Naturally small neutrino mass from asymptotic safety

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Based on
JHEP 08 (2022) 262 (2204.00866)
and 2308.06114

in collaboration with
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Corfu Summer Institute

31.08.2023

National Science Centre Poland

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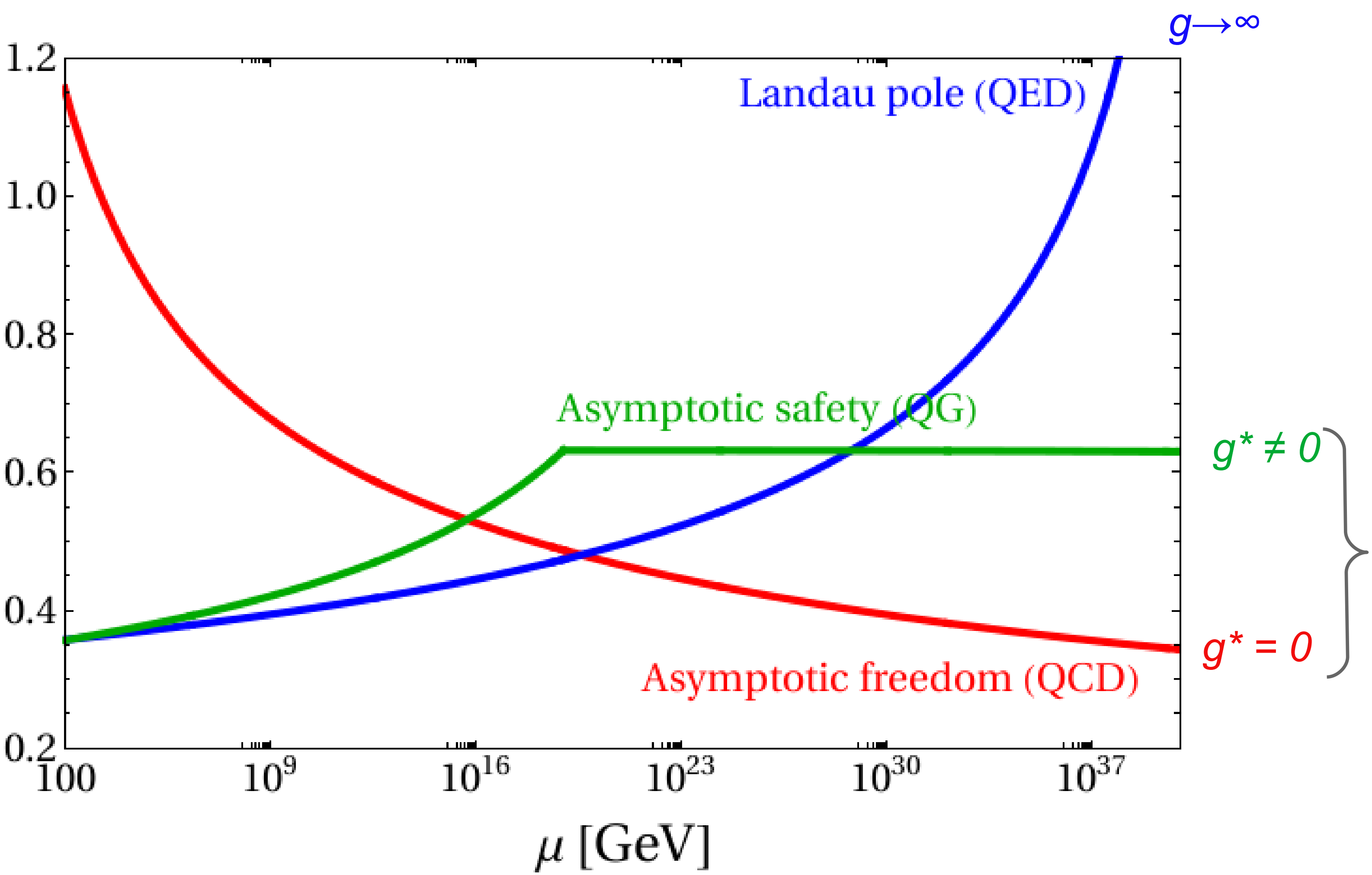
Asymptotic behaviors



$$\beta_g = \frac{dg}{dt} = \frac{dg}{d\ln\mu}$$

$$\beta_g(g^*) = 0$$

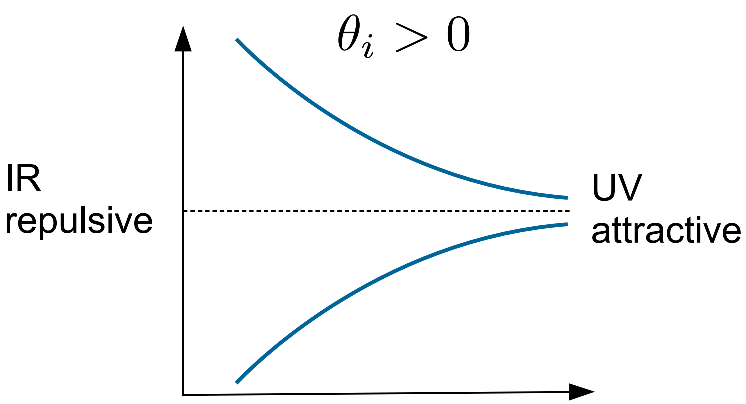
fixed point g^*
in the RGE flow

- Asymp. safety originally advocated by Weinberg for gravity (non-perturbative renormalizability)
- Applied to other field theories too, addresses triviality

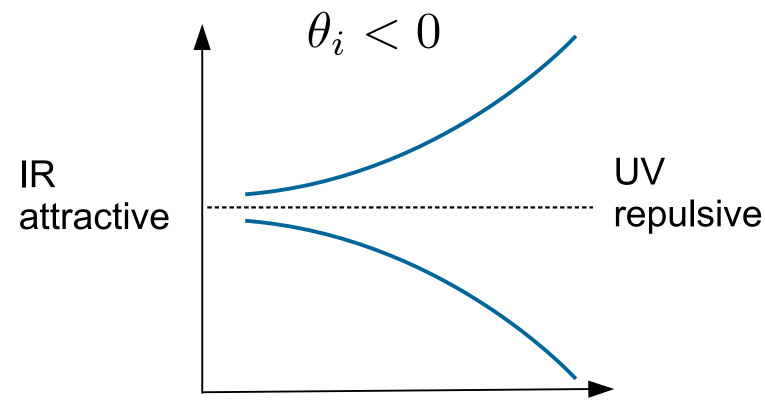
Fixed points

$$\beta_i(\{\alpha_j^*\}) = 0 \longrightarrow M_{ij} = \partial\beta_i/\partial\alpha_j|_{\{\alpha_i^*\}} \longrightarrow \{-\theta_i\}$$

fixed point — stability matrix — critical exponents



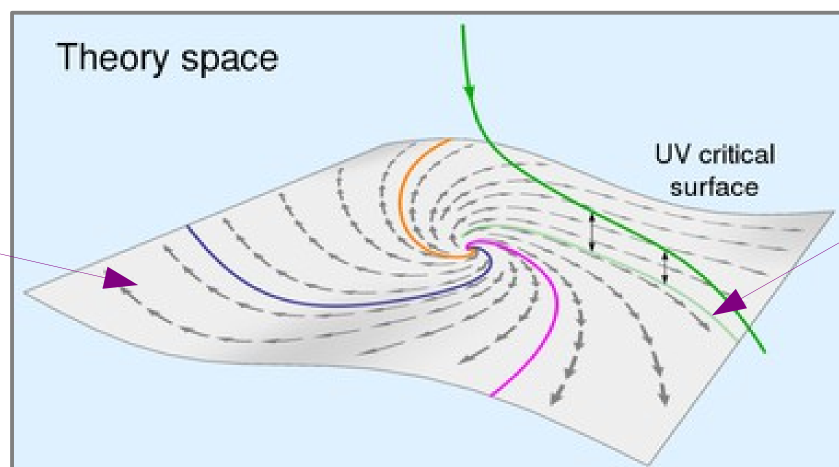
Relevant couplings are **free parameters** of the theory



Irrelevant couplings provide **predictions**

span the
UV critical surface

They are determined by experiment ...



from Wikipedia

can only deviate from the FP along the critical surface

… they are functions of relevant pars

Asymptotically safe gravity

Quantum gravity might feature interactive UV fixed points

Reuter '96, Reuter, Saueressig '01, Litim '04, Codello, Percacci, Rahmede '06, Benedetti, Machado, Saueressig '09, Narain, Percacci '09, Manrique, Rechenberger, Saueressig '11, Falls, Litim, Nikolakopoulos '13, Dona', Eichhorn, Percacci '13, Daum, Harst, Reuter '09, Folkerst, Litim, Pawlowski '11, Harst, Reuter '11, Zanusso *et al.* '09 … many more

e.g. Einstein-Hilbert action

$$\Gamma_k = \frac{1}{16\pi G}\int d^4x\sqrt{g}\,[-R(g)+2\Lambda]$$

FRG (Wetterich equation)

