Evaluating Feynman Integrals with the Help of the Landau Equations

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CLUSTER OF EXCELLENCE
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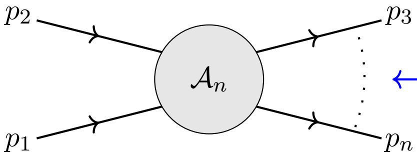
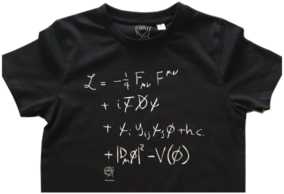
`2304.02629 with Christoph Dlapa, Martin Helmer, Felix Tellander`

Motivation: Scattering Amplitudes $\mathcal{A}_n$ in Quantum Field Theory



$\longleftrightarrow$



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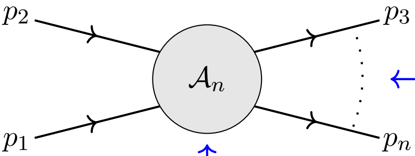
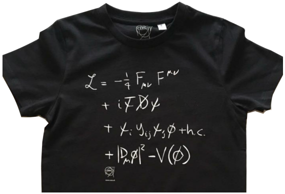
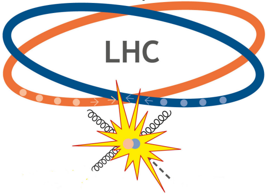




Collider Experiments



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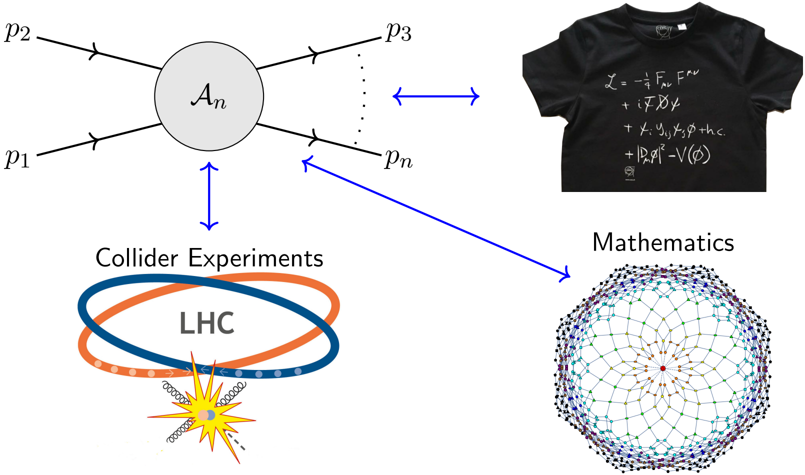

- Theoretical predictions for outcome of elementary particle collisions, central for experiments such as the LHC & High-Luminosity upgrade
- Exhibit remarkably deep mathematical structures

Amplitude calculation workflow

E.g. for $n = 4$ gluons: $\mathcal{A}_4 = g_{YM}^2 \sum_{L=0,1\ldots} g_{YM}^{2L} \mathcal{A}_4^{(L)}$, g_{YM} coupling const.

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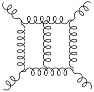
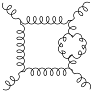
1. Draw contributing Feynman graphs $\quad \mathcal{A}_4^{(2)} =$  $+$  $+ \dots$

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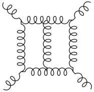
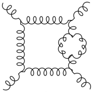
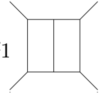
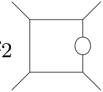
1. Draw contributing Feynman graphs $\mathcal{A}_4^{(2)} =$  $+$  $+ \dots$
2. Decompose to scalar Feynman integrals (FI) $= c_1$  $+ c_2$  $+ \dots$

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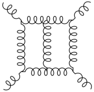
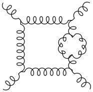
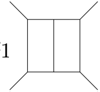
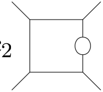
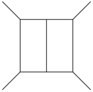
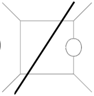
1. Draw contributing Feynman graphs $\qquad \mathcal{A}_4^{(2)} =$  $+$  $+ \dots$
2. Decompose to scalar Feynman integrals (FI) $\qquad = c_1$  $+ c_2$  $+ \dots$
3. Reduce to a basis $\vec{f}$ (master FI) $\qquad = c_1'$  $+ 0$  $+ \dots$

