An alternative subtraction scheme for next-to-leading order QCD calculations

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Introduction and Motivation: SM at NLO

- **SM of particle physics**: accurate description of current collider data (particle data group; http://pdg.lbl.gov)

Why higher orders ??

- many processes governed by strong interactions: next to leading (and even higher) order contributions can be **huge** (NLO: $\mathcal{O}(100\,\%)$ for many processes; NLO Les Houches report 2010, arXiv:1003.1241)
- also important: reduction of scales dependencies (usually huge for LO calculations)

$\Rightarrow$ match experimental accuracies, make meaningful statements: need **at least** NLO

- also important: have differential corrections; use of **overall global** corrections (K-factors) can lead to wrong predictions for differential quantitites

$\Rightarrow$ wanted: **NLO corrections for differential distributions**

Tools: Monte Carlo event generators

Way to do this: MC Event Generators

MC event generators: Generate event samples

(same form as experimental outcome)

- experiments: see **final decay products**
- need to compare with **simulated event samples**

$\Rightarrow$ too complicated for analytic computation, need **numerical simulations**

- **LO tools**: Pythia (Sjöstrand ea), Herwig (Gieseke ea), Sherpa (Krauss ea), Alpgen (Mangano ea), ..
- **NLO tools**: MC@NLO (Frixione ea), Powheg (Nason ea), VBFNLO (Arnold ea), MCFM (Campbell ea), NLOJet++ (Nagy), + more (non-public) codes...

Infrared divergencies and NLO subtraction schemes: ingredients

$$\sigma_{\text{NLO,tot}} = \underbrace{\int d\Gamma_m \, |\mathcal{M}^{(m)}_{\text{Born}}|^2}_{\sigma_{\text{LO}}} + \underbrace{2\int d\Gamma_m Re(\mathcal{M}^{(m)}_{\text{Born}} (\mathcal{M}^{(m)}_{\text{virt}})^*)}_{\sigma_{\text{virt}}(\varepsilon)} + \underbrace{\int d\Gamma_{m+1} |\mathcal{M}^{(m+1)}|^2}_{\sigma_{\text{real}}(\varepsilon)}$$

- **infrared poles** $1/\varepsilon$, $1/\varepsilon^2$ cancel in $\sum \sigma_{\text{real}} + \sigma_{\text{virt}}$
- **matrix elements factorize in singular limits, unique behaviour** (depending in nature of splitting)

$$|\mathcal{M}^{(m+1)}|^2 \longrightarrow D_{ij}(p_i, p_j)\,|\mathcal{M}^{(m)}|^2, \quad D_{ij} \sim \frac{1}{p_i p_j}$$

- D_{ij}: **dipoles**, contain complete singularity structure

$$\Longrightarrow \quad \int d\Gamma_{m+1} \left(|\mathcal{M}^{(m+1)}|^2 - \sum_{ij} D_{ij} |\mathcal{M}^{(m)}|^2 \right) = \textbf{finite}$$

- general idea of dipole subtraction:
  shift singular parts from $m+1$ to m particle phase space

Dipole subtraction for total cross sections

Master formula

$$\sigma = \sigma^{LO} + \sigma^{NLO}$$

$$\sigma^{NLO} = \int_{m+1} d\sigma^R + \int_m d\sigma^V + \int d\sigma^C$$

$$= \int_{m+1} (d\sigma^R - d\sigma^A) + \int_m \left(d\sigma^{\tilde{A}} + d\sigma^V + d\sigma^C\right),$$

$\Rightarrow$ effectively added "0"; both integrals finite

$$\sigma^{\mathrm{NLO}}_m(s) = \int_m \Big\{ |\widetilde{\mathcal{M}}_{\mathrm{virt}}(s;\varepsilon)|^2 + \mathbf{I}(\varepsilon)|\mathcal{M}_{\mathrm{Born}}(s)|^2 + \int_0^1 dx\, (\mathbf{K}(x) + \mathbf{P}(x;\mu_F))\, |\mathcal{M}_{\mathrm{Born}}(x,s)|^2 \Big\}$$

Dipole subtraction for total cross sections

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$$\sigma_m^{\mathrm{NLO}}(s) = \int_m \left\{ |\widetilde{\mathcal{M}}_{\mathrm{virt}}(s;\varepsilon)|^2 + \mathbf{I}(\varepsilon)|\mathcal{M}_{\mathrm{Born}}(s)|^2 + \int_0^1 dx \left(\mathbf{K}(x) + \mathbf{P}(x;\mu_F)\right) |\mathcal{M}_{\mathrm{Born}}(x,s)|^2 \right\}$$

Dipole subtraction for total cross sections

Master formula

$$
\begin{aligned}
\sigma &= \sigma^{LO} + \sigma^{NLO} \\
\sigma^{NLO} &= \int_{m+1} d\sigma^{R} + \int_{m} d\sigma^{V} + \int d\sigma^{C} \\
&= \int_{m+1} \left(d\sigma^{R} - d\sigma^{A}\right) + \int_{m} \left(d\sigma^{\tilde{A}} + d\sigma^{V} + d\sigma^{C}\right),
\end{aligned}
$$

$\Rightarrow$ effectively added "0"; both integrals finite

$$
\begin{aligned}
\sigma^{\text{NLO}}_{m}(s) = \int_{m} \Big\{ & |\widetilde{\mathcal{M}}_{\text{virt}}(s;\varepsilon)|^2 + \mathbf{I}(\varepsilon)|\mathcal{M}_{\text{Born}}(s)|^2 \\
& + \int_0^1 dx \, \left(\mathbf{K}(x) + \mathbf{P}(x;\mu_F)\right) |\mathcal{M}_{\text{Born}}(x,s)|^2 \Big\}
\end{aligned}
$$

Dipole subtraction for total cross sections

Master formula

$$\sigma = \sigma^{LO} + \sigma^{NLO}$$

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Ingredient for subtraction schemes: momentum matching

- previous slide: add and subtract "0" in terms of

$$\int d\Gamma_m \widetilde{F}_{\text{sing}} |\mathcal{M}^{(m)}_{\text{Born}}|^2 - \int d\Gamma_{m+1} F_{\text{sing}} |\mathcal{M}^{(m)}_{\text{Born}}|^2$$

- addition and subtraction takes place in different phase spaces

$$p^{(m)}_{\tilde{a}} = F\left(p^{(m+1)}_a, p^{(m+1)}_b, ...\right)$$

This function is highly scheme dependent !!!

