Cylindrically symmetric gravitational collapse

Louise Nolan, NUI Galway

Joint work with Brien Nolan, DCU.

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Cylindrical symmetry

Pressureless dust is used to model the matter content $T_{ij} = \rho(r,t)u_i u_j$. A dust spacetime with cylindrical symmetry will have a line element:

$$ds^2 = \mathrm{d}t^2 - \mu^2(r,t)\mathrm{d}r^2 - \nu^2(r,t)\mathrm{d}z^2 - \sigma^2(r,t)\mathrm{d}\phi^2,$$

We also want to impose self similarity on our system, We define a similarity (self-similar) solution of the field equations as one for which the resulting spacetime admits the conformal Killing vector k^i satisfying

$$\mathcal{L}_{\vec{k}} g_{ij} = k_{i;j} + k_{j;i} = 2g_{ij},$$

this implies that

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mu(\xi)^2\mathrm{d}r^2 - r^2\left(\nu(\xi)^2\mathrm{d}z^2 + \sigma(\xi)^2\mathrm{d}\phi^2\right).$$

where $\xi = \frac{r}{t}, \zeta = \frac{t}{r}$.

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Einstein's field equations

We can now write out the field equations for this spacetime, $G_{ij} = T_{ij}$. The solution is given by the solution to a third order non-linear ODE

$$\begin{aligned}
-\mu'''\xi^2 + 2\xi\mu'' + 2\mu' + 2\xi^2\frac{(\mu'')^2}{\mu'} &+ \frac{2\xi^2\mu^2(\xi\mu'' + 2\mu')}{(\mu^2\xi^2 - 1)} + \\
+\left(-\frac{\xi(\mu')^2}{\mu} - \frac{\xi^2\mu''\mu'}{\mu}\right)\frac{\mu^2\xi^2+1}{\mu^2\xi^2-1} &= 0.
\end{aligned}$$

Converting into a dynamical system

- Rewrite the third order non-linear ODE as an autonomous three dimensional dynamical system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}s} &= y,\\ \frac{\mathrm{d}y}{\mathrm{d}s} &= z,\\ \frac{\mathrm{d}z}{\mathrm{d}s} &= y - \frac{2(y-z)^2}{(x-y)} - \frac{2x^2(y-z)}{(x^2-1)}\\ &+ \frac{(x-y)(x-2y+z)(x^2+1)}{x(x^2-1)},\end{aligned} \tag{1}$$

  where $x = \mu(\xi)|\xi|, \quad y = \frac{\mathrm{d}x}{\mathrm{d}s}, \quad z = \frac{\mathrm{d}y}{\mathrm{d}s}, \quad s = \ln|\xi|$.
- Imposing regularity condition on the axis $r = 0, t < 0$ provides initial data for our dynamical system.

$$\lim_{|\xi|\to 0} \vec{x} = \vec{0}, \quad \vec{x} = (x, y, z)^T$$

$|\xi| \in (0, 1)$

- The regular axis $\xi = 0$ is shown to correspond to one of the equilibrium points of (1), this is a hyperbolic equilibrium point, so we can find a solution in the 3-dim stable manifold of this point.
- By deriving properties of this solution we can eliminate all but one possible equilibrium set that the solution will evolve to. We show that this equilibrium set corresponds to the past null cone, at $\xi = -1$.
- $\mu(\xi) = \frac{1}{|\xi|}\left(1 + k \ln|\xi| + \mathcal{O}(\ln|\xi|^2)\right)$ as $|\xi| \to 1$
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$|\xi| \in (1, \infty)$

- The solution emanating from the past null cone is found to be a solution in the 2-dim unstable manifold of a hyperbolic equilibrium point at the axis.
- Evolution of solution to $\xi = -\infty$, $\zeta = 0$, $(t = 0, r > 0)$: Using this local solution and (1) we can derive properties of the unique solution, hence all but one possible equilibrium set E_1 is eliminated.
  $E_1 : (x, y, z) \to (x_0, x_0, x_0), x_0 > 1$ as $\tau \to \infty$.
- A Liapunov function for this equilibrium set is constructed.
- This equilibrium set is non hyperbolic.
- We obtain the local solution $\mu = c_0 + c_1|\zeta| + \mathcal{O}(|\zeta|^2)$.