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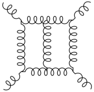
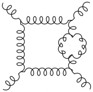
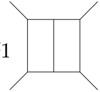
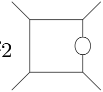
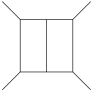
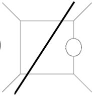
1. Draw contributing Feynman graphs $\quad \mathcal{A}_4^{(2)} =$  $+$  $+ \ldots$
2. Decompose to scalar Feynman integrals (FI) $\quad = c_1$  $+ c_2$  $+ \ldots$
3. Reduce to a basis $\vec{f}$ (master FI) $\quad = c_1'$  $+ 0$  $+ \ldots$
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Of the form
$$f_1 = \int \prod_{l=1}^{L} \frac{d^D k_l}{i\pi^{D/2}} \prod_{i=1}^{E} \frac{1}{D_i^{\nu_i}},$$

where $D_i = -q_i^2 + m_i^2$ and $D = D_0 - 2\epsilon$.

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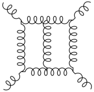
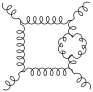
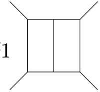
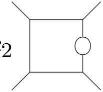
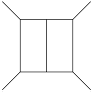
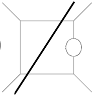
1. Draw contributing Feynman graphs $\quad \mathcal{A}_4^{(2)} =$  $+$  $+\ldots$
2. Decompose to scalar Feynman integrals (FI) $\quad = c_1$  $+ c_2$  $+\ldots$
3. Reduce to a basis $\vec{f}$ (master FI) $\quad = c_1'$  $+ 0$  $+\ldots$
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Evaluation of Feynman Integrals
Method of choice: Canonical differential equations

For polylogarithmic FI, find basis transformation $\vec{g} = T \cdot \vec{f}$ such that
[Gehrmann,Remiddi'99] [Henn'13]

$$d\vec{g} = \epsilon \, d\widetilde{M} \, \vec{g}, \qquad \widetilde{M} \equiv \sum_i \overbrace{\tilde{a}_i}^{\text{constant matrices}} \log \underbrace{W_i}_{\text{letters}},$$

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In line with strategy of state of the art precision calculations, e.g.
[Abreu,Ita,Moriello,Page,Tschernow,Zeng'20]

The Landau equations

Yield specific values of (kinematic) parameters of any (Feynman) integral, for which it may become singular. [Landau'59]

$$f_1 = \int \prod_{l=1}^{L} \frac{d^D k_l}{i\pi^{D/2}} \int_0^\infty \prod_{i=1}^{E} dx_i \frac{\delta(\sum_j x_j - 1)}{\left(\sum_j x_j D_j\right)^{\sum_k \nu_k}}$$

where $D_i = -q_i^2 + m_i^2$.

Landau equations:

$$x_i D_i = 0 \ \forall i = 1, \dots E$$

$$\frac{\partial}{\partial k_l} \sum_{i=1}^{E} x_i D_i = 0, \ \forall l = 1, \dots, L.$$

Formulated as conditions for the contour of integration to become trapped between two poles of integrand.

Believed for long to only provide information on where $W_i = 0$.

This work

Evidence through two loops: Rational letters of polylogarithmic FI captured by Landau equations, when recast as polynomial of the kinematic variables of integral, known as the *principal A-determinant* E_A!

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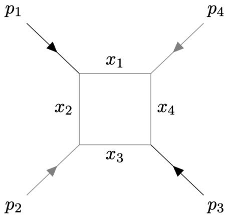
Example: 'Two-mass easy' box with $p_2^2 = p_4^2 = 0$, $p_1^2, p_3^2 \neq 0$:



E_A equipped with natural factorization, $(s = (p_1 + p_2)^2,\ t = (p_1 + p_4)^2)$

$$E_A = (p_1^2 p_3^2 - st) p_1^2 p_3^2 st (p_1^2 + p_3^2 - s - t)(p_3^2 - t)(p_3^2 - s)(p_1^2 - t)(p_1^2 - s).$$

where each factor is indeed a letter of the integral!

Outline

Feynman integrals in the Lee-Pomeransky representation:

$$f_1 = \frac{\Gamma(D/2)}{\Gamma((L+1)D/2 - \sum_i \nu_i)} \int_0^\infty \prod_{i=1}^{E} \left(\frac{x^{\nu_i - 1} dx_i}{\Gamma(\nu_i)} \right) \frac{1}{\mathcal{G}^{D/2}}$$

where $\mathcal{G} = \mathcal{U} + \mathcal{F}$, and for graph G associated to integral f_1,

$$\mathcal{U} = \sum_{\substack{T \text{ a spanning} \\ \text{tree}^1 \text{ of } G}} \prod_{e \notin T} x_e,$$

$$\mathcal{F} = \mathcal{U} \sum_{e \in E} m_e^2 x_e - \sum_{\substack{F \text{ a spanning} \\ 2\text{-forest}^2 \text{ of } G}} p(F)^2 \prod_{e \notin F} x_e,$$

are the 1^{st} and 2^{nd} Symanzik polynomials, of degree $L, L+1$ in the x_i.

