# The θ-angle physics at finite baryon density

Jahmall Bersini

August 29, 2023

Based on: JB, A. D'Alise, F.Sannino, M. Torres, JHEP 11 (2022) 080

wpi World Premier International Research Center Initiative

KAVLI IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

## *The QCD ϑ-angle*

$$\mathcal{L}_\theta = \theta \frac{g^2}{32\pi^2} F_a^{\mu\nu} \tilde{F}_{a\mu\nu}$$

It leads to CP-symmetry violation.

Chiral transformations, because of the anomaly, change the θ-term and physics depends only on:

$$\bar{\theta} = \theta + \mathrm{Arg}\det M$$

M: quark mass matrix.

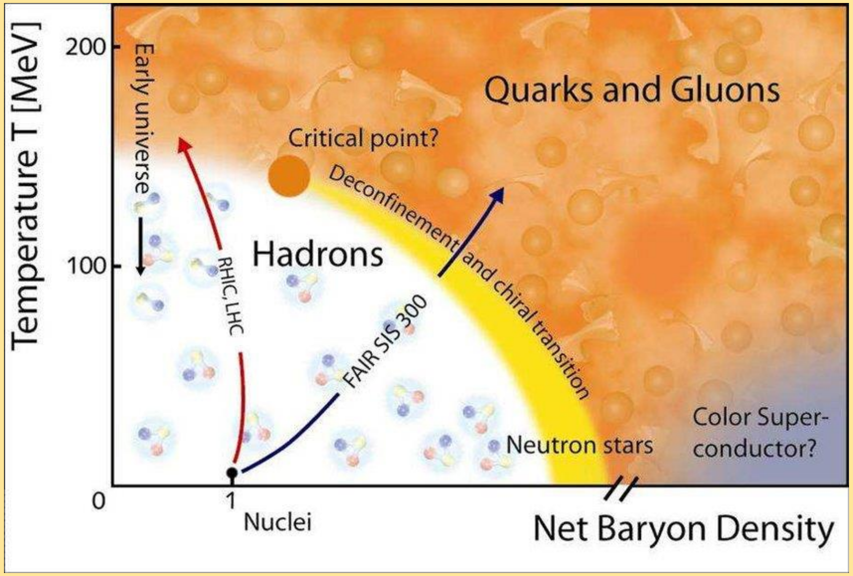
Experimentally constrained by measurements of the neutron electric dipole moment:

$$\bar{\theta} < 10^{-10}$$

**STRONG CP PROBLEM**

## *QCD Thermodynamics*

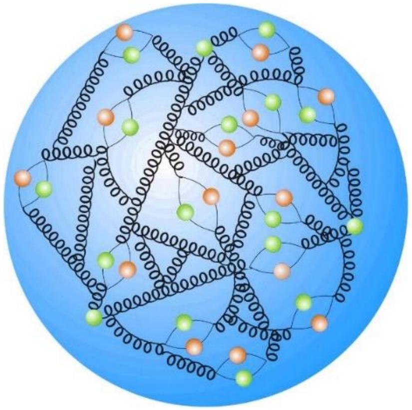
QCD at finite baryon density and temperature.




Many phases: QGP, color superconductivity…

*This talk: θ-angle physics at finite baryon density in two-color QCD.*

# Motivations



- Understand the QCD phase diagram at finite density and θ-angle.
- Focus on the SSB of CP symmetry at θ=π.
- Understand cosmological phase transitions from nonzero to zero θ.

Finite density QCD cannot be efficiently studied on lattice due to the **sign problem:** the determinant of the Dirac operator is not real.

**Two-color QCD**: no sign problem thanks to the pseudo-reality of the quark representations. Similar to QCD at finite isospin density (work in progress!).

# Two-color QCD

Two-color QCD exhibits an enhanced U(2$N_f$) symmetry, as compared to the U($N_f$)xU($N_f$) chiral symmetry of QCD.

