unimodular gravity

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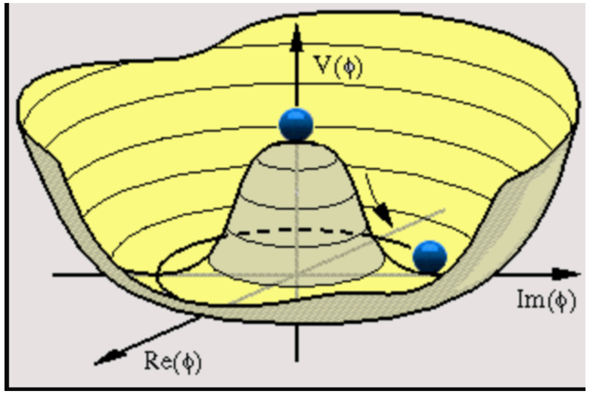
• e.a. and eduardo velasco-aja, 2301.07641 cosmological constant

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa^2 T_{\mu\nu}$$

$$T^{vac}_{\mu\nu} = \rho g_{\mu\nu}$$



SSB

$$V(\phi) = -\frac{1}{4!}\left(|\phi|^2 - v^2)\right)^2$$

Estimates of vacuum energy \sim v^4 (at least 40 orders of magnitude higher than observed)

Can we consistently achieve that the constant vacuum energy density does not weigh at all?

Yes.
UG does it

Einstein (1919); Pauli (1921)

$$\lambda \sim 10^{-122}\, m_p^2$$

$$\Omega_\lambda \equiv \frac{\lambda}{3H_0^2} \sim 0.7$$

Nothing new to say about that!

Linear Field Theory.

The most general spin 2 quadratic lagrangian

$$\mathcal{L} \equiv \sum_{i=0}^{4} C_i \mathcal{O}^{(i)},$$

$$\mathcal{O}^{(1)} \equiv \frac{1}{4}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} \qquad \mathcal{O}^{(2)} \equiv -\frac{1}{2}\partial_\lambda h^{\mu\lambda}\partial_\rho h_\mu{}^\rho$$

$$\mathcal{O}^{(3)} \equiv \frac{1}{2}\partial_\mu h \partial_\lambda h^{\mu\lambda} \qquad \mathcal{O}^{(4)} \equiv -\frac{1}{4}\partial^\mu h \partial_\mu h$$

Fierz-Pauli: $C_i = 1$

ABGV

$$C_1 = C_2 = 1.$$

$$C_3 = \frac{2}{n} \quad \text{and} \quad C_4 = \frac{n+2}{n^2}.$$

Propagates also only spin 2 in a unitary way. Why?

massive to massless

Fierz and Pauli constructed the most general free massive spin 2 theory assuming Lorentz invariance and unitarity

In the massless limit

DOF: 5 -> 2

needs 3 gauge symmetries (4 is an overkill)

Fierz-Pauli $h_{\mu\nu}(x) \equiv h_{\mu\nu}(x) + 2\partial_{(\mu}\xi_{\nu)}(x),$

TDiff (VPD) $\partial_\mu \xi^\mu(x) = 0.$

Transverse diffs (TD)= volume preserving diffs (VPD)

FROM FIERZ-PAULI TO THIS NEW THEORY

$$h_{\mu\nu} \to h_{\mu\nu} - \frac{1}{n} h\, \eta_{\mu\nu}.$$

(not a field redefinition, because it is not invertible)

The trace is not recovered from the traceless piece

$$\partial_\mu \xi^\mu = 0$$

New Weyl invariance under TDiff

$$\delta_W h_{\mu\nu} = \phi(x) \eta_{\mu\nu}$$

Non-Linear Field Theory.

$$S[\hat{g}_{\mu\nu}] = -\frac{1}{2\kappa^2}\int d^n x \sqrt{\gamma}\, R\left[\gamma_{\mu\nu}\right].$$

restrict to unimodular metrics

variations are complicated:

$$\delta\gamma_{\mu\nu} \equiv M^{\alpha\beta}_{\mu\nu}\delta g_{\alpha\beta} = g^{-\frac{1}{n}}\left(\frac{1}{2}\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}\right) - \frac{1}{n}g^{\alpha\beta}g_{\mu\nu}\right)\delta g_{\alpha\beta}.$$

work with arbitrary metrics

$$\gamma_{\alpha\beta} = g^{-\frac{1}{n}} g_{\mu\nu}$$

$$S[g_{\mu\nu}] = -\frac{1}{2\kappa^2}\int d^n x\, g^{\frac{1}{n}}\left(R\left[g_{\mu\nu}\right] + \frac{(n-1)(n-2)}{4n^2}\frac{\nabla_\mu g \nabla^\mu g}{g^2}\right).$$

Note that this is the unique generalization to the FP-> UG non-invertible transformation

$$h_{\mu\nu} \to h_{\mu\nu} - \frac{1}{n} h\,\eta_{\mu\nu}.$$

$$\gamma_{\alpha\beta} = g^{-\frac{1}{n}} g_{\mu\nu}$$

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \kappa h_{\alpha\beta}$$

$$g = 1 + \kappa h$$

$$\therefore\; g^{-\frac{1}{n}}\, g_{\alpha\beta} = \eta_{\alpha\beta} + \kappa\left(h_{\alpha\beta} - \tfrac{1}{n}\,h\,\eta_{\alpha\beta}\right)$$

Constraint can also be imposed through a Lagrange multiplier

$$\Delta S_{UG} \equiv \int d^n x \sqrt{|g|}\, \frac{1}{\kappa} \lambda(x) \left(g+1\right)$$

The multiplier gets nontrivial dynamics by quantum corrections

EA, Anero and Martin, 2023

Exponential parametrization

(Background metric carries the signature)

$$g_{\mu\nu} = \bar{g}_{\mu\lambda} \left(e^{M}\right)^{\lambda}_{\ \nu}$$

$$M^{\lambda}_{\lambda} = 0$$

EA, Anero and Martin, 2023


The
Elephant
in the Room

we have lost Diff invariance!

$$J \equiv \det \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} = \pm 1.$$

$$S[g,\phi] \equiv \int d(vol)\, \mathcal{L}[g,\phi],$$

$$d(vol) = \omega \epsilon_{\mu_1 \dots \mu_d} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_d}, \qquad \sqrt{g} = \omega.$$

Volume preserving diffs

$$\nabla_\mu \xi^\mu = 0. \qquad TDiff_0$$

$VPD \subset Diff$

This is indeed a subgroup: $\left[\xi_1^\mu, \xi_2^\mu\right] \in VPD\,.$

This is also the residual gauge invariance of General Relativity in the gauge g=1

But the EM are not the same!

$$\xi_1 \equiv \xi_\mu dx^\mu \qquad \Omega_2 \equiv \frac{1}{2}\Omega_{\mu\nu}\, dx^\mu \wedge dx^\nu .$$

$$\delta \equiv *^{-1}\, d\, *, \qquad \delta^2 = 0$$

$$\xi_1 = -2\delta\Omega_2,$$

$$(\delta\Omega_2)_\rho = -\frac{1}{2}\nabla^\nu\, (\Omega_2)_{\nu\rho}\, .$$

The number of independent components of a two-form is $\binom{n}{2}$

withdraw $\Omega_2 = \delta\Omega_3,$ **(those are in the kernel: they give zero)**

final counting of gauge parameters

$$\binom{n}{2} - \left(\binom{n}{3} - \left(\binom{n}{4} - \cdots\right)\right) = n - 1,$$

$$(1-1)^n = 0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots = 1 - n + \binom{n}{2} - \binom{n}{3} + \ldots$$