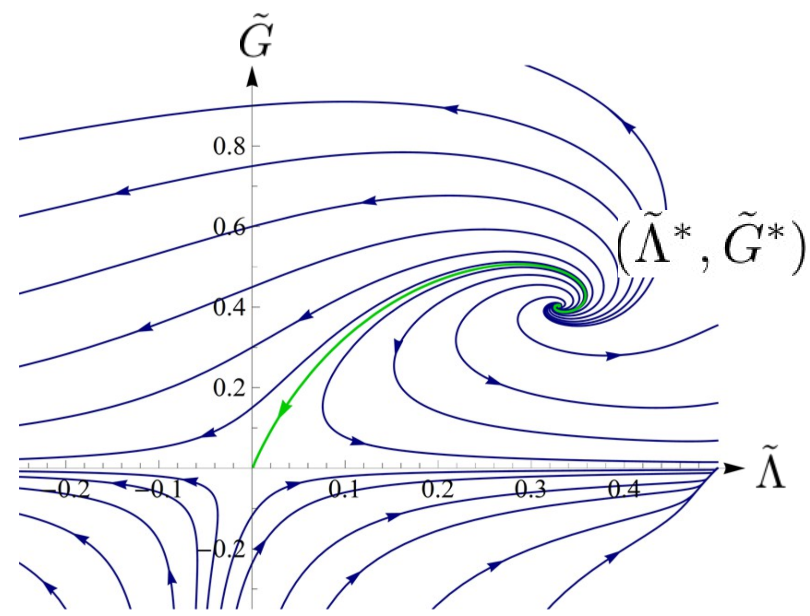
$$\partial_t\Gamma_k = k\,\partial_k\Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\frac{1}{\Gamma_k^{(2)}+\mathcal{R}_k}\partial_t\mathcal{R}_k\right)$$

Beta functions of grav. couplings

$$\frac{d\tilde{G}}{dt} = \left[2+\tilde{G}\,\eta_1(\tilde{G},\tilde{\Lambda})\right]\tilde{G} \quad = 0$$

$$\frac{d\tilde{\Lambda}}{dt} = -2\tilde{\Lambda}+\tilde{G}\,\eta_2(\tilde{G},\tilde{\Lambda}) \quad = 0$$

Reuter, Saueressig, hep-th/0110054



2 relevant fixed points

... fixed points persist under the addition of gravity and matter interactions

Matter RGEs with quantum gravity

Christiansen, Eichhorn '17, Christiansen *et al.* '17, Shaposhnikov, Wetterich '09, Oda, Yamada '15, Eichhorn, Held, Pawlowski '16, Wetterich, Yamada '16, Hamada, Yamada '17, Pawlowski *et al.* '18, Eichhorn, Versteegen '17, Eichhorn, Held '17-'18, Pastor-Gutiérrez, Pawlowski, Reichert '22, ...

Trans-Planckian corrections of matter RGEs $k > M_{\rm Pl}$

SM gauge couplings

$$\frac{dg_Y}{dt} = \frac{g_Y^3}{16\pi^2}\frac{41}{6} - f_g\, g_Y$$

$$\frac{dg_2}{dt} = -\frac{g_2^3}{16\pi^2}\frac{19}{6} - f_g\, g_2$$

$$\frac{dg_3}{dt} = -\frac{g_3^3}{16\pi^2}7 - f_g\, g_3$$

universal corrections depend on gravity fixed points

$$f_g = \tilde{G}^* \frac{1-4\tilde{\Lambda}^*}{4\pi\left(1-2\tilde{\Lambda}^*\right)^2}, \qquad f_y = -\tilde{G}^* \frac{96+\tilde{\Lambda}^*\left(-235+103\tilde{\Lambda}^*+56\tilde{\Lambda}^{*2}\right)}{12\pi\left(3-10\tilde{\Lambda}^*+8\tilde{\Lambda}^{*2}\right)^2}$$

A. Eichhorn, A. Held, 1707.01107
A. Eichhorn, F. Versteegen, 1709.07252

SM Yukawa couplings

$$\frac{dy_t}{dt} = \frac{y_t}{16\pi^2}\left(\frac{9}{2}y_t^2 + \frac{3}{2}y_b^2 + 3y_s^2 + 3y_c^2 - \frac{9}{4}g_2^2 - 8g_3^2 - \frac{17}{12}g_Y^2 + y_e^2 + y_\mu^2 + y_\tau^2\right) - f_y\, y_t$$

$$\frac{dy_b}{dt} = \frac{y_b}{16\pi^2}\left(\frac{9}{2}y_b^2 + \frac{3}{2}y_t^2 + 3y_s^2 + 3y_c^2 - \frac{9}{4}g_2^2 - 8g_3^2 - \frac{5}{12}g_Y^2 + y_e^2 + y_\mu^2 + y_\tau^2\right) - f_y\, y_b \quad \ldots$$

... same for other quarks and leptons

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get fixed points

... same for other quarks and leptons

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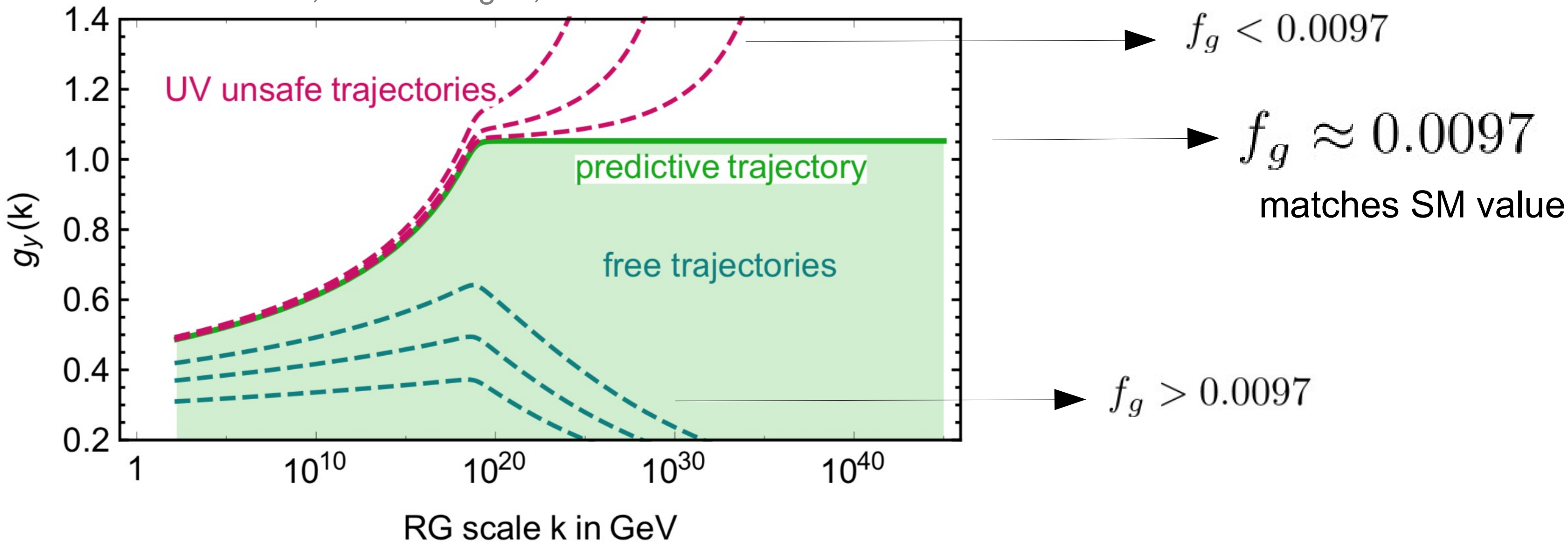
(hypercharge case)

A. Eichhorn, F. Versteegen, 1709.07252



IR / UV interplay has consequences for the pheno of many BSM models...

… see K.Kowalska's talk on Sunday

Naturalness
with
asymptotic safety

Neutrinos

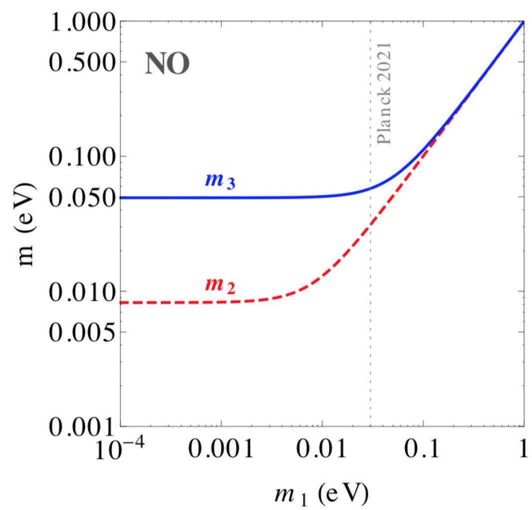
Neutrino masses are very small !

