 k$  $\sim \int^{k_{\text{cut}}} \frac{dk}{k} \sim \ln k_{\text{cut}}$

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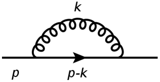

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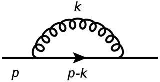
$\int d^4k$ 

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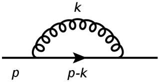
$\int d^Dk$ 

$$= i\frac{\alpha_s}{4\pi} C_F \left(\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E\right)\left(-3m + \xi(\not{p} - m)\right)$$
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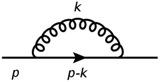

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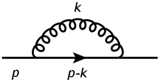

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UV divergence is a property of QCD (and many other QFTs):

- it arises because we extend our theory up to infinite energies, but each theory is valid only up to a certain scale Λ.

Renormalization

Divergences can be attributed a meaning and removed via the procedure of renormalization, which amounts to the following redefinitions

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\begin{aligned}
A^\mu &= Z_3^{1/2} A_R^\mu \\
q &= Z_2^{1/2} q_R \\
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The Z_i coefficients contain divergences that cancel the divergences of the bare objects (A^μ, q, η, g_s, m^2) giving the finite renormalized objects (A_R^μ, q_R, η, g_{sR}, m_R^2).

Renormalization

The QCD Lagrangian (just the classical part for simplicity) takes the following form in terms of the renormalized fields

$$\begin{aligned}\mathcal{L}_{\text{classical}} &= Z_2 \bar{q}_R \left(i\partial\!\!\!/ - Z_m m_R \right) q_R - Z_2 Z_3^{1/2} Z_g g_{sR} \mu^{\epsilon} \bar{q}_R A\!\!\!/_R q_R \\ &\quad - \frac{Z_3}{4} \left(\partial_\mu A^a_{R\nu} - \partial_\nu A^a_{R\mu} \right)^2 - \frac{Z_3^2 Z_g^2 g_{sR}^2 \mu^{2\epsilon}}{4} \left(f^{abc} A^b_{R\mu} A^c_{R\nu} \right)^2 \\ &\quad + \frac{Z_3^{3/2} Z_g g_{sR} \mu^{\epsilon}}{2} f^{abc} \left(\partial^\mu A^{a\nu}_R - \partial^\nu A^{a\mu}_R \right) A^b_{R\mu} A^c_{R\nu}\end{aligned}$$

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This can be rewritten as a sum of the original Lagrangian + counterterms

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\mathcal{L}_{\text{classical}} &= \bar{q}_R \left(i\partial\!\!\!/ - m_R \right) q_R - g_{sR} \mu^{\epsilon} \bar{q}_R A\!\!\!/_R q_R + \ldots \\
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Hence, when doing computations one proceeds as follows

- use the Feynman rules discussed earlier (1^{st} line above, now with all objects renormalized)
- supplement that with the set of counterterm vertices (2^{nd} line above)

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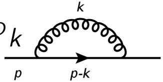

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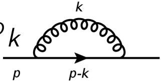

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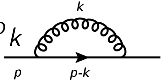
PP $\int d^D k$ 

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Requirement of vanishing of the sum leads to the conditions:

$$i\not{p}\left[\frac{\alpha_s}{4\pi} C_F S_\epsilon \xi + Z_2 - 1\right] = 0,$$

$$im\left[\frac{\alpha_s}{4\pi} C_F S_\epsilon (3m + \xi m) + (Z_2 Z_m - 1)\right] = 0,$$

and this fixes Z_2 and Z_m.

Renormalization: $\overline{\text{MS}}$ scheme

But wait! If we take e.g. the first condition with explicit S_ϵ

$$i\not{p}\left[\frac{\alpha_s}{4\pi}C_F\left(\frac{1}{\epsilon}+\ln(4\pi)-\gamma_E\right)\xi+Z_2-1\right]=0\,,$$

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Z_i coefficients of the MS and the $\overline{\text{MS}}$ schemes are $\frac{m}{\mu}$ independent

- Z_is, by construction, cancel only the part singular at high momentum. But in this limit all masses are negligible and cannot appear in residues of the pole.

