# Using an Effective Charges Method to Find $\alpha_s(M_z)$

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# Outline

# Motivation

**Scheme dependence problem**:

- Fixed-order results depend on renormalisation scheme (RS) and scale, $\mu$.
- Standard method:
  - Use $\overline{\text{MS}}$, set $\mu =$Q (e.g. com energy) and vary by factor of 2.
  - Large theoretical error.

# Motivation

**Scheme dependence problem**:

- Fixed-order results depend on renormalisation scheme (RS) and scale, $\mu$.
- Standard method:
  - Use $\overline{\text{MS}}$, set $\mu$ =Q (e.g. com energy) and vary by factor of 2.
  - Large theoretical error.
- Effective Charges Method (ECH):
  - Don't have to argue for a particular value of $\mu$.
  - Only use $\mu$ in intermediate steps of derivation. Final result is $\mu$ independent.
  - Links variables to universal $\Lambda_{\text{QCD}}$.

# Method [Burby, Maxwell; 2001], [Grunberg; 1984], [Dhar, Gupta; 1984]

Consider a generic, dimensionless QCD quantity $\mathcal{R}(Q)$. By dimensional analysis:

$$\mathcal{R}(Q) = f\left(\frac{\Lambda}{Q}\right)$$

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Consider a generic, dimensionless QCD quantity $\mathcal{R}(Q)$. By dimensional analysis:

$$\mathcal{R}(Q) = f\left(\frac{\Lambda}{Q}\right)$$

Invert:

$$\frac{\Lambda}{Q} = f^{-1}(\mathcal{R}(Q)) \tag{1}$$

Want to find $f^{-1}$.
Take $\mathcal{R}(Q)$ to be an **effective charge**:

$$\mathcal{R}(Q) = \mathrm{a}(1 + \mathrm{r}_1\mathrm{a} + \mathrm{r}_2\mathrm{a}^2 + \ldots) \tag{2}$$

where $\mathrm{a} = \alpha_s(\mu)/\pi$

# Method [Burby, Maxwell; 2001], [Grunberg; 1984], [Dhar, Gupta; 1984]

QCD $\beta$-function equation:

$$\frac{\partial a}{\partial \ln \mu} = \beta(a) = -ba^2(1 + ca + c_2 a^2 + ...) \qquad (3)$$

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For $\mathcal{R}(Q)$:

$$\frac{d\mathcal{R}}{d \ln Q} = \rho(\mathcal{R}(Q)) = -b\mathcal{R}^2(1 + c\mathcal{R} + \rho_2\mathcal{R}^2 + \rho_3\mathcal{R}^3 + \ldots) \tag{4}$$

where $\rho_2 = c_2 + r_2 - r_1 c - r_1^2$ is RS-invariant.

# Method [Burby, Maxwell; 2001], [Grunberg; 1984], [Dhar, Gupta; 1984]

Exponentiate and rearrange to get:

$$\Lambda_{\mathcal{R}} = \; Q\,\mathcal{F}(\mathcal{R}(Q))\,\mathcal{G}(\mathcal{R}(Q)) \tag{5}$$

where $\mathcal{F}(\mathcal{R}(Q)) = \exp[-\frac{1}{b\mathcal{R}} - \frac{c}{b}\ln\frac{c\mathcal{R}}{1+c\mathcal{R}}]$.

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where $\mathcal{F}(\mathcal{R}(Q)) = \exp[-\frac{1}{b\mathcal{R}} - \frac{c}{b}\ln\frac{c\mathcal{R}}{1+c\mathcal{R}}]$.
$\Lambda_{\mathcal{R}}$ is scheme independent but specific to the effective charge. Relate to $\Lambda_{\overline{MS}}$ which is universal.
Can move between Λ's in different schemes exactly.

$$\Lambda_{\overline{MS}} = \Lambda_{\mathcal{R}}\,\exp\left(\frac{-r}{b}\right)\left(\frac{2c}{b}\right)^{\frac{c}{b}} \tag{6}$$

where $r = r_1^{\overline{MS}}(Q)$.