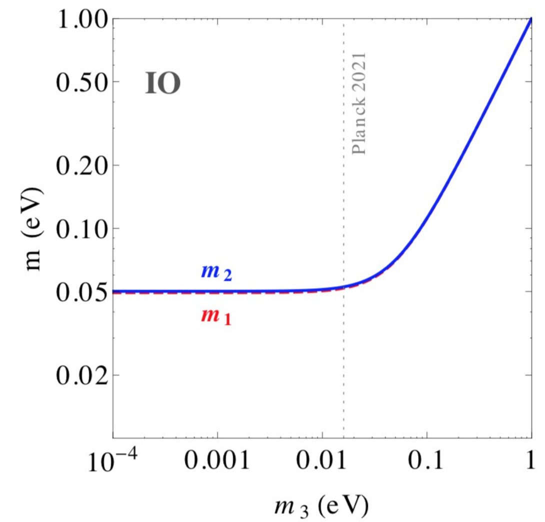
NuFIT5.1 (2021)   2007.14792

$$\begin{aligned} &\Delta m^2_{21} = 7.42^{+0.21}_{-0.20} \times 10^{-5}\,\text{eV}^2, \\ \text{NO:}\quad &\Delta m^2_{31} = 2.515^{+0.028}_{-0.028} \times 10^{-3}\,\text{eV}^2, \\ \text{IO:}\quad &\Delta m^2_{32} = -2.498^{+0.028}_{-0.029} \times 10^{-3}\,\text{eV}^2, \end{aligned}$$

Planck (2021)   1807.06209

$$\sum_{i=1,2,3} m_i < 0.12\,\text{eV}$$



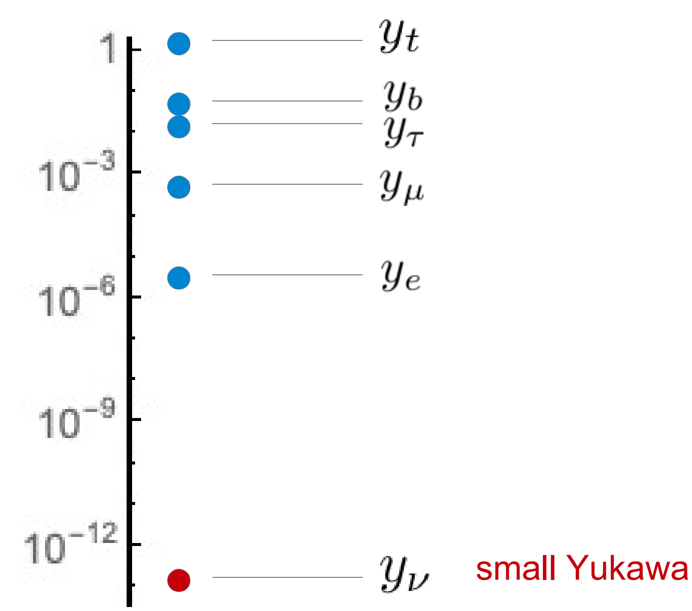


… either Dirac neutrino ...

Small Yukawa → Higgs mechanism

$$\mathcal{L}_D = -y_\nu^{ij}\,\nu_{R,i}\,(H^c)^\dagger L_j + \text{H.c.}$$

$$m_\nu = \frac{y_\nu v_H}{\sqrt{2}}$$



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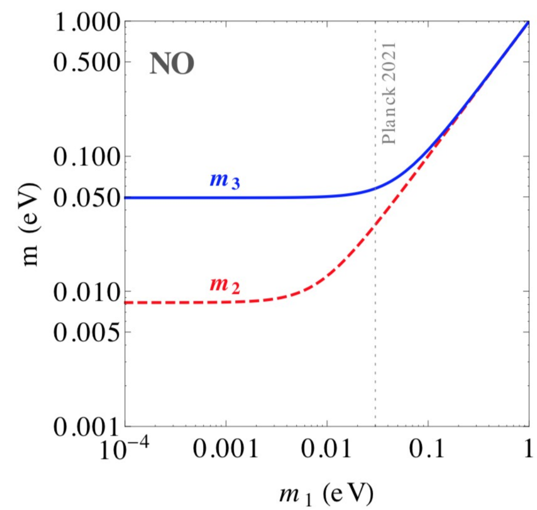
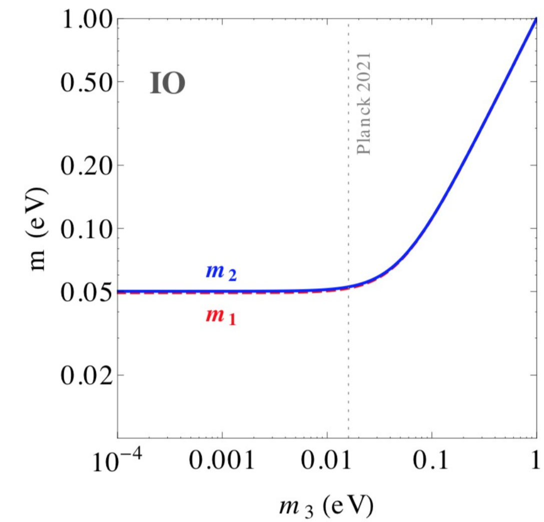


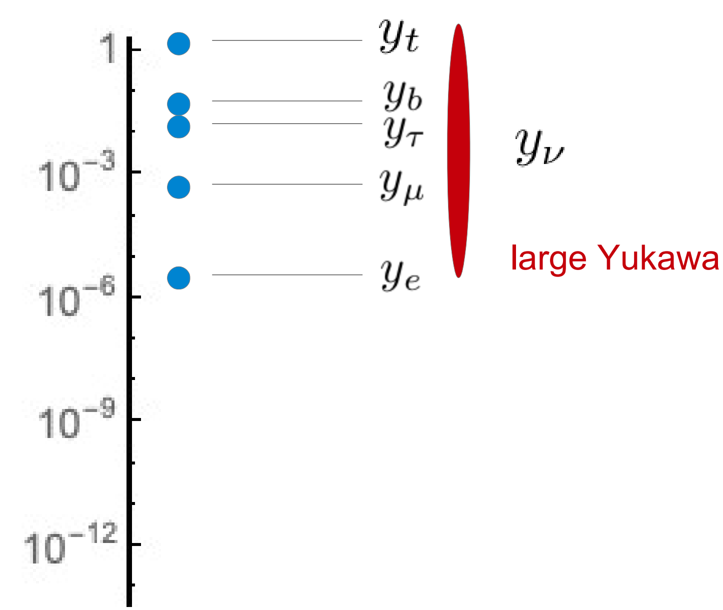


… or Majorana neutrino ...

see-saw mechanism

$$\mathcal{L}_M = \mathcal{L}_D - \frac{1}{2} M_N^{ij} \nu_{R,i} \nu_{R,j} + \mathrm{H.c.}$$

$$m_\nu = \frac{y_\nu^2 v_H^2}{\sqrt{2} M_N}$$



Fixed points of SM + RHN:

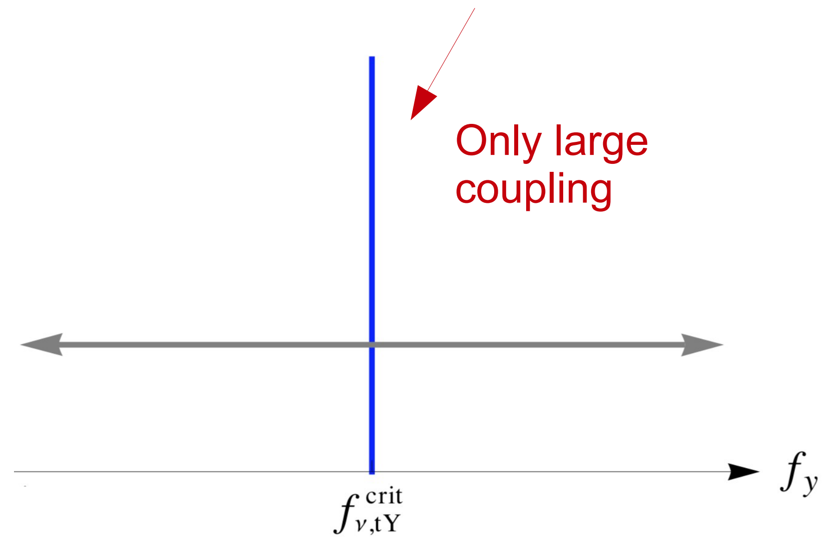
K.Kowalska, S.Pramanick, EMS, 2204.00866

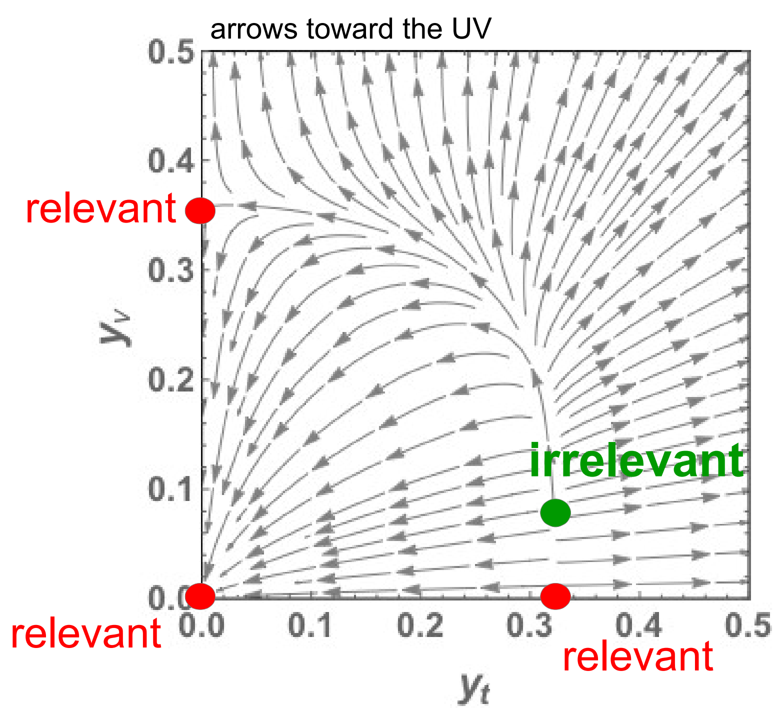
$$\frac{dg_Y}{dt} = \frac{g_Y^3}{16\pi^2}\frac{41}{6} - f_g\, g_Y \;\; = 0$$

$$\frac{dy_t}{dt} = \frac{y_t}{16\pi^2}\left[\frac{9}{2}y_t^2 + y_\nu^2 - \frac{17}{12}g_Y^2\right] - f_y\, y_t \;\; = 0$$

$$\frac{dy_\nu}{dt} = \frac{y_\nu}{16\pi^2}\left[3y_t^2 + \frac{5}{2}y_\nu^2 - \frac{3}{4}g_Y^2\right] - f_y\, y_\nu \;\; = 0$$