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$\zeta \in (0, \zeta_F)$

- We define new dependent and independent variables:,

$$a = x, b = x - y, c = \frac{y - z}{y - x}, \tau$$

  and find the equilibrium set of this system corresponding to our solution, when we write out this solution we get a one parameter solution.

- We recast our dynamical system again, there are 3 equilibrium sets that the solution can evolve to.
- We can eliminate all but one $E_3 : (a, b, c) = (1, B_0, \frac{-B_0}{1+B_0}), B_0 \in [0, 1]$.

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$\zeta \to \zeta_F$

We can prove that the two parameter solution emanating from $\zeta = 0$ reaches the future null cone at $\zeta = \zeta_F < \infty$.

$$\mu = \zeta \left(1 - (B_0 - 1) c \ln\left(\frac{\zeta}{\zeta_F}\right) + \mathcal{O}\left(c \ln\left(\frac{\zeta}{\zeta_F}\right)^2 \right) \right)^{-1}$$

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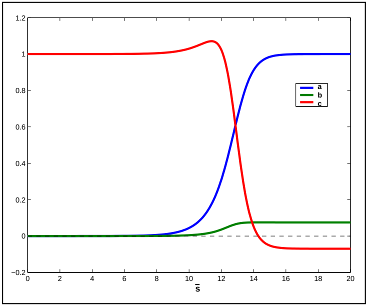
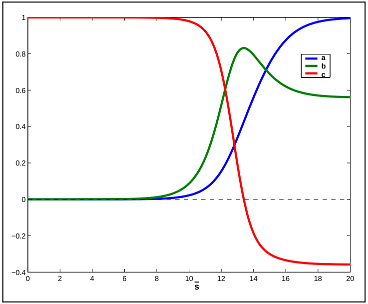
Numerical approximation of solution approaching E_3



Initial data: $\mu_0 = 1$, $\mu_1 = 2$, $\mu_2 = -1$.



Initial data: $\mu_0 = 0.5$, $\mu_1 = 0.1$, $\mu_2 = 0.1$.

Numerical demonstration of solution approaching equilibrium point E_3.

y_0	μ_0	μ_1	μ_2	b_{num}	c_{num}	$\frac{-b_{num}}{1+b_{num}}$	δ
0.000001	0.5	0.1	0.1	0.0748508	-0.069638	-0.069638	-8.2179×10^{-8}
0.000001	0.5	0.1	-0.1	0.035328	-0.034124	-0.034123	-7.8888×10^{-7}
0.000001	0.5	0.2	0.1	0.11138	-0.100219	-0.10021	-1.7904×10^{-7}
0.000001	0.5	0.2	-0.1	0.078752	-0.073003	-0.0730035	-2.2279×10^{-7}
0.000001	0.5	0.2	-0.2	0.068672	-0.064259	-0.064259	-2.5560×10^{-7}
0.000001	1	1	1	0.62717	-0.38562	-0.38544	-0.000183556
0.000001	1	1	-1	0.37598	-0.27322	-0.27325	0.000023724
0.000001	1	2	-1	0.55796	-0.35924	-0.35813	-0.0011123
0.000001	1	2	1	0.61980	-0.38286	-0.38264	-0.00022887
0.000001	3	2.5	5	0.63618	-0.38985	-0.38882	-0.0010271

Physical interpretation of the solution

- A singularity forms at the point $p_0 : r = 0, t = 0$.
- Outgoing radial null geodesics through p_0 are represented by the first positive value of ξ satisfying $\xi = \frac{1}{\mu}$, this singularity is naked.
- Existence of a solution in this spacetime agrees with the self similarity hypothesis.

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