Can convert $\Lambda_{\overline{MS}}$ to $\alpha_s$ using the $\beta$-function eqn at an appropriate order.

# Event Shape (ES) Observables

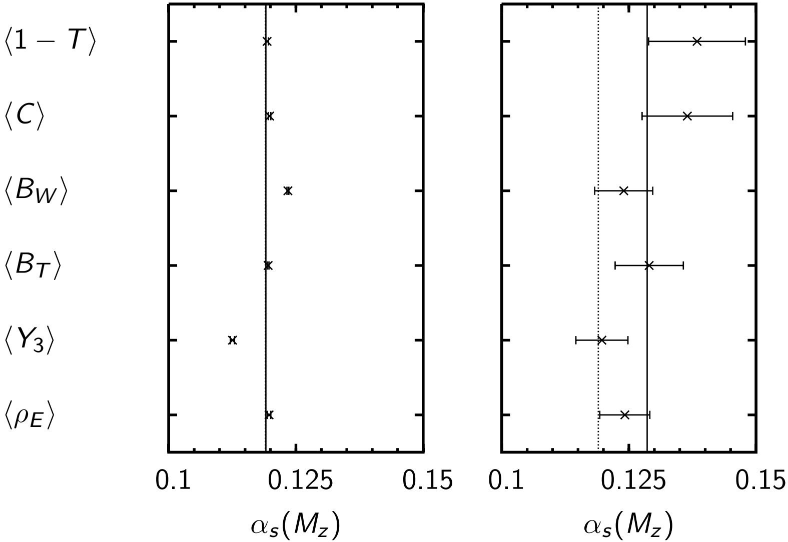
Using data from LEP (DELPHI, OPAL, L3) and PETRA (JADE).
Moments of: 1-T, C, $B_W$, $B_T$, $Y_3$, $\rho_E$
ES moments have PT expansion:

$$\langle y^n \rangle = \left(\frac{\alpha_s(Q)}{2\pi}\right)\overline{\mathcal{A}}_{y,n} + \left(\frac{\alpha_s(Q)}{2\pi}\right)^2\overline{\mathcal{B}}_{y,n} + \left(\frac{\alpha_s(Q)}{2\pi}\right)^3\overline{\mathcal{C}}_{y,n} + \dots \quad (7)$$

DELPHI [DELPHI coll., Feb 2003, hep-ex/0307048] have looked at means (n=1) with NLO ECH. Very promising results.

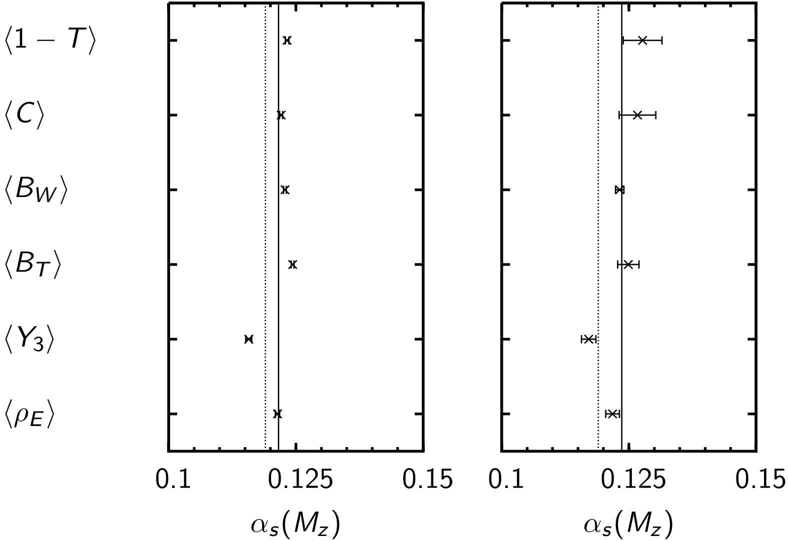
# Results: Means

NLO ECH and $\overline{\text{MS}}$ PT results for the means:



# Results: Means

NNLO ECH and $\overline{\text{MS}}$ PT results for the means:



