QCD and Feynman Diagrams

Lecture 2

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Our plan for lecture 2

Let's try to do some complete, useful and interesting calculation

- $e^+ + e^- \to q + \bar{q}$ at leading order (aka Born, tree level)
- $e^+ + e^- \to q + \bar{q}$ at one loop
- $e^+ + e^- \to q + \bar{q} + g$

We will calculate the above processes and take baby steps to show exactly all the elements involved

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We shall consider quarks to be massless, hence we can obtain the following relations

$$p_1 = p_2 + p_3$$
$$Q^2 \equiv p_1^2 = 2p_2 p_3$$

Predictions in perturbative QCD

- In the region where the strong coupling $\alpha_s \ll 1$, fixed-order perturbative expansions is expected to work well

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- N3LO
  - $pp \to H,\ Z/\gamma^*,\ W^\pm$ [Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger '15 - '21]

Feynman rules

- External particles:

$u^s(p)$	incoming fermion
$\bar{u}^s(p)$	outgoing fermion
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$v^s(p)$	outgoing antifermion
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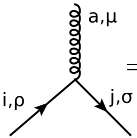
- Each fermion loop brings a factor -1
- Four-momentum is conserved at each vertex
- Average over spins/polarizations of incoming particles and sum the spins/polarizations of outgoing particles

Feynman rules

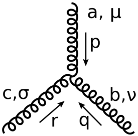
$$= \quad \delta^{ab} \left[-g^{\mu\nu} + (1-\xi) \frac{p^\mu p^\nu}{p^2 + i\epsilon} \right] \frac{i}{p^2 + i\epsilon}$$

$$= \quad \delta^{ij} \frac{i(\not{p} + m)_{\mu\nu}}{p^2 - m^2 + i\epsilon}$$

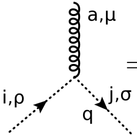
$$= \quad \delta^{ab} \frac{i}{p^2 + i\epsilon}$$



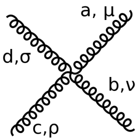
$$= -ig_s (T^a_{ji})_F \gamma^\mu_{\sigma\rho}$$



$$= \quad -g_s f^{abc} \big[(p-q)^\sigma g^{\mu\nu} + (q-r)^\mu g^{\nu\sigma} + (r-p)^\nu g^{\sigma\mu} \big]$$



$$= \quad g_s f^{aij} q^\mu$$



$$\begin{aligned} = & -ig_s^2 f^{xac} f^{xbd} \left[g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\gamma} \right] \\ & -ig_s^2 f^{xad} f^{xbc} \left[g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} \right] \\ & -ig_s^2 f^{xab} f^{xcd} \left[g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\mu\rho} \right] \end{aligned}$$

Tools

All calculations presented here are done with

- Mathematica
  `httops://wolfram.com`
- FeynArts
  `https://feynarts.de/`
- FeynCalc
  `https://feyncalc.github.io`

$\gamma^* \to q\bar{q}$ at LO

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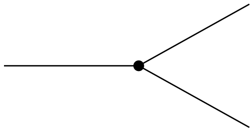
Step 1: Draw all possible topologies



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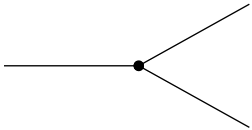
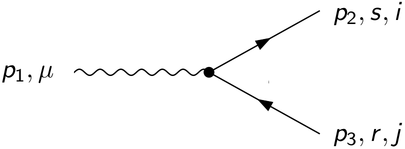


- A set of possible topologies is restricted by the number of external particles and types of vertices (*e.g.* 3- and 4-point vertices in QCD)

$\gamma^* \to q\bar{q}$ at LO

Step 2: Figure out which particles can run through diagrams



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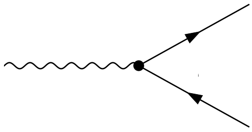
Step 3: Obtain the amplitude (aka matrix element)



$$\mathcal{M}^{(0)}_{\gamma^* \to q\bar{q}} = \varepsilon_\mu(p_1)\bar{u}^s(p_2)\Big(- iQ_f\, e\, \gamma^\mu \delta_{ij}\Big) v^r(p_3)$$

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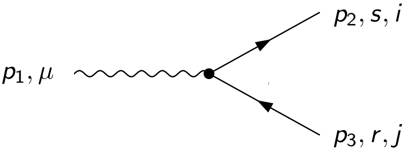