Renormalization

Applying similar procedure to other Green functions gives us the full set, of Zs, which, in $\overline{\text{MS}}$, to the first order in α_s, read

$$
\begin{aligned}
Z_2 &= 1 - \frac{\alpha_s\, S_\epsilon}{4\pi\epsilon} \xi C_F + \mathcal{O}\left(\alpha_s^2\right) \\
Z_3 &= 1 - \frac{\alpha_s\, S_\epsilon}{4\pi\epsilon} \left[\left(\frac{\xi}{2} - \frac{13}{6} \right) C_A + \frac{4}{3} T_R n_f \right] + \mathcal{O}\left(\alpha_s^2\right) \\
\tilde{Z} &= 1 - \frac{\alpha_s\, S_\epsilon}{4\pi\epsilon} C_A \left(\frac{3}{4} - \frac{\xi}{4} \right) + \mathcal{O}\left(\alpha_s^2\right) \\
Z_m &= 1 - \frac{\alpha_s\, S_\epsilon}{4\pi\epsilon} 3 C_F + \mathcal{O}\left(\alpha_s^2\right) \\
Z_g &= 1 - \frac{\alpha_s\, S_\epsilon}{4\pi\epsilon} \left(\frac{11}{6} C_A - \frac{2}{3} T_R n_f \right) + \mathcal{O}\left(\alpha_s^2\right)
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Renormalization group

[To simplify notation, from now on, $g_s \to g_0$ (bare) and $g_{sR} \to g$ (renormalized)]

$$g_0 = g\mu^\epsilon Z_g = g\mu^\epsilon \left[1 - \frac{\alpha_s\, S_\epsilon}{4\pi\epsilon}\left(\frac{11}{6}C_A - \frac{2}{3}T_R n_f\right)\right]$$

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- coupling runs with the scale μ^2
- $11C_A - 4T_R n_f = 21 > 0$, hence $\beta(\alpha_s) < 0$ in QCD

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The renormalization group equation (here at lowest order):

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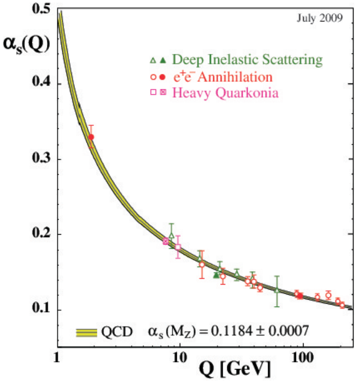

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The one-loop running coupling diverges at low scales

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Let us denote the scale at which this happens as $\mu^2 = \Lambda^2$. Solving the equation on r.h.s. above gives

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We've introduced the parameter Λ defined as the scale at which $\alpha_s = \infty$.

- $\Lambda \simeq 200\,\text{MeV}$ is measurable but it is not an observable as its value depends on: perturbative order, renorm. scheme, number of flavours.
- The order of magnitude of Λ indicates a scale at which α_s becomes large and perturbative theory is not applicable any longer.
- Notice that for massless QCD there is no mass scale in the theory as g_s is dimensionless. Mass scale however emerges via renormalization group and the appearance of Λ parameter (dimensional transmutation).

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  and this term violates CP symmetry.
  - This term is a total divergence so it does not contribute to perturbation theory (that is why it is absent in Feynman rules).
  - It turns out that due to non-trivial topological structure of the QCD vacuum it can however contribute via non-perturbative effects.
  - Experimental limit $\theta < 10^{-9}$. This raises the question: what makes it so small? - the so called strong CP problem. One popular solution is introduction of Axion particle.

Approximate symmetries of QCD

Isospin SU(2) symmetry

- $m_u \simeq 2.3\,\text{MeV}, m_d \simeq 4.8\,\text{MeV} \ll m_p$ hence u and d quarks have approx. equal masses and form a doublet of the SU(2) isospin group

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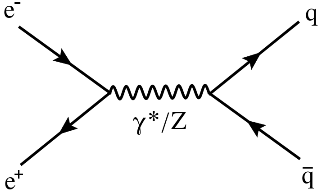
- This results in the appearance of three (number of broken generators) pseudo-Goldstone bosons: π^0, π^+ and π^-, which are indeed very light with $m \simeq 140\,\mathrm{MeV}$ (they are not exactly massless as the chiral symmetry is not exact).