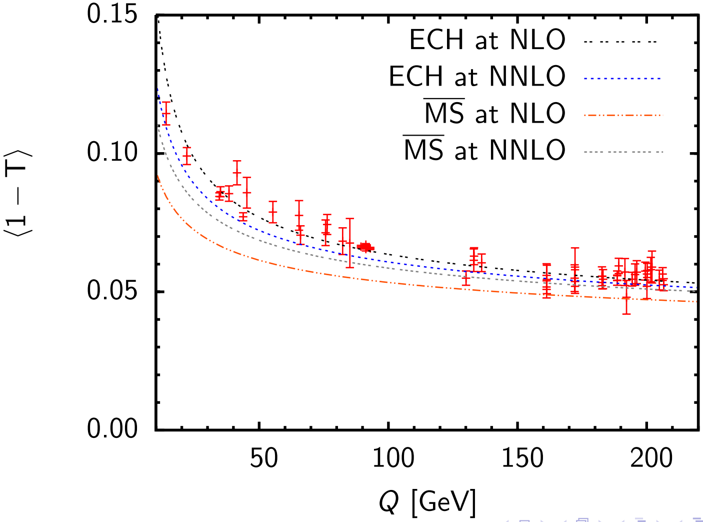
# Results: Plots approximating the data

Look at a different way of implementing ECH method:

- Take a fixed value of $\alpha_s(M_z) = 0.1190$ [Baikov, Chetyrkin, Kühn; Jan 2008]
- Work backwards through ECH method to get an approximation to data.
- Can compare with $\overline{\text{MS}}$ results for fixed $\alpha_s$ using (7).

# ECH plots: $\langle 1 - T \rangle$

ECH and $\overline{\text{MS}}$ curves at NLO and NNLO against data for $\langle 1 - T \rangle$:



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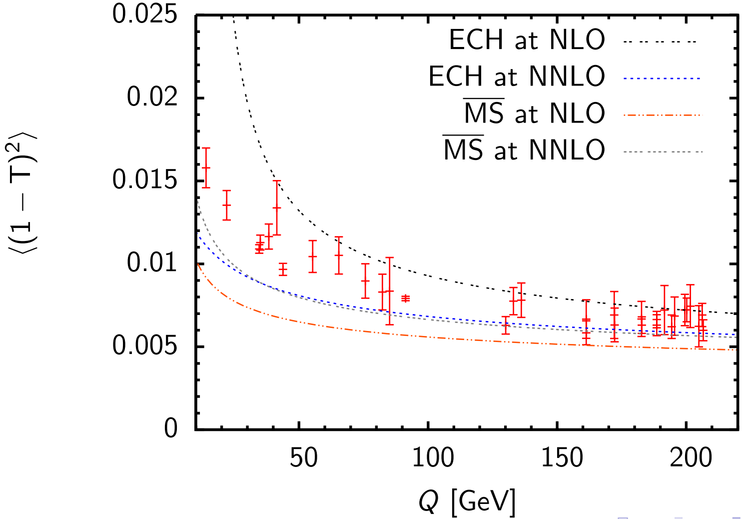

Look at values of $\Lambda_{\mathcal{R}}$ to help see where the perturbative regime is:

$$\begin{aligned} \langle 1-\mathrm{T}\rangle : &\qquad \Lambda_{\mathcal{R}} = 4\,\mathrm{GeV} \\ \langle (1-\mathrm{T})^2\rangle : &\qquad \Lambda_{\mathcal{R}} = 24\,\mathrm{GeV} \end{aligned}$$

Have data over the range 14.0-206.6 GeV.

# Conclusions and Future work

**Conclusions**:

- ECH at NLO works well for ES means - better than $\overline{\text{MS}}$
- ECH at NNLO often works less well than NLO
- Some higher moments are not suited to this method

**In progress**:

- Inclusion of non-perturbative power corrections

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**In progress**:

- Inclusion of non-perturbative power corrections

Thanks for listening!