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And the conjugate amplitude

$$\mathcal{M}^{*(0)}_{\gamma^* \to q\bar{q}} = \varepsilon^*_\nu(p_1)\bar{v}^r(p_3)\Big(iQ_f\, e\,\gamma^\nu \delta_{ji}\Big) u^s(p_2)$$

Complex conjugate amplitude

$$\left(\bar{u}\gamma^{\mu}v\right)^{*} = v^{\dagger}\left(\gamma^{\mu}\right)^{\dagger}\left(\gamma^{0}\right)^{\dagger}u = v^{\dagger}\left(\gamma^{\mu}\right)^{\dagger}\gamma^{0}u = v^{\dagger}\gamma^{0}\gamma^{\mu}u = \bar{v}\gamma^{\mu}u$$

Hence

$$\mathcal{M}^{*(0)}_{\gamma^*\to q\bar{q}} = \Big(\varepsilon_{\nu}(p_1)\bar{u}(p_2)\Big(-iQ_f\, e\,\gamma^{\nu}\delta_{ij}\Big)v(p_3)\Big)^{*}$$

$$= \varepsilon^{*}_{\nu}(p_1)\bar{v}(p_3)\Big(i\,Q_f\, e\,\gamma^{\nu}\delta_{ji}\Big)u(p_2)$$

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Step 3: Obtain the amplitude squared

$$|\mathcal{M}^{(0)}_{\gamma^* \to q\bar{q}}|^2 = Q_f^2\, e^2\, \delta_{ij}\delta_{ji}\, \varepsilon_\mu(p_1)\, \varepsilon^*_\nu(p_1)\, \bar{v}^r(p_3)\gamma^\nu u^s(p_2)\bar{u}^s(p_2)\gamma^\mu v^r(p_3)$$

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Step 4: Perform colour algebra

$$\begin{aligned} \delta_{ij}\delta_{ji} &= \delta_{ii} = \mathrm{Tr}[\delta] \qquad \text{(fundamental representation)} \\ &= N_c = C_A \end{aligned}$$

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Step 5: Perform fermion spin sum

$$|\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}|^2 = Q_f^2\, e^2\, C_A \varepsilon_\mu(p_1)\, \varepsilon^*_\nu(p_1)\, \bar{v}^r(p_3)\gamma^\nu u^s(p_2)\bar{u}^s(p_2)\gamma^\mu v^r(p_3)$$

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$$= Q_f^2\, e^2\, C_A \varepsilon_\mu(p_1)\, \varepsilon^*_\nu(p_1)\, \mathrm{Tr}\left[\bar{v}^r(p_3)\gamma^\nu u^s(p_2) \bar{u}^s(p_2)\gamma^\mu v^r(p_3)\right]$$

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&= Q_f^2\, e^2\, C_A \varepsilon_\mu(p_1)\, \varepsilon^*_\nu(p_1)\, \mathrm{Tr}\left[u^s(p_2)\bar{u}^s(p_2)\gamma^\mu v^r(p_3)\bar{v}^r(p_3)\gamma^\nu\right]
\end{aligned}
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Spin sums

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Hence, in our case

$$\sum_{s,r} \mathrm{Tr}\left[u^s(p_2)\bar{u}^s(p_2)\gamma^\mu v^r(p_3)\bar{v}^r(p_3)\gamma^\nu\right] = \mathrm{Tr}[\not{p}_2\gamma^\mu \not{p}_3\gamma^\nu]$$

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Step 6: Compute traces of γ matrices

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\end{aligned}
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Clifford algebra

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&\quad + 2\mathrm{Tr}[\gamma^\rho\gamma^\mu]g^{\sigma\nu}
\end{aligned}$$

$$\mathrm{Tr}[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu] = 4\ \left(g^{\mu\sigma}g^{\nu\rho} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\mu}g^{\sigma\nu}\right)$$

$\gamma^* \to q\bar{q}$ at LO

Step 6: Compute traces of γ matrices

$$
\begin{aligned}
|\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}|^2 &= Q_f^2\, e^2\, C_A\, \varepsilon_\mu(p_1)\, \varepsilon^*_\nu(p_1)\, \mathrm{Tr}\left[\not{p}_2 \gamma^\mu \not{p}_3 \gamma^\nu\right] \\
&= Q_f^2\, e^2\, C_A\, \varepsilon_\mu(p_1)\, \varepsilon^*_\nu(p_1)\, p_{2\rho} p_{3\sigma}\, \mathrm{Tr}\left[\gamma^\rho \gamma^\mu \gamma_\sigma \gamma^\nu\right]
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&= 4 Q_f^2\, e^2\, C_A\, \varepsilon_\mu(p_1)\, \varepsilon^*_\nu(p_1)\, p_{2\rho} p_{3\sigma} \left(g^{\mu\sigma} g^{\nu\rho} - g^{\rho\sigma} g^{\mu\nu} + g^{\rho\mu} g^{\sigma\nu}\right)
\end{aligned}
$$