Transverse diffeomorphisms vanishing at infinity

$$\xi^{\mu}_{T} = \partial_{\nu}\left(\frac{\Omega^{\mu\nu}_{(1)}}{r} + \frac{\Omega^{\mu\nu}_{(2)}}{r^2} + \dots\right) = \frac{\partial_{\nu}\Omega^{\mu\nu}_{(1)}}{r} - \frac{1}{r^3}\Omega^{\mu\nu}_{(1)}x_{\nu} + \frac{\partial_{\nu}\Omega^{\mu\nu}_{(2)}}{r^2} - 2\frac{\Omega^{\mu\nu}_{(2)}}{r^4}x_{\nu} + \dots$$

$$\Omega^{\mu\nu}_{(i)} = -\Omega^{\nu\mu}_{(i)}$$

$$x^{\lambda}\partial_{\lambda}\Omega^{\mu\nu}_{(i)} = 0$$

bianchi is softened

$$E_{\mu\nu} \equiv \frac{\delta S}{\delta g^{\mu\nu}},$$

$$H^3_{dR}(M) = 0 \qquad \nabla_\mu E^\mu{}_\nu = \nabla_\nu S,$$

Noether 2

$$\delta S = \int d(vol) E_{\mu\nu} \left(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu\right) =$$

$$= \int d(vol) E_{\mu\nu} \left(\nabla^\mu \eta^{\nu\alpha\beta\gamma} \nabla_\alpha \Omega_{\mu\gamma} + \nabla^\nu \eta^{\mu\alpha\beta\gamma} \nabla_\alpha \Omega_{\beta\gamma}\right),$$

$$\Omega_{\mu\nu} = -\Omega_{\nu\mu}, \qquad \eta_{\alpha\beta\gamma\delta} \equiv \sqrt{g}\epsilon_{\alpha\beta\gamma\delta}, \qquad \nabla_{[\lambda|} \nabla^\mu E_{\mu|\nu]} = 0,$$

$$\nabla^{\rho}\nabla_{\alpha}\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{\alpha\beta}}=\nabla^{\beta}\nabla_{\alpha}\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{\alpha\rho}}.$$

$$\Theta^{\beta}\equiv\nabla_{\alpha}\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{\alpha\beta}},$$

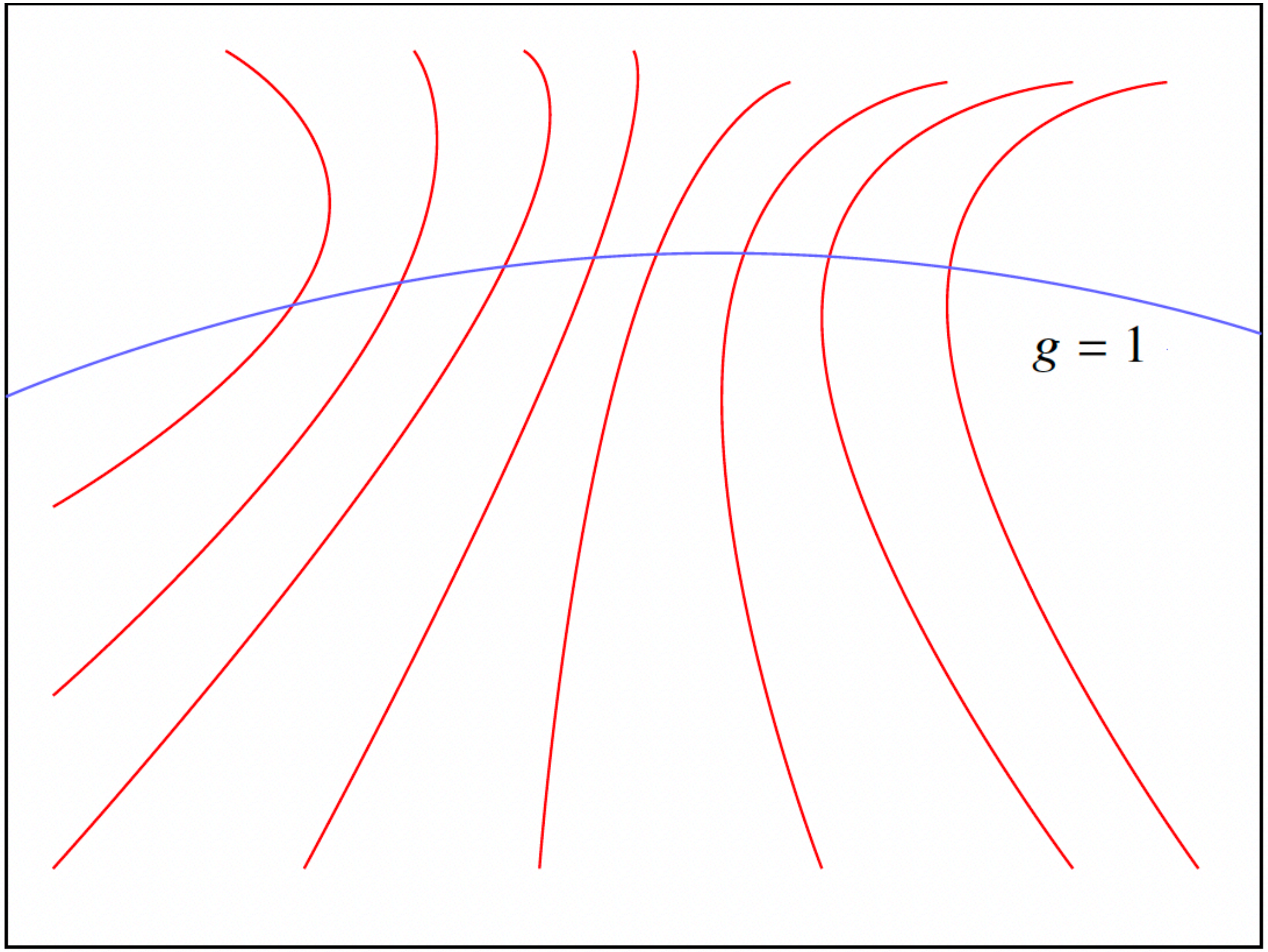
(This vanishes in any Diff invariant theory)

$$\nabla^{\rho}\Theta^{\beta}=\nabla^{\beta}\Theta^{\rho},$$

$$\Theta_{\rho}=\nabla_{\rho}\Phi+\gamma_{\rho},$$

If there are any harmonic forms)

$$\mathrm{WTDiff} = \mathrm{Weyl} \ltimes \mathrm{TDiff}.$$



Equations of motion (EM)

$$R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} + \frac{(2-n)(2n-1)}{4n^2} \left(\frac{\nabla_\mu g \nabla_\nu g}{g^2} - \frac{1}{n} \frac{(\nabla g)^2}{g^2} g_{\mu\nu} \right) +$$
$$+ \frac{n-2}{2n} \left(\frac{\nabla_\mu \nabla_\nu g}{g} - \frac{1}{n} \frac{\nabla^2 g}{g} g_{\mu\nu} \right) = 2\kappa^2 \left(T_{\mu\nu} - \frac{1}{n} T g_{\mu\nu} \right),$$

(This is the price to pay to be able to work with general metrics)

In the Weyl gauge g=1

$$R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} = 2\kappa^2 \left(T_{\mu\nu} - \frac{1}{n} T g_{\mu\nu} \right).$$

(these are almost Einstein's 1919 EM)

$E_{vac}\, g_{\mu\nu}$

does not source Einstein's equations

In contrast, in GR even in the unimodular gauge does source
because the second member
of Einstein's equations is not traceless.