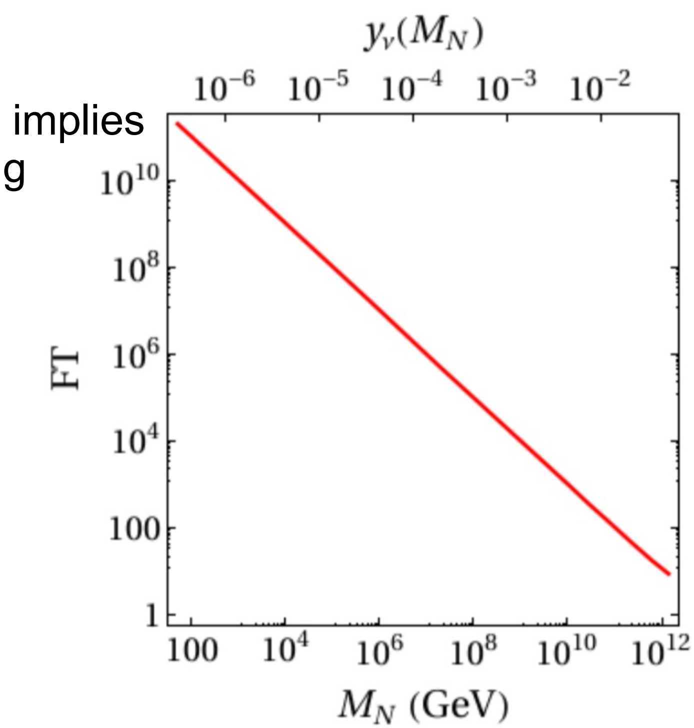
fy > fcrit $\sim 8\times 10^{-4}$





$y_\nu^* \sim \mathcal{O}(1)$

Small coupling implies large fine tuning



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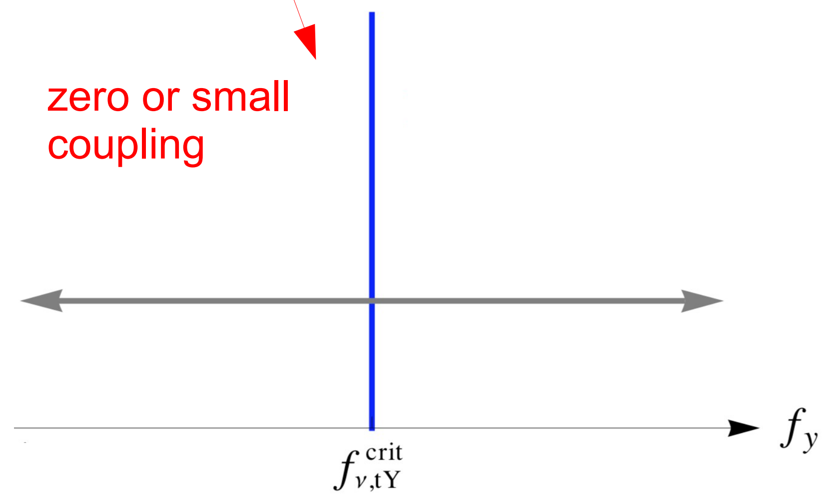
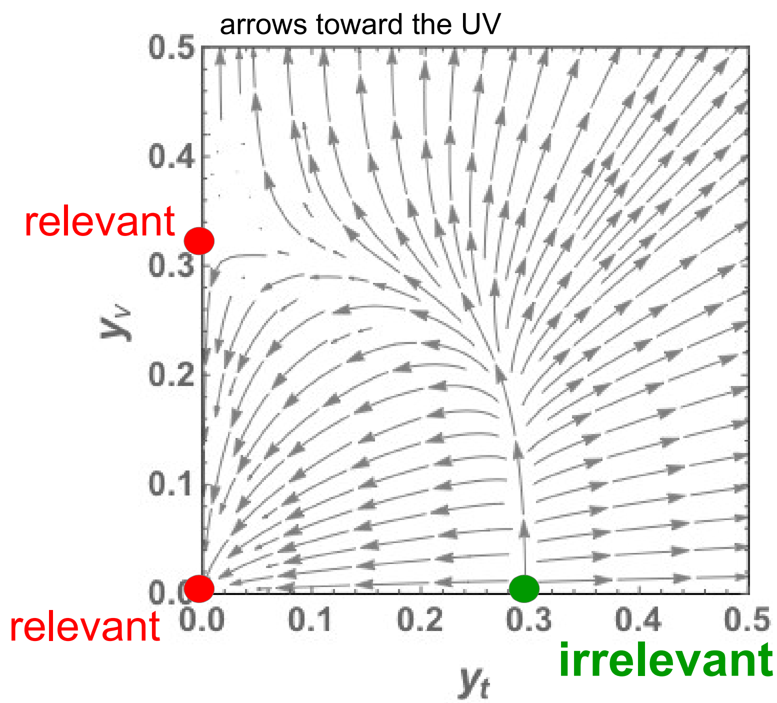
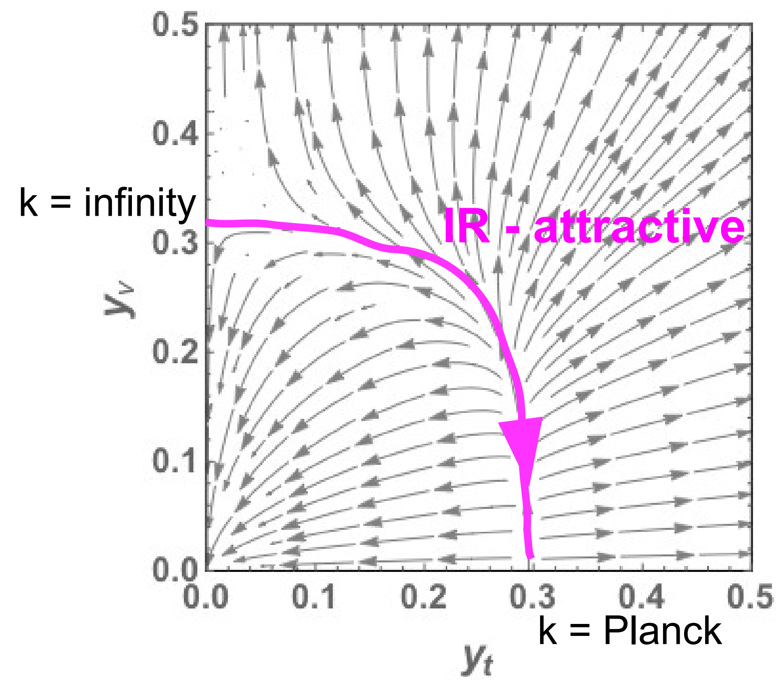




$y_\nu^* = 0$



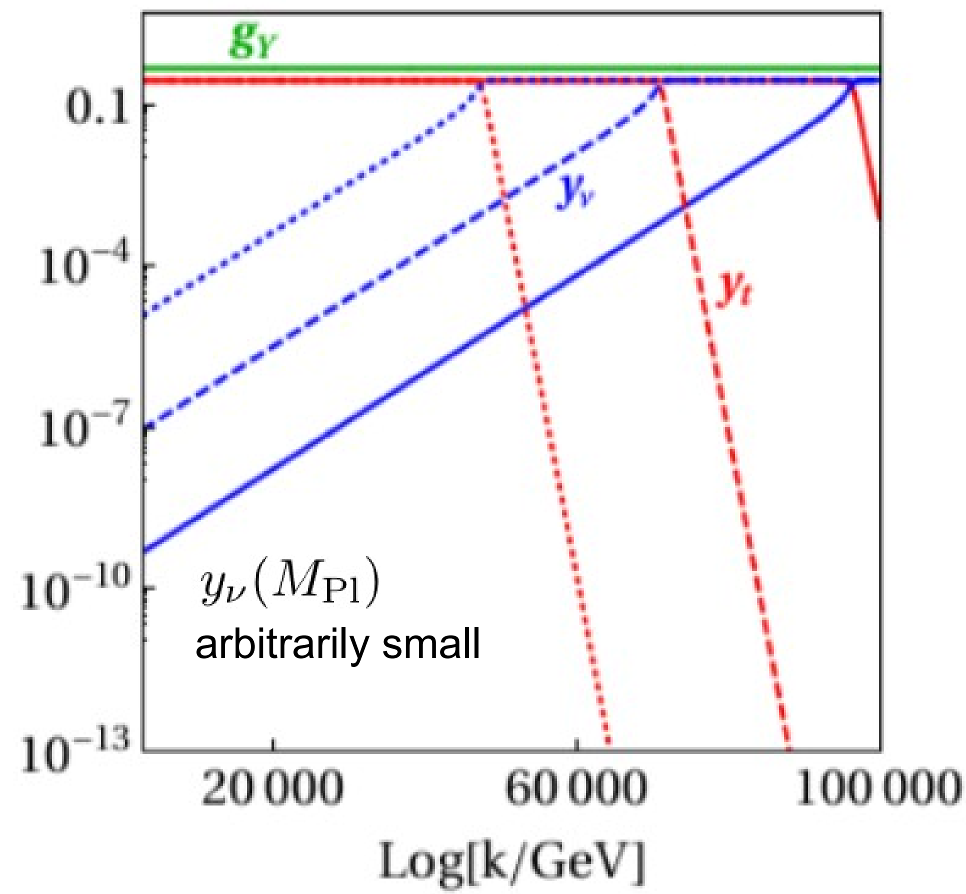
A dynamical mechanism!