- requirement: energy/ momentum conservation, onshellness of external particles:

$$\sum_m p_{\tilde{a}} \stackrel{!}{=} \sum_{m+1} p_a, \; p^2_{i,\tilde{i}} = 0$$

(sum over outgoing particles only)

Nagy Soper subtraction scheme

- many different subtraction schemes are around (best known: Catani, Seymour, 1996)
- all schemes: poles have to be the same; finite parts can differ

Main motivation for new scheme

- basic idea: can use the splitting functions in the parton shower as dipole subtraction terms

  $\Rightarrow$ have same behaviour in singular limits
- introduce new matching between m and $m+1$ phase spaces

$\Rightarrow$ leads to a much smaller number of subtraction terms

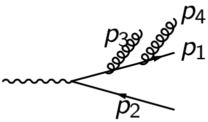
especially important for large number of external particles

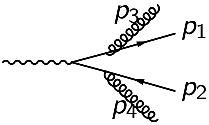
(same dipoles in shower and subtraction scheme: facilitates matching with NLO calculations)

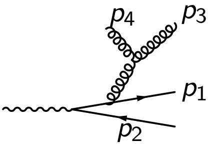
Shifting momenta: Example

$\gamma^* \longrightarrow q(p_1)\bar{q}(p_2)\,g(p_3)$ **(@ NLO)**

$q\,\bar{q}\,g\,g$ **real emission contributions:**







CS: 1 momentum shift/ spectator

$p_2,\ p_3$: 2 transformations

NS: 1 total transformation

$\Rightarrow$ **from simple counting:** $(+(p_1 \leftrightarrow p_2))$

12 transformations using CS vs 6 using NS dipoles !!

Deep inelastic scattering (subprocess of...) (hard)

$$e(p_{in})\, q(p_1) \longrightarrow e(p_{out})\, q(p_4)\; [g(p_3)]$$

- CS: spectator for final state gluon emission:

  initial state quark

- NS: spectator for final state gluon emission:

  final state lepton

⇒ first nontrivial check of NS scheme ⇐

- major difference in integrated subtraction terms
- contains **integrals** which need to be evaluated numerically
- for some integrals, $m+1$ variables have to be reconstructed

⇒ HUGE difference wrt standard scheme(s) ⇐

DIS: Nagy Soper - subtraction in the virtual contribution

⇒ **major difference in integrated subtraction terms**

$$\int_0^1 \mathbf{dx}\,|\mathcal{M}|_2^2 = \int_0^1 dx \left\{ \frac{\alpha_s}{2\pi} C_F\,\delta(1-x)\left[-9 + \frac{1}{3}\pi^2 - \frac{1}{2}\mathrm{Li}_2[(1-\tilde{z}_0)^2] + 2\ln 2\ln\tilde{z}_0 + 3\ln\tilde{z}_0 + 3\,\mathrm{Li}_2(1-\tilde{z}_0) + \mathbf{I}_{\text{fin}}^{\text{tot},0}(\tilde{\mathbf{z}}_0) + \mathbf{I}_{\text{fin}}^{1}(\tilde{\mathbf{a}})\right] + K_{\text{fin}}^{\text{tot}}(x;\tilde{z}) + P_{\text{fin}}^{\text{tot}}(x;\mu_F^2)\right\}|\mathcal{M}|^2_{\text{Born}}(x\,p_1),$$

$$\mathbf{K}_{\text{fin}}^{\text{tot}}(\mathbf{x};\tilde{\mathbf{z}}) = \frac{\alpha_s}{2\pi} C_F \left\{\frac{1}{x}\left[2\,(1-x)\ln(1-x) - \left(\frac{1+x^2}{1-x}\right)_+ \ln x + 4x\left(\frac{\ln(1-x)}{1-x}\right)_+\right] + \mathbf{I}_{\text{fin}}^{1}(\tilde{\mathbf{z}},\mathbf{x})\right\},$$

⇒ **contains integrals which need to be evaluated numerically** ⇐

DIS: Nagy Soper - integrals to be evaluated numerically

⇒ **Integrals contain nontrivial functions depending on m and $m+1$ four-momenta** ⇐

$$\mathbf{I}^{\text{tot},0}_{\text{fin}}(\tilde{\mathbf{z}}_0) = 2\int_0^1 \frac{dy}{y}\left\{\frac{\tilde{z}_0}{\sqrt{4\,y^2\,(1-\tilde{z}_0)+\tilde{z}_0^2}} \times \ln\left[\frac{2\,z\,\sqrt{4\,y^2\,(1-\tilde{z}_0)+\tilde{z}_0^2}\,(1-y)}{\left(2\,y+\tilde{z}_0-2\,y\,\tilde{z}_0+\sqrt{4\,y^2\,(1-\tilde{z}_0)+\tilde{z}_0^2}\right)^2}\right]+\ln 2\right\}.$$

$$\mathbf{I}^{1}_{\text{fin}}(\tilde{\mathbf{a}}) = 2\int_0^1 \frac{du}{u}\int_0^1 \frac{dx}{x} \times \left[\frac{\mathbf{x}\,(\mathbf{1}-\mathbf{x}+\mathbf{u}\,\mathbf{x}\,[(\mathbf{1}-\mathbf{u}\,\mathbf{x})\,\tilde{\mathbf{a}}+\mathbf{2}])}{\mathbf{k}(\mathbf{u},\mathbf{x},\tilde{\mathbf{a}})}-\frac{1}{\sqrt{1+4\,\tilde{a}_0\,u^2\,(1+\tilde{a}_0)}}\right].$$

$$\mathbf{I}^{1}_{\text{fin}}(\tilde{\mathbf{z}},\mathbf{x}) = \frac{2}{(1-x)_+}\,\frac{1}{\pi}\int_0^1 \frac{dy'}{y'}\left[\int_0^1 \frac{dv}{\sqrt{v\,(1-v)}}\,\frac{\tilde{\mathbf{z}}}{\mathbf{N}(\mathbf{x},\mathbf{y}',\tilde{\mathbf{z}},\mathbf{v})}-1\right],$$