(EM are derived BEFORE gauge fixing)

Self-consistency:

$$\nabla_\mu \left(T^{\mu\nu} - \frac{1}{n} T \, g^{\mu\nu}\right) = -\frac{1}{n}\nabla^\nu T \rightarrow$$

$$\nabla^\mu \left(R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu}\right) = \frac{n-2}{2n}\nabla^\nu R = -\frac{2\kappa^2}{n}\nabla^\nu T.$$

$$\nabla^\mu \left(2\kappa^2 T + R\right) = 0 \rightarrow 2\kappa^2 T + R = -C.$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + C g_{\mu\nu} = 2\kappa^2 T_{\mu\nu}$$

The constant C is NOT determined by the constant vacuum density

Same static limit as GR

No Birkhoff theorem in UG

Vacuum solutions with accelerated expansion

Quantum corrections do not generate a coupling CC/graviton

Tree amplitudes identical to GR (in spite of very different graviton propagator

backup

Physical sources.

Free Energy with external sources in a flat background.

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}.$$

Kinetic energy piece of the lagrangian:

$$K^{U}_{\mu\nu\rho\sigma} = \frac{1}{8}k^2\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\right) - \frac{1}{8}\left(k_\nu k_\sigma \eta_{\mu\rho} + \eta_{\mu\sigma}k_\nu k_\rho + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho}\right) - \frac{1}{2n}\left(\eta_{\mu\nu}k_\rho k_\sigma + \eta_{\rho\sigma}k_\mu k_\nu\right) - \frac{n+2}{4n^2}k^2\eta_{\mu\nu}\eta_{\rho\sigma}. \quad (69$$

It is convenient to use spin projectors:

Box 1 Starting with the longitudinal and transverse projectors

$$\theta_{\alpha\beta} \equiv \eta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}, \quad \omega_{\alpha\beta} \equiv \frac{k_\alpha k_\beta}{k^2}.$$

They obey

$$\begin{aligned}
&\theta + \omega \equiv \theta_\mu^\nu + \omega_\mu^\nu = \delta_\mu^\nu \equiv 1, \\
&\theta^2 \equiv \theta_\alpha^\beta \theta_\beta^\gamma = \theta_\alpha^\gamma \equiv \theta, \\
&\omega^2 \equiv \omega_\alpha^\beta \omega_\beta^\gamma = \omega_\alpha^\gamma \equiv \omega, \\
&tr\ \theta = n - 1, \\
&tr\ \omega = 1.
\end{aligned}$$

The four-indices projectors are

$$P_2 \equiv \frac{1}{2}\left(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}\right) - \frac{1}{n-1}\theta_{\mu\nu}\theta_{\rho\sigma},$$

$$P_1 \equiv \frac{1}{2}\left(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}\right),$$

$$P_0^s \equiv \frac{1}{n-1}\theta_{\mu\nu}\theta_{\rho\sigma},$$

$$P_0^w \equiv \omega_{\mu\nu}\omega_{\rho\sigma},$$

$$P_0^{sw} \equiv \frac{1}{\sqrt{n-1}}\theta_{\mu\nu}\omega_{\rho\sigma},$$

$$P_0^{ws} \equiv \frac{1}{\sqrt{n-1}}\omega_{\mu\nu}\theta_{\rho\sigma}. \quad \text{(I}$$

$$
\begin{aligned}
P_i^a P_j^b &= \delta_{ij}\delta^{ab} P_i^b , \\
P_i^a P_j^{bc} &= \delta_{ij}\delta^{ab} P_j^{ac} , \\
P_i^{ab} P_j^c &= \delta_{ij}\delta^{bc} P_j^{ac} , \\
P_i^{ab} P_j^{cd} &= \delta_{ij}\delta^{bc}\delta^{ad} P_j^a . \qquad (
\end{aligned}
$$

as well as

$$
\begin{aligned}
& tr\ P_2 \equiv \eta^{\mu\nu}(P_2)_{\mu\nu\rho\sigma} = 0, \\
& tr\ P_0^s = \theta_{\rho\sigma}, \\
& tr\ P_0^w = \omega_{\rho\sigma}, \\
& tr\ P_1 = 0, \\
& tr\ P_0^{sw} = \sqrt{n-1}\ \omega_{\rho\sigma}, \\
& tr\ P_0^{ws} = \frac{1}{\sqrt{n-1}}\ \theta_{\rho\sigma}, \\
& P_2 + P_1 + P_0^w + P_0^s = \frac{1}{2}\left(\delta_\mu^\nu \delta_\rho^\sigma + \delta_\mu^\sigma \delta_\rho^\nu\right) . \qquad (
\end{aligned}
$$

Any symmetric operator can be written as

$$K = a_2 P_2 + a_1 P_1 + a_w P_0^w + a_s P_0^s + a_\times P_0^\times, \quad ($$

(where $P_0^\times \equiv P_0^{ws} + P_0^{sw}$). Then

$$K^{-1} = \frac{1}{a_2} P_2 + \frac{1}{a_1} P_1 + \frac{a_s}{a_s a_w - a_\times^2} P_0^w + \frac{a_w}{a_s a_w - a_\times^2} P_0^s - \frac{a_\times}{a_s a_w - a_\times^2} P_0^\times . \quad ($$

Define the trace-free projector

$$(P_{tr})_{\rho\sigma}{}^{\lambda\delta} \equiv \frac{1}{2}\left(\delta_\rho^\lambda \delta_\sigma^\delta + \delta_\rho^\delta \delta_\sigma^\lambda\right) - \frac{1}{n} \eta_{\rho\sigma} \eta^{\lambda\delta} . \quad ($$

Then, the following relations hold:

Orthogonality properties

$$(P_2)_{\mu\nu}{}^{\rho\sigma}\,(P_{tr})_{\rho\sigma}{}^{\lambda\delta} = P_2,$$

$$P_0^s P_{tr} = P_0^s - \frac{n-1}{n} P_0^s - \frac{\sqrt{n-1}}{n} P_0^{sw},$$

$$P_0^w P_{tr} = P_0^w - \frac{\sqrt{n-1}}{n} P_0^{ws} - \frac{1}{n} P_0^w,,$$

$$P_1 P_{tr} = P_1,$$

$$P_0^{sw} P_{tr} = P_0^{sw} - \frac{\sqrt{n-1}}{n} P_0^{ws} - \frac{1}{n} P_0^w,$$

$$P_0^{ws} P_{tr} = P_0^{ws} - \frac{\sqrt{n-1}}{n} P_0^{sw} - \frac{n-1}{n} P_0^s.$$

$$K^U = \frac{k^2}{8}\left(2P_2 + 2P_0^s + 2P_1 + 2P_0^w\right) - \frac{k^2}{8}\left(2P_1 + 4P_0^w\right) + \frac{k^2}{2n}\left(\sqrt{n-1}P_0^\times + 2P_0^w\right) - \frac{n+2}{4n^2}k^2\left((n-1)P_0^s + \sqrt{n-1}P_0^\times + P_0^w\right) = k^2\left\{\frac{1}{4}P_2 - \frac{n-2}{4n^2}P_0^s - \frac{n^2-3n+2}{4n^2}P_0^w + \frac{n-2}{4n^2}\sqrt{n-1}P_0^\times\right\}.$$