In this form, f_1 is special case[3] of $\mathcal{A}$-hypergeometric function as defined by Gelfand, Graev, Kapranov & Zelevinsky (GKZ). [de la Cruz'19][Klausen'19]

[1] Connected subgraph of G containing all vertices but no loops.
[2] Defined similarly, but with 2 connected components.
[3] Generic case: All $\mathcal{G}$ polynomial coefficients are variables, different from each other.

Singularities of GKZ-systems

Let $\mathcal{G} = \sum_{j=1}^{m} c_j \prod_{i=1}^{E} x_i^{a_{ij}}$, c_j all independent variables.
Values of c_i for which GKZ-system becomes singular are solutions to

$$E_A(\mathcal{G}) = 0$$

where $E_A(\mathcal{G})$ is the *principal A-determinant of $\mathcal{G}$*: Polynomial in c_j with integer coefficients, that vanishes whenever the system of equations

$$\mathcal{G} = x_1 \frac{\partial \mathcal{G}}{\partial x_1} = \ldots = x_E \frac{\partial \mathcal{G}}{\partial x_E} = 0 \text{ has a solution for } \vec{x} \in \left(\mathbb{C}^*\right)^E .$$

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In practice, compute via theorem factorizing it into contributions from each face Γ of polytope with vertices $(a_{1j}, \ldots, a_{Ej})$, $j = 1, \ldots, m$

$$E_A(\mathcal{G}) = \prod_{\Gamma} \Delta_\Gamma(\mathcal{G})$$

where the *A-discriminant* $\Delta_\Gamma(\mathcal{G})$ also polynomial in c_i, that vanishes when

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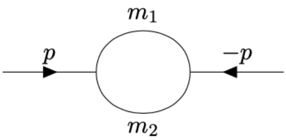
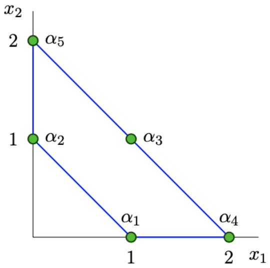
Example: Principal A-determinant of bubble





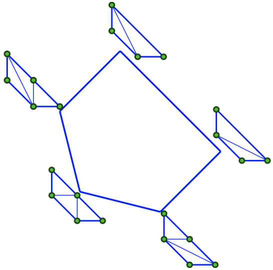
(Newton) polytope of $\mathcal{G}$ polynomial exponents, Newt($\mathcal{G}$)

$$\mathcal{G} = x_1 + x_2 + (m_1^2 + m_2^2 - p^2)x_1x_2 + m_1^2x_1^2 + m_2^2x_2^2\,,$$

$$\begin{aligned} E_A(\mathcal{G}) &= \Delta_{\alpha_4}\Delta_{\alpha_5}\Delta_{\alpha_4\alpha_5}\Delta_{\alpha_1\alpha_2\alpha_4\alpha_5} \\ &= m_1^2m_2^2(p^4 + m_1^4 + m_2^4 - 2p^2m_1^2 - 2p^2m_2^2 - 2m_1^2m_2^2)p^2\,, \end{aligned}$$

Interpretation of $E_A(\mathcal{G})$ polytope

Newt($E_A(\mathcal{G})$), built out of exponents of $E_A(\mathcal{G})$ polynomial: Keeps track of *triangulations* of Newt($\mathcal{G}$).