Infrared and collinear safety

Let's take the process $e^+e^- \to$ hadrons. At LO (*i.e.* Born level) we have



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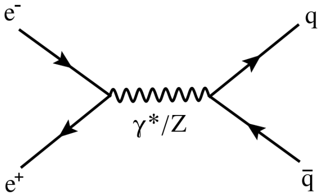


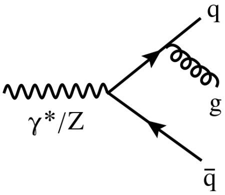
$$\sigma_{\text{Born}} = \frac{1}{\text{flux}} \int \sum |\mathcal{M}_{e^+e^- \to q\bar{q}}|^2 d\Phi_2 = \frac{4\pi\alpha_{\text{em}}^2}{3s} e_q^2 \, N_c$$

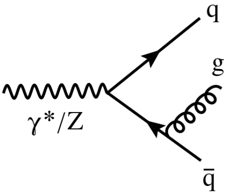
where s is the center-of-mass energy of the incoming e^+e^- pair and e_q is a quark charge.

- Factor N_c comes from sum over colours.

Infrared and collinear safety

NLO real correction to the $e^+e^- \to$ hadrons annihilation





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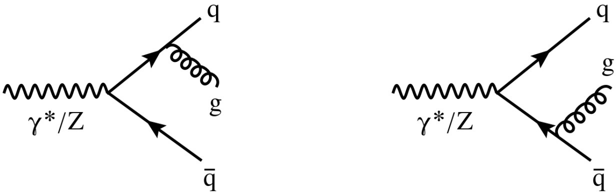

$$|M_{\gamma\to q\bar{q}g}|^2 \propto \frac{q\cdot\bar{q}}{(q\cdot g)(\bar{q}\cdot g)} \sim \frac{1}{E_g^2}\frac{1}{(1-\cos\theta_{qg})}\frac{1}{(1-\cos\theta_{\bar{q}g})}$$

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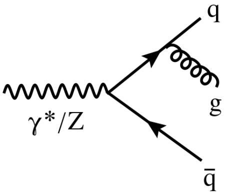

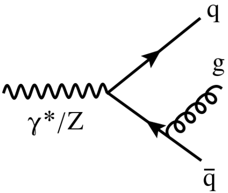

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Hence

$$|M_{\gamma\to q\bar{q}g}|^2 \to \infty \quad \text{if} \quad \begin{cases} E_g^2 \to 0 & \text{soft limit} \\ \theta_{qg} \to 0 & \text{collinear limit} \\ \theta_{\bar{q}g} \to 0 & \text{collinear limit} \end{cases}$$

- real correction to e^+e^- annihilation has soft and collinear divergences

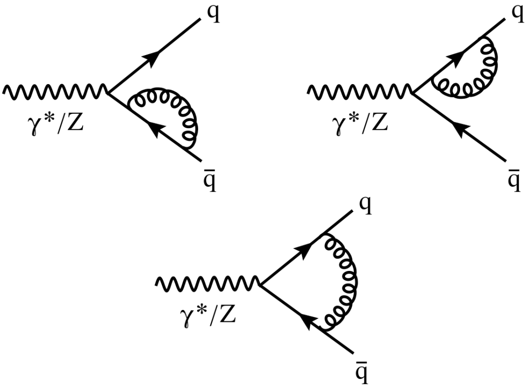
Infrared and collinear safety

The integral $\int |M_{\gamma\to q\bar{q}g}|^2 d\Omega$ can be performed in $D = 4 - 2\epsilon > 4$ dimensions and yields

$$\sigma_R^{e^+e^-\to q\bar{q}g} = \sigma_{\text{Born}} \left\{ \frac{\alpha_s}{2\pi} C_F \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} \right) + \mathcal{O}(\epsilon) \right\}$$

- $\frac{1}{\epsilon^2}$ term corresponds to the soft + collinear divergence
- $\frac{1}{\epsilon}$ term corresponds to the collinear divergence
- $\frac{19}{2}$ is a finite term
- $\mathcal{O}(\epsilon)$ are terms vanishing in the limit $D \to 4$