$$K^{WT}_{\mu\nu\rho\sigma}\eta^{\rho\sigma} = 0,$$
$$\xi.k = 0 \;\Rightarrow\; K_{\mu\nu\rho\sigma}\xi^\rho k^\sigma = 0.$$

$$\mathcal{L}_{gf} = h_{\mu\nu}\, K_{gf}^{\mu\nu\rho\sigma}\, h_{\rho\sigma},$$

$$K_{gf}^{\mu\nu\rho\sigma} = \frac{k^6}{4\Lambda^4}P_1.$$

$$\mathcal{L}_{gf} = \frac{1}{2\Lambda^4} F_\alpha^2 \equiv \frac{1}{2\Lambda^4} \left(\partial_\alpha \partial^\mu \partial^\nu h_{\mu\nu} - \Box \partial^\mu h_{\alpha\mu}\right)^2,$$

$$4K^U_{tot} = k^2 P_2 + \left((n-1)m^2 - \frac{n-2}{n^2}k^2\right) P_0^s + \left(m^2 - \frac{n^2-3n+2}{n^2}k^2\right) P_0^w + \left(m^2 + \frac{n-2}{n^2}k^2\right)\sqrt{n-1}P_0^\times + \frac{k^6}{M^4}P_1. \quad ($$

Propagator:

$$k^2\Delta = P_2 + \frac{M^4}{k^4}P_1 - \frac{1}{(n-2)m^2}\left\{\left(m^2 - \frac{n^2-3n+2}{n^2}k^2\right)P_0^s + \left((n-1)m^2 - \frac{n-2}{n^2}k^2\right)P_0^w - \left(m^2 + \frac{n-2}{n^2}k^2\right)\sqrt{n-1}P_0^\times\right\}.$$

Traceless sources:

$$J_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{n} T \eta_{\mu\nu}.$$

$$W[J] \equiv \frac{1}{2} \int d^n x d^n y J^*_{\mu\nu}(x)\, \Delta^{\mu\nu\rho\sigma}(x,y)\, J_{\rho\sigma}(y) =$$

$$= \frac{1}{2} (2\pi)^{2n} \int d^n k J^*_{\mu\nu}(k)\, \Delta^{\mu\nu\rho\sigma}(k)\, J_{\rho\sigma}(k),$$

$$(P_1 J)_{\mu\nu} = 0,$$

$$tr\; J P_1 J = 0,$$

$$(P_2)_{\mu\nu\rho\sigma} J^{\rho\sigma} = T_{\mu\nu} - \frac{1}{n-1} \theta_{\mu\nu} T,$$

$$tr\; (J P_2 J) = |T_{\mu\nu}|^2 - \frac{1}{n-1}\, |T|^2,$$

$$W[J] = \frac{1}{2} (2\pi)^{2n} \int d^n k \left\{ J^*_{\mu\nu}(k)\, \frac{1}{k^2}\, P_2^{\mu\nu\rho\sigma}(k)\, J_{\rho\sigma}(k) + C(k) |T(k)|^2 \right\},$$

$$C(k) = -\frac{1}{(n-1)(n-2)k^2}.$$

$$W[T] = \frac{1}{2}(2\pi)^{2n} \int d^n k \frac{1}{k^2}\left(|J_{\mu\nu}|^2 - \frac{2}{n(n-2)}|T(k)|^2\right) =$$
$$= \frac{1}{2}(2\pi)^{2n} \int d^n k \frac{1}{k^2}\left(|T_{\mu\nu}|^2 - \frac{1}{n-2}|T(k)|^2\right).$$

how do we recover the newtonian potential in GR?

$$\hat{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{4} T\, \eta^{\mu\nu} \quad \text{and} \quad \hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4} h\, \eta_{\mu\nu}.$$

$$\mathcal{A}_{12} = -i\frac{\kappa^2}{4} T^1_{\mu\nu}(p_1, p_1') < \hat{h}^{\mu\nu}(k)\hat{h}^{\rho\sigma}(-k) > T^2_{\rho\sigma}(p_2, p_2'),$$

static sources

$$k_\mu = (0, \mathbf{k})_\mu.$$

$$T^i_{\mu\nu}(p_i, p'_i) = p_{i\,\mu} p'_{i\,\nu} + p_{i\,\nu} p'_{i\,\mu} + \frac{1}{2} k^2 \eta_{\mu\nu}.$$

ow, for very massive particles and for $k_\mu = (0, \mathbf{k})$,

$$\frac{1}{2M_i} T^i_{\mu\nu}(p_i, p'_i) = M_i\, \eta^{\mu 0} \eta^{\nu 0}, \quad i = 1, 2$$

that, in the static limit,

$$\frac{1}{2M_1\, 2M_2} \mathcal{A}_{12} = -i \frac{\kappa^2}{4} m_1 m_2 < \hat{h}^{00}(k) \hat{h}^{00}(-k) >,$$

$$V_{N\,w}(\mathbf{k}) = -\frac{\kappa^2}{8} \frac{M_1 M_2}{\mathbf{k}^2}.$$

What pappens with Unimodular Gravity?

$$\langle h_{\mu\nu}(k)h_{\rho\sigma}(-k)\rangle = \frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\sigma}-\eta_{\mu\nu}\eta_{\rho\sigma}\right)-\frac{a(k^2)}{2k^2}\eta_{\mu\nu}\eta_{\rho\sigma}+$$

$$+\frac{b(k^2)}{(k^2)^2}\left(k_\rho k_\sigma\eta_{\mu\nu}+k_\mu k_\nu\eta_{\rho\sigma}\right)+\frac{c(k^2)}{(k^2)^3}k_\mu k_\nu k_\rho k_\sigma, \quad ($$

$$\frac{1}{2}\left(\eta_{\mu\sigma}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\sigma}-\eta_{\mu\nu}\eta_{\rho\sigma}\right).$$

ɔllowing sum over polarizations,

$$\sum_{\lambda=\pm 2}\epsilon^{(\lambda)}_{\mu\nu}\,\epsilon^{(-\lambda)}_{\rho\sigma}.$$

$$\langle \hat{h}_{\mu\nu}(k)\hat{h}_{\rho\sigma}(-k)\rangle = \frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} + \left(-\frac{1}{2} + \frac{c(k^2)}{8}\right)\eta_{\mu\nu}\eta_{\rho\sigma}\right) + \\ + \frac{c(k^2)}{4(k^2)^2}\left(k_\rho k_\sigma \eta_{\mu\nu} + k_\mu k_\nu \eta_{\rho\sigma}\right) + \boxed{\frac{c(k^2)}{(k^2)^3} k_\mu k_\nu k_\rho k_\sigma.} \tag{95}$$

ubstituting the previous result in eq. (90) –recall that $k_\mu = (0, \mathbf{k})_\mu$,

$$-i\frac{\kappa^2}{4} m_1 m_2 \langle \hat{h}^{00}(k)\hat{h}^{00}(-k)\rangle = -\frac{\kappa^2}{8} M_1 M_2 \left(\frac{3}{2} + \frac{c(-\mathbf{k}^2)}{8}\right)\frac{1}{\mathbf{k}^2}. \tag{96}$$

his expression will match the Newtonian potential in (91) if, and only if, $c(-\mathbf{k}^2) = 4$, which, by Lorentz invariance leads to

