Trace anomaly and induced action for metric-scalar backgrounds

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CENTRO DE CIENCIAS DE BENASQUE PEDRO PASCUAL

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Particle Physics XXI:

Science of measuring 0 with increasing levels of precision

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- The quantum effective action of scalar fields have relevant implication on the stability of th SM
- Efective action of gravity can have cosmological implications

Conformal anomaly in $SU(3)$ theory

$$\mathcal{L} = -\frac{1}{4e^2} G^a_{\mu\nu} G^{a\mu\nu} + i \sum_{k=1}^{N} \bar{\Psi}^a_k \left(\gamma^\mu \mathcal{D}^{ab}_\mu - h \varepsilon^{acb} \Phi^c \right) \Psi^b_k$$

$$+ \frac{1}{2} (\mathcal{D}^\mu \Phi)^a (\mathcal{D}_\nu \Phi)^a + \frac{1}{6} R \Phi^a \Phi^a - \frac{1}{4!} \lambda (\Phi^a \Phi^a)^2 + \tau \Box \Phi^a \Phi^a$$

+ N fermions +1 scalar field in the adjoint representation

The action

$$S(\Phi, \Psi, g) = \int d^4x \sqrt{-g}\, \mathcal{L} = \int_{\mathcal{M}} \mathcal{L}$$

is invariant under Weyl transformations

$$g_{\mu\nu} = e^{2\sigma} \bar{g}_{\mu\nu}, \;\; \Phi = e^{-\sigma} \bar{\Phi}, \;\; \Psi = e^{-\frac{3}{2}\sigma} \Psi, \;\; \bar{\Psi}^* = e^{-\frac{3}{2}\sigma} \bar{\Psi}^*, \;\; A = \bar{A},$$

Conformal anomaly in a $SU(3)$ model

$$\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{n-4}\int d^n x\,\sqrt{-g}\,\Big\{ wC^2 + bE_4 + c\,\Box\,R$$

$$+\,\gamma_\Phi\left[(D\Phi)^2 + \frac{1}{6}R\Phi^2\right] - \frac{1}{4!}\widetilde{\beta}_\lambda \Phi^4 \;+ \beta_\tau \Box\,\Phi^2\Big\},$$

where

$$C^2(4) \;=\; R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 2R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{3}R^2$$

is the square of the Weyl tensor and

$$E_4 \;=\; R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2$$

is the Gauss-Bonnet density topological term.

Conformal anomaly in $SU(3)$ model

$$w = \frac{1}{(4\pi)^2}\left(\frac{1}{15} + \frac{2N}{5} + \frac{4}{5}\right)$$

$$b = -\frac{1}{(4\pi)^2}\left(\frac{1}{45} + \frac{11\,N}{45} + \frac{62}{45}\right),$$

$$c = \frac{1}{(4\pi)^2}\left(\frac{2}{45} + \frac{4N}{15} - \frac{4}{5}\right).$$

$$\gamma_\Phi = -\frac{8}{(4\pi)^2}(h^2 - e^2),$$

$$\widetilde{\beta}_\lambda = \beta_\lambda + 4\lambda\gamma_\Phi = \frac{1}{(4\pi)^2}\left(\frac{11}{3}\lambda^2 - 8\lambda e^2 + 72e^4 - 96h^4\right),$$

$$\beta_\tau = \frac{1}{(4\pi)^2}\left(\frac{5}{36}\lambda^2 + \frac{11}{3}e^2 - 4h^2\right).$$

Conformal anomaly with scalar fields

Renormalization requires to add a vacuum term.

$$S_{cv} = \int d^4x \sqrt{-g}\, \left\{ a_1 C^2 + a_2 E_4 + a_3 \Box R \right\}$$ [M. Duff]

Renormalization

$$\Gamma^{(1)}_{\mathrm{ren}} \;=\; S + S_{cv} + \bar{\Gamma}^{(1)}_{div} + \bar{\Gamma}^{(1)}_{fin} + \Delta S^{(1)},$$

The surface terms $\Box\, \Phi^2$, E_4, and $\Box\, R$ are not Weyl invariant, but Noether identity holds.