# Back ups: ES definitions

Thrust:

$$\mathrm{T} = \max \frac{\sum_i |\mathbf{p_i} \cdot \mathbf{n_T}|}{\sum_i |\mathbf{p_i}|}$$

C-parameter - defined by the eigenvalues of the linear momentum tensor:

$$\begin{aligned} \theta_{m,n} &= \frac{\sum_i p_i^m p_i^n / |\mathbf{p_i}|}{\sum_i |\mathbf{p_i}|} \\ \mathrm{C} &= 3(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) \end{aligned}$$

Wide jet broadening - define jet broadening in two hemispheres divided by plane perpendicular to $\mathbf{n_T}$:

$$\begin{aligned} \mathrm{B}_x &= \frac{\sum_{\mathbf{p_k} \in \mathbf{H_x}} |\mathbf{p_i} \times \mathbf{n_T}|}{2 \sum_{\mathbf{p_k} \in \mathbf{H_x}} |\mathbf{p_i}|} \\ \mathrm{B_W} &= \max(\mathrm{B}_a, \mathrm{B}_b) \end{aligned}$$

# Back ups: ES definitions cont.

Total jet broadening - also defined by $\mathrm{B}_x$:

$$\mathrm{B}_\mathrm{T} = \mathrm{B}_a + \mathrm{B}_b$$

Three-to-two jet transition parameter in the Durham algorithm:

$$y_{ij}^{D} = \frac{2\min(E_i^2, E_j^2)(1-\cos\theta_{ij})}{E_{\mathrm{vis}}^2}$$

Heavy jet mass in the E-scheme - hemisphere observable:

$$\rho = \max\frac{M_x^2}{E_{\mathrm{vis}}^2} = \max\frac{1}{E_{\mathrm{vis}}^2}\left(\sum_{\mathbf{p}_k\in\mathbf{H}_x} p_k\right)^2 \tag{8}$$

The use of the E-scheme corrects for the effect of hadron masses. In the E-scheme the four-momentum $p = (\mathbf{p}, E)$ is replaced by $(\hat{p}E, E)$ where $\hat{p}$ is a unit vector in the **p**-direction. The definition in the E-scheme is identical to (8) for massless hadrons.

# Back ups: power corrections [Gehrmann, Jaquier, Luisoni; 2010]

Hadronisation corrections for event shape moments are expected to be additive:

$$\langle y^n \rangle = \langle y^n \rangle_{pt} + \langle y^n \rangle_{np} \qquad (9)$$

Dispersive model for power corrections: accounts for NP behaviour at low energies by replacing $\alpha_s$ with an $\alpha_{eff}$ below an IR cutoff scale, $\mu_I$. Done s.t. the integral of the coupling up to $\mu_I$ is finite:

$$\frac{1}{\mu_I}\int_0^{\mu_I} dQ\, \alpha_{\text{eff}}(Q^2) = \alpha_0(\mu_I) \qquad (10)$$

Leads to:

$$\langle y \rangle_{np} = a_y P$$

# Back ups: power corrections (cont.)

with:

$$P = \frac{4C_F}{\pi^2}\mathcal{M}\left(\alpha_0 - \left[\alpha_s(\mu_R) + \frac{b}{\pi}\left(1 + \ln\frac{\mu_R}{\mu_I} + \frac{K}{2b}\right)\alpha_s^2(\mu_R) + \left(4bc\left(1 + \ln\frac{\mu_R}{\mu_I} + \frac{L}{4bc}\right) + 8b^2\left(1 + \ln\frac{\mu_R}{\mu_I} + \frac{K}{2b}\right) + 4b^2\ln\frac{\mu_R}{\mu_I}\left(\ln\frac{\mu_R}{\mu_I} + \frac{K}{b}\right)\right)\frac{\alpha_s^3(\mu_R)}{4\pi^2}\right]\right)\frac{\mu_I}{Q}$$