$$c(k^2) = -4, \tag{97}$$

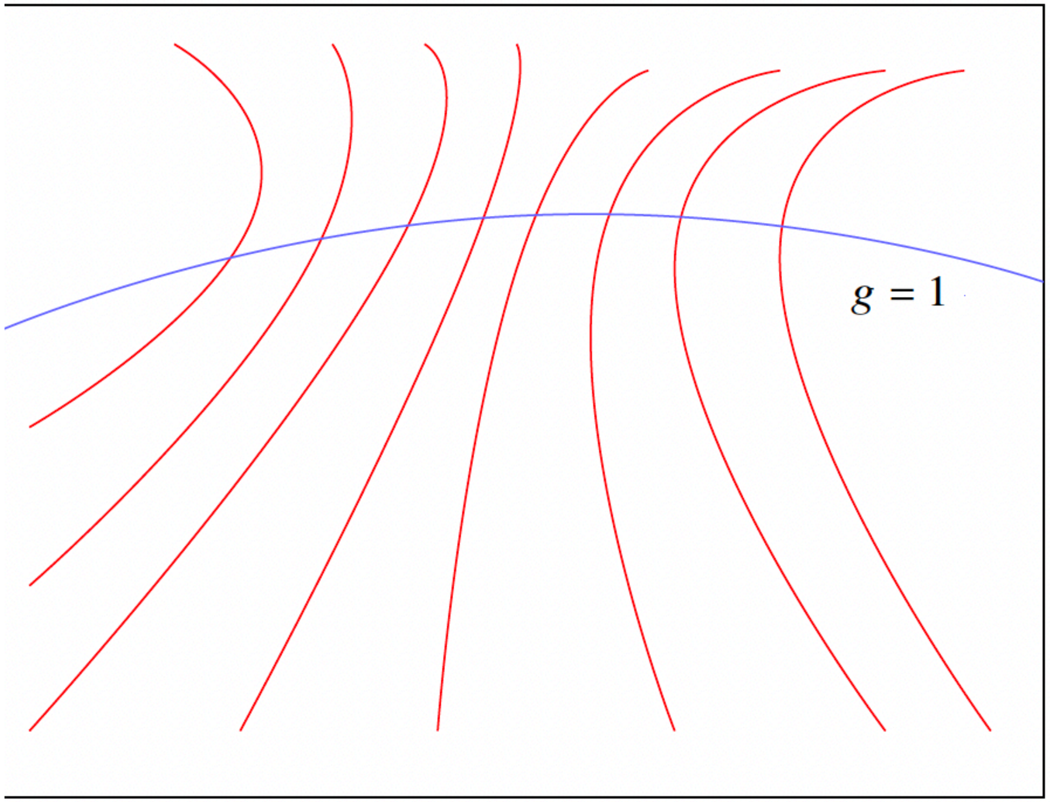
It works! In fact all PPN tests are kosher.

No Birkhoff theorem in UG.

weyl gauge g=1

$$R_{\mu\nu} = \frac{1}{n} R\, g_{\mu\nu}.$$

any Einstein space is a solution of the vacuum UG equations of motion



$$\gamma_{\alpha\beta} = g^{-\frac{1}{n}} g_{\mu\nu}$$ **(one to one)**

$$g_{\alpha\beta} = g^{\frac{1}{4}}\, \gamma_{\alpha\beta}$$ **(maps into the whole Weyl Orbit)**

Exponential expansion without vacuum energy

schwarzschild de sitter as a vacuum solution

Unimodular Cosmology.

$$ds^2 = b(t)^{-3/2}\, dt^2 - b(t)^{1/2}\, \delta_{ij} dx^i dx^j,$$

$$u^\mu = \left(b^{3/4}, 0, 0, 0\right).$$

$$u^\nu \nabla_\nu u^\mu = 0,$$

$$\theta \equiv \nabla_\mu u^\mu = \frac{3}{4} b^{-1/4} \frac{db}{dt}.$$

$$R_{\mu\nu} - \frac{1}{4} R\, g_{\mu\nu} = 2\kappa^2 \left(T_{\mu\nu} - \frac{1}{4} T\, g_{\mu\nu}\right),$$

$$T_{\mu\nu} \equiv (\rho + p)\, u_\mu u_\nu - p g_{\mu\nu},$$

$$u^\mu \nabla_\mu \rho + (\rho + p)\, \theta = 0.$$

$$u^\mu \nabla_\mu \theta + \frac{1}{n-1}\theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta} - \omega_{\alpha\beta}\omega^{\alpha\beta} + \frac{1}{n} R + \frac{2(n-1)}{n}\kappa^2(\rho + p) = 0.$$

Vacuum EM

$$u^\mu \nabla_\mu \theta = u^\nu \nabla_\nu \nabla_\mu u^\mu = -\frac{3}{16\sqrt{b}}\left[\left(\frac{db}{dt}\right)^2 - 4b\frac{d^2b}{dt^2}\right] = 0.$$

$$b(t) = b_0$$

flat spacetime

$$b(t) = H_0^{\frac{4}{3}}\,(3t - t_0)^{\frac{4}{3}}\,,$$

de sitter expansion

$$ds^2 = \left(\frac{dt}{3Ht}\right)^2 - (3Ht)^{2/3}\delta_{ij}\,dx^i dx^j$$

$$b(t) \sim t^{\frac{4}{3}}$$

UG \neq GR in the g=1 gauge

no synchronous gauge

$$ds^2 = a(t)dt^2 - R^2(t)\delta_{ij}dx^i dx^j, \qquad aR^6 = 1,$$

$$G_{00} = 3\frac{\dot{R}^2}{R^2} = \kappa^2 a\rho,$$

$$G_{ij} = \frac{\dot{a}\dot{R}R - 2a\ddot{R}R - a\dot{R}^2}{a^2}\delta_{ij} = -\kappa^2\rho R^2\delta_{ij}.$$

synchronous time

$$R = \left[R_0^3 + \kappa\sqrt{\frac{\rho}{3}}\,(t - t_0)\right]^{\frac{1}{3}},$$

$$a = \left[R_0^3 + \kappa\sqrt{\frac{\rho}{3}}\,(t - t_0)\right]^{-2}.$$

$$\tau = \int^t \frac{dx}{R_0^3 + \kappa\sqrt{\frac{\rho}{3}}\,(x - t_0)} = \frac{1}{\kappa}\sqrt{\frac{3}{\rho}}\log\left(R_0^3 + \kappa\sqrt{\frac{\rho}{3}}\,(t - t_0)\right)$$

$$R(\tau) = e^{\frac{\kappa}{3}\sqrt{\frac{\rho}{3}}(\tau - \tau_0)},$$

Tree diagrams.

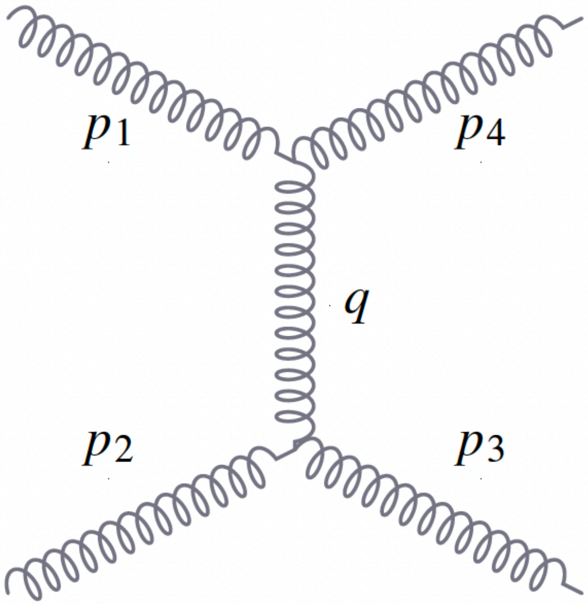


Fig. 2 *u* channel.