Noether identities

$$-\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta S(g_{\mu\nu}, \Phi, \Psi, A)}{\delta g_{\mu\nu}} + \frac{1}{\sqrt{-g}}\, \Phi\, \frac{\delta S(g_{\mu\nu}, \Phi, \Psi, A)}{\delta \Phi}$$

$$+\frac{3}{2}\frac{1}{\sqrt{-g}}\, \Psi\, \frac{\delta S(g_{\mu\nu}, \Phi, \Psi, A)}{\delta \Psi} = 0.$$

$$-\frac{1}{\sqrt{-\bar{g}}}\, e^{-4\sigma}\, \frac{\delta \bar{\Gamma}^{(1)}_{\rm div}}{\delta \sigma}\bigg| = -\frac{1}{\sqrt{-\bar{g}}}\, e^{-4\sigma}\, \frac{\delta \Delta S^{(1)}}{\delta \sigma}\bigg| = 0$$

|

Conformal anomaly

$$\langle \mathcal{T} \rangle = -\frac{1}{\sqrt{-\bar{g}}}\, e^{-4\sigma}\, \frac{\delta \bar{\Gamma}^{(1)}_{\mathrm{ren}}}{\delta \sigma}\bigg| = -\frac{1}{\sqrt{-\bar{g}}}\, e^{-4\sigma}\, \frac{\delta \bar{\Gamma}^{(1)}_{\mathrm{fin}}}{\delta \sigma}\bigg|$$

$$= -wC^2 - bE_4 - c\,\Box\, R + \frac{1}{4!}\widetilde{\beta}_\lambda \Phi^4 - \beta_\tau \Box\, \Phi^2$$

$$-\gamma_\Phi \left[(\nabla \Phi)^2 + \frac{1}{6} R \Phi^2 \right].$$

Scalar Structures of Conformal Anomaly

i) Real conformal terms

$$X_c = (\nabla \Phi)^2 + \frac{1}{6} R \Phi^2.$$

$$Y(g_{\mu\nu}, \Phi) = wC^2 + \gamma_\Phi X_c - \frac{1}{4!} \widetilde{\beta}_\lambda \Phi^4.$$

ii) Topological term

$$E_4$$

iii) Total derivative terms

$$\Box R \qquad \Box \Phi^2$$

Effective Induced Action

$$-\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta\Gamma_{ind}}{\delta g_{\mu\nu}} + \frac{1}{\sqrt{-g}}\, \Phi\, \frac{\delta\Gamma_{ind}}{\delta\Phi} = -\frac{1}{\sqrt{-\bar{g}}}\, e^{-4\sigma}\, \frac{\delta\Gamma_{ind}}{\delta\sigma}\bigg| = \langle\mathcal{T}\rangle,$$

i)Total derivative terms

$$-\frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \sqrt{-g}\, R^2 = 12\Box R$$

$$\left(-\frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} + \frac{1}{\sqrt{-g}} \Phi \frac{\delta}{\delta\Phi}\right) \int d^4x \sqrt{-g}\, R\Phi^2 = 6\Box\Phi^2.$$

Paneitz operator

$$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}(\nabla^\mu R)\nabla_\mu,$$

Effective Induced Action

$$g_{\mu\nu} = e^{2\sigma}\,\bar{g}_{\mu\nu}, \qquad \Phi = e^{-\sigma}\,\bar{\Phi},$$

$$\sqrt{-g}\left(E_4 - \frac{2}{3}\,\Box R\right) \;=\; \sqrt{-\bar{g}}\left(\bar{E}_4 - \frac{2}{3}\,\bar{\Box}\bar{R} + 4\bar{\Delta}_4\sigma\right),$$