DIS: Nagy Soper - variables in integrals to be evaluated numerically

for some integrals, $m+1$ variables have to be reconstructed

⇒ HUGE difference wrt standard scheme(s) ⇐

in initial state subtraction terms

$$\mathbf{N} = \frac{\hat{p}_3 \cdot \hat{p}_4}{\hat{p}_4 \cdot \hat{Q}} \frac{1}{1-x} + y', \tilde{z} = \frac{1}{x} \frac{p_1 \cdot \hat{p}_4}{\hat{p}_4 \cdot \hat{Q}}$$

$$\hat{\mathbf{p}}_3 = \underbrace{\frac{(1-\mathbf{x})(1-\mathbf{y}')}{\mathbf{x}}}_{\alpha} \mathbf{p}_1 + \underbrace{(1-\mathbf{x})\,\mathbf{y}'}_{\beta} \mathbf{p}_i - \mathbf{k}_\perp, \hat{\mathbf{p}}_4^\mu = \Lambda^\mu{}_\nu(\hat{\mathbf{K}}, \mathbf{K}) \hat{\mathbf{p}}_4^\nu$$

$$k_\perp^2 = -2\,\alpha\,\beta\,p_1 \cdot p_i,\; k_\perp = -|k_\perp| \begin{pmatrix} 0 \\ 1-2\,v \\ 2\sqrt{v\,(1-v)} \\ 0 \end{pmatrix},$$

in final state subtraction terms

$$\mathbf{k}^2(\mathbf{x}, \mathbf{u}, \tilde{\mathbf{a}}) = \left[(1+ux-x)(z-z') + ux\left((1-ux)\,\tilde{a}+1\right)\right]^2 + 4\,u\,x\,z'\,(1-z)\,(1+u\,x-x)\,\left((1-ux)\,\tilde{a}+1\right)$$

$$\tilde{a} = \frac{p_1 \cdot p_o}{p_1 \cdot (p_i - (1-y)p_o)}$$

DIS: Nagy Soper - variables in integrals to be evaluated numerically

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$$\hat{\mathbf{p}}_3 = \underbrace{\frac{(1-x)(1-y')}{x}}_{\alpha} \mathbf{p}_1 + \underbrace{(1-x)\, y'}_{\beta} \mathbf{p}_i - \mathbf{k}_\perp, \hat{\mathbf{p}}_4^\mu = \Lambda^\mu{}_\nu(\hat{\mathbf{K}}, \mathbf{K}) \hat{\mathbf{p}}_4^\nu$$

$$k_\perp^2 = -2\,\alpha\,\beta\, p_1 \cdot p_i,\; k_\perp = -|k_\perp| \begin{pmatrix} 0 \\ 1-2\,v \\ 2\sqrt{v\,(1-v)} \\ 0 \end{pmatrix},$$

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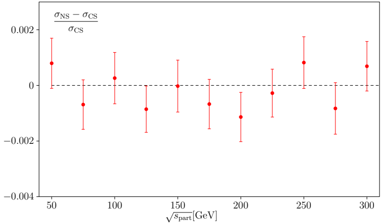
$$\tilde{a} = \frac{p_1 \cdot p_o}{p_1 \cdot (p_i - (1-y)p_o)}$$

DIS: Catani Seymour vs Nagy Soper - numerical result

consistency check: get the same result



relative difference between CS and NS: $\frac{\sigma_{CS} - \sigma_{NS}}{\sigma_{CS}}$

agree on the sub-permill level ✓

Catani Seymour vs Nagy Soper - Summary and upshot

- **NS checked for**: single W, Dijet, $pp \longrightarrow H$, $H \longrightarrow gg$, DIS, $ee \longrightarrow$ 3 jets (singular regions only)

NS upshots:

\+ **less transformations** in the subtraction terms (leading to faster codes for higher multiplicity final states)

− **more complicated expressions**, and **numerically evaluable integrals**, in the subtraction terms (already for "easy" processes)

- both due to **different mapping procedure**

$\Rightarrow$ need to test on higher multiplicity final states to see net result $\Leftarrow$

next step on the road...

Appendix

Catani Seymour vs Nagy Soper: Shifting momenta

- matching between m and $m+1$ particle spaces requires reshuffling of momenta
- for
$$p^{(m)}_{\text{mother}} = p^{(m+1)}_{\text{daughter, 1}} + p^{(m+1)}_{\text{daughter,2}},$$
not all particles can be onshell simultaneously

$\Rightarrow$ need additional spectators to take over additional momenta

- Catani Seymour: define emitter-spectator pair, momentum goes to 1 additional particle only

$\Rightarrow$ quite easy integrations; however, for increasing number of particles, huge number of transformations necessary

- Nagy Soper:
shift momenta to **all** non-emitting external particles
- number of transformations = number of emitters
- leads to more complicated integrals during framework setup
- in general: # of transformations: CS $\sim N^3_{\text{jets}}/2$, NS$\sim N^2_{\text{jets}}/2$

Difference 2: Matching with parton showers

- double counting: hard real emissions are described in both shower and "real emission" matrix element
- avoid double counting

$$-\int_{m+1} d\sigma^{\rm PS}|_{m+1} + \int_{m+1} d\sigma^{\rm PS}|_m$$

details eg in hep-ph/0204244: "Matching NLO QCD computations and parton shower simulations" (Frixione, Webber), MC@NLO

- important: have new terms in $m+1$ phase space

$$\int_{m+1}\left(d\sigma^R \underbrace{-d\sigma^A + d\sigma^{PS}|_m}_{=0} - d\sigma^{PS}|_{m+1}\right)$$

- same splitting functions: second and third term cancel analytically !!