In fact, thanks to the pseudoreality of the two-color Dirac operator the quark fields $q_L$ and $\sigma_2 \tau_2 q_R^*$ transform in the same color representation. Hence we can introduce

$$\mathcal{Q} = \begin{pmatrix} q_L \\ i\sigma_2\tau_2 q_R^* \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \mathbf{1}_{N_f}$$

and write the Lagrangian as

$$\mathcal{L} = -\frac{1}{4g^2}\vec{G}_{\mu\nu}\cdot\vec{G}^{\mu\nu} + i\bar{\mathcal{Q}}\bar{\sigma}^\nu\left[\partial_\nu - i\vec{G}_\nu\cdot\frac{\vec{\tau}}{2}\right]\mathcal{Q} - \frac{1}{2}m_q\mathcal{Q}^T\tau_2 E\mathcal{Q} + \mathrm{h.c.}\ .$$

In this form the U(2$N_f$) symmetry becomes manifest. The symmetry is broken to SU(2$N_f$) by the ABJ anomaly. The baryon charge is one of the generators of SU(2$N_f$) and baryons are diquark.

# Two-color chiral Lagrangian

The infrared dynamics of the theory can be described by the following **chiral Lagrangian**

$$\mathcal{L}_{\rm eff} = \nu^2 Tr\{\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\} + m_\pi^2 \nu^2 Tr\{M\Sigma + M^\dagger \Sigma^\dagger\}$$

The chiral symmetry breaking is SU(2$N_f$) → Sp(2$N_f$).

For the sake of simplicity, we consider a democratic mass matrix

$$M = -i\sigma_2 \otimes \mathbf{1}_{N_f} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}_{N_f}$$

and introduce the chemical potential μ in the covariant derivative as:

$$\partial_\mu \to D_\mu = \partial_\mu - i\mu\delta_\mu^0 B \qquad B \equiv \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \otimes \mathbf{1}_{N_f}$$

# Adding the θ-angle

We introduce the topological charge: $q(x) = \frac{g^2}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$

$$\mathcal{L}_{q(x)} = \frac{i}{4}q(x)Tr[\log\Sigma - \log\Sigma^\dagger] - \theta q(x) + \frac{q(x)^2}{4a\nu^2}$$

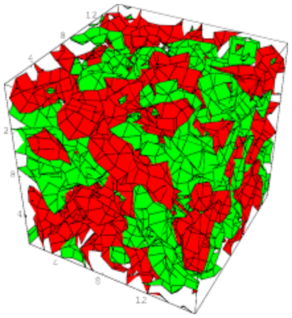
The coefficient of the quadratic term is the topological susceptibility of the Yang-Mills theory. The coefficients reproduce the axial anomaly:

$$\partial_\mu J^\mu_5 = 4N_f q(x)$$

We can integrate out q(x) via its EOM to get

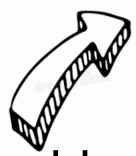
$$\mathcal{L}_\theta = \nu^2 Tr\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\} + 4\mu\nu^2 Tr\{B\Sigma^\dagger\partial_0\Sigma\} + m_\pi^2\nu^2 Tr\{M\Sigma + M^\dagger\Sigma^\dagger\} + 2\mu^2\nu^2\left[Tr\{\Sigma B^T\Sigma^\dagger B\} + Tr\{BB\}\right] - a\nu^2\left(\theta - \frac{i}{4}Tr\{\log\Sigma - \log\Sigma^\dagger\}\right)^2$$

# Vacuum structure



In the absence of the θ-angle we can look for a ground state of the form

$$\Sigma_c = \begin{pmatrix} 0 & \mathbf{1}_{N_f} \\ -\mathbf{1}_{N_f} & 0 \end{pmatrix} \cos\varphi + i \begin{pmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{I} \end{pmatrix} \sin\varphi \qquad \mathcal{I} = \begin{pmatrix} 0 & -\mathbf{1}_{N_f/2} \\ \mathbf{1}_{N_f/2} & 0 \end{pmatrix}$$



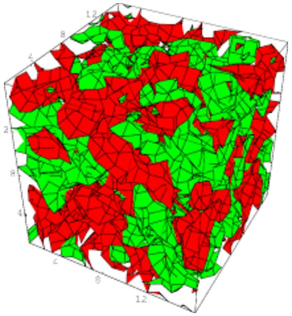
Competition of mass and baryon chemical potential (chiral and diquark condensates).