… smallness of neutrino Yukawa due to "distance" of Planck scale from infinity
(no fine tuning)

$$y_Z(t,\kappa) = \left[\frac{16\pi^2 c_X \left(f_{Z,XY}^{\rm crit} - f_y \right)}{e^{2c_X \left(f_{Z,XY}^{\rm crit} - f_y \right)(16\pi^2\kappa - t)} + \alpha'_Z} \right]^{1/2}$$

κ = "distance" in e-foldings

K.Kowalska, S.Pramanick, EMS, 2204.00866



Neutrinos can naturally be Dirac

Couple of comments...

1. Asymp. safe SM full fit works (with normal ordering)

PMNS parametrization

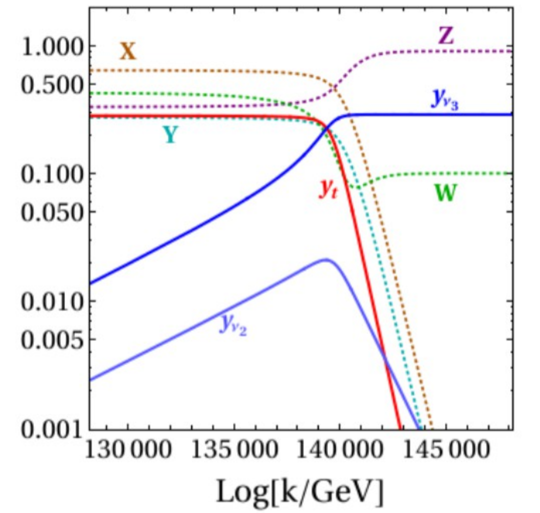
$$U_2 = |U_{\alpha i}|^2 = \begin{bmatrix} X & Y & 1-X-Y \\ Z & W & 1-Z-W \\ 1-X-Z & 1-Y-W & X+Y+Z+W-1 \end{bmatrix}$$

$$\theta_{12} = \arctan\sqrt{\frac{Y}{X}}$$
$$\theta_{13} = \arccos\sqrt{X+Y}$$
$$\theta_{23} = \arcsin\sqrt{\frac{1-W-Z}{X+Y}}$$

$$\delta = \arccos\frac{(X+Y)^2 Z - Y(X+Y+Z+W-1) - X(1-W-Z)(1-X-Y)}{2\sqrt{XY(1-X-Y)(1-Z-W)(X+Y+Z+W-1)}}$$

PMNS fit

$X \in [0.64 - 0.71]$ $\quad Y \in [0.26 - 0.34]$ $\quad Z \in [0.05 - 0.26]$ $\quad W \in [0.21 - 0.48]$



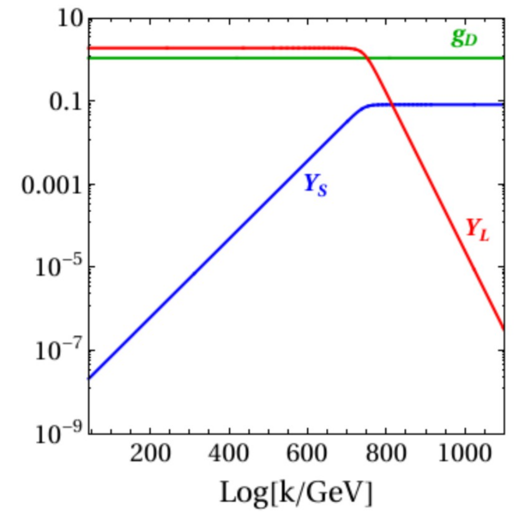
2. The mechanism is more generic than SM

e.g. dark gauge coupling g_D + Yukawa interactions

$$\mathcal{L} \supset Y_S\, \chi_R \Phi \chi_L + Y_L\, \psi_R \Phi \psi_L + \text{H.c.}$$

$Q_\psi \gg Q_\chi$ (dark abelian charge)

Can use it to justify freeze-in, feebly interacting models, etc...

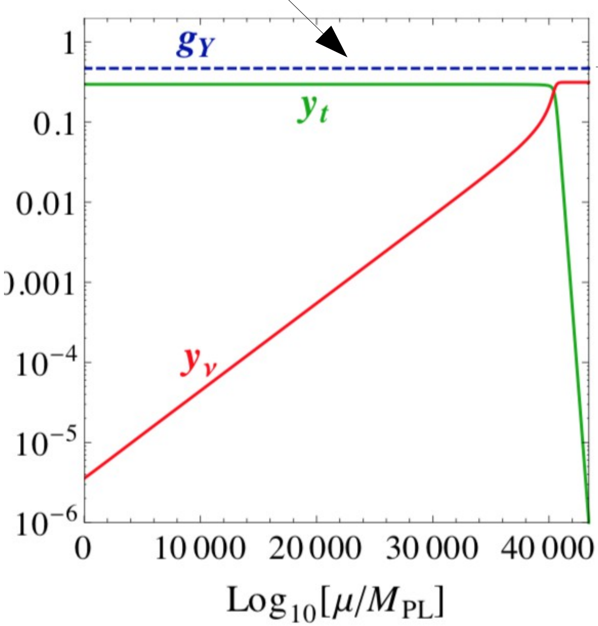


Connections to quantum gravity

In SM+QG low-scale pheno constrains gravity fixed points

neutrino crit. exponent < 0:

$$16\pi^2\theta_\nu \approx -\frac{2}{3}g_Y^{*2} + \frac{3}{2}y_t^{*2} < 0$$

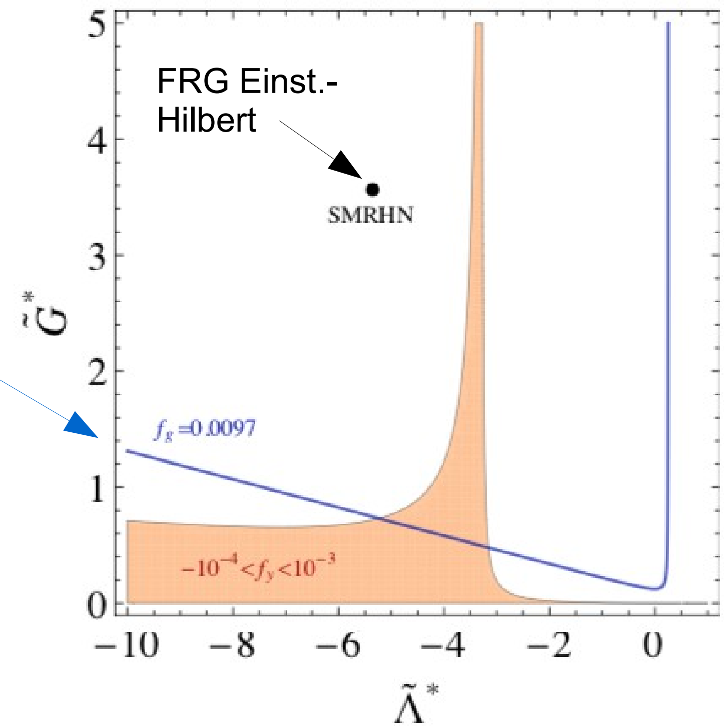


$f_g \approx 0.0097$

to match SM value ...

... It is a line in fixed points of gravity

A. Chikkaballi, K.Kowalska, EMS, 2308.06114



FRG calculation following
A. Eichhorn, F.Versteegen, 1709.07252

Quantum gravity calculation should eventually match the blue line

Connections to quantum gravity

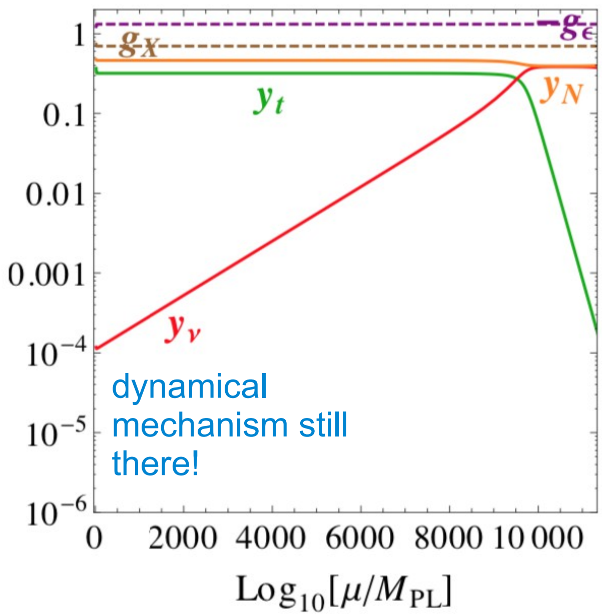
Gauged U(1)$_{B-L}$ vs SM:

extended gauge sector

$$\mathcal{L} \supset -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}X_{\mu\nu}X^{\mu\nu} - \frac{\epsilon}{2}B_{\mu\nu}X^{\mu\nu} + i\bar{f}\left(\partial^\mu - ig_Y Q_Y \tilde{B}^\mu - ig_{B-L}Q_{B-L}\tilde{X}^\mu\right)\gamma_\mu f$$