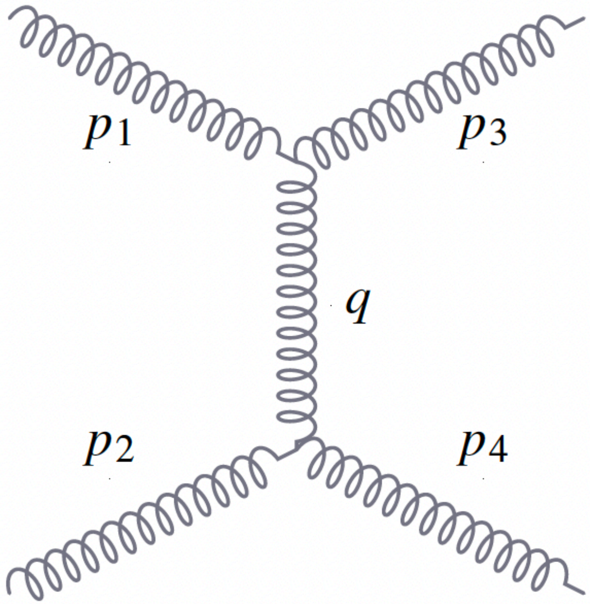


Fig. 3 *t* channel.

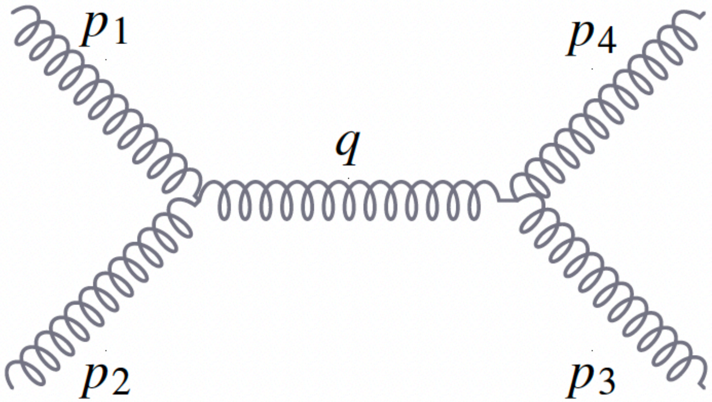


Fig. 4 *s* channel.

$$P^{\mathrm{UG}}_{\mu\nu,\rho\sigma} = \frac{1}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma}\right) - \frac{1}{k^2}\frac{\alpha^2 n^2 - n + 2}{\alpha^2 n^2 (n-2)}\eta_{\mu\nu}\eta_{\rho\sigma} + \\ + \frac{2}{n-2}\left(\frac{k_\rho k_\sigma \eta_{\mu\nu}}{k^4} + \frac{k_\mu k_\nu \eta_{\rho\sigma}}{k^4}\right) - \frac{2n}{n-2}\frac{k_\mu k_\nu k_\rho k_\sigma}{k^6}.$$

Recall that the usual GR graviton propagator in the de Donder gauge,

$$P^{\mathrm{GR}}_{\mu\nu\rho\sigma} = \frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \frac{2}{n-2}\eta_{\mu\nu}\eta_{\rho\sigma}\right),$$

$$V^{\mu\nu,\rho\rho,\alpha\beta}_{(p1,p2,p3)} = i2\kappa\, \mathcal{S}\Bigg\{ \frac{(2+n)(p_1.p_2)}{2n}\left[\frac{\eta^{\alpha\beta}\eta^{\mu\nu}\eta^{\rho\rho}}{n^2} - \frac{2\eta^{\alpha\rho}\eta^{\beta\rho}\eta^{\mu\nu}}{n} - \frac{\eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\nu\rho}}{2+n}\right] +$$

$$+\frac{2\eta^{\beta\nu}\eta^{\rho\rho}p_1^m p_2^\alpha}{n} + \tfrac{1}{2}\eta^{mr}\eta^{\nu\rho}p_1^\alpha p_2^\beta - \frac{\Big(2+n\Big)\eta^{\mu\nu}\eta^{\rho\rho}p_1^\alpha p_2^\beta}{2n^2} - 2\eta^{\beta\rho}\eta^{\nu\rho}p_1^\alpha p_2^\mu - \eta^{\alpha\nu}\eta^{\beta\rho}p_1^\rho p_2^\mu +$$

$$+\frac{\eta^{\alpha\beta}\eta^{\nu\rho}p_1^\rho p_2^\mu}{n} + \frac{2\eta^{\beta\mu}\eta^{\rho\rho}p_1^\alpha p_2^\nu}{n} - \frac{2\eta^{\alpha\beta}\eta^{\rho\rho}p_1^\mu p_2^\nu}{n^2} + \frac{2\eta^{\alpha\mu}\eta^{\beta\nu}p_1^\rho p_2^\rho}{\nu} + (p_1.p_2)\eta^{\alpha\nu}\eta^{\beta\rho}\eta^{\mu\rho}\Bigg\}. \tag{170}$$