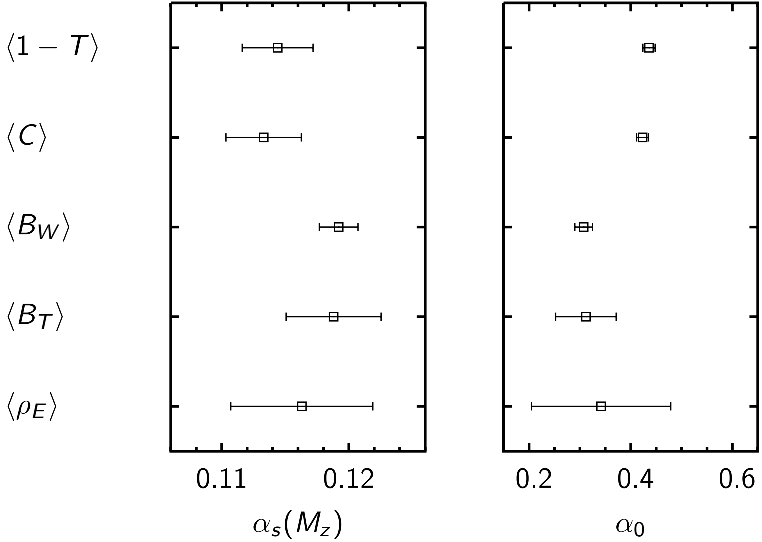
where $\mathcal{M}$ is the Milan factor and:

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{5}{9}N_F$$

$$L = C_A^2\left(\frac{245}{24} - \frac{67}{9}\frac{\pi^2}{6} + \frac{11}{6}\zeta_3 + \frac{11}{5}\left(\frac{\pi^2}{6}\right)^2\right) + C_F N_F\left(-\frac{55}{24} + 2\zeta_3\right) + C_A N_F\left(-\frac{209}{108} + \frac{10}{9}\frac{\pi^2}{6} - \frac{7}{3}\zeta_3\right) - \frac{1}{27}N_F^2$$

# Back ups: power corrections (cont.)

ECH and power corrections at NLO:



# Back ups: Method [Burby, Maxwell; 2001], [Grunberg; 1984], [Dhar, Gupta; 1984]

Integrate up (4):

$$
\begin{aligned}
\mathrm{b} \ln \frac{Q}{\Lambda_{\mathcal{R}}} &= \int_0^{\mathcal{R}(Q)} \frac{\mathrm{b}dx}{\rho(x)} + \kappa \\
&= \int_0^{\mathcal{R}(Q)} \frac{\mathrm{b}dx}{\rho(x)} + \int_0^{\infty} \frac{dx}{x^2(1+\mathrm{c}x)}
\end{aligned}
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&= \int_0^{\mathcal{R}(Q)} \frac{\mathrm{b}dx}{\rho(x)} + \int_0^{\infty} \frac{dx}{x^2(1+\mathrm{c}x)} \\
&= \int_{\mathcal{R}(Q)}^{\infty} \frac{dx}{x^2(1+\mathrm{c}x)} + \int_0^{\mathcal{R}(Q)} dx\left[\frac{\mathrm{b}}{\rho(x)} + \frac{1}{x^2(1+\mathrm{c}x)}\right] \\
&= \mathrm{F}(\mathcal{R}) + \mathrm{G}(\mathcal{R})
\end{aligned}
$$

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&= \int_0^{\mathcal{R}(Q)}\frac{\mathrm{b}dx}{\rho(x)}+\int_0^{\infty}\frac{dx}{x^2(1+\mathrm{c}x)} \\
&= \int_{\mathcal{R}(Q)}^{\infty}\frac{dx}{x^2(1+\mathrm{c}x)} \quad +\int_0^{\mathcal{R}(Q)}dx\left[\frac{\mathrm{b}}{\rho(x)}+\frac{1}{x^2(1+\mathrm{c}x)}\right] \\
&= \qquad \mathrm{F}(\mathcal{R}) \qquad + \qquad \mathrm{G}(\mathcal{R}) \\
&= \frac{1}{\mathcal{R}}+\mathrm{c}\ln\left[\frac{\mathrm{c}\mathcal{R}}{1+\mathrm{c}\mathcal{R}}\right]+ \qquad \mathrm{G}(\mathcal{R}) \qquad (11)
\end{aligned}$$

Result for $\mathrm{G}(\mathcal{R})$ depends on the order you work to. At NLO $\mathrm{G}(\mathcal{R})=0$. At NNLO there is an analytic result.