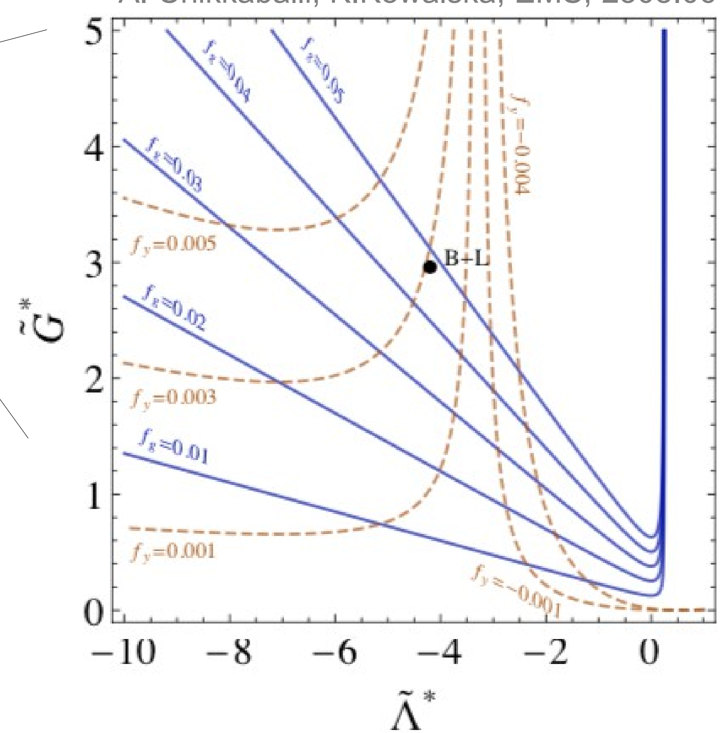
g_Y role played here by

$$g_X = \frac{g_{B-L}}{\sqrt{1-\epsilon^2}}, \quad g_\epsilon = -\frac{\epsilon\, g_Y}{\sqrt{1-\epsilon^2}}$$



$f_g =$ any

A. Chikkaballi, K.Kowalska, EMS, 2308.06114



Quantum gravity calculation provides predictions for *g_X*, *g_ϵ*

FRG calculation following
A. Eichhorn, F.Versteegen, 1709.07252

Predictions *B-L*

A. Chikkaballi, K.Kowalska, EMS, 2308.06114

f_g, f_y lead to *predictive* (irrel.) fixed points for g_X, g_ε, y_N :

(all BPs have $y_\nu^* = 0$ irrel.)

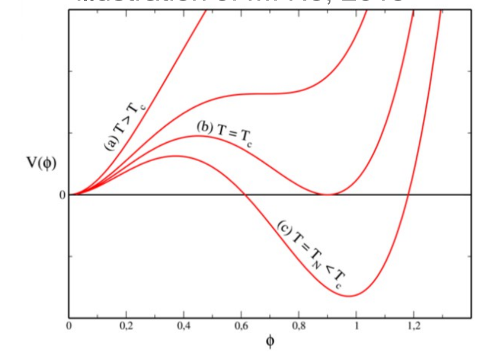
$$\mathcal{L}_M = -y_N^{ij} S\, \nu_{R,i}\, \nu_{R,j} + \text{H.c.}$$

	f_g	f_y	g_X^*	g_ϵ^*	y_N^*	g_X ($10^{5,7,9}$ GeV)	g_ϵ ($10^{5,7,9}$ GeV)	y_N ($10^{5,7,9}$ GeV)	
BP1	0.01	0.0005	0.10	-0.55	0.12	0.29, 0.29, 0.30	-0.26, -0.27, -0.28	0.16, 0.16, 0.16	**Majorana**
BP2	0.05	-0.005	0.70	-1.32	0.47	0.40, 0.41, 0.44	-0.52, -0.56, -0.61	0.42, 0.44, 0.45	**Majorana**
BP3	0.02	-0.0015	0.10	-0.75	0.0	0.12, 0.12, 0.12	-0.33, -0.35, -0.37	0.0	**Dirac**
BP4	0.03	-0.004	0.10	0.75	0.0	0.09, 0.09, 0.09	0.23, 0.25, 0.28	0.0	**Dirac**

(large kinetic mixing implies $v_S \gg v_H$)

Predictions *B-L*

illustration J. M. No, 2015



… possible gravitational-wave (GW) signatures from FOPT?

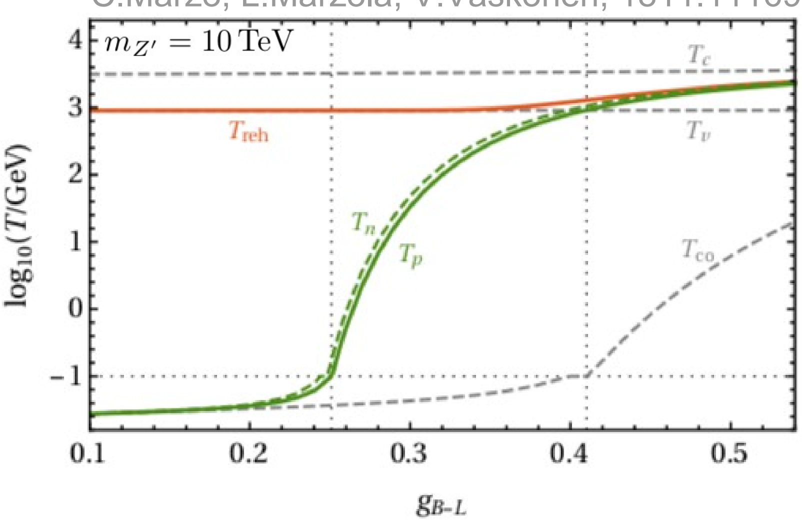
predictions have strong discriminating features... may show up in GW amplitude!

	g_X ($10^{5,7,9}$ GeV)	y_N ($10^{5,7,9}$ GeV)	
BP1	0.29, 0.29, 0.30	0.16, 0.16, 0.16	Majorana
BP2	0.40, 0.41, 0.44	0.42, 0.44, 0.45	Majorana
BP3	0.12, 0.12, 0.12	0.0	Dirac

...but, if C-W potential is "conformal" $V_{\rm CW} = \frac{1}{2} m_S^2 \phi^2 + \frac{1}{4}\lambda_S \phi^4 + \frac{1}{128\pi^2}\left(20\lambda_S^2 + 96 g_X^4 - 48 y_N^4\right)\phi^4 \left(-\frac{25}{6} + \ln\frac{\phi^2}{k^2}\right)$

NO GW SIGNAL! … nucleation/percolation *T* is too low

C.Marzo, L.Marzola, V.Vaskonen, 1811.11169



Scale-invariant potential confronts asymptotic safety...

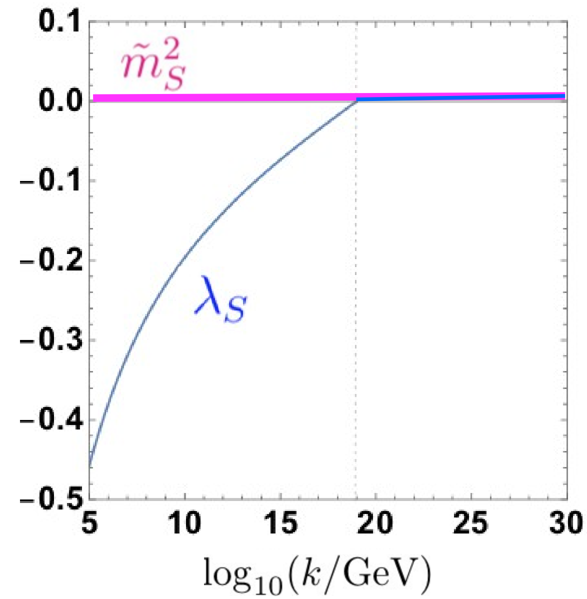
$$\frac{d\tilde{m}_S^2}{dt} \approx (-2 - f_\lambda)\,\tilde{m}_S^2$$

$$\frac{d\lambda_S}{dt} \approx -f_\lambda \lambda_S + \frac{6 g_X^{*4}}{\pi^2} + \ldots$$

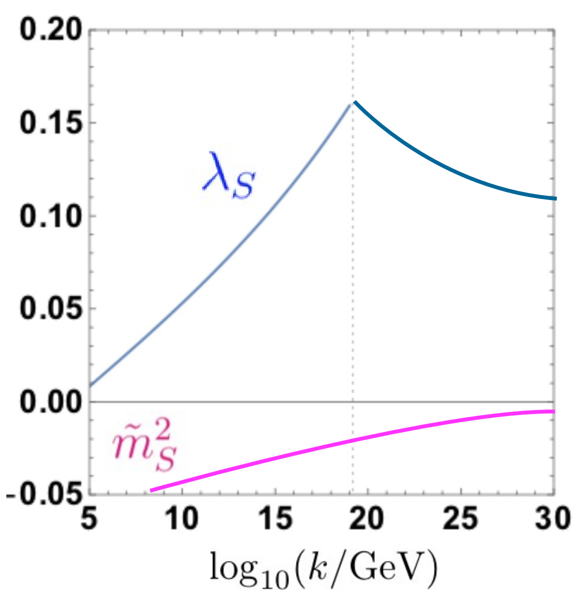
$\tilde{m}_S^{2*} = 0$ irrelevant

implies predictive $\lambda_S(t)$

… potential destabilized!



viceversa...