Polarization sums

In the axial gauge

$$n \cdot A = 0$$

with $n^2 = 0$ (light-cone gauge), polarization vectors fulfill the relation

$$n \cdot \varepsilon(k) = 0$$

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In Feynman gauge

$$\sum_{\text{pol.}} \varepsilon_\mu(p)\, \varepsilon^*_\nu(p) = -g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}$$

but this sum runs also through non-physical polarizations and one needs to introduce ghosts to account for that

$\gamma^* \to q\bar{q}$ at LO

Step 7: Perform sum over polarizations

$$g^{\mu\nu} g_{\nu\sigma} = \delta^{\mu}_{\ \sigma}$$

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$$\begin{aligned} |\mathcal{M}^{(0)}_{\gamma^* \to q\bar{q}}|^2 &= 4 Q_f^2\, e^2\, C_A\, \varepsilon_\mu(p_1)\, \varepsilon^*_\nu(p_1)\, p_{2\rho} p_{3\sigma} \left(g^{\mu\sigma} g^{\nu\rho} - g^{\rho\sigma} g^{\mu\nu} + g^{\rho\mu} g^{\sigma\nu} \right) \\ &= 4 Q_f^2\, e^2\, C_A\, (-g_{\mu\nu}) p_{2\rho} p_{3\sigma} \left(g^{\mu\sigma} g^{\nu\rho} - g^{\rho\sigma} g^{\mu\nu} + g^{\rho\mu} g^{\sigma\nu} \right) \end{aligned}$$

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$$= 4Q_f^2\, e^2\, C_A\, (-g_{\mu\nu}) p_{2\rho} p_{3\sigma} \left(g^{\mu\sigma} g^{\nu\rho} - g^{\rho\sigma} g^{\mu\nu} + g^{\rho\mu} g^{\sigma\nu}\right)$$

$$= 4Q_f^2\, e^2\, C_A\, p_{2\rho} p_{3\sigma} \left(g^{\mu\sigma} \delta^{\rho}_{\mu} - g^{\rho\sigma} \delta^{\mu}_{\mu} + g^{\rho\mu} \delta^{\sigma}_{\mu}\right)$$

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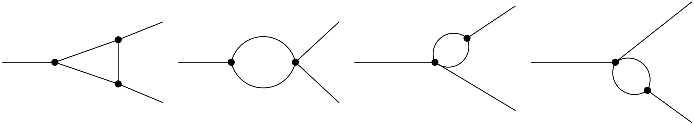
|\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}|^2 &= 4Q_f^2\, e^2\, C_A\, \varepsilon_\mu(p_1)\, \varepsilon^*_\nu(p_1)\, p_{2\rho} p_{3\sigma} \left(g^{\mu\sigma} g^{\nu\rho} - g^{\rho\sigma} g^{\mu\nu} + g^{\rho\mu} g^{\sigma\nu}\right) \\
&= 4Q_f^2\, e^2\, C_A\, (-g_{\mu\nu}) p_{2\rho} p_{3\sigma} \left(g^{\mu\sigma} g^{\nu\rho} - g^{\rho\sigma} g^{\mu\nu} + g^{\rho\mu} g^{\sigma\nu}\right) \\
&= 4Q_f^2\, e^2\, C_A\, p_{2\rho} p_{3\sigma} \left(g^{\mu\sigma} \delta^{\rho}_{\mu} - g^{\rho\sigma} \delta^{\mu}_{\mu} + g^{\rho\mu} \delta^{\sigma}_{\mu}\right) \\
&= 4Q_f^2\, e^2\, C_A\, (1-\epsilon) Q^2
\end{aligned}$$

Done!