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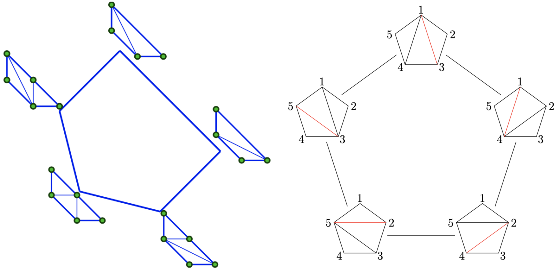
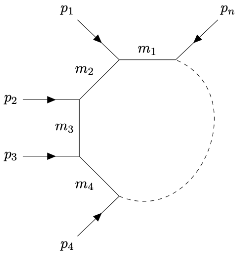


Cluster algebras also describe triangulations of geometric spaces [Fomin,Zelevinsky'01] [Felikson,Shapiro,Tumarkin'11]

First-principle derivation of observed cluster-algebraic structure of Feynman integrals? [Chicherin,Henn,Papathanasiou'20]... [He,Liu,Tang,Yang'22]

Generic n-point 1-loop integrals

All $m_i, p_i^2 \neq 0$ and different from each other



All Landau singularity information captured in *modified Cayley matrix* $\mathcal{Y}$,[Melrose'65]

$$\mathcal{Y} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & Y_{11} & Y_{12} & \cdots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & Y_{1n} & Y_{2n} & \cdots & Y_{nn} \end{pmatrix} \qquad \begin{aligned} Y_{ii} &= 2m_i^2 \\ Y_{ij} &= m_i^2 + m_j^2 - s_{ij-1} \\ s_{ij} &= (p_i + \ldots + p_j)^2 \end{aligned}$$

[1]Where all $x_i \neq 0$

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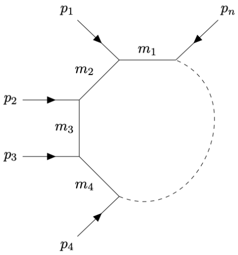

All Landau singularity information captured in *modified Cayley matrix $\mathcal{Y}$*, [Melrose'65]

$$\mathcal{Y} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & Y_{11} & Y_{12} & \cdots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & Y_{1n} & Y_{2n} & \cdots & Y_{nn} \end{pmatrix} \qquad \begin{aligned} Y_{ii} &= 2m_i^2 \\ Y_{ij} &= m_i^2 + m_j^2 - s_{ij-1} \\ s_{ij} &= (p_i + \ldots + p_j)^2 \end{aligned}$$

because A-discriminants reduce to usual determinants:

- $\Delta_{\mathsf{Newt}(\mathcal{F})}(\mathcal{F}) = \det Y$: Leading[1] Landau singularity of type I (finite k)
- $\Delta_{\mathsf{Newt}(\mathcal{G})}(\mathcal{G}) = \det \mathcal{Y}$: Leading[1] Landau singularity of type II ($k \to \infty$)
- Subleading Landau singularity where $x_{i_1}, \ldots, x_{i_m} = 0 \sim$ Leading singularity of subgraph where internal edges $i_1, \ldots, i_m$ removed [Klausen'21]

[1] Where all $x_i \neq 0$

Minors of modified Cayley matrix

For any matrix A with elements a_{mn}, let (j,k)-th minor of A be

$$A\begin{bmatrix} j \\ k \end{bmatrix} \equiv \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & k & a_{1,k+1} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,k-1} & & a_{2,k+1} & \cdots & a_{2,N} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{j-1,1} & a_{j-1,2} & a_{j-1,3} & \cdots & a_{j-1,k-1} & & a_{j-1,k+1} & \cdots & a_{j-1,N} \\ j & & & & & & & & \\ a_{j+1,1} & a_{j+1,2} & a_{j+1,3} & \cdots & a_{j+1,k-1} & & a_{j+1,k+1} & \cdots & a_{j+1,N} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{N,1} & a_{N,2} & a_{N,3} & \cdots & a_{N,k-1} & & a_{N,k+1} & \cdots & a_{N,N} \end{vmatrix},$$

where shading indicates removal of row and column. Similarly $A\begin{bmatrix} i_1 \ldots i_k \\ j_1 \ldots j_k \end{bmatrix}$,

$A\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} = \det A$.

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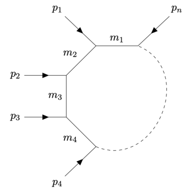
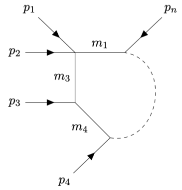
$$\mathcal{Y}\begin{bmatrix} 3 \\ 3 \end{bmatrix}\left(\vphantom{\Big|} \right. \quad$$



$$\left. \vphantom{\Big|} \right) = \mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\left(\vphantom{\Big|} \right. \quad$$



$$\left. \vphantom{\Big|} \right)$$

Principal A-determinant of generic 1-loop graphs

Gathering previous bits of information, arrive at

$$E_A(\mathcal{G}) = \mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \prod_{i=1}^{n+1} \mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix} \ldots \prod_{i_{n-1}>\ldots>i_1=1}^{n+1} \mathcal{Y}\begin{bmatrix} i_1 \ldots i_{n-1} \\ i_1 \ldots i_{n-1} \end{bmatrix} \prod_{i=2}^{n+1} \mathcal{Y}_{ii} \,.$$

Product of all diagonal k-dimensional minors of $\mathcal{Y}$ with $k = 1, \ldots, n+1$, except $\mathcal{Y}_{11} = 0$.