To take into account the θ-angle: we introduce the Witten variables $\alpha_i$

$$\Sigma_0 = U(\alpha_i)\Sigma_c \,, \qquad U(\alpha_i) \equiv \mathrm{diag}\{e^{-i\alpha_1}, \ldots, e^{-i\alpha_{N_f}}, e^{-i\alpha_1}, \ldots, e^{-i\alpha_{N_f}}\}$$

Each phase $\alpha_i$ is an overall axial transformation for each left-right quark pair.

# Vacuum structure



The Lagrangian evaluated on the vacuum ansatz reads

$$\mathcal{L}_\theta[\Sigma_0] = \nu^2\left[4m_\pi^2 X\cos\varphi + 2\mu^2 N_f\sin^2\varphi - a\bar{\theta}^2\right]$$

$$\bar{\theta} = \theta - \sum_i^{N_f}\alpha_i\,, \qquad X = \sum_i^{N_f}\cos\alpha_i$$

The equations of motion are

$$\sin\varphi\left(N_f\cos\varphi - \frac{m_\pi^2}{\mu^2}X\right) = 0$$

$$2m_\pi^2\sin\alpha_i\cos\varphi = a\bar{\theta}\,, \quad i = 1, .., N_f$$

# Superfluid phase transition

Consider the first EOM: $\sin\varphi\left(N_f\cos\varphi-\frac{m_\pi^2}{\mu^2}X\right)=0$

Two solutions:

- $\text{normal phase}\quad(\varphi=0)$
- $\text{superfluid phase}\quad\left(\cos\varphi=\frac{m_\pi^2}{N_f\mu^2}X\right)$

The superfluid phase transition is of the second order and is associated with diquark (baryon) condensation. The energy reads

- $E=-\nu^2\left[4m_\pi^2X-a\bar{\theta}^2\right]\ ,\quad \text{normal phase}$
- $E=-\nu^2\left[2\frac{N_f^2\mu^4+m_\pi^4X^2}{N_f\mu^2}-a\bar{\theta}^2\right]\ ,\quad \text{superfluid phase}$

**θ=0:** $X=N_f$ : superfluid phase transition at $\mu=m_\pi$.

**θ≠0:** We need to know the θ-dependence in both phases: <u>the energy is minimized when X (normal phase) and $X^2$ (superfluid phase) is maximized.</u>

# θ-dependence: normal phase

In the normal phase we have the well-known equation

$$2m_\pi^2 \sin\alpha_i = a\bar{\theta} = a\left(\theta - \sum_i^{N_f} \alpha_i\right)$$

Then: $\sin\alpha_i = \sin\alpha_j$. We solve in powers of $m_\pi^2/a$. Leading order:

$$\alpha_i = \begin{cases} \pi - \alpha, & i = 1, \ldots, n \\ \alpha, & i = n+1, \ldots, N_f \end{cases} \qquad n(\pi - \alpha) + (N_f - n)\alpha = \theta \text{ Mod } 2\pi$$

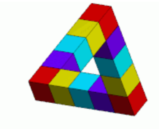
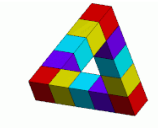
Solution:

$$\alpha = \frac{\theta + (2k - n)\pi}{(N_f - 2n)}, \quad k = 0, \ldots, N_f - 2n - 1, \quad n = 0, \ldots, \left[\frac{N_f - 1}{2}\right]$$

The solutions with **n≠0** spontaneously break $Sp(2N_f)$ because of the different phases for each flavour.

# CP symmetry

CP is conserved when $\bar{\theta} = \theta - \sum^{N_f} \alpha_i = 0$

This happens if:  **θ=0**  **$m_\pi^2 = 0$**

For **θ=π** the Lagrangian is CP invariant. However, the vacua lie at

$$U(\alpha_i) = e^{\frac{i\pi}{N_f}} \mathbf{1}_{2N_f} \qquad U(\alpha_i) = e^{-\frac{i\pi}{N_f}} \mathbf{1}_{2N_f}$$

The two solutions are related by a CP transformation $U \to U^\dagger$ and thus CP is spontaneously broken by the vacuum.