$\gamma^* \to q\bar{q}$ at NLO (virtual)

$\gamma^* \to q\bar{q}$ at NLO (virtual)

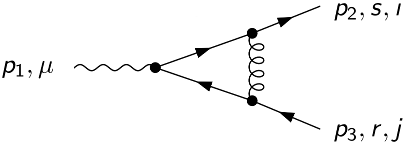
Step 1: Draw all possible topologies



- A set of possible topologies is restricted by the number of external particles and types of vertices (*e.g.* 3- and 4-point vertices in QCD)

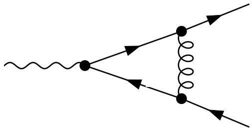
$\gamma^* \to q\bar{q}$ at NLO (virtual)

Step 2: Figure out which particles can run through diagrams



$\gamma^* \to q\bar{q}$ at NLO (virtual)

Step 3: Obtain the amplitude



$$\mathcal{M}^{(1)}_{\gamma^*\to q\bar{q}} = -\int \frac{d^d k_1}{(2\pi)^d} \varepsilon^\rho(p_1) \frac{g^{\mu\nu}}{(k_1-p_2)^2}$$

$$\bar{u}(p_2)\left(-ig_s\gamma^\mu T^a_{ik}\right)\frac{\not{k}_1}{k_1^2}\left(-iQ_f e\gamma^\rho\right)\frac{\not{k}_1-\not{p}_2-\not{p}_3}{(k_1-p_2-p_3)^2}\left(-ig_s\gamma^\nu T^a_{kj}\right)v(p_3)$$

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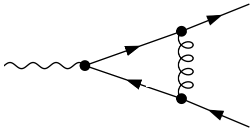

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$$\bar{u}(p_2) \left(-ig_s \gamma^\mu T^a_{ik}\right) \frac{\not{k}_1}{k_1^2} \left(-iQ_f e \gamma^\rho\right) \frac{\not{k}_1 - \not{p}_2 - \not{p}_3}{(k_1 - p_2 - p_3)^2} \left(-ig_s \gamma^\nu T^a_{kj}\right) v(p_3)$$

$$= iQ_f e\, g_s^2\, T^a_{ik} T^a_{kj} \int \frac{d^d k_1}{(2\pi)^d} \varepsilon^\rho(p_1) \frac{1}{(k_1 - p_2)^2}$$

$$\times\ \bar{u}(p_2) \gamma^\mu \frac{\not{k}_1}{k_1^2} \gamma^\rho \frac{\not{k}_1 - \not{p}_2 - \not{p}_3}{(k_1 - p_2 - p_3)^2} \gamma_\mu v(p_3)\,,$$

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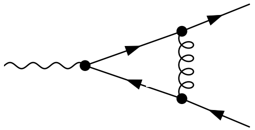

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and, similarly, the conjugate amplitude $\mathcal{M}^*$.

$\gamma^* \to q\bar{q}$ at NLO (virtual)

Step 3: Obtain the amplitude squared

In fact, what we are interested in is an interference term, since

$$
\begin{aligned}
|\mathcal{M}|^2 &= |\mathcal{M}^{(0)} + \mathcal{M}^{(1)} + \cdots|^2 \\
&= \left(\mathcal{M}^{*(0)} + \mathcal{M}^{*(1)} + \cdots\right)\left(\mathcal{M}^{(0)} + \mathcal{M}^{(1)} + \cdots\right) \\
&= |\mathcal{M}^{(0)}|^2 + \mathcal{M}^{*(0)}\mathcal{M}^{(1)} + \mathcal{M}^{*(1)}\mathcal{M}^{(0)} + \mathcal{O}\left(g_s^4\right)
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Step 4: Perform colour algebra

$$
\delta_{ij} T^a_{ik} T^a_{kj} = T^a_{ik} T^a_{ki} = \mathrm{Tr}[T^a T^a] = T_F \delta^{aa} = T_F(N_c^2 - 1) = C_A C_F
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- Recall that the Casimir operators read

$$
C_F = \frac{N^2 - 1}{2N}, \qquad C_A = N .
$$

Colour algebra

The $SU(N)$ Lie algebra is defined by

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The last equation follows from

$$\left(\boldsymbol{T}^a\boldsymbol{T}^b\right)^* = \left(\boldsymbol{T}^a\boldsymbol{T}^b\right)^*_{ij} = T^{a\,*}_{ik}T^{b\,*}_{kj} = T^a_{ki}T^b_{jk} = \left(\boldsymbol{T}^b\boldsymbol{T}^a\right)_{ji} = \boldsymbol{T}^b\boldsymbol{T}^a\,.$$