$\Rightarrow$ improves numerical efficiency

Processes at hadron colliders: general

- hadron colliders (as Tevatron, LHC) collide **hadrons**
- QCD: talks about **partons**
- transition: parton distribution functions (PDFs) $f_l(x, \mu_F)$; $l = q, \bar{q}, g$ flavour, x momentum fraction, (μ_F factorization scale)

masterformula

$$\sigma_{\text{hadr}}(p\,\bar{p} \to X) = \sum_{l_1, l_2} \int dx_1 \int dx_2\, f_{l_1}(x_1) f_{l_2}(x_2)\, \sigma_{\text{part}}(x_1, x_2;\, l_1 l_2 \to X)$$

- perturbative, nonperturbative part

Second ingredient: Parametrization of integration variables

- again: remember you have

$$F_{\text{sing}} \propto D_{ij},\ \widetilde{F}_{sing} = \int d\Gamma_1\, D_{ij},\ d\Gamma_1 \propto d^4 p_j\, \delta(p_j^2)$$

$$\Longrightarrow \widetilde{F}_{sing} \propto \int d^4 p_j\, \delta(p_j^2)\, D_{ij}$$

- 3 free variables (in D dimensions: $D - 1$)

  !! need to be written in terms of m particle variables !!
- now all ingredients:
  total energy momentum conservation, onshellness of external particles, need for integration variables

NS integration measures (1)

Initial state

$$d\xi_p = dx \int_0^1 dy' \int_0^1 dv \frac{(2p_a \cdot p_b)^{1-\varepsilon} x^{\varepsilon-1}}{(4\pi)^2} \frac{\pi^{\varepsilon-\frac{1}{2}}}{\Gamma\left(\frac{1-2\varepsilon}{2}\right)} \times \left[y'(1-y')\right]^{-\varepsilon} \left[v\,(1-v)\right]^{-\frac{1+2\,\varepsilon}{2}} \Theta\left[(1-x)x\right]$$

Final state, $2 \to 2$ processes

$$d\xi_p = \frac{(2p_\ell \cdot Q)^{1-\varepsilon}}{16} \frac{\pi^{-\frac{5}{2}+\varepsilon}}{\Gamma\left(\frac{1}{2}-\varepsilon\right)} \times \int_0^1 du u^{-\varepsilon}(1-u)^{-\varepsilon} \int_0^1 dx x^{1-2\varepsilon}(1-x)^{-\varepsilon} \int_0^1 dv \left[v\,(1-v)\right]^{-\frac{1+2\varepsilon}{2}}$$

NS integration measures (2)

Final state, $2 \rightarrow n$ processes

$$d\xi_p = \frac{(2\,p_{\tilde{i}}Q)^{1-\varepsilon}}{16}\,\frac{\pi^{-\frac{5}{2}+\varepsilon}}{\Gamma\left(\frac{1}{2}-\varepsilon\right)}$$
$$\int_0^1 du u^{-\varepsilon}\int_0^1 dx\delta^{1-\varepsilon}\gamma^{1-2\varepsilon}\,[(1-x)(x-x_0)]^{-\varepsilon}\int_0^1 dv\,[v(1-v)]^{-\frac{1+2\,\varepsilon}{2}}\,.$$

Variables

$$\lambda = \sqrt{(1+y)^2-4\,a\,y},\; a = \frac{Q^2}{2\,p_{\tilde{i}}Q},\; y = \frac{\hat{p}_i\hat{p}_j}{p_{\tilde{i}}Q}$$
$$\gamma = \frac{1}{2}\,(1+y+\lambda)\,,\; x_0(y,a) = \frac{1-\lambda+y}{1+\lambda+y},$$

Maximal number of transformations

emitter, spectator	CS	NS
fin,fin	$N'\,(N'-1)\,(N'-2)/2$	$N'\,(N'-1)/2$
fin,ini	$N'\,(N'-1)$	–
ini,fin	$2\,(N'-1)\,N'$	$2\,N'$
ini,ini	$2\,N'$	–
total	$N'^2(N'+3)/2 = (N+1)^2(N+4)/2$	$N'(N'+3)/2 = (N+1)(N+4)/2$

Table: Maximal number of transformations needed for N particles in the born final state ($N' = N+1$ in the real radiation contribution) using Catani Seymour or Nagy Soper prescriptions. Formula is exact for processes where all partons are gluons; for quarks, number can actually be smaller. Note: number of transformations is not equal to number of dipoles.

Dipole subtraction: Real master formula

Real Masterformula $(s = (p_a + p_b)^2)$

$$\begin{aligned}\sigma(s) =& \int_m d\Phi^{(m)}(s)\frac{1}{n_c(a)n_c(b)}\,|\mathcal{M}^{(m)}|^2(s)F_J^{(m)}\\ &+\int d\Phi^{(m+1)}(s)\left\{\frac{1}{n_c(a)n_c(b)}\,|\mathcal{M}^{(m+1)}|^2(s))F_J^{(m+1)} - \sum_{\text{dipoles}}(\mathcal{D}\cdot F_J^{(m)})\right\}\\ &+\int d\Phi^{(m)}(s)\left\{\frac{1}{n_c(a)n_c(b)}|\mathcal{M}^{(m)}|^2_{1\text{ loop}}(p_a,p_b)+\mathbf{I}(\varepsilon)|\mathcal{M}^{(m)}|^2(s)\right\}_{\varepsilon=0} F_J^{(m)}\\ &+\left\{\int dx_a\,dx_b\delta(x-x_a)\,\delta(x_b-1)\int d\Phi^{(m)}(x_a p_a, x_b p_b)|\mathcal{M}^{(m)}|^2(x_a p_a, x_b p_b)\right.\\ &\left.\times\left(\mathbf{K}^{a,a'}(x)+\mathbf{P}^{a,a'}(x_a p_a, x_b p_b, x;\mu_F^2)\right)\right\} + (a\leftrightarrow b)\end{aligned}$$

where all colour/ phase space factors have been accounted for

Integrated Dipoles in more details: I, K, P (1)