Infrared and collinear safety: virtual correction



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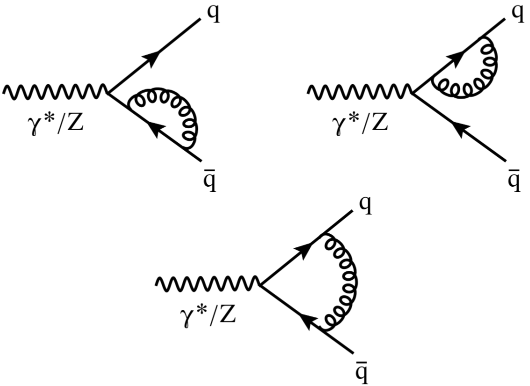

$$\sigma_V^{e^+e^-\to q\bar{q}} = M_{q\bar{q}}^{(\text{Born})} M_{q\bar{q}}^{(\text{virt})\,\dagger} = \sigma_{\text{Born}} \left\{ \frac{\alpha_s}{2\pi} C_F \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 \right) + \mathcal{O}(\epsilon) \right\}$$

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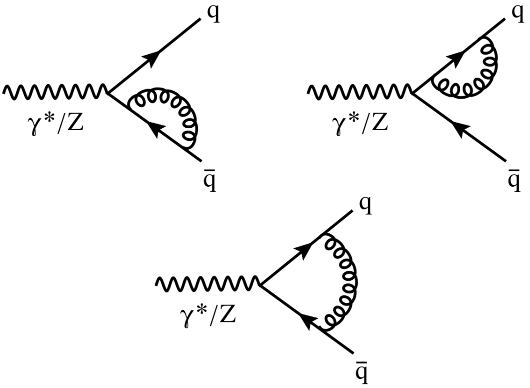

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- The same structure as in the case of real cross section: double and single poles in ϵ + regular term.

$e^+e^- \to$ hadrons: combined result

$$\begin{aligned}
\sigma^{e^+e^-\to\text{hadrons}} &= \sigma_{\text{Born}} + \sigma_R^{e^+e^-\to q\bar{q}g} + \sigma_V^{e^+e^-\to q\bar{q}} \\
&= \sigma_{\text{Born}}\left\{1 + \frac{\alpha_s}{2\pi}C_F\left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2}\right) + \frac{\alpha_s}{2\pi}C_F\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8\right) + \mathcal{O}(\epsilon)\right\} \\
&\overset{\epsilon\to 0}{=} \sigma_{\text{Born}}\left\{1 + \frac{\alpha_s}{2\pi}\frac{3}{4}C_F + \mathcal{O}\left(\alpha_s^2\right)\right\}
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- Collinear and soft divergences cancelled between real and virtual diagram emissions. We could safely take the $\epsilon \to 0$ limit.

Total cross section for e^+e^- annihilation to hadrons is a collinear and infrared safe observable.

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- This is a manifestation of a more general theorem by Kinoshita, Lee and Nauenberg (KLN), which states that the soft and collinear singularities, present in real and virtual corrections, must cancel each other in the sum, for sufficiently inclusive observables.

Infrared and collinear safe observables

$$\frac{d\sigma}{dX} = \frac{1}{\text{flux}} \sum_n d\Phi^n |M^{(n)}|^2 \delta\big(X - f_X(p_1, \ldots, p_n)\big)$$

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An observable X is called infrared and collinear safe if

$$f_X^{(n+1)}(p1, \ldots, p_n, p_{n+1}) \to \begin{cases} f_X^{(n)}(p1, \ldots, p_n) & \text{if} \;\; p_{n+1} \to 0 \\ f_X^{(n)}(p1, \ldots, p_n + p_{n+1}) & \text{if} \;\; p_n \parallel \;\; p_{n+1} \end{cases}$$

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Physics behind this requirement:

- one is not able to distinguish between configurations $|q\bar{q}\rangle$ and $|q\bar{q} + ng\,(\text{soft or collinear})\rangle$
- the results of measurements should not be dependent on detector's energy resolution and granularity

Summary of the features of QCD

- Quantum field theory with spin-$\frac{1}{2}$ quarks as fundamental degrees of freedom.