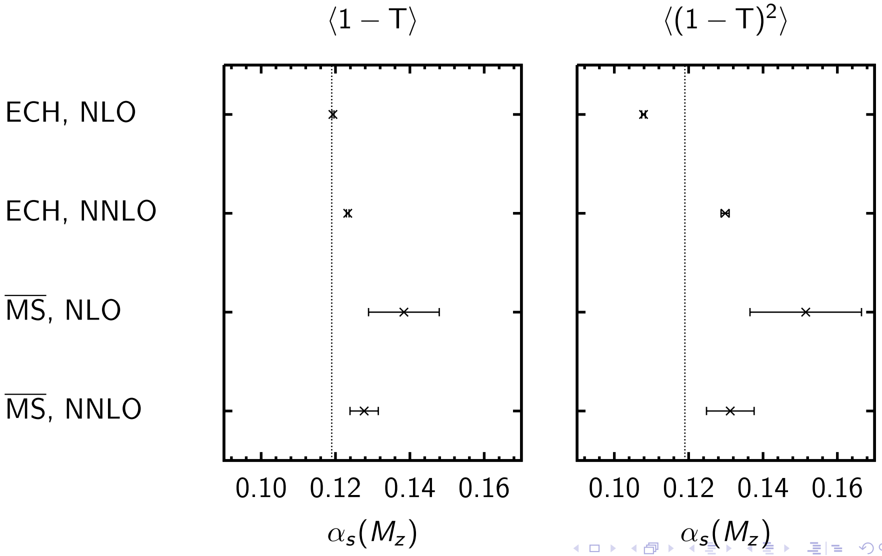
# Back ups: ES PT expansion

ES moments have PT expansion:

$$\langle y^n \rangle = \left(\frac{\alpha_s(\mu_R)}{2\pi}\right)\overline{\mathcal{A}}_{y,n} + \left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^2\left(\overline{\mathcal{B}}_{y,n} + \overline{\mathcal{A}}_{y,n}\mathrm{b}\ln\frac{\mu_R^2}{Q^2}\right)$$
$$+\left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^3\left(\overline{\mathcal{C}}_{y,n}+2\overline{\mathcal{B}}_{y,n}\mathrm{b}\ln\frac{\mu_R^2}{Q^2}+\overline{\mathcal{A}}_{y,n}\left(\mathrm{b}^2\ln^2\frac{\mu_R^2}{Q^2}+2\mathrm{bc}\ln\frac{\mu_R^2}{Q^2}\right)\right)$$
$$+\ldots \qquad (12)$$

# Results: $\langle (1-T)^n \rangle$

NLO and NNLO ECH results for $\langle (1-T)^n \rangle$:



# Asymptotic Series

Look at PT expansions of ESs. These are asymptotic series.

**Definition of Asymptotic Series**

$$f(g) = \sum_{n=0}^{\infty} f_n g^n$$

is asymptotic if the series diverges for all $g \neq 0$ and satisfies the bound:

$$|f(g) - \sum_{n=0}^{N} f_n g^n| \leq f_{N+1}|g|^{N+1}$$

So error is **bounded by the size of the first term you neglect**.

# Asymptotic Series

So look at PT expansion for $\rho(\mathcal{R}) = -\mathrm{b}\mathcal{R}^2(1 + \mathrm{c}\mathcal{R} + \rho_2\mathcal{R}^2 + ...)$:

$\langle 1 - \mathrm{T} \rangle$:

- $\rho(\mathcal{R}) = -\mathrm{b}\mathcal{R}^2(1 + 0.0785 - 0.0494)$

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$\langle 1 - T\rangle$:

- $\rho(\mathcal{R}) = -b\mathcal{R}^2(1 + 0.0785 - 0.0494)$

$\langle (1 - T)^2\rangle$:

- $\rho(\mathcal{R}) = -b\mathcal{R}^2(1 + 0.1049 - 0.3227)$