$m+1$ **phase space: in principle easy**

$$\int d\Gamma_{m+1} \left(|\mathcal{M}_{\text{real}}|^2 - \sum D \right), \quad \text{finite}$$

m **particle phase space: more complicated**

need integration variables (emission from p_1):

$$x = 1 - \frac{p_4(p_1+p_2)}{p_1 p_2} \text{ softness, } \tilde{v} = \frac{p_1 p_4}{p_1 p_2} \text{ collinearity}$$

Integrated Dipoles in more details: I, K, P (2)

- in principle, obtain $\int d\Gamma_1 \, D \,=\, \int_0^1 dx \left(\mathbf{I}(\varepsilon) + \tilde{\mathbf{K}}(x,\varepsilon)\right)$
- $\mathbf{I}(\varepsilon) \propto \delta(1-x)$: corresponds to loop part
- $\tilde{\mathbf{K}}(x,\varepsilon)$ contains finite parts as well as **collinear singularities**
- latter need to be cancelled by adding **collinear counterterm**

$$\frac{1}{\epsilon}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon} P^{qq}(x)$$

  depends on factorization scale μ_F ($P^{qq}(x)$ splitting function)
- PDFs come in again: term already accounted for by folding w PDF, needs to be subtracted
- for $q\,g \,\rightarrow\, W\,q$ like processes, only singularity which appears

$q \longrightarrow q\,g$ for initial state quarks: Catani Seymour (1)

- $q(\tilde{p}_1) \longrightarrow q(p_1) + g(p_4)$, q enters hard interaction
- Dipole:

$$D^{14,2} = -\frac{8\,\pi\,\mu^2\,\alpha_s\,C_F}{s+t+u}\left(\frac{2\,s\,(s+t+u)}{t\,(t+u)} + (1-\varepsilon)\,\frac{t+u}{t}\right)$$

- matching ($\tilde{p}_2 = p_2$)

$$\tilde{p}_1 = x\,p_1, \quad x = 1 - \frac{p_4\,(p_1+p_2)}{(p_1\,p_2)}$$

$$\tilde{p}_k^{\mu} = \Lambda^{\mu}{}_{\nu}\,p_k^{\nu}, \quad (k\text{: final state particles})$$

$$\Lambda^{\mu\nu} = -g^{\mu\nu} - \frac{2\,(K+\widetilde{K})^{\mu}(K+\widetilde{K})^{\nu}}{(K+\widetilde{K})^2} + \frac{2\,K^{\mu}\,\widetilde{K}^{\nu}}{K^2}$$

$$K = p_1 + p_2 - p_4, \quad \widetilde{K} = \tilde{p}_1 + p_2$$

$q \longrightarrow q\,g$ for initial state quarks: Catani Seymour (2)

- integration variables:

$$v = \frac{p_1 p_4}{p_1 p_2},\; x = 1 - \frac{p_4\,(p_1 + p_2)}{(p_1 p_2)}$$

- in p_1, p_2 cm system: $E_4 \longrightarrow 0 \Rightarrow x \longrightarrow 1$ (softness)
  $\cos\theta_{14} \longrightarrow 1 \Rightarrow v \longrightarrow 0$ (collinearity)
- Dipole in terms of integration variables

$$D^{14,2} = -\frac{8\,\pi\,\alpha_s\,C_F}{v\,x\,s}\left(\frac{1+x^2}{1-x} - \varepsilon(1-x)\right)$$

- integration measure

$$[dp_j] = \frac{(2\,p_1 p_2)^{1-\varepsilon}}{16\,\pi^2}\,\frac{d\Omega_{d-3}}{(2\,\pi)^{1-\varepsilon}}\,dv\,dx\,(1-x)^{-2\,\varepsilon}\left[\frac{v}{1-x}\left(1-\frac{v}{1-x}\right)\right]^{-\varepsilon}$$

where $v \leq 1 - x$ and all integrals between 0 and 1

$q \longrightarrow q\,g$ for initial state quarks: Catani Seymour (3)

- result

$$\mu^{2\varepsilon} \int [dp_j]\, D^{14,2} = \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} C_F \left(\frac{2\mu^2\pi}{p_1 p_2}\right)^{\varepsilon}$$

$$\times \int_0^1 dx \left(\mathbf{I}(\varepsilon)\delta(1-x) + \tilde{\mathbf{K}}(x,\varepsilon) \underbrace{-\frac{1}{\varepsilon} P^{qq}(x)}_{\text{killed by coll CT}} \right)$$

with

$$\mathbf{I}(\epsilon) = \frac{1}{\epsilon^2} + \frac{3}{2\epsilon} - \frac{\pi^2}{6}$$

$$\mathbf{K}(x) = (1-x) - 2\,(1+x)\ln(1-x) + \left(\frac{4}{1-x}\ln(1-x)\right)_+$$

$$P^{qq}(x) = \left(\frac{1+x^2}{1-x}\right)_+ \quad \text{regularized splitting function}$$

$q \longrightarrow q\,g$ for initial state quarks: Nagy Soper (1)

- $q(\tilde{p}_1) \longrightarrow q(p_1) + g(p_4)$, q enters hard interaction
- Dipole:

$$D^{14,2} = -\frac{8\,\pi\,\mu^2\,\alpha_s\,C_F}{s+t+u}\left(\frac{2\,s\,u\,(s+t+u)}{t\,(t^2+u^2)} + (1-\varepsilon)\,\frac{u}{t}\right)$$

as CS, same pole structure as CS

- matching, integration variables, integration measure: as Catani Seymour($v \leftrightarrow y$)
- Dipole in terms of integration variables

$$D^{14,2} = -\frac{8\pi\alpha_s\,C_F}{x\,s} \times \left(\frac{1-x-y}{y}(1-\varepsilon) + \frac{2\,x}{y\,(1-x)} - \frac{2\,x\,[2\,y-(1-x)]}{(1-x)[y^2+(1-x-y)^2]}\right)$$

$q \longrightarrow q\,g$ for initial state quarks: Nagy Soper (2)