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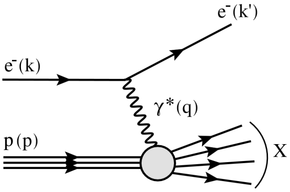
- Quarks and gluons build bound states that are singlets of SU(3).

How do we know that QCD is the right theory?

Deep inelastic scattering (DIS) process



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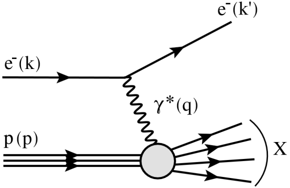

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General form of the cross section

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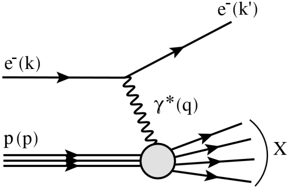

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where F_1 and F_2 are the structure functions.

Hypothesis: proton consists of pointlike, spin-$\frac{1}{2}$, free objects called partons. $\gamma^* p$ interaction happens via γ^* interacting with exactly one parton. $\hookrightarrow$ That goes under the name of the parton model.

How do we know that QCD is the right theory?

Parton model hypothesis implies the so called Bjorken scaling

$$F_i(x, Q^2) \to F_i(x)$$

- If γ^* was scattering off non-pointlike constituents of size Q_0, F_i, which is dimensionless, would need to depend on the ratio Q/Q_0.

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More specifically, in the parton model:

$$\begin{aligned} F_1(x) &= \frac{1}{2}\sum_i e_i^2 f_i(x)\,, \\ F_2(x) &= \sum_i e_i^2 x f_i(x)\,, \end{aligned}$$

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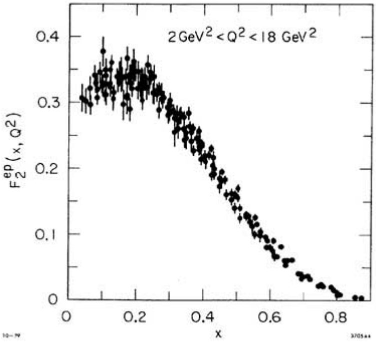
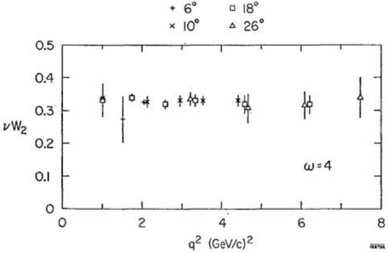
- Seeing Bjorken scaling in the data would provide a strong evidence in favour of the parton model.

Bjorken scaling

And what does the data tell us? Back then in 1970...

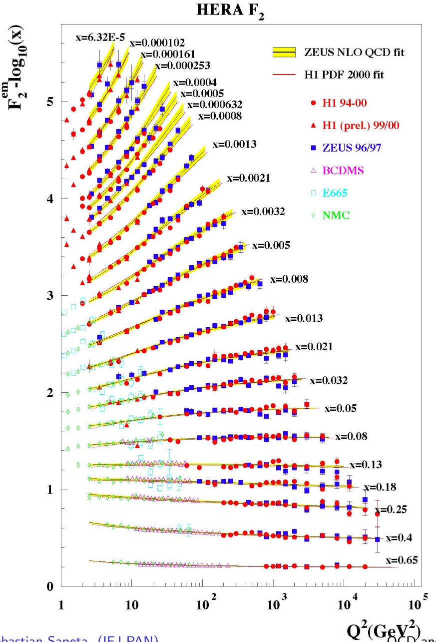




Conclusion: experimental evidence in favour of the parton model proves that the proton consists of objects that are

- pointlike ⇒ today we identify them with quarks
- free ⇒ that requires that the theory behind is asymptotically free

Bjorken scaling now



- data from DIS experiments: fixed target and HERA
- clearly visible region of Bjorken scaling for $x \gtrsim 0.1$
- we will come back to the region $x < 0.1$ in a minute

Callan-Gross relation

In the parton model

$$F_2(x) = 2xF_1(x) \qquad \text{(Callan-Gross relation)}$$

which follows from spin-$\frac{1}{2}$ property of partons.