Fundamental relation

$$\frac{\delta}{\delta\sigma}\int_{\mathcal{M}} \mathcal{F}[g_{\mu\nu},\,\Phi]\left(E_4 - \frac{2}{3}\Box R\right)\Bigg| = 4\sqrt{-g}\,\Delta_4\mathcal{F}[g_{\mu\nu},\,\Phi],$$

where

$$\mathcal{F}[g_{\mu\nu},\,\Phi] = \mathcal{F}[\bar{g}_{\mu\nu},\,\bar{\Phi}]$$

Effective Induced Action

$$\Gamma_{ind} = S_c[\bar{g}_{\mu\nu}, \bar{\Phi}] - \int_{\mathcal{M}} \left\{ \frac{2b+3c}{36} R^2 + \frac{\beta_\tau}{6} R \Phi^2 \right\} + \int_{\mathcal{M}} \left\{ \sigma Y\left(\bar{g}_{\mu\nu}, \bar{\Phi}\right) + b\sigma\left(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R}\right) + 2b\sigma\bar{\Delta}_4\sigma \right\},$$

or the non-local solution

$$\Gamma_{ind} = S_c + \frac{b}{8}\int_{\mathcal{M}} \left(E_4 - \frac{2}{3}\Box R\right) \Delta_4^{-1} \left(E_4 - \frac{2}{3}\Box R\right) + \frac{1}{4}\int_{\mathcal{M}} Y \Delta_4^{-1}\left(E_4 - \frac{2}{3}\Box R\right) - \int_{\mathcal{M}} \left(\frac{2b+3c}{36} R^2 + \frac{\beta_\tau}{6} R \Phi^2\right).$$

[Riegert]

Effective Induced Action

Local expression in terms of two extra auxiliary scalar fields φ and ψ

$$\Gamma_{ind} = S_c[g_{\mu\nu}, \Phi] - \int_{\mathcal{M}} \left\{ \frac{2b+3c}{36} R^2 + \frac{\beta_\tau}{6} R \Phi^2 \right\}$$
$$+ \int_{\mathcal{M}} \left\{ \frac{1}{2} \varphi \Delta_4 \varphi - \frac{1}{2} \psi \Delta_4 \psi + \frac{\sqrt{-b}}{2} \varphi \left(E_4 - \frac{2}{3} \Box R + \frac{1}{b} Y \right) \right.$$

Universality and Ambiguities

i) Real Conformal terms	UNIVERSAL
ii) Topological terms	UNIVERSAL
iii) Total derivative terms	NON-UNIVERSAL

where the ambiguities associated to total derivative terms come from?

Pauli-Villars Ambiguities

Pauli-Villars regulators (scalar)

$$S^{(i)}_{reg} = \int_{\mathcal{M}} \Big\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi_i \partial_\nu \varphi_i + \frac{\xi_i}{2} R \varphi_i^2 - \frac{m_i^2}{2} \varphi_i^2 - \frac{\varkappa}{2} \Phi^2 \varphi_i^2 \Big\},$$

$$\begin{aligned} \bar{\Gamma}^{(1)}_{scal} = \frac{1}{2(4\pi)^2} \int_{\mathcal{M}} \Big\{ & m_i^4 \Big(\frac{1}{2\varepsilon} + \frac{3}{4} \Big) + \tilde{\xi}_i \, m_i^2 R \Big(\frac{1}{\varepsilon} + 1 \Big) \\ & + C_{\mu\nu\alpha\beta} \Big[\frac{1}{120\varepsilon} + \frac{1}{2} k_W(\tau_i) \Big] C^{\mu\nu\alpha\beta} + R \Big[\frac{1}{2\varepsilon} \tilde{\xi}_i^2 + k_R(\tau_i) \Big] R \\ & - \frac{\varkappa}{2\varepsilon} m_i^2 \Phi^2 + \Phi^2 \Big[\frac{\varkappa^2}{8\varepsilon} + k_\varkappa(\tau_i) \Big] \Phi^2 + \Phi^2 \Big[- \frac{\varkappa}{2\varepsilon} \tilde{\xi}_i + k_\xi(\tau_i) \Big] R \Big\} \end{aligned}$$