- result

$$\mu^{2\varepsilon}\int [dp_j]\,D^{14,2} = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\,C_F\left(\frac{2\,\mu^2\pi}{p_1 p_2}\right)^{\varepsilon} \times \int_0^1 dx\left(\mathbf{I}(\varepsilon)\delta(1-x) + \tilde{\mathbf{K}}(x,\varepsilon)\underbrace{-\frac{1}{\varepsilon}P^{qq}(x)}_{\text{killed by coll CT}}\right)$$

with

$$\mathbf{K}(x) = (1-x) - 2\,(1+x)\ln(1-x) + \left(\frac{4}{1-x}\ln(1-x)\right)_+ -(1-x)$$

- equivalence of dipoles schemes checked analytically

Final state $g \longrightarrow q\,\bar{q}$: Catani Seymour vs Nagy Soper (1)

- $g(\tilde{p}_i) \longrightarrow q(p_i) + \bar{q}(p_j)$,
  spectator: any other final state parton, p_k
- Dipole (in terms of integration variables):

$$D^{ij,k}_{\text{NS, CS}} \propto \underbrace{\frac{1}{y}}_{\text{sing}} \left[1 - \frac{z\,(1-z)}{1-\varepsilon}\right]$$

- NS definitions

$$y_{\text{NS}} = \frac{p_i p_j}{(p_i+p_j)Q - p_i p_j}, \quad z_{\text{NS}} = \frac{p_j \tilde{n}}{p_i \tilde{n} + p_j \tilde{n}}$$

$$\tilde{n} = \frac{1+y+\lambda}{2\lambda} Q - \frac{a}{\lambda}\left(p_i + p_j\right), \; \lambda = \sqrt{(1+y)^2 - 4\,a\,y}, \; a = \frac{Q^2}{(p_i+p_j)Q - p_i p_j}$$

- CS definitions:

$$y_{CS} = \frac{p_i p_j}{p_i p_j + p_i p_k + p_j p_k}, \quad z_{CS} = \frac{p_i p_k}{p_i p_k + p_j p_k}$$

Final state $g \longrightarrow q\,\bar{q}$: Catani Seymour vs Nagy Soper (2)

- CS matching (all other final state particles untouched)

$$\tilde{p}_i \;=\; p_i + p_j - \frac{y}{1-y} p_k,\;\; \tilde{p}_k^{\mu} \;=\; \frac{1}{1-y} p_k$$

- NS matching

$$\tilde{p}_i \;=\; \frac{1}{\lambda}(p_i + p_j) - \frac{1-\lambda+y}{2\,\lambda\,a}\,Q,\;\; \tilde{p}_k^{\mu} \;=\; \Lambda^{\mu}{}_{\nu} p_k^{\nu} \quad \textbf{all fs particles}$$

$$\Lambda^{\mu\nu} = g^{\mu\nu} - \frac{2\,(K+\widetilde{K})^{\mu}(K+\widetilde{K})^{\nu}}{(K+\widetilde{K})^2} + \frac{2\,K^{\mu}\widetilde{K}^{\nu}}{K^2},\; K = Q - p_i - p_j,\;\; \widetilde{K} = Q - \tilde{p}_i$$

- integration measure (identical, same pole structure)

$$[dp_j]_{\text{CS}} \;=\; \frac{(2\,\tilde{p}_i\tilde{p}_k)^{1-\varepsilon}}{16\,\pi^2}\,\frac{d\Omega_{d-3}}{(2\,\pi)^{1-\varepsilon}}\,dz\,dy\,(1-y)^{1-2\,\varepsilon}\,y^{-\varepsilon}\,\left[z\,(1-z)\right]^{-\varepsilon},$$

$$[dp_j]_{\text{NS}} \;=\; \frac{(2\,\tilde{p}_i\,Q)^{1-\varepsilon}}{16\,\pi^2}\,\frac{d\Omega_{d-3}}{(2\,\pi)^{1-\varepsilon}}\,dz\,dy\,\lambda^{1-2\,\varepsilon}\,y^{-\varepsilon}\,\left[z\,(1-z)\right]^{-\varepsilon}$$

Final state $g \longrightarrow q\,\bar{q}$: Catani Seymour vs Nagy Soper (3)

- result CS

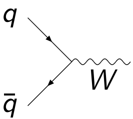
$$\mu^{2\varepsilon}\int [dp_j]\, D^{ij,k} = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\, T_R \left(\frac{2\mu^2\pi}{\tilde{p}_i\tilde{p}_k}\right)^{\varepsilon}\left[-\frac{2}{3\,\varepsilon}-\frac{16}{9}\right]$$

- result NS

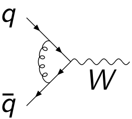
$$\mu^{2\varepsilon}\int [dp_j]\, D^{ij} = T_R\,\frac{\alpha_s}{2\pi}\frac{\alpha_s}{\Gamma(1-\varepsilon)}\left(\frac{2\,\pi\,\mu^2}{p_{\tilde{i}}Q}\right)^{\varepsilon}$$
$$\times\left[-\frac{2}{3\,\varepsilon}-\frac{16}{9}+\frac{2}{3}\left[(a-1)\ln(a-1)-a\ln a\right]\right],$$

- for $a=1$, reduces completely to Catani Seymour result
- (reason: $a=1$ implies only 2 particles in the final state, $\tilde{n}\longrightarrow p_k$, $\Rightarrow$ complete equivalence)
- tradeoff: all final state particles get additional momenta: integral more complicated, but fewer transformations necessary

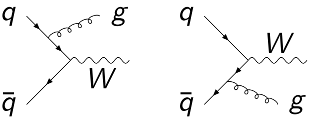
Single W production (slide by C. Chung)



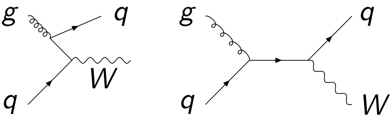
Tree level: $q\bar{q} \to W$



Virtual corrections: $q\bar{q} \to W$



Real corrections: $q\bar{q} \to Wg$



$gq \to Wq$ (+ 2 more diagrams)

$$\frac{1}{4}\frac{1}{9}|\mathcal{M}_B|^2 = \frac{g^2}{12}|V_{qq'}|^2 M_W^2, \quad \frac{1}{4}\frac{1}{9}\sum|\mathcal{M}_R|^2 = \frac{8g^2\pi\alpha_s}{9}|V_{qq'}|^2\frac{\hat{t}^2+\hat{u}^2+2M_W^2\hat{s}}{\hat{t}\hat{u}}$$

$$|\mathcal{M}_V|^2 = |\mathcal{M}_B|^2 \frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\left\{-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\pi^2+\mathcal{O}(\epsilon)\right\}$$