One can construct the longitudinal structure function, corresponding to the absorption of the longitudinally polarized photons

$$F_L(x) = F_2(x) - 2xF_1(x)\,.$$

Callan-Gross relation means that $F_L = 0$ in the parton model.

- Follows from the fact that spin-$\frac{1}{2}$ parton cannot absorb a longitudinally polarized photon.
- In the experiment, we indeed see that F_L is very small. That confirms that partons are spin-$\frac{1}{2}$ particles.

Colour

Spin-statistics

The wave function of particles like like Δ^{++}:

$$|\Delta^{++}; +\tfrac{3}{2}\rangle = |u\uparrow\rangle|u\uparrow\rangle|u\uparrow\rangle$$

is totally symmetric in spin and flavour. That violates Pauli-principle unless there is an addition degree of freedom, in which the wave function is fully anti-symmetric. This is colour.

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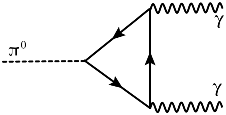
But how many colours?

$N_c = 3$

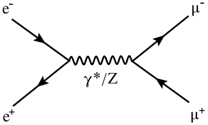
- $\pi^0 \to \gamma\gamma$ decay rate

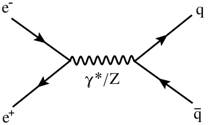


$$\Gamma(\pi^0 \to \gamma\gamma) = 7.63\,\text{eV}\left(\frac{N_c}{3}\right)^2$$

Experimental value: $\Gamma(\pi^0 \to \gamma\gamma) = 7.84 \pm 0.56\text{eV}$.

- e^+e^- decay ratio





$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c\,\Sigma_q e_q^2 = N_c\frac{11}{9}$$

Experimental value: $N_c \simeq 3.2$.

What about gluons?

- Electron-nucleus DIS allows us to measure the momentum weighted probability density of quarks and anti-quarks in the nucleon

$$\frac{18}{5}\int_0^1 dx\, F_2^{eN}(x) = \int_0^1 dx\, x[u(x) + d(x) + \bar{u}(x) + \bar{d}(x)] \simeq 0.5$$

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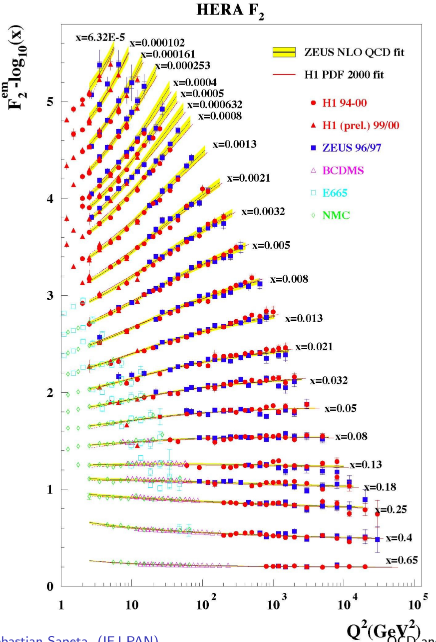
- Bjorken scaling holds only approximately and F_2 starts to depend on Q^2 as we go to lower values of x
  - this happens because of gluons which are produced in abundance at low-x
  - gluons go beyond the simple picture of the naive parton model

Violation of Bjorken scaling is indeed seen in the data at low-x!

Scaling violation seen in the data!



$$\lambda(x,Q^2) = -\left.\frac{\partial F_2(x,Q^2)}{\partial \ln x}\right|_{Q^2}$$

$$F_2(x,Q^2) \simeq c(Q^2)\,x^{-\lambda(Q^2)}$$

From HERA data:

$\lambda \simeq 0.2 - 0.4$ for the range $x < 0.01$ and $Q^2 > 10\,\mathrm{GeV}^2$

- large x: proton consists mostly of valence quarks and looks like in the naive parton model
- small x: proton consist mostly of gluons and the parton model needs to be improved with QCD

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+ Countless tests from several generations of experiments!
(some of them covered in lectures 2 and 3)

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- QCD has withstood an enormous number of experimental tests.