$$
\begin{aligned}
V^{\mu\nu,\rho\sigma,\alpha\beta,\eta\lambda}_{(p1,p2,p3,p4)} &= i2s\kappa\,\mathcal{S}\Bigg\{\frac{(2+n)(p_3.p_4)}{n^2}\left[\frac{g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta}g^{\eta\lambda}}{4n^2}-\frac{g^{\mu\rho}g^{\alpha\beta}g^{\eta\lambda}g^{\nu\sigma}}{4n}+g^{\mu\rho}g^{\alpha\beta}g^{\eta\sigma}g^{\nu\lambda}\right]- \\
&-\tfrac{1}{2}(p_3.p_4)g^{\mu\eta}g^{\rho\lambda}g^{\alpha\nu}g^{\sigma\beta}+\frac{\left(2+n\right)(p_3.p_4)g^{\mu\eta}g^{\rho\alpha}g^{\nu\lambda}g^{\sigma\beta}}{2n^2}-\frac{\left(2+n\right)(p_3.p_4)g^{\mu\nu}g^{\rho\eta}g^{\alpha\beta}g^{\sigma\lambda}}{n^3}- \\
&-\frac{(p_3.p_4)}{n}\left[\frac{g^{\mu\nu}g^{\rho\sigma}g^{\alpha\eta}g^{\beta\lambda}}{4n}+g^{\mu\nu}g^{\rho\eta}g^{\alpha\sigma}g^{\beta\lambda}-ng^{\mu\rho}g^{\alpha\nu}g^{\eta\sigma}g^{\beta\lambda}+\frac{g^{\mu\rho}g^{\alpha\eta}g^{\nu\sigma}g^{\beta\lambda}}{4}\right]+ \\
&+g^{\mu\eta}g^{\alpha\sigma}g^{\beta\lambda}p_3^{\nu}p_4^{\rho}+\frac{2+n}{2n^2}\left[g^{\mu\rho}g^{\alpha\beta}g^{\eta\lambda}p_3^{\sigma}p_4^{\nu}-\frac{g^{\mu\nu}g^{\alpha\beta}g^{\eta\lambda}p_3^{\rho}p_4^{\sigma}}{n}+2g^{\mu\alpha}g^{\eta\lambda}g^{\nu\beta}p_3^{\rho}p_4^{\sigma}\right]- \\
&-\tfrac{1}{2}g^{\mu\rho}g^{\alpha\eta}g^{\beta\lambda}p_3^{\sigma}p_4^{\nu}+\frac{g^{\mu\nu}g^{\alpha\eta}g^{\beta\lambda}p_3^{\rho}p_4^{\sigma}}{2n}-g^{\mu\alpha}g^{\eta\nu}g^{\beta\lambda}p_3^{\rho}p_4^{\sigma}-2\frac{g^{\mu\alpha}g^{\rho\beta}g^{\eta\lambda}p_3^{\nu}p_4^{\sigma}}{n}+ \\
&+2g^{\mu\rho}g^{\alpha\lambda}g^{\eta\sigma}p_3^{\nu}p_4^{\beta}-2\frac{g^{\mu\rho}g^{\alpha\sigma}g^{\eta\lambda}p_3^{\nu}p_4^{\beta}}{n}+2g^{\mu\eta}g^{\rho\lambda}g^{\alpha\nu}p_3^{\sigma}p_4^{\beta}-2\frac{g^{\mu\nu}g^{\rho\eta}g^{\alpha\lambda}p_3^{\sigma}p_4^{\beta}}{n}+ \\
&+2\frac{g^{\mu\nu}g^{\rho\alpha}g^{\eta\lambda}p_3^{\sigma}p_4^{\beta}}{n^2}-2\frac{g^{\mu\eta}g^{\rho\alpha}g^{\nu\lambda}p_3^{\sigma}p_4^{\beta}}{n}+\frac{g^{\mu\nu}g^{\rho\sigma}g^{\alpha\eta}p_3^{\lambda}p_4^{\beta}}{2n^2}-\frac{g^{\mu\nu}g^{\rho\eta}g^{\alpha\sigma}p_3^{\lambda}p_4^{\beta}}{n} \\
&+g^{\mu\rho}g^{\alpha\nu}g^{\eta\sigma}p_3^{\lambda}p_4^{\beta}-\frac{g^{\mu\rho}g^{\alpha\eta}g^{\nu\sigma}p_3^{\lambda}p_4^{\beta}}{2n}-2\frac{g^{\mu\alpha}g^{\eta\sigma}g^{\nu\beta}p_3^{\rho}p_4^{\lambda}}{n}-\frac{g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta}p_3^{\eta}p_4^{\lambda}}{n^3}+ \\
&+\frac{g^{\mu\rho}g^{\alpha\beta}g^{\nu\sigma}p_3^{\eta}p_4^{\lambda}}{n^2}-2\frac{g^{\mu\rho}g^{\alpha\sigma}g^{\nu\beta}p_3^{\eta}p_4^{\lambda}}{n}+2\frac{g^{\mu\nu}g^{\rho\alpha}g^{\sigma\beta}p_3^{\eta}p_4^{\lambda}}{n^2}- \\
&-2\frac{g^{\mu\rho}g^{\alpha\beta}g^{\eta\sigma}p_3^{\nu}p_4^{\lambda}}{n}+2\frac{g^{\mu\nu}g^{\rho\eta}g^{\alpha\beta}p_3^{\sigma}p_4^{\lambda}}{n^2}\Bigg\}. \qquad (171)
\end{aligned}
$$

To summarize, it has been shown that the Maximal Helicity Violating (MHV), three, four, and five graviton tree amplitudes give the same contribution in GR and UG. Moreover, this result holds for each diagram independently and not only for the total amplitude. Therefore, at least at tree level, and with three, four, or five external legs, the MHV contribution to the S matrix for pure gravity without coupling to other fields is the same in both theories.

A remarkable fact is that all the terms that involve the double and triple poles in the propagator of UG (167) do not contribute to any diagram.

One-Loop Unimodular Gravity.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}.$$

$$\int [\mathcal{D} g_{\mu\nu}] \ \exp\left[i\, S[g] + T\cdot g\right] = \int [\mathcal{D} h_{\mu\nu}] \ \exp\left[i\, S[h] + T\cdot g\right],$$

$$T\cdot g = \int d^4x \sqrt{g}\, T^{\mu\nu} g_{\mu\nu}.$$

$$S \longrightarrow S + S_{\text{G.F.}} + S_{\text{Gh}},$$

BRST

$$\mathfrak{s} = \mathfrak{s}_W + \mathfrak{s}_D,$$

where

$$\mathfrak{s}\overline{g}_{\mu\nu} = 0,$$

$$\mathfrak{s}h_{\mu\nu} = \mathfrak{s}_D h_{\mu\nu} + \mathfrak{s}_W h_{\mu\nu} = \mathcal{L}_{c^\mu}(\overline{g}_{\mu\nu} + h_{\mu\nu}) + 2c(\overline{g}_{\mu\nu} + h_{\mu\nu}).$$

$$S_{\text{BRST}} = \int d^4x\, \mathcal{L} + \mathfrak{s}\Psi.$$

$$D_\mu c^\mu = 0,$$

projector
(implies a U(1) symmetry)

$$c^\nu \to D^\nu f.$$

$$h_{\mu\nu}{}^{(0,0)},\; c_\mu^{(1,1)},\; b_\mu^{(1,-1)},\; f_\mu^{(0,0)},\; \phi^{(0,2)}$$ **TDiff**

$$\pi^{(1,-1)},\; \pi'^{(1,1)},\; \overline{c}^{(0,-2)},\; c'^{(0,0)}$$ **U(1)**

$$c^{(1,1)},\; b^{(1,-1)},\; f^{(0,0)},$$ **Weyl**

(n,m)=(Grassmann number,Ghost number)

$$\begin{aligned}
S_{\rm BRST}^{\rm TDiff} + S_{\rm BRST}^{\rm Weyl} = \int d^n x\, b^\mu \Big(& \Box^2 c_\mu^{(1,1)} - 2R_{\mu\rho}\nabla^\rho\nabla^\nu c_\nu^{(1,1)} - \Box R_\mu{}^\rho c_\rho^{(1,1)} - \\
& - 2\nabla_\sigma R_\mu{}^\rho \nabla^\sigma c_\rho^{(1,1)} - R_{\mu\rho}R^{\rho\nu} c_\nu^{(1,1)} \Big) - \overline{c}^{(0,-2)}\Box\phi^{(0,2)} + \pi^{(1,-1)}\Box\pi'^{(1,1)} - \\
& - \frac{1}{\rho_1}\left(F_\mu F^\mu + \nabla_\mu c'^{(0,0)}\nabla^\mu c'^{(0,0)} + 2F_\mu\nabla^\mu c'^{(0,0)} \right) - f^{(0,0)}\Box f^{(0,0)} + \frac{\alpha}{2} f^{(0,0)}\Box h + \\
& + \frac{\alpha}{2} h\Box f^{(0,0)} + 2n\alpha b^{(1,-1)}\Box c^{(1,1)}, \qquad \text{where} \quad F_\mu \equiv \nabla^\nu h_{\mu\nu} - \frac{1}{n}\nabla_\mu h. \qquad (188)
\end{aligned}$$

$$W_\infty = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x\left(\frac{119}{90}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \left(\frac{1}{6\alpha^2} - \frac{359}{90}\right)R_{\mu\nu}R^{\mu\nu} + \right.$$

$$\left. + \frac{1}{72}\left(22 - \frac{3}{\alpha^2}\right)R^2\right). \qquad (\text{:}$$

on shell conditions

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = E_4,$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{1}{4}R^2,$$

$$R = \text{constant}.$$

$$W_\infty^{\text{on-shell}} = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x\left(\frac{119}{90}E_4 - \frac{83}{120}R^2\right).$$

$$W_\infty^{GR} \equiv \frac{1}{16\pi^2(n-4)}\int \sqrt{g}\, d^4 x\left(\frac{53}{45}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - \frac{1142}{135}\Lambda^2\right).$$

Is it a problem the fact that the background metric is unimodular?