DASHEN PHENOMENON

R. F. Dashen Phys.Rev.D 3 (1971) 1879-1889

# θ-dependence: superfluid phase

In the superfluid phase the equation of motion is

$$\frac{2m_\pi^4}{N_f\mu^2} X \sin\alpha_i = a\bar{\theta}\,, \qquad\qquad i = 1, .., N_f\,.$$

In this case the natural expansion parameter is

$$\frac{m_\pi^4}{a\mu^2}$$

**We now proceed by considering fixed values of $N_f$.**

# $N_f = 2$

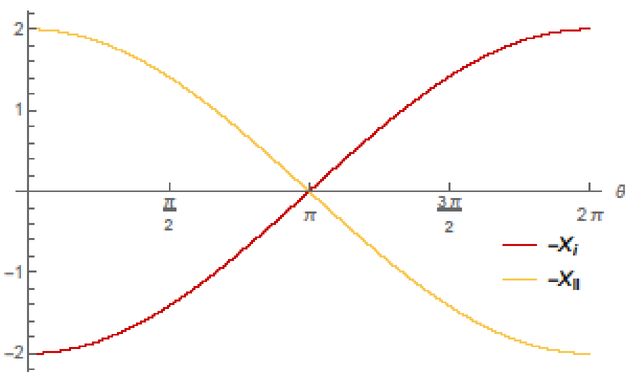
At the leading order (in $m_\pi^2/a$ or $m_\pi^4/(a\,\mu^2)$) the EOM is

$$\alpha_1 + \alpha_2 = \theta + 2k\pi \qquad \sin\alpha_1 = \sin(\theta + 2k\pi - \alpha_1)$$

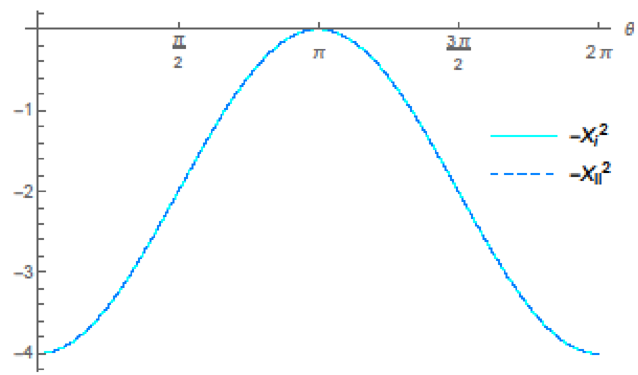
There are two solutions

$$\{\alpha_1, \alpha_2\} = \{\frac{\theta}{2}, \frac{\theta}{2}\} \qquad \{\alpha_1, \alpha_2\} = \{\frac{\theta + 2\pi}{2}, \frac{\theta + 2\pi}{2}\}$$

The energy is minimized when X (normal phase) or $X^2$ (superfluid phase) is maximized:




X (normal phase)




$X^2$ (superfluid phase

# $N_f = 2$

The energy in the two phases is

$$E(\theta) = -8m_\pi^2\nu^2\left(\left|\cos\frac{\theta}{2}\right| + \frac{1}{2}\frac{m_\pi^2}{a}\sin^2\frac{\theta}{2} - \frac{1}{4}\frac{m_\pi^4}{a^2}\left|\sin\frac{\theta}{2}\sin\theta\right|\right), \qquad \text{normal phase}$$

$$E(\theta) = -\nu^2\left(\frac{4\left(m_\pi^4\cos^2\frac{\theta}{2} + \mu^4\right)}{\mu^2} + \frac{m_\pi^8\sin^2\theta}{a\mu^4} - \frac{m_\pi^{12}\sin^2\theta\cos\theta}{a^2\mu^6}\right), \qquad \text{superfluid phase}$$