Colour algebra

Some useful color algebra identities

$$
\begin{aligned}
\mathrm{Tr}(T^a) &= 0\,, \\
\mathrm{Tr}(T^a T^b) &= T_F \delta_{ab}\,, \\
\delta_{aa} &= N^2 - 1\,, \\
\mathrm{Tr}(T^a T^b T^a T^c) &= -\frac{1}{4N}\delta_{bc}\,, \\
\mathrm{Tr}(T^a T^b T^c) &= \frac{1}{4}\left(d^{abc} + i f^{abc}\right)\,, \\
f_{acd} d_{bcd} &= 0\,, \\
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\end{aligned}$$

From the above we have

$$\begin{aligned}
\mathrm{Tr}(T^a T^b T^b T^a) &= C_A C_F^2\,,\\
\mathrm{Tr}(T^a T^b T^a T^b) &= -\frac{1}{2} C_F\,.\\
f^{abc} T^b T^c &= \frac{i}{2} C_A T^a\,.
\end{aligned}$$

Colour algebra

[Cvitanovic '76]

$$if^{abc} = \frac{1}{T_F}\mathrm{Tr}([T^a, T^b]T^c)$$

$$T^a_{ij}T^a_{kl} = T_F\left(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$$

- Allows one to evaluate all the colour factors algorithmically

$\gamma^* \to q\bar{q}$ at NLO (virtual)

Step 5: Perform fermion spin sum

Step 6: Compute traces of γ matrices

Step 7: Perform sum over polarizations

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At this point, our interference term reads

$$\mathcal{M}^{*(0)}_{\gamma^*\to q\bar{q}}\mathcal{M}^{(1)}_{\gamma^*\to q\bar{q}} = -iQ_f^2 e^2(2-2\epsilon)C_A C_F\, g_s^2$$

$$\int\frac{d^d k_1}{(2\pi)^d}\,\frac{p_2\cdot p_3(-2\epsilon k_1^2 - 4k_1\cdot p_3) + k_1\cdot p_2(4k_1\cdot p_3 + 4\epsilon)p_2\cdot p_3}{k_1^2(k_1-p_2)^2(k_1-p_2-p_3)^2}$$

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- Now we need to integrate over the loop momentum k_1

$\gamma^* \to q\bar{q}$ at NLO (virtual)

We have the following factors in the denominator (let's call them propagators)

$$k_1^2 \equiv P_1$$
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Now, we can write all the scalar products as combinations of denominators

$$k_1^2 = P_1$$
$$2\,k_1 \cdot p_2 = k_1^2 - (k_1 - p_2)^2 = P_1 - P_2$$
$$2\,k_1 \cdot p_3 = (k_1 - p_2)^2 - (k_1 - p_2 - p_3)^2 + Q^2 = P_2 - P_3 + Q^2$$

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- Our entire expression can be written in terms of the propagators P_1, P_2, P_3 and the external parameter Q^2

$\gamma^* \to q\bar{q}$ at NLO (virtual)

Topology

$$T_1(a_1, a_2, a_3) = \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{k_1^{2a_1}(k_1 - p_2)^{2a_2}(k_1 - p_2 - p_3)^{2a_3}}$$

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- All integrals in our expression can be written with the above topology T_1
- Note that positive a_i corresponds to a propagator in the denominator

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It turns out that the following integrals appear in our expression

$$T_1(0,0,1)$$
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- We have, in total 8 integrals to evaluate, some of them with only positive, others with positive and negative indices

$\gamma^* \to q\bar{q}$ at NLO (virtual)

IBP reduction

In dimensional regularization the integral over total derivative is zero

$$\int d^d k_1 \, \ldots d^d k_L \frac{\partial}{\partial k_i^\mu} \left(\frac{q^\mu}{P_1^{a_1} \cdots P_N^{a_N}} \right) = 0 \,,$$

where q is an arbitrary loop or external momentum.

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IBP reduction

In dimensional regularization the integral over total derivative is zero

$$\int d^d k_1 \, \dots d^d k_L \frac{\partial}{\partial k_i^\mu} \left(\frac{q^\mu}{P_1^{a_1} \cdots P_N^{a_N}} \right) = 0 \,,$$

where q is an arbitrary loop or external momentum.

This generates a set of relations between integrals

$$\sum_k c_k I_k = 0 \,.$$

known as integration by parts (IBP) identities.