No unless there is a conformal anomaly

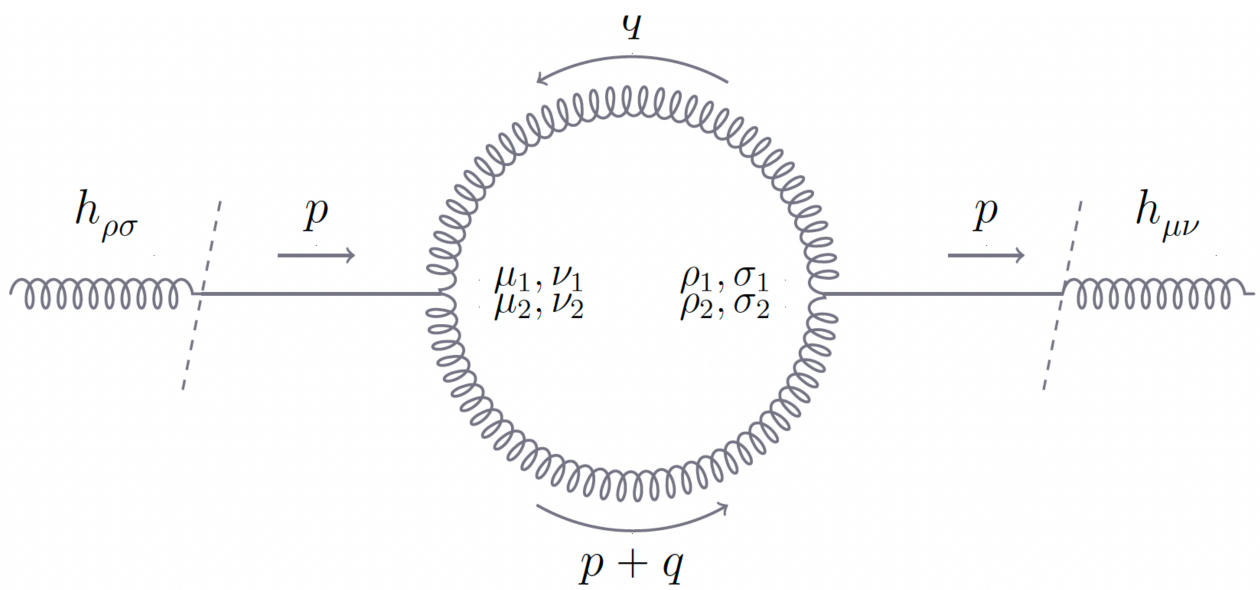
we have argued that there is none

tautological= pseudo weyl (Duff)

dimension dependent counterterms?

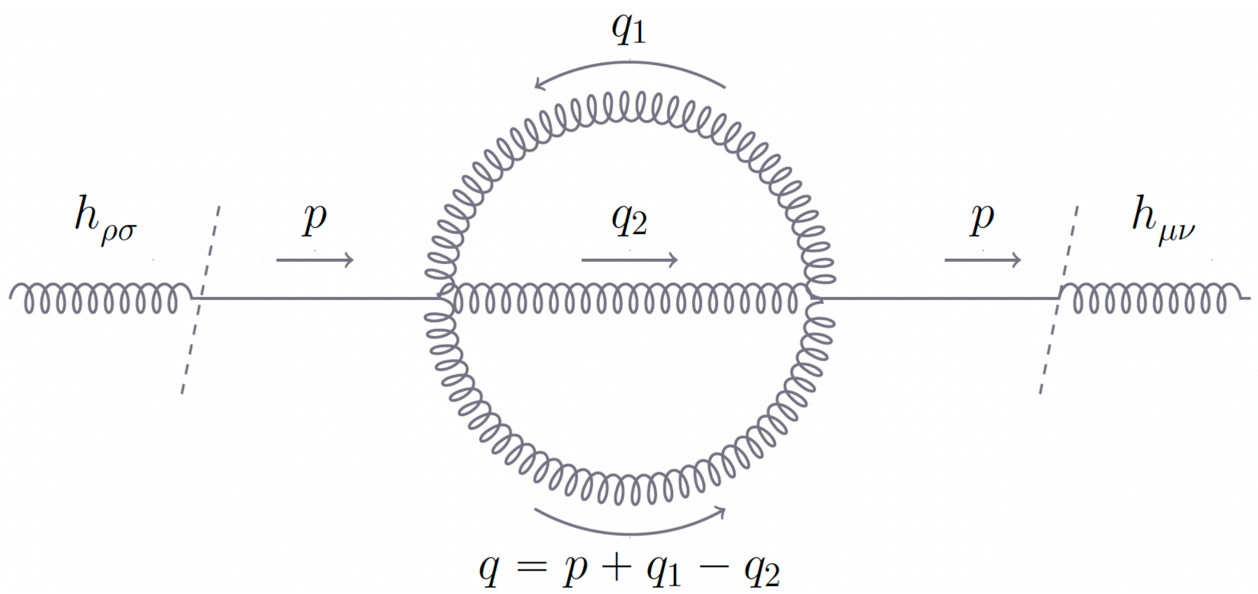
No conformal anomaly?

backup



$$-\frac{9i}{2\pi^2\epsilon} - \frac{(9i)(4\log(p) + 4\gamma + 1)}{8\pi^2} + O(\epsilon),$$

$$\frac{9i}{16\pi^2} \log \frac{\Lambda_{UV}}{\Lambda} \kappa^2 \int d^4x \, \lambda_{low}(x)^2$$



Two-loop Diagram generating kinetic energy for the multiplier

Exponential parametrization

$$Z[T_{\mu\nu}(x), j(x)] = e^{iW[T_{\mu\nu}(x),j(x)]} = \int \mathcal{D}h_{\mu\nu}\mathcal{D}\lambda\, e^{iS+i\int d^4x\lambda(x)h(x)+i(T^{\mu\nu}(x)h_{\mu\nu}(x)+j(x)\lambda(x))}$$

$$h(x) \equiv \bar{g}^{\mu\nu}h_{\mu\nu}.$$

Introducing sources for the graviton and for the multiplier

Ward identity

$$0 = \int \mathcal{D}h_{\mu\nu}\mathcal{D}\lambda\,\frac{\delta}{\lambda(x)}\left(e^{iS+i\int d^4x\lambda(x)h(x)+i(T^{\mu\nu}(x)h_{\mu\nu}(x)+j(x)\lambda(x))}\right) = i\langle h(x)+j(x)\rangle$$

$$-i\bar{g}^{\mu\nu}\frac{\delta Z}{\delta T^{\mu\nu}}+j(x)Z = 0 = \bar{g}^{\mu\nu}\frac{\delta W}{\delta T^{\mu\nu}}+j(x)$$

In terms of the 1PI

$$\Gamma[h_{\mu\nu},\lambda] \equiv W[T_{\mu\nu},j] - \int d^4x\,(T^{\mu\nu}h_{\mu\nu}+j\lambda)$$

$$\bar{g}^{\mu\nu}h_{\mu\nu}(x) - \frac{\delta\Gamma}{\delta\lambda(x)} = 0$$

The whole dependence of the effective action on the multiplier reads

$$\Gamma = \tilde{\Gamma} + \int d^4x \lambda(x) \bar{g}^{\mu\nu} h_{\mu\nu}(x)$$

$$\frac{\delta \tilde{\Gamma}}{\delta \lambda(x)} = 0$$