The superfluid phase transition occurs at

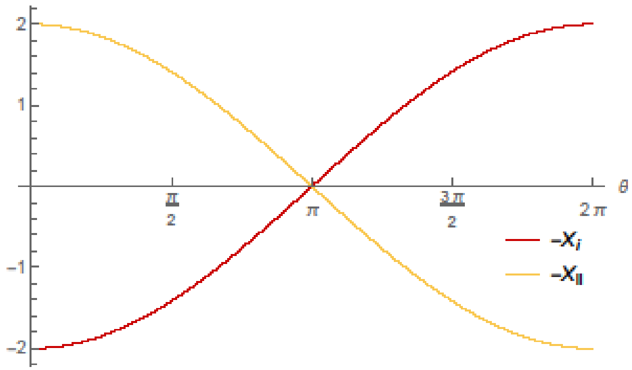
$$\mu_c = m_\pi(\theta) = m_\pi\left[\sqrt{\left|\cos\frac{\theta}{2}\right|} + \mathcal{O}\left(\frac{m_\pi^2}{a}\right)\right]$$

Hence it can be realized for tiny values μ when θ≈π. We have

$$\mu_c \sim m_\pi\sqrt{\frac{m_\pi^2}{a} + \frac{|\phi|}{2}} \qquad\qquad \theta = \pi + \phi$$

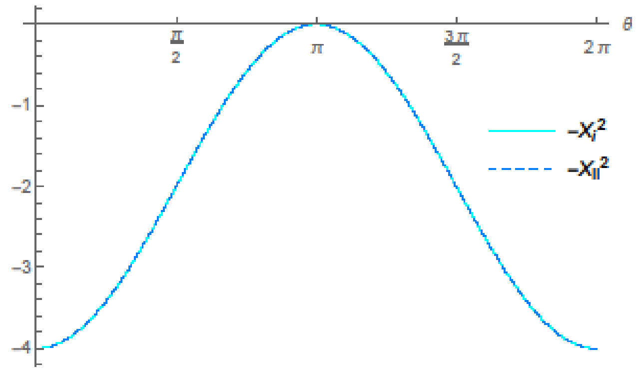
# $N_f = 2$

Normal phase: the solutions cross at θ=π where I have spontaneous breaking of CP symmetry.




$$\bar{\theta} = \frac{2m_\pi^2}{a}\sin\frac{\theta}{2} \underset{\theta=\pi}{=} \frac{2m_\pi^2}{a} + \mathcal{O}\left(\frac{m_\pi^6}{a^3}\right)$$

Superfluid phase: the energy is an analytic function of θ. No spontaneous breaking of CP symmetry at θ=π.




$$\bar{\theta} = \frac{m_\pi^4}{a\mu^2}\sin\theta \underset{\theta=\pi}{=} 0$$

This is exact to all orders in $m_\pi^2/a$.
In fact at θ=π the EOM is

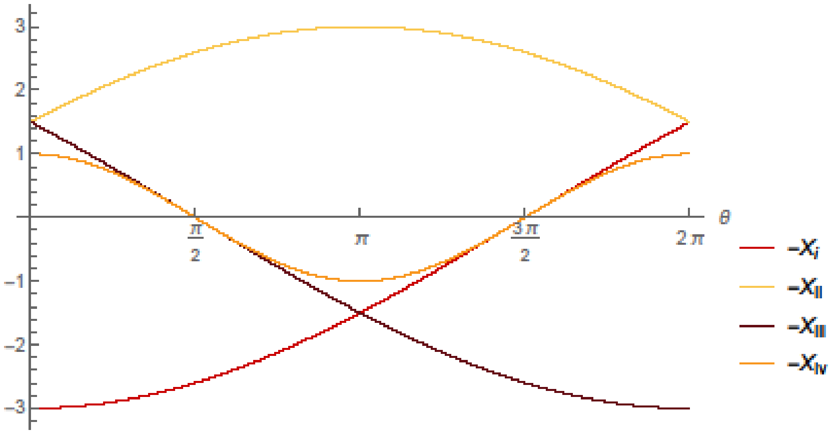
$$\frac{m_\pi^4}{a\mu^2}\sin(2\alpha) = \pi - 2\alpha$$