Single W production: Nagy Soper vs Catani Seymour subtraction (easy)

NS, CS-NS, CS= NS+CS-NS

- **2 particle phase space (real emission)**

$$\mathcal{D}^{14,2} + \mathcal{D}^{24,1} = \underbrace{\frac{1}{4}\frac{1}{9}\sum|\mathcal{M}_{\text{real}}|^2}_{\text{singular}} + \underbrace{\frac{16}{9}g^2\,\alpha_s\,\pi}_{\text{finite}}$$

- **1 particle phase space (virtual contribution)**

$$\mathbf{I}(\epsilon)|\mathcal{M}_b|^2 = \underbrace{\frac{2\alpha_s}{3\pi}\frac{1}{\Gamma(1-\epsilon)}(-8+\frac{2}{3}\pi^2)|\mathcal{M}_b|^2}_{\text{finite}} - \underbrace{|\widetilde{\mathcal{M}}_v|^2}_{\text{singular (+finite)}}$$

$$\begin{aligned}\mathbf{K}^a(xp_a) &= \tfrac{\alpha_s}{2\pi}C_F\tfrac{1}{\Gamma(1-\epsilon)}\Big[-(1-x)\ln x + 2(1-x)\ln(1-x)\\ &+4x\left(\tfrac{\ln(1-x)}{1-x}\right)_+ - \tfrac{2x\ln x}{(1-x)_+} - \left(\tfrac{1+x^2}{1-x}\right)_+\ln\left(\tfrac{4\pi\mu^2}{2xp_a\cdot p_b}\right)\\ &+(1-x)\Big]\end{aligned}$$

compare to Nagy Soper :
pole structure the same, finite terms differ ✓

Single W production: Nagy Soper vs Catani Seymour subtraction (easy)

NS, CS-NS, CS= NS+CS-NS

- 2 **particle phase space (real emission)**

$$\mathcal{D}^{14,2} + \mathcal{D}^{24,1} = \underbrace{\frac{1}{4}\frac{1}{9}\sum|\mathcal{M}_{\text{real}}|^2}_{\text{singular}} + \underbrace{\frac{16}{9}g^2\,\alpha_s\,\pi}_{\text{finite}}$$

- 1 **particle phase space (virtual contribution)**

$$\mathbf{I}(\epsilon)|\mathcal{M}_b|^2 = \underbrace{\frac{2\alpha_s}{3\pi}\frac{1}{\Gamma(1-\epsilon)}(-8+\frac{2}{3}\pi^2)|\mathcal{M}_b|^2}_{\text{finite}} - \underbrace{|\widetilde{\mathcal{M}_v}|^2}_{\text{singular (+finite)}}$$

$$\begin{aligned}\mathbf{K}^a(xp_a) = \;& \frac{\alpha_s}{2\pi}C_F\frac{1}{\Gamma(1-\epsilon)}\Big[-(1-x)\ln x + 2(1-x)\ln(1-x) \\ & +4x\left(\frac{\ln(1-x)}{1-x}\right)_+ - \frac{2x\ln x}{(1-x)_+} - \left(\frac{1+x^2}{1-x}\right)_+\ln\left(\frac{4\pi\mu^2}{2xp_a\cdot p_b}\right) \\ & +(1-x)\Big]\end{aligned}$$

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NS, CS-NS, CS= NS+CS-NS

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$$\mathbf{K}^a(xp_a) = \frac{\alpha_s}{2\pi}C_F\frac{1}{\Gamma(1-\epsilon)}\Big[-(1-x)\ln x + 2(1-x)\ln(1-x) + 4x\left(\frac{\ln(1-x)}{1-x}\right)_+ - \frac{2x\ln x}{(1-x)_+} - \left(\frac{1+x^2}{1-x}\right)_+\ln\left(\frac{4\pi\mu^2}{2xp_a\cdot p_b}\right) + (1-x)\Big]$$

compare to Nagy Soper :

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NS, CS-NS, CS= NS+CS-NS

- **2 particle phase space (real emission)**

$$\mathcal{D}^{14,2} + \mathcal{D}^{24,1} = \underbrace{\frac{1}{4}\frac{1}{9}\sum |\mathcal{M}_{\text{real}}|^2}_{\text{singular}} + \underbrace{\frac{16}{9} g^2 \alpha_s \pi}_{\text{finite}}$$

- **1 particle phase space (virtual contribution)**

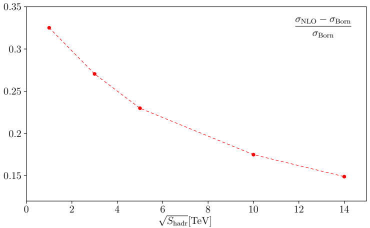
$$\mathbf{I}(\epsilon)|\mathcal{M}_b|^2 = \underbrace{\frac{2\alpha_s}{3\pi}\frac{1}{\Gamma(1-\epsilon)}(-8+\frac{2}{3}\pi^2)|\mathcal{M}_b|^2}_{\text{finite}} - \underbrace{|\widetilde{\mathcal{M}}_v|^2}_{\text{singular (+finite)}}$$

$$\mathbf{K}^a(xp_a) = \frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)}\Big[-(1-x)\ln x + 2(1-x)\ln(1-x) + 4x\left(\frac{\ln(1-x)}{1-x}\right)_+ - \frac{2x\ln x}{(1-x)_+} - \left(\frac{1+x^2}{1-x}\right)_+ \ln\left(\frac{4\pi\mu^2}{2xp_a\cdot p_b}\right) + (1-x)\Big]$$