$\gamma^* \to q\bar{q}$ at NLO (virtual)

That's what we get from IBP reduction

$$
\begin{aligned}
T_1(0,0,1) &= 0\\
T_1(0,1,0) &= 0\\
T_1(1,0,0) &= 0\\
T_1(1,-1,1) &= -\frac{1}{2} s_{23}\, T_1(1,0,1)\\
T_1(0,1,1) &= 0\\
T_1(1,0,1) &= T_1(1,0,1)\\
T_1(1,1,0) &= 0\\
T_1(1,1,1) &= -\frac{2(d-3)}{(d-4)s_{23}}\, T_1(1,0,1)
\end{aligned}
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All integrals can be expressed in terms of the single master integral

$$T_1(1,0,1)$$

Scaleless integrals

Notice that some integrals from our list have been reduced to zero by the IBP identities, *e.g.*

$$T_1(1,0,0) = \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{k_1^2} = 0\,,$$

$$T_1(1,1,0) = \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{k_1^2 (k_1 - p_2)^2} = 0\,.$$

These are so-called scaleless integrals.

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These are so-called scaleless integrals.

Notice that the only external scale available in our problem is

$$Q^2 = 2 p_2 \cdot p_3\,.$$

However, integrals like $T_1(1,0,0)$ or $T_1(1,1,0)$ cannot depend on this scale, hence, they must vanish.

$\gamma^* \to q\bar{q}$ at NLO (virtual)

Our only master integral is is called a bubble integral and it reads

$$T_1(1,0,1) \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{k_1^2 (k_1 - p_2 - p_3)^2} \equiv \mathrm{Bub}(s_{23}),$$

where

$$s_{23} = (p_2 + p_3)^2 = 2 p_2 \cdot p_3 = Q^2$$

$\gamma^* \to q\bar{q}$ at NLO (virtual)

In summary:

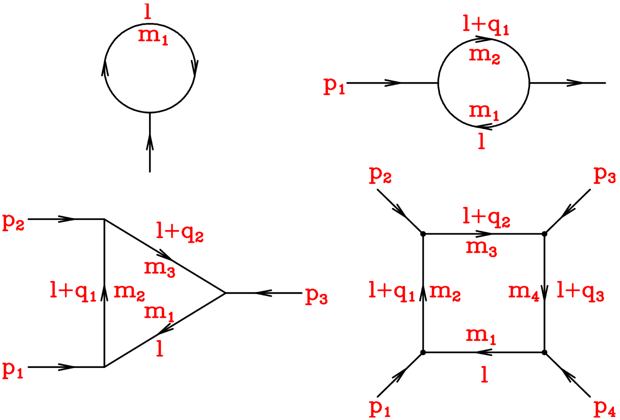
Our interference term $\mathcal{M}^{*(0)}\mathcal{M}^{(1)}$ with 8 loop integrals can be expressed in terms of only one, bubble integral, which is known exactly

$$\text{Bub}(Q^2) = i\pi^{\frac{d}{2}} \frac{\Gamma(\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)(-Q^2)^{\epsilon}}$$

- We shall calculate this integral in lecture 3

Spoiler

All NLO master integrals in 4 dimensions can be expressed by ....



- Considering all combinations of massless/massive internal/external particles, this gives in total 24 integrals at NLO [Ellis, Zanderighi '08]

$\gamma^* \to q\bar{q}$ at NLO (virtual)

The final result (exact) reads

$$\mathcal{M}^{*(0)}_{\gamma^*\to q\bar{q}}\mathcal{M}^{(1)}_{\gamma^*\to q\bar{q}} = i\pi^{d/2} c_\Gamma Q_f^2 e^2 g_s^2 C_A C_F \frac{(1-\epsilon)(2-\epsilon+2\epsilon^2)}{(1-2\epsilon)\,\epsilon^2} \left(-Q^2\right)^{-\epsilon}$$

where

$$c_\Gamma = \frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}$$

- Notice the $\dfrac{1}{\epsilon^2}$ pole. It corresponds to the gluon being both soft and collinear.

$\gamma^* \to q\bar{q}$ at NLO (real)

$$\gamma^*(p_1) \to q(p_2) + \bar{q}(p_3) + g(p_4)$$

$\gamma^* \to q\bar{q}$ at NLO (real)

$$\gamma^*(p_1) \to q(p_2) + \bar{q}(p_3) + g(p_4)$$

Now, this is a $1 \to 3$ process and it is characterized by the following Mandelstam variables

$$s_{23} = (p_2 + p_3)^2 = 2p_2 \cdot p_3$$

$$s_{24} = (p_2 + p_4)^2 = 2p_2 \cdot p_4$$

$$s_{34} = (p_3 + p_4)^2 = 2p_3 \cdot p_4$$

$$s_{234} = (p_2 + p_3 + p_4)^2 \equiv Q^2$$

$\gamma^* \to q\bar{q}$ at NLO (real)