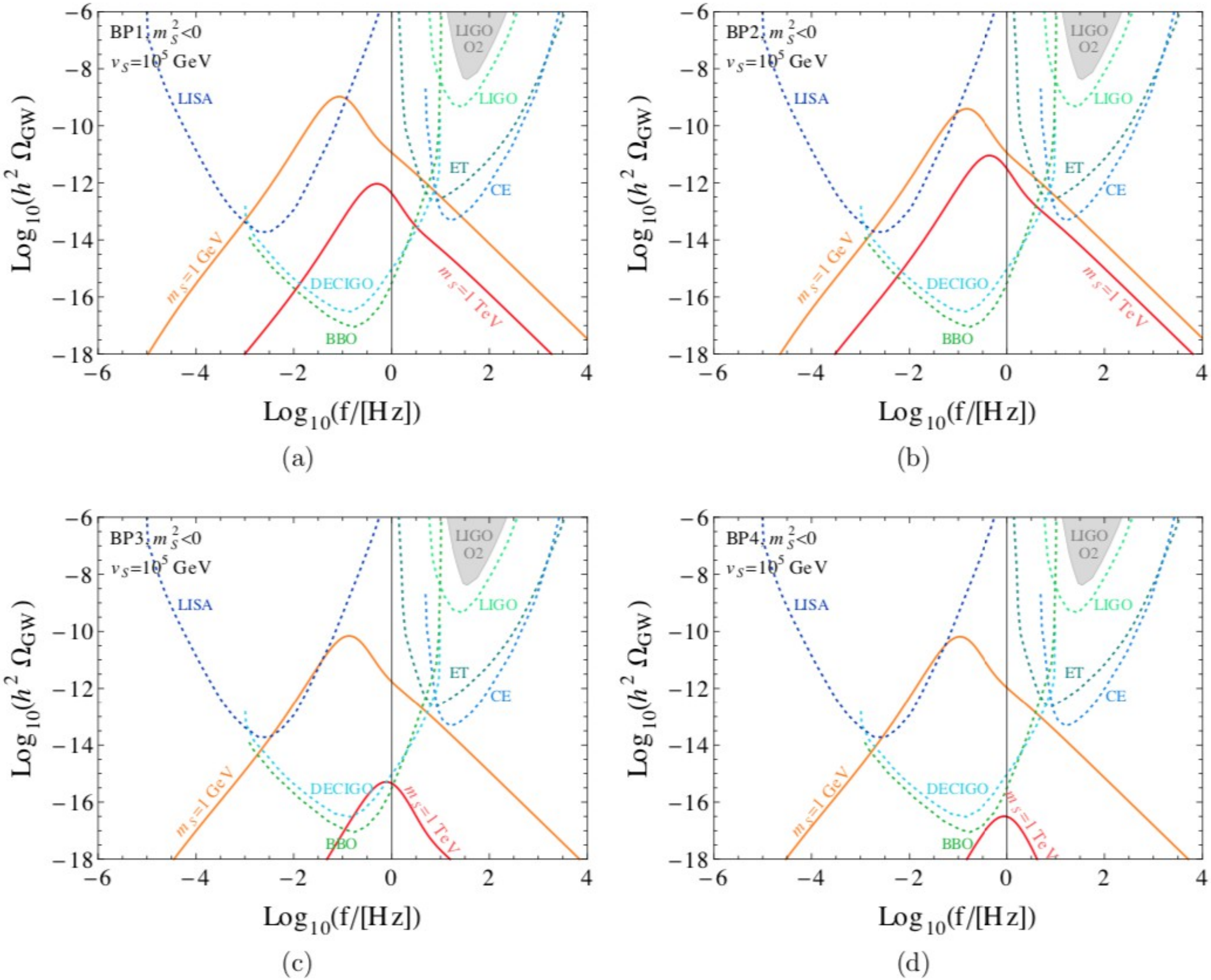
$\lambda_S(t)$ consistent with C-W

implies $\tilde{m}_S^{2*} = 0$ relevant

… tree-level mass is allowed

no conformal potential!

Signal is now visible... A. Chikkaballi, K.Kowalska, EMS, 2308.06114



… but all discriminating features washed-out by scalar potential masses

To take home...

- We used AS to make the neutrino (or other) Yukawa coupling arbitrarily small dynamically

- Mechanism relies on an *irrelevant*, *Gaussian* fixed point of the trans-Planckian RG flow of Yukawa coupling

- In the SM + QG the UV calculation is expected to be very constrained, but perhaps not so in gauged *B-L*

- AS extremely predictive in several BSM models ...
  … other times predictivity is washed out by IR / UV consistency (*e.g.* case gravitational waves from FOPTs in gauged *B-L)*

Backup

Lepton sector RGEs

$$\begin{aligned}\frac{dy_e}{dt} &= \frac{y_e}{16\pi^2}\left\{\frac{3}{2}y_e^2 - \frac{3}{2}\left[X y_{\nu1}^2 + Y y_{\nu2}^2 + (1-X-Y)y_{\nu3}^2\right] + y_e^2 + y_\mu^2 + y_\tau^2 + y_{\nu1}^2 + y_{\nu2}^2 + y_{\nu3}^2\right.\\ &\left. - \left(\frac{15}{4}g_Y^2 + \frac{9}{4}g_2^2\right) + 3\left(y_t^2 + y_b^2\right)\right\} - f_y\, y_e \qquad \text{(A.9)}\end{aligned}$$

$$\begin{aligned}\frac{dy_\mu}{dt} &= \frac{y_\mu}{16\pi^2}\left\{\frac{3}{2}y_\mu^2 - \frac{3}{2}\left[Z y_{\nu1}^2 + W y_{\nu2}^2 + (1-Z-W)y_{\nu3}^2\right] + y_e^2 + y_\mu^2 + y_\tau^2 + y_{\nu1}^2 + y_{\nu2}^2 + y_{\nu3}^2\right.\\ &\left. - \left(\frac{15}{4}g_Y^2 + \frac{9}{4}g_2^2\right) + 3\left(y_t^2 + y_b^2\right)\right\} - f_y\, y_\mu \qquad \text{(A.10)}\end{aligned}$$

$$\begin{aligned}\frac{dy_\tau}{dt} &= \frac{y_\tau}{16\pi^2}\left\{\frac{3}{2}y_\tau^2 - \frac{3}{2}\left[(1-X-Z)y_{\nu1}^2 + (1-Y-W)y_{\nu2}^2 + (X+Y+Z+W-1)y_{\nu3}^2\right]\right.\\ &\left. + y_e^2 + y_\mu^2 + y_\tau^2 + y_{\nu1}^2 + y_{\nu2}^2 + y_{\nu3}^2 - \left(\frac{15}{4}g_Y^2 + \frac{9}{4}g_2^2\right) + 3\left(y_t^2 + y_b^2\right)\right\} - f_y\, y_\tau \qquad \text{(A.11)}\end{aligned}$$

$$\begin{aligned}\frac{dy_{\nu1}}{dt} &= \frac{y_{\nu1}}{16\pi^2}\left\{\frac{3}{2}y_{\nu1}^2 - \frac{3}{2}\left[X y_e^2 + Z y_\mu^2 + (1-X-Z)y_\tau^2\right] + y_e^2 + y_\mu^2 + y_\tau^2 + y_{\nu1}^2 + y_{\nu2}^2 + y_{\nu3}^2\right.\\ &\left. - \left(\frac{3}{4}g_Y^2 + \frac{9}{4}g_2^2\right) + 3\left(y_t^2 + y_b^2\right)\right\} - f_y\, y_{\nu1} \qquad \text{(A.12)}\end{aligned}$$

$$\begin{aligned}\frac{dy_{\nu2}}{dt} &= \frac{y_{\nu2}}{16\pi^2}\left\{\frac{3}{2}y_{\nu2}^2 - \frac{3}{2}\left[Y y_e^2 + W y_\mu^2 + (1-Y-W)y_\tau^2\right] + y_e^2 + y_\mu^2 + y_\tau^2 + y_{\nu1}^2 + y_{\nu2}^2 + y_{\nu3}^2\right.\\ &\left. - \left(\frac{3}{4}g_Y^2 + \frac{9}{4}g_2^2\right) + 3\left(y_t^2 + y_b^2\right)\right\} - f_y\, y_{\nu2} \qquad \text{(A.13)}\end{aligned}$$

$$\begin{aligned}\frac{dy_{\nu3}}{dt} &= \frac{y_{\nu3}}{16\pi^2}\left\{\frac{3}{2}y_{\nu3}^2 - \frac{3}{2}\left[(1-X-Y)y_e^2 + (1-Z-W)y_\mu^2 + (X+Y+Z+W-1)y_\tau^2\right]\right.\\ &\left. + y_e^2 + y_\mu^2 + y_\tau^2 + y_{\nu1}^2 + y_{\nu2}^2 + y_{\nu3}^2 - \left(\frac{3}{4}g_Y^2 + \frac{9}{4}g_2^2\right) + 3\left(y_t^2 + y_b^2\right)\right\} - f_y\, y_{\nu3} \qquad \text{(A.14)}\end{aligned}$$