# $N_f = 3$

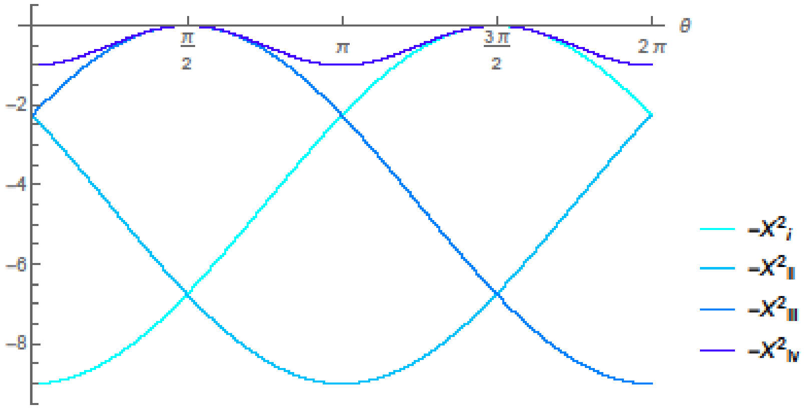
We have four solutions:

$$\mathbf{i.}\left\{\frac{\theta}{3},\frac{\theta}{3},\frac{\theta}{3}\right\},\quad \mathbf{ii.}\left\{\frac{\theta+2\pi}{3},\frac{\theta+2\pi}{3},\frac{\theta+2\pi}{3}\right\},\quad \mathbf{iii.}\left\{\frac{\theta+4\pi}{3},\frac{\theta+4\pi}{3},\frac{\theta+4\pi}{3}\right\},\quad \mathbf{iv.}\{\theta-\pi\,,\theta-\pi\,,2\pi-\theta\}$$

Normal phase: the ground state is given by solutions 1. and 3. that cross at θ=π where I have CP SSB.




Superfluid phase: No CP SSB at θ=π but two novel first-order phase transitions at θ=π/2, 3π/2.




The non-minimum solutions represent metastable vacua which can be long-lived. Later we will estimate their decay rate.

# General $N_f$

Solutions of the EOM are generally not periodic of 2π for θ.
The periodicity condition can be satisfied only if at least two solutions cross. Consider

$$U = e^{-i\alpha}\mathbf{1}_{2N_f}$$

and ask when crossing can happen. We have

$$\cos\left(\frac{\theta+2\pi k_1}{N_f}\right) = \cos\left(\frac{\theta+2\pi k_2}{N_f}\right) \quad \text{normal phase}$$

$$\cos^2\left(\frac{\theta+2\pi k_1}{N_f}\right) = \cos^2\left(\frac{\theta+2\pi k_2}{N_f}\right) \quad \text{superfluid phase}$$

- Near $\theta=0$ the ground state is $k_1=0$.
- Both conditions can be satisfied at $\theta=\pi$. For $k_1=0$ we have $k_2=N_f-1$.
- In the normal phase there is only this solution.

# Superfluid phase: *even* $N_f$

**In the superfluid phase we have other solutions.**

When $N_f$ is even we have the solution: $k_1 = k_2 + N_f/2$

Which does not depend on θ: the solutions organize themselves in pairs (α and α+π) with the same energy for every θ. This holds to all orders in $m_\pi^2/a$. In fact given the EOM for a certain α:

$$\frac{m_\pi^4}{a\mu^2}\sin(2\alpha) = \theta - N_f\alpha$$

we have the same EOM for α + π upon shifting $\theta \to \theta + N_F\pi$ .
Then given the general solution

$$\alpha = \frac{\theta + (2k-n)\pi}{(N_f - 2n)}, \quad k = 0, \ldots, N_f - 2n - 1\,, \quad n = 0, \ldots, \left[\frac{N_f - 1}{2}\right]$$

The ground state has n=k=0 on (0, π) and n=0, k= $N_f$-1 on (π, 2π) along with their degenerate partners. SSB of CP at θ=π except for $N_f$ =2.