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- Recall that for $\gamma^* \to q\bar{q}$ we had only one variable Q^2

$\gamma^* \to q\bar{q}$ at NLO (real)

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- Recall that for $\gamma^* \to q\bar{q}$ we had only one variable Q^2

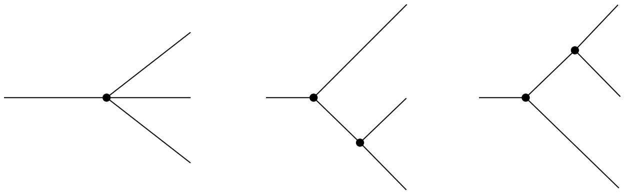
Again, we shall consider quarks to be massless, hence we can obtain the following relations

$$p_1 = p_2 + p_3 + p_4$$

$$Q^2 \equiv p_1^2 = s_{23} + s_{24} + s_{34}$$

$\gamma^* \to q\bar{q}$ at NLO (real)

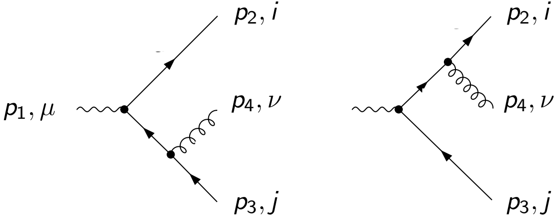
Step 1: Draw all possible topologies



- A set of possible topologies is restricted by the number of external particles and types of vertices (*e.g.* 3- and 4-point vertices in QCD)

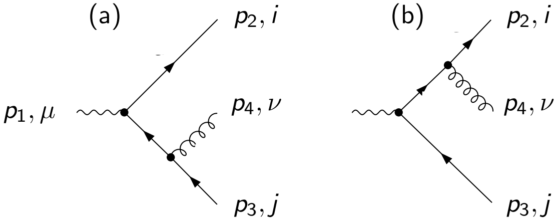
$\gamma^* \to q\bar{q}$ at NLO (real)

Step 2: Figure out which particles can run through diagrams



$\gamma^* \to q\bar{q}$ at NLO (real)

Step 3: Obtain the amplitude



$$\mathcal{M}_a^{(0)} = -\varepsilon^\mu(p_1)\varepsilon^\nu(p_4)\bar{u}(p_2)\big(iQ_f e\gamma^\mu\big)\frac{\not{p}_3 + \not{p}_4}{(p_3+p_4)^2}\left(-ig_s\gamma^\nu T^a_{ij}\right) v(p_3)$$

$\gamma^* \to q\bar{q}$ at NLO (real)

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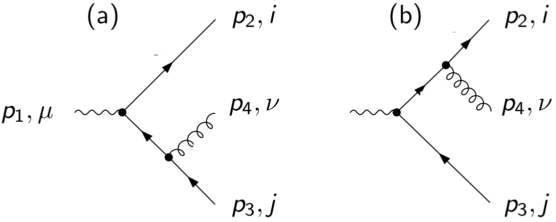

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$$\mathcal{M}_b^{(0)} = -\varepsilon^\mu(p_1)\varepsilon^\nu(p_4)\bar{u}(p_2)\left(-ig_s\gamma^\nu T^a_{ij}\right)\frac{\not{p}_2+\not{p}_4}{(p_2+p_4)^2}\left(iQ_f e\gamma^\mu\right)v(p_3)$$

$\gamma^* \to q\bar{q}$ at NLO (real)

Step 3: Obtain the amplitude (aka matrix element) squared

$$|\mathcal{M}|^2 = |\mathcal{M}_a + \mathcal{M}_b|^2 = |\mathcal{M}_a|^2 + |\mathcal{M}_b|^2 + \mathcal{M}_a^* \mathcal{M}_b + \mathcal{M}_a \mathcal{M}_b^*$$

$\gamma^* \to q\bar{q}$ at NLO (real)

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Step 4: Perform colour algebra

$$T^a_{ij}\left(T^a_{ij}\right)^* = T^a_{ij}T^a_{ji} = \mathrm{Tr}[T^aT^a] = T_F\delta^{aa} = T_F(N_c^2-1) = C_AC_F$$

$\gamma^* \to q\bar{q}$ at NLO (real)

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Step 4: Perform colour algebra

$$T_{ij}^a \left(T_{ij}^a\right)^* = T_{ij}^a T_{ji}^a = \mathrm{Tr}[T^a T^a] = T_F \delta^{aa} = T_F(N_c^2 - 1) = C_A C_F$$