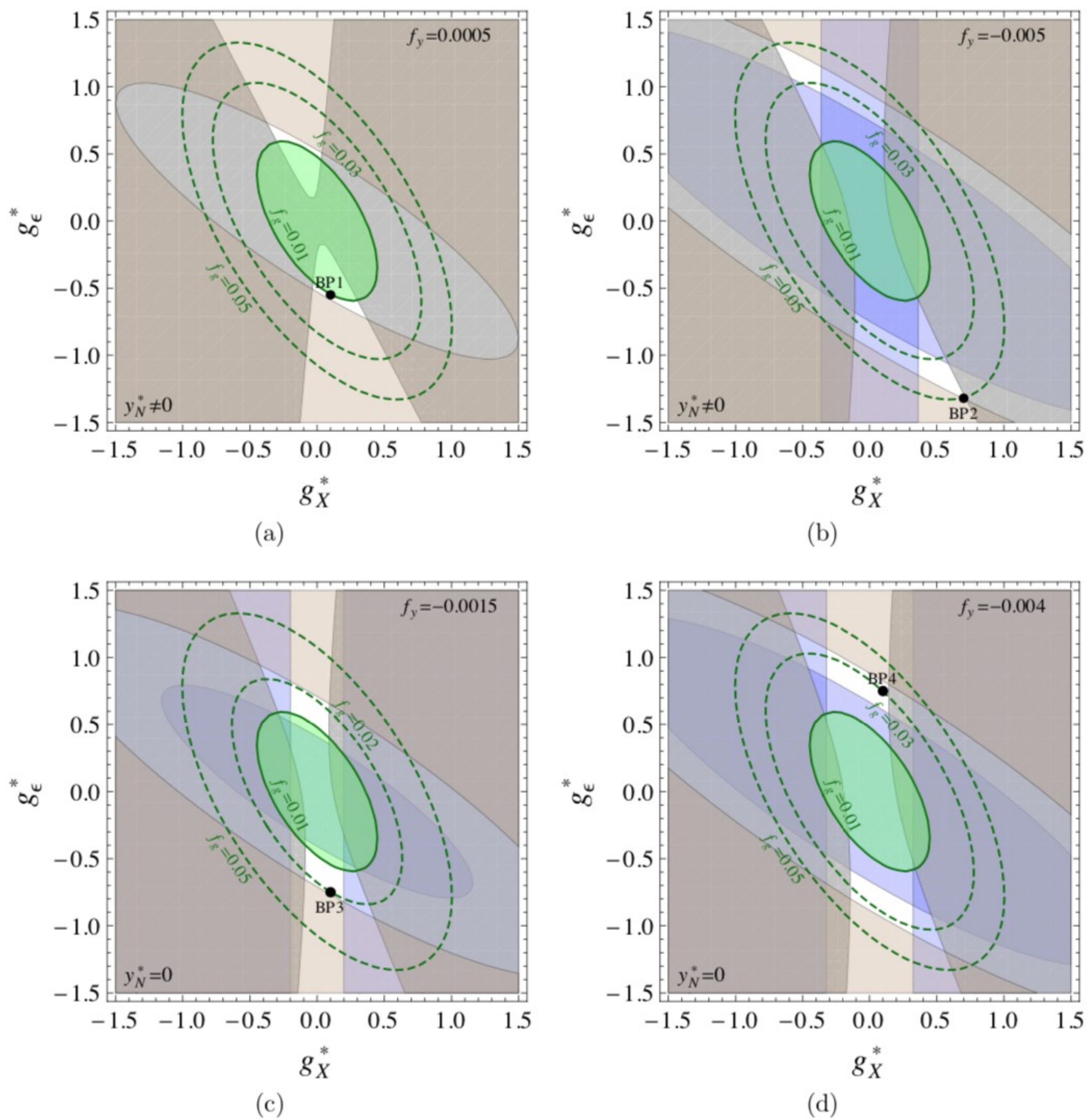
$$\begin{aligned}\frac{dX}{dt} &= -\frac{3}{(4\pi)^2}\left[\left(\frac{y_e^2+y_\mu^2}{y_e^2-y_\mu^2}\right)\left\{(y_{\nu1}^2-y_{\nu3}^2)XZ + \frac{(y_{\nu3}^2-y_{\nu2}^2)}{2}[W(1-X)+X-(1-Y)(1-Z)]\right\}\right.\\ &+ \left(\frac{y_e^2+y_\tau^2}{y_e^2-y_\tau^2}\right)\left\{(y_{\nu1}^2-y_{\nu3}^2)X(1-X-Z) + \frac{(y_{\nu3}^2-y_{\nu2}^2)}{2}[(1-Y)(1-Z)-X(1-2Y)-W(1-X)]\right\}\\ &+ \left(\frac{y_{\nu1}^2+y_{\nu2}^2}{y_{\nu1}^2-y_{\nu2}^2}\right)\left\{(y_e^2-y_\tau^2)XY + \frac{(y_\tau^2-y_\mu^2)}{2}[W(1-X)+X-(1-Y)(1-Z)]\right\}\\ &\left.+ \left(\frac{y_{\nu1}^2+y_{\nu3}^2}{y_{\nu1}^2-y_{\nu3}^2}\right)\left\{(y_e^2-y_\tau^2)X(1-X-Y) + \frac{(y_\tau^2-y_\mu^2)}{2}[(1-Y)(1-Z)-X(1-2Z)-W(1-X)]\right\}\right] \qquad \text{(A.15)}\end{aligned}$$

$$\begin{aligned}\frac{dY}{dt} &= -\frac{3}{(4\pi)^2}\left[\left(\frac{y_e^2+y_\mu^2}{y_e^2-y_\mu^2}\right)\left\{\frac{(y_{\nu3}^2-y_{\nu1}^2)}{2}[W(1-X)+X-(1-Y)(1-Z)] + (y_{\nu2}^2-y_{\nu3}^2)YW\right\}\right.\\ &+ \left(\frac{y_e^2+y_\tau^2}{y_e^2-y_\tau^2}\right)\left\{\frac{(y_{\nu3}^2-y_{\nu1}^2)}{2}[(1-Y)(1-Z)-W(1-X)-X(1-2Y)] + (y_{\nu2}^2-y_{\nu3}^2)Y(1-Y-W)\right\}\\ &+ \left(\frac{y_{\nu2}^2+y_{\nu1}^2}{y_{\nu2}^2-y_{\nu1}^2}\right)\left\{(y_e^2-y_\tau^2)XY + \frac{(y_\tau^2-y_\mu^2)}{2}[W(1-X)+X-(1-Y)(1-Z)]\right\}\\ &\left.+ \left(\frac{y_{\nu2}^2+y_{\nu3}^2}{y_{\nu2}^2-y_{\nu3}^2}\right)\left\{(y_e^2-y_\tau^2)Y(1-X-Y) + \frac{(y_\mu^2-y_\tau^2)}{2}[W(1-X-2Y)+X-(1-Z)(1-Y)]\right\}\right] \qquad \text{(A.16)}\end{aligned}$$

$$\begin{aligned}\frac{dZ}{dt} &= -\frac{3}{(4\pi)^2}\left[\left(\frac{y_\mu^2+y_e^2}{y_\mu^2-y_e^2}\right)\left\{(y_{\nu1}^2-y_{\nu3}^2)XZ + \frac{(y_{\nu3}^2-y_{\nu2}^2)}{2}[W(1-X)+X-(1-Y)(1-Z)]\right\}\right.\\ &+ \left(\frac{y_\mu^2+y_\tau^2}{y_\mu^2-y_\tau^2}\right)\left\{(y_{\nu1}^2-y_{\nu3}^2)Z(1-X-Z) + \frac{(y_{\nu2}^2-y_{\nu3}^2)}{2}[W(1-X-2Z)+X-(1-Y)(1-Z)]\right\}\\ &+ \left(\frac{y_{\nu1}^2+y_{\nu2}^2}{y_{\nu1}^2-y_{\nu2}^2}\right)\left\{\frac{(y_e^2-y_\tau^2)}{2}[(1-Y)(1-Z)-X-W(1-X)] + (y_\mu^2-y_\tau^2)ZW\right\}\\ &\left.+ \left(\frac{y_{\nu1}^2+y_{\nu3}^2}{y_{\nu1}^2-y_{\nu3}^2}\right)\left\{\frac{(y_\tau^2-y_e^2)}{2}[(1-Z)(1-Y)-W(1-X)-X(1-2Z)] + (y_\mu^2-y_\tau^2)Z(1-Z-W)\right\}\right] \qquad \text{(A.17)}\end{aligned}$$

$$\begin{aligned}\frac{dW}{dt} &= -\frac{3}{(4\pi)^2}\left[\left(\frac{y_\mu^2+y_e^2}{y_\mu^2-y_e^2}\right)\left\{(y_{\nu2}^2-y_{\nu3}^2)WY + \frac{(y_{\nu3}^2-y_{\nu1}^2)}{2}[W(1-X)+X-(1-Y)(1-Z)]\right\}\right.\\ &+ \left(\frac{y_\mu^2+y_\tau^2}{y_\mu^2-y_\tau^2}\right)\left\{(y_{\nu2}^2-y_{\nu3}^2)W(1-Y-W) + \frac{(y_{\nu3}^2-y_{\nu1}^2)}{2}[(1-Y)(1-Z)-X-W(1-X-2Z)]\right\}\\ &+ \left(\frac{y_{\nu2}^2+y_{\nu1}^2}{y_{\nu2}^2-y_{\nu1}^2}\right)\left\{(y_\mu^2-y_\tau^2)WZ + \frac{(y_\tau^2-y_e^2)}{2}[(1-X)W+X-(1-Y)(1-Z)]\right\}\\ &\left.+ \left(\frac{y_{\nu2}^2+y_{\nu3}^2}{y_{\nu2}^2-y_{\nu3}^2}\right)\left\{(y_\mu^2-y_\tau^2)W(1-Z-W) + \frac{(y_\tau^2-y_e^2)}{2}[(1-Y)(1-Z)-X-W(1-X-2Y)]\right\}\right]. \qquad \text{(A.18)}\end{aligned}$$

Benchmark points of *B-L*



GWs at different scales