# Superfluid phase: *odd* $N_f$

**In the superfluid phase we have other solutions.**

When $N_f$ is odd we have the solution $k_1 = N_f/2 - k_2 - \theta/\pi$

It can be realized for θ=π/2 and θ=3π/2.

The ground state is:

| | |
|---|---|
| $\alpha=\theta/N_f$ | $(0, \pi/2)$ |
| $\alpha=\pi+(\theta-\pi)/N_f$ | $(\pi/2, 3\pi/2)$ |
| $\alpha=(\theta-2\pi)/N_f$ | $(3\pi/2, 2\pi)$ |

No spontaneous symmetry breaking of CP at θ=π.

Two novel first order phase transitions at θ=π/2 and θ=3π/2.

# Domain walls

The tension of the domain wall between the two degenerate vacua at $\theta = \pi$ for even $N_f$ in the superfluid phase reads

$$T = 2\nu^2 \int_{-\infty}^{\infty} dx \left[ (N_f - 1) N_f \alpha'(x)^2 - \frac{m_\pi^4}{\mu^2 N_f} \left( (N_f - 1) \cos\left(\alpha(x) + \frac{\pi}{N_f}\right) + \cos\left(\frac{\pi}{N_f} - (N_f - 1)\alpha(x)\right)\right)^2 \right]$$

Regardless of the exact form of the wall's profile, its tension scales as

$$T \sim \frac{\nu^2 m_\pi^2}{\mu}$$

To be compared with $T \sim \nu^2 \, m_\pi$ in the normal phase.

[A. V. Smilga, Phys.Rev.D 59, 114021 (1999)]

The decay rate of the metastable vacua near $\theta = \pi$ is

$$\Gamma \propto \exp\left(-C \frac{T^4}{m_\pi^6 \nu^6 |\phi|^3}\right) \begin{cases} \sim \exp\left(-\dfrac{\nu^2}{m_\pi^2 |\phi|^3}\right) \\ \sim \exp\left(-\dfrac{\nu^2 m_\pi^2}{\mu^4 |\phi|^3}\right) \end{cases}$$

Here C is a positive constant and $\theta = \pi + \phi$ .

# Symmetry breaking pattern

$$U(2N_f) \to SU(2N_f) \to Sp(2N_f)$$

ANOMALY χSB

We have $2N_f^2 - N_f - 1$ (pseudo)Goldstone modes from the χSB plus the „anomalous" singlet with a mass of order *a*.

$m_\pi = 0$

$$Sp(2N_f) \to SU(N_f)_L \times SU(N_f)_R \times U(1)_B \rightsquigarrow Sp(N_f)_L \times Sp(N_f)_R$$

μ VACUUM

We have $N_f^2 - N_f - 1$ massless Goldstone modes while the other modes get a mass of order μ.

$m_\pi \neq 0$

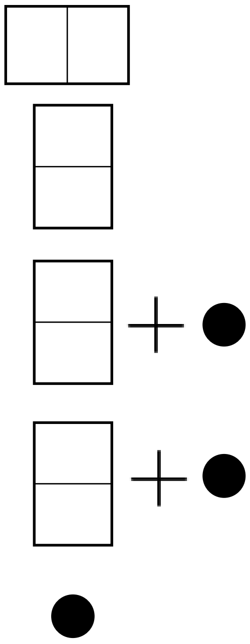
$$Sp(2N_f) \to SU(N_f)_L \times SU(N_f)_R \times U(1)_B \to SU(N_f)_V \times U(1)_B \to Sp(N_f)_V$$

μ $m_\pi$ VACUUM

We have $\frac{1}{2}N_f(N_f - 1)$ massless Goldstone modes.