Step 5: Perform fermion spin sum

Step 6: Compute traces of γ matrices

Step 7: Perform sum over polarizations for the photon and the gluon

$\gamma^* \to q\bar{q}$ at NLO (real)

Recall that

$$Q^2 = s_{23} + s_{24} + s_{34}$$

$\gamma^* \to q\bar{q}$ at NLO (real)

Recall that

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Let's introduce

$$s_{ij} = \frac{Q^2}{2}(1 - x_{ij})$$

where

$$x_{ij} \in [0, 1]$$

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where

$$x_{ij} \in [0, 1]$$

Then

$$2 = x_{23} + x_{24} + x_{34}$$

- We can use the above to eliminate x_{23}

$\gamma^* \to q\bar{q}$ at NLO (real)

Then, our final result reads

$$|\mathcal{M}_{\gamma^* \to q\bar{q}g}|^2 = 8 C_A C_F Q_f^2 e^2 g_s^2 \,(1-\epsilon)\frac{x_{24}^2 + x_{34}^2 + \epsilon(2 - x_{24} - x_{34})}{(1-x_{24})(1-x_{34})}$$

$\gamma^* \to q\bar{q}$ at NLO (real)

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If we now try to integrate this matrix element in order to obtain the total cross section for the process $\gamma^* \to q + \bar{q} + g$, we shall encounter singularities.

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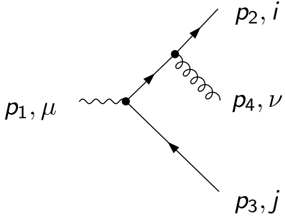
Structure of singularities familiar from e^+e^-

$$|\mathcal{M}|^2 \to \infty \quad \text{if} \quad \begin{cases} x_{24} \to 1 \text{ and } x_{34} \to 1 & \text{soft limit} \\ x_{24} \to 1 & \text{collinear limit} \\ x_{34} \to 1 & \text{collinear limit} \end{cases}$$

$\gamma^* \to q\bar{q}$ at NLO (real)



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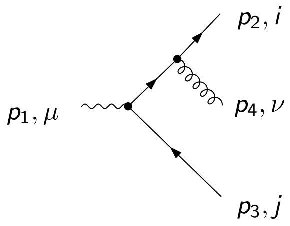
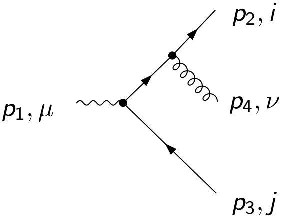


$$p_{24} = p_2 + p_4 = p + \frac{s_{24}}{2p_{24} \cdot n} n \,, \quad \text{where} \qquad p^2 = n^2 = 0\,, \quad p \cdot n \neq 0$$

$\gamma^* \to q\bar{q}$ at NLO (real)



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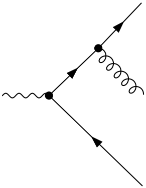
- p defines the collinear direction and n is an auxiliary light-like vector

$\gamma^* \to q\bar{q}$ at NLO (real)



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In the collinear limit:

$$p_2 \to z_2 p\,, p_4 \to z_4 p\,, \qquad z_i = \frac{p_i \cdot n}{p \cdot n}$$

$$z_2 + z_4 = 1$$

$\gamma^* \to q\bar{q}$ at NLO (real)

$$|\mathcal{M}_{\gamma^*\to q\bar{q}g}|^2 = 8\,C_A C_F Q_f^2 e^2 g_s^2\,(1-\epsilon)\frac{x_{24}^2 + x_{34}^2 + \epsilon(2 - x_{24} - x_{34})}{(1-x_{24})(1-x_{34})}$$

$$\frac{|\mathcal{M}_{\gamma^*\to q\bar{q}g}|^2}{|\mathcal{M}_{\gamma^*\to q\bar{q}}|^2} \propto \frac{2\,C_F g_s^2}{Q^2}\left[\frac{1+z^2}{1-z} - \epsilon\,(1-z)\right]$$

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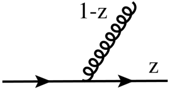
Compare to



$$\hat{P}_{qq}^{(0)}(z) \quad = \quad C_F\left[\frac{1+z^2}{1-z} - \epsilon\,(1-z)\right]$$

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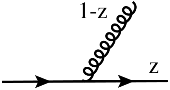

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- Other splitting functions can be obtained by the same procedure