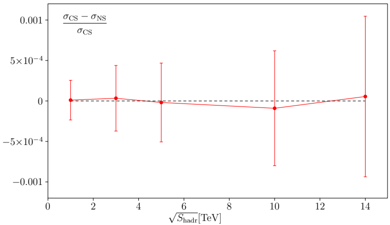
compare to Nagy Soper :
pole structure the same, finite terms differ ✓

Numerical results for single W (slide by C. Chung)

input: $M_W = 80.35$ GeV, PDF $\Rightarrow$ cteq6m, $\alpha_s(M_W) = 0.120299$



$\frac{\sigma_{NLO} - \sigma_{LO}}{\sigma_{LO}}$ as a function of $\sqrt{S_{\rm hadr}}$
corrections up to 30%



relative difference between CS and NS: $\frac{\sigma_{CS} - \sigma_{NS}}{\sigma_{CS}}$
agree on the sub-permill level ✓

DIS: Catani Seymour

Real emission subtraction terms

$$\mathbf{D}_{43,1} = \frac{4\pi\alpha_s}{p_3 p_4}\frac{1}{x_{43,1}} C_F \left[\frac{2}{1-\tilde{z}_4 + (1-x_{43,1})} - (1+\tilde{z}_4)\right] |\mathcal{M}|^2_{\text{Born}}(\tilde{p}_1, \tilde{p}_4)$$

$$\mathbf{D}_{13,4} = \frac{4\pi\alpha_s}{p_1 p_3}\frac{1}{x_{34,1}} C_F \left[\frac{2}{1- x_{34,1} + u_3} - (1+x_{34,1})\right] |\mathcal{M}|^2_{\text{Born}}(\tilde{p}_1, \tilde{p}_4)$$

$$\tilde{z}_4 = \frac{p_1 p_4}{(p_3+p_4)p_1},\; x_{43,1} = x_{34,1} = \frac{p_i p_o}{p_1 p_4 + p_1 p_3},\; u_3 = \frac{p_1 p_3}{(p_3+p_4)p_1}$$

Mapping

$$\tilde{\mathbf{p}}_1 = x_{43,1} p_1,\; \tilde{\mathbf{p}}_4 = p_3 + p_4 - (1-x_{43,1})p_1$$

Integrated subtraction terms

$$\int_0^1 \mathbf{dx} |\mathcal{M}|^2_{2,\text{tot}} = \int_0^1 \frac{dx}{x}\left\{-\frac{9}{2}\frac{\alpha_s}{2\pi} C_F \delta(1-x) + K^{\text{eff}}_{\text{fin}}(x) + P^{\text{eff}}_{\text{fin}}(x;\mu_F^2)\right\} |\mathcal{M}|^2_{\text{Born}}(x p_1)$$

$$\mathbf{K}^{\text{eff}}(\mathbf{x}) = \frac{\alpha_s}{2\pi} C_F \left\{\left(\frac{1+x^2}{1-x}\ln\frac{1-x}{x}\right)_+ + \frac{1}{2}\delta(1-x) + (1-x) - \frac{3}{2}\frac{1}{(1-x)_+}\right\}$$

DIS: Nagy Soper - real emission terms

Initial state real emission subtraction

$$\mathbf{D}^{1,3} = \frac{4\,\pi\,\alpha_s}{x\,y\,\hat{p}_1\cdot\hat{p}_i}\,C_F\left(1-x-y+\frac{2\,\tilde{z}\,x}{v\,(1-x)+y}\right)|\mathcal{M}_{\text{Born}}(p)|^2$$

$$x = \frac{\hat{p}_o\cdot\hat{p}_4}{\hat{p}_i\cdot\hat{p}_1},\; y = \frac{\hat{p}_1\cdot\hat{p}_3}{\hat{p}_1\cdot\hat{p}_i},\; \tilde{z} = \frac{\hat{p}_1\cdot\hat{p}_4}{\hat{p}_4\cdot\hat{Q}},\; v = \frac{(\hat{p}_1\cdot\hat{p}_i)\,(\hat{p}_3\cdot\hat{p}_4)}{(\hat{p}_4\cdot\hat{Q})\,(\hat{p}_3\cdot\hat{Q})}.$$

Initial state: mapping

$$\mathbf{p_1} = x\,\hat{p}_1,\; \mathbf{p_i} = \hat{p}_i,\; \mathbf{p}^{\mu}_{\mathbf{o,4}} = \mathbf{\Lambda}^{\mu}{}_{\nu}(\mathbf{K},\hat{\mathbf{K}})\hat{p}^{\nu}_{o,4},\, K = x\,\hat{p}_1+\hat{p}_i,\; \hat{K} = \hat{p}_1+\hat{p}_i-\hat{p}_3\,.$$

Final state real emission subtraction

$$\mathbf{D}^{4,3} = \frac{4\pi\alpha_s\,C_F}{y(\hat{p}_i\cdot\hat{p}_1)}\left[\frac{y}{1-y}F_{\text{eik}}+z+2\frac{(1-v)(1-z(1-y))}{v\,[1-z(1-y)]+y\,[(1-y)\tilde{a}+1]}\right]|\mathcal{M}_{\text{Born}}(p)|^2$$

$$y = \frac{\hat{p}_3\cdot\hat{p}_4}{\hat{p}_1\cdot\hat{p}_i},\; z = \frac{\hat{p}_3\cdot\hat{p}_o}{\hat{p}_3\cdot\hat{p}_o+\hat{p}_4\cdot\hat{p}_o},\; v = \frac{\hat{p}_1\cdot\hat{p}_3}{\hat{p}_1\cdot\hat{p}_3+\hat{p}_1\cdot\hat{p}_4},\; F_{\text{eik}} = 2\,\frac{(\hat{p}_3\cdot\hat{p}_o)\,(\hat{p}_4\cdot\hat{p}_o)}{(\hat{p}_3\cdot\hat{Q})^2}.$$

Final state: mapping

$$\mathbf{p_i} = \hat{p}_i,\; \mathbf{p_1} = \hat{p}_1,\; \mathbf{p_4} = \frac{1}{1-y}\left[\hat{p}_3+\hat{p}_4-y\,(\hat{p}_1+\hat{p}_i)\right],\; \mathbf{p_o} = \frac{\hat{p}_o}{1-y}.$$