# Spectrum

| | | Sp($N_f$) representations |
|---|---|---|
| $\omega_1^2 = k^2 + \mu^2$ | $\frac{1}{2}N_f(N_f+1)$ | |
| $\omega_2^2 = k^2 + \frac{m_\pi^4 X^2}{\mu^2 N_f^2}$ | $\frac{1}{2}N_f(N_f-1)-1$ | |
| $\omega_3^2 = k^2 + \frac{2\left(\mu^4 N_f^2 + 3m_\pi^4 X^2\right)}{N_f^2\mu^2} + A$ | $\frac{1}{2}N_f(N_f-1)$ | |
| $\omega_4^2 = k^2 + \frac{2\left(\mu^4 N_f^2 + 3m_\pi^4 X^2\right)}{N_f^2\mu^2} - A$ | $\frac{1}{2}N_f(N_f-1)$ | |
| $\omega_5^2 = k^2 + M_S^2$ | $1$ | |



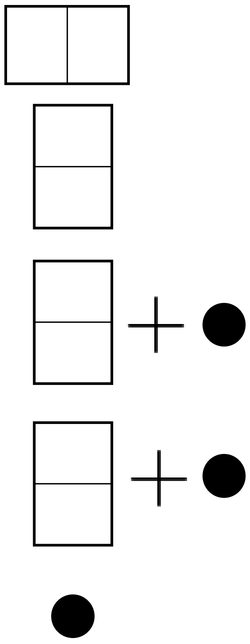
$$A = \frac{2}{N_f^2\mu^2}\sqrt{\left(N_f^2\mu^4 + 3m_\pi^4 X^2\right)^2 + 4N_f^2\mu^2 m_\pi^4 X^2 k^2}$$

$$M_S^2 = \frac{a\mu^4 N_f^3 + 2\mu^2 m_\pi^4 X^2}{2\mu^4 N_f^2 - 2m_\pi^4 X^2}\left(1 - \frac{m_\pi^4 X^2}{\mu^2 N_f^2}\right)$$

# Spectrum

| Dispersion relation | Multiplicity |
| --- | --- |
| $\omega_1^2 = k^2 + \mu^2$ | $\frac{1}{2}N_f(N_f+1)$ |
| $\omega_2^2 = k^2 + \frac{m_\pi^4 X^2}{\mu^2 N_f^2}$ | $\frac{1}{2}N_f(N_f-1)-1$ |
| $\omega_3^2 = k^2 + \frac{2\left(\mu^4 N_f^2 + 3m_\pi^4 X^2\right)}{N_f^2\mu^2} + A$ | $\frac{1}{2}N_f(N_f-1)$ |
| $\omega_4^2 = k^2 + \frac{2\left(\mu^4 N_f^2 + 3m_\pi^4 X^2\right)}{N_f^2\mu^2} - A$ | $\frac{1}{2}N_f(N_f-1)$ |
| $\omega_5^2 = k^2 + M_S^2$ | $1$ |

Sp($N_f$) representations



- The number of d.o.f sum to $\dim\left(\frac{U(2N_f)}{Sp(2N_f)}\right) = N_f(2N_f-1)$
- $\omega_4$ describes Goldstone modes with speed $v_G$=1.
- For $m_\pi$=0, $\omega_2$ describes Goldstone modes with speed $v_G$=1.

# The η’

The Sp($N_f$) singlet with dispersion relation

$$\omega_5^2 = k^2 + M_S^2 \qquad M_S^2 = \frac{a\mu^4 N_f^3 + 2\mu^2 m_\pi^4 X^2}{2\mu^4 N_f^2 - 2m_\pi^4 X^2}\left(1 - \frac{m_\pi^4 X^2}{\mu^2 N_f^2}\right)$$

is analogous to the η’ meson of QCD.

For $m_\pi$=0 its mass is: $M_S^2 = \frac{aN_f}{2}$

At the same time, the topological susceptibility is: $\frac{d^2E}{d\theta^2}|_{\theta=0} = 2\nu^2 a$

We, therefore, have

$$M_S^2 = \frac{N_f}{4\nu^2}\frac{d^2E}{d\theta^2}|_{\theta=0}$$

This is the **Witten-Veneziano relation** which still holds at finite density in the chiral limit.

# Conclusions

- Two-color QCD displays a rich phase diagram in the μ-θ plane depending on the number of flavours (even VS odd).
- For a odd number of flavours there is no CP breaking at θ=π. However there are two novel first order phase transition at θ=π/2 and θ=3π/2.
- For every phase we determined the related symmetry breaking pattern and the resulting spectrum of the theory.