Pauli-Villars Ambiguities

where $\tau_i = \Box / m_i^2$ and we use the compact notations

$$\tilde{\xi}_i = \left(\xi_i - \frac{1}{6}\right), \qquad \frac{1}{\varepsilon} \equiv \frac{2}{4-n} + \ln\left(\frac{4\pi\mu^2}{m_i^2}\right) - \gamma$$

In the conformal limit, $m_i \to 0$ and $\tilde{\xi}_i \to 0$, the finite part boils down to

$$\bar{\Gamma}^{(1)}_{UV} = -\frac{1}{2(4\pi)^2} \int_{\mathcal{M}} \left\{ \frac{1}{120} C_{\mu\nu\alpha\beta} \ln\left(\frac{\Box}{4\pi\mu^2}\right) C^{\mu\nu\alpha\beta} + \frac{\varkappa^2}{8} \Phi^2 \ln\left(\frac{\Box}{4\pi\mu^2}\right) \Phi^2 + \frac{1}{1080} R^2 + \frac{\varkappa}{36} \Phi^2 R \right\}$$

Pauli-Villars Ambiguities

Then the Pauli-Villars regularized effective action can be defined as

$$\bar{\Gamma}^{(1)}_{\mathrm{reg}} = \sum_{i=0}^{N} s_i \bar{\Gamma}^{(1)}_i(m_i,\, \widetilde{\xi}_i,\, n).$$

Cancelation of divergences conditions ($\mu_i = m_i/M$)

$$\sum_{i=1}^{N} s_i = -s_0 = -1; \sum_{i=1}^{N} s_i \mu_i^2 = 0, \qquad \sum_{i=1}^{N} s_i \widetilde{\xi}_i = 0;$$

$$\sum_{i=1}^{N} s_i \mu_i^4 = 0, \qquad \sum_{i=1}^{N} s_i \widetilde{\xi}_i^2 = 0.$$

Pauli-Villars Ambiguities

A possible solution to these conditions corresponds to $N = 5$ and

$$s_1 = 1, \qquad s_2 = 4, \qquad s_3 = s_4 = s_5 = -2;$$
$$\mu_1^2 = \mu_5^2 = 4, \qquad \mu_2^2 = \mu_4^2 = 3, \qquad \mu_3^2 = 1;$$
$$\widetilde{\xi}_i = \mu_i^2.$$

$$\langle \mathcal{T} \rangle = \frac{\beta_\lambda}{4!}\Phi^4 - \gamma_\Phi X_c - wC^2 - bE_4 - (c - 6\delta)\Box R - (\beta_\tau + 3\rho)\Box \Phi^2$$

where we define

$$\rho = \frac{1}{2(4\pi)^2}\sum_{i=1}^{N} s_i\,\widetilde{\xi}_i \ln \mu_i^2; \qquad \delta = \frac{1}{2(4\pi)^2}\sum_{i=1}^{N} s_i\,\widetilde{\xi}_i^2 \ln \mu_i^2.$$

Effective Induced Action

$$\Gamma_{ind} = S_c + \frac{b}{8}\int_{\mathcal{M}} \left(E_4 - \frac{2}{3}\Box R\right) \Delta_4^{-1} \left(E_4 - \frac{2}{3}\Box R\right) + \frac{1}{4}\int_{\mathcal{M}} Y\, \Delta_4^{-1} \left(E_4 - \frac{2}{3}\Box R\right) - \int_{\mathcal{M}} \left(\frac{2b + 3(c - 6\delta)}{36} R^2 + \frac{\beta_\tau + 3\rho}{6} R\Phi^2\right).$$

The ambiguities in the R^2 and $R\Phi^2$ terms allows a very large R^2 terms which will drive cosmological inflation according to Starobinsky scenario.

