Cauchy Horizon Stability in the Self Similar LTB Spacetime

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Outline

The Self-Similar LTB Spacetime

- Inhomogeneous dust collapsing into a singularity.
- Self-similarity can be imposed by requiring the spacetime to admit a homothetic Killing vector field $\vec{\xi}$ such that

$$\mathcal{L}_{\vec{\xi}} g_{\mu\nu} = 2 g_{\mu\nu}. \tag{1}$$

- The resulting line-element is given by

$$ds^2 = e^{2p}(-dz^2 + (e^{\nu}(z) - z^2)dp^2 - 2zdpdz + S^2(z)d\Omega^2), \tag{2}$$

where $z = -t/r$ and $p = \ln(r)$.

- We get a naked singularity for $\lambda \in (0, 0.09)$, where $m(r) = \lambda r$ is the Misner-Sharp mass.
- Gerlach-Sengupta formalism - gives gauge invariant linear perturbations for the stress-energy and the metric using a spherical decomposition. Even parity only.

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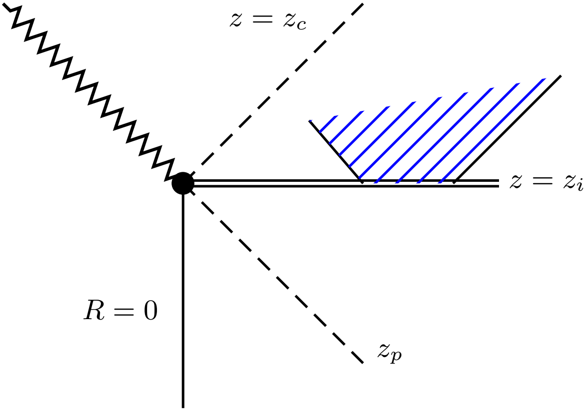

First Order Reduction

- The 5 dimensional symmetric hyperbolic system

$$\tau \frac{\partial \vec{u}}{\partial \tau} + A(\tau) \frac{\partial \vec{u}}{\partial p} + B(\tau)\vec{u} = \vec{\Sigma}(\tau, p). \tag{3}$$

Theorem 1: Existence and Uniqueness

The IVP consisting of the system (3) along with the initial data

$$\vec{u}|_{\tau_1} = \vec{f}(p), \tag{4}$$

where $\vec{f} \in C_0^\infty(\mathbb{R}, \mathbb{R}^5)$, possesses a unique solution $\vec{u}(\tau, p)$, $\vec{u} \in C^\infty(\mathbb{R} \times (0, \tau_1], \mathbb{R}^5)$. For all $\tau \in (0, \tau_1]$, $\vec{u}(\tau, \cdot) : \mathbb{R} \to \mathbb{R}^5$ has compact support.

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Behaviour of the Averaged Perturbation

- We introduce $\hat{u} := \int_{\mathbb{R}} \vec{u}\, dp$ which obeys

$$\tau \frac{d\hat{u}}{d\tau} = -B(\tau)\hat{u} + \hat{\Sigma}(\tau). \qquad (5)$$

Theorem 2: Behaviour of $\hat{u}$

Let $\vec{u}$ be a solution of (3) obeying theorem 1. Then the behaviour of $\hat{u}$ is given by

$$\hat{u}_i \text{ is } O(1) \text{ as } \tau \to 0, \qquad \hat{u}_5 \text{ is } O(\tau^{-b}) \text{ as } \tau \to 0 \qquad (6)$$

for $i = 1, ..4$, where b is the only non-zero eigenvalue of the matrix $B(\tau = 0)$, after it is put in Jordan canonical form. $b > 0$.

Theorem 3: L^q Divergence

$\lim_{\tau \to 0} ||\vec{u}||_q = \infty$ for $1 \leq q < \infty$.

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Behaviour of $\vec{x}$ on the Cauchy horizon

Corollary 1: Bound on $\vec{x} = \tau^b \vec{u}$

$\vec{x} := \tau^b \vec{u}$ is uniformly bounded in the range $\tau \in (0, \tau^*)$ for some $\tau^* > 0$. That is

$$|\vec{x}| = |\tau^b \vec{u}| \leq \lambda + \epsilon \tau^{2b-1} + O(\tau^{2b}), \tag{7}$$

where λ and ϵ are constants.

Note that $\hat{x}$ is non-zero on the Cauchy horizon. Define

$$\vec{x}^{(n)} := \vec{x}(\tau^{(n)}, p), \tag{8}$$

where $\tau^{(n)}$ is a sequence of τ-values such that as $n \to \infty$, $\tau^{(n)} \to 0$.

Theorem 4: $\vec{x}$ exists on the Cauchy horizon

$\vec{x}_{ch} := \lim_{\tau \to 0} \vec{x}(\tau, p)$ exists on the Cauchy horizon.

Theorem 5: $\vec{x} \in L^1$

$\vec{x}(\cdot, p) \in L^1(\mathbb{R}, \mathbb{R}^5)$ for all fixed τ.

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Behaviour of $\vec{x}$ on the Cauchy horizon

Lemma 1: Existence of a Dominated Subsequence

Define $\vec{x}^{(n)}$ as in (8). Then there exists a subsequence

$$\vec{x}^{(n_m)} := \vec{x}(\tau^{(n_m)}, p) \tag{9}$$

which converges to $\vec{x}(0,p)$ as $n_m \to \infty$, such that $\vec{x}^{(n_m)}$ is dominated, that is, there exists some $h(p) \in L^1(\mathbb{R})$ such that

$$|\vec{x}^{(n_m)}| \leq h(p) \qquad \forall \quad m. \tag{10}$$

Theorem 6: $\vec{x}$ is non-zero at $\tau = 0$

Define $\vec{x}^{(n)}$ as in (8). Then there exists some interval $(a,b) \subset \mathbb{R}$ such that $\vec{x}(0,p) \neq 0$ within this interval.

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Physical Interpretation of Results

- The perturbed Weyl scalars are given by

$$\delta\Psi_0 = \frac{Q}{2r^2}\bar{l}^A\bar{l}^B k_{AB}, \tag{11}$$

$$\delta\Psi_4 = \frac{Q^*}{2r^2}\bar{n}^A\bar{n}^B k_{AB}, \tag{12}$$

and the scalar δP_{-1} is given by $\delta P_{-1} = |\delta\Psi_0\delta\Psi_4|^{1/2}$.

Theorem 7: Perturbed Weyl Scalars

Given a solution $\vec{u}$ to the same IVP, the perturbed Weyl scalars $\delta\Psi_0$ and $\delta\Psi_4$, as well as the scalar δP_{-1}, generically diverge on the Cauchy horizon.

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Conclusions

- Unsolved mystery: How to handle the problem of scaling?
- Results so far show that the even parity perturbations diverge on the Cauchy horizon for all l.
- In the odd parity case, perturbations were found to remain finite on the Cauchy horizon for all l.
- Apply these methods to the self-similar perfect fluid spacetime.

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