Effective action in the infrared

Assumptions (10^{10}Gev < E < 10^{19} Gev):

i) All matter field are massless and $\xi \approx \frac{1}{6}$

ii) Scalar terms Φ^4 and X_c are dominating over the curvature terms

$$\left|\Phi^2\right| \gg \left|R_{....}\right| \qquad \text{and} \qquad \left|(\nabla\Phi)^2\right| \gg \left|R^2_{....}\right|$$

iii) Gravitational field is weak $|\Box R| \gg |R^2_{....}|$ for all curvature contractions.

Effective action in the infrared

$$\Delta_4^{-1} = \left(\Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}R^{;\mu}\nabla_\mu\right)^{-1} \approx \Box^{-2}$$

The leading terms in the expression E_4 are those with Φ and $\Box R$,

$$E_4 - \frac{2}{3}\Box R + \frac{1}{b}Y \approx -\frac{2}{3}\Box R + \frac{1}{b}\left(\gamma_\Phi X_c - \frac{1}{4!}\widetilde{\beta}_\lambda \Phi^4\right)$$

$$\Gamma_{ind,nl} \approx \frac{3(c-6\delta)}{36}\int_{\mathcal{M}} R^2 + \frac{\widetilde{\beta}_\tau}{6}\int_{\mathcal{M}} R\Phi^2 + \frac{1}{6}\int_{\mathcal{M}}\left(\frac{1}{4!}\widetilde{\beta}_\lambda\Phi^4 - \gamma_\Phi X_c\right)\frac{1}{\Box^2}\Box R$$

Effective action in the infrared

$$\Gamma_2 \;\approx\; \frac{1}{6}\int_{\mathcal{M}} \left(\frac{1}{4!}\widetilde{\beta}_\lambda \Phi^4 - \gamma_\Phi X_c\right) \frac{1}{\Box}\, R\,.$$

[E. Mottola and R. Vaulin]

$$\frac{1}{6}\int_{\mathcal{M}} \left(\frac{1}{4!}\widetilde{\beta}_\lambda \Phi^4 \;-\; \frac{1}{6}\,\gamma_\Phi R \Phi^2\right)\frac{1}{\Box}R.$$

Using scaling relations

$$\frac{1}{\Box} \;=\; e^{2\sigma}\frac{1}{\bar{\Box}} \qquad R \;=\; e^{-2\sigma}\left[\bar{R} - 6\bar{\Box}\sigma + O(\sigma^2)\right]$$

$$\Gamma_2 \;=\; \int_{\mathcal{M}} \left(\gamma_\Phi \bar{X}_c \;-\; \frac{1}{4!}\widetilde{\beta}_\lambda \bar{\Phi}^4\right)\sigma$$

Effective action in the infrared

Scalar field effective action

$$X_c \longrightarrow \bar{X}_c(1+\gamma_\Phi\sigma), \qquad \lambda\Phi^4 \longrightarrow \bar{\Phi}^4(\lambda+\widetilde{\beta}_\lambda\sigma)$$

as it should be under the renormalization group - based improvement.

Effective potential

$$\bar{\Phi} \longrightarrow \Phi, \qquad \sigma \longrightarrow \ln\frac{\Phi}{\mu},$$

$$V^{(1)}_{eff} = \frac{1}{4!}\Big(\lambda + \frac{1}{2}\widetilde{\beta}_\lambda \ln\frac{\Phi^2}{\mu^2}\Big)\Phi^4 - \frac{1}{12}\Big(1 + \gamma_\Phi \ln\frac{\Phi^2}{\mu^2}\Big)R\Phi^2,$$

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Coleman-Weinberg

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- The ambiguous terms correspond to pure derivative terms in the anomaly
- Room for stable Starobinsky inflation
- At low energies the effective potential can be fully recovered
- Beyond one loop the situation is less clear