$2^{n+1} - n - 2$ factors, e.g. $1, 4, 11, 26, 57, 120$ factors for $n = 1, \ldots, 6\,.$

From 1-loop rational to square-root letters

Working assumption: Square-root letters produced by re-factorizing E_A using Jacobi determinant identities of the form

$$p \cdot q = f^2 - g = (f - \sqrt{g})(f + \sqrt{g}),$$

where

1. p, q factors of E_A, i.e. rational letters.
2. Square-root letters $f \pm \sqrt{g}$ obtained contain leading singularity of the Feynman integral considered in second term. [Cachazo'08]

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Motivated by interpretation of 1-loop integrals as volumes of spherical simplices. [Davydychev,Delbourgo'99] Jacobi identities,

$$A\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} A\begin{bmatrix} i & j \\ i & j \end{bmatrix} = A\begin{bmatrix} i \\ i \end{bmatrix} A\begin{bmatrix} j \\ j \end{bmatrix} - A\begin{bmatrix} i \\ j \end{bmatrix} A\begin{bmatrix} j \\ i \end{bmatrix} \overset{A=A^T}{=} A\begin{bmatrix} i \\ i \end{bmatrix} A\begin{bmatrix} j \\ j \end{bmatrix} - A\begin{bmatrix} i \\ j \end{bmatrix}^2$$

crucial for their computation. Point 2 adopts widely observed pattern in 1- and 2-loop computations.

All 1-loop letters I

Need only ratio $\frac{f-\sqrt{g}}{f+\sqrt{g}}$, as product already contained in rational alphabet. Letting $D = D_0 - 2\epsilon$, obtain N letters of type,

$$W_{1,\ldots,(i-1),\ldots,n} = \begin{cases} \dfrac{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} - \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} + \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}}}, & D_0 + n \text{ odd}, \\ \\ \dfrac{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} - \sqrt{\mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} + \sqrt{\mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}, & D_0 + n \text{ even}. \end{cases}$$

All 1-loop letters II

In addition, $n(n-1)/2$ letters of type,

$$W_{1,\ldots,(i-1),\ldots,(j-1),\ldots,n} = \begin{cases} \dfrac{\mathcal{Y}\begin{bmatrix} i \\ j \end{bmatrix} - \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\mathcal{Y}\begin{bmatrix} i & j \\ i & j \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} i \\ j \end{bmatrix} + \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\mathcal{Y}\begin{bmatrix} i & j \\ i & j \end{bmatrix}}}, & D_0 + n \text{ odd}, \\ \\ \dfrac{\mathcal{Y}\begin{bmatrix} 1 & j \\ 1 & i \end{bmatrix} - \sqrt{-\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 & i & j \\ 1 & i & j \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} 1 & j \\ 1 & i \end{bmatrix} + \sqrt{-\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 & i & j \\ 1 & i & j \end{bmatrix}}}, & D_0 + n \text{ even}, \end{cases}$$

All 1-loop letters III

Our procedure also predicts $\mathcal{Y}\left[\begin{smallmatrix}\cdot\\\cdot\end{smallmatrix}\right]$ and $\mathcal{Y}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ as individual rational letters, but in fact only the ratio

$$W_{1,2,\ldots,n} = \frac{\mathcal{Y}\begin{bmatrix}\cdot\\\cdot\end{bmatrix}}{\mathcal{Y}\begin{bmatrix}1\\1\end{bmatrix}}\,,$$

appears, as we'll get back to in next slide.

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Total letter count: Assuming $n \leq d+1$ for external kinematics dimension d,

$$|W| = 2^{n-3}\left(n^2+3n+8\right) - \frac{1}{6}\left(n^3+5n+6\right)\,,$$

e.g. $|W| = 1, 5, 18, 57, 166$ for $n = 1, \ldots, 5$ and D_0 even.

Verification through differential equations & comparison with literature

From letter prediction, derived canonical differential equations through numeric IBP identities $\Rightarrow$ confirmation.

By explicit computation up to $n = 10$, infer general form, e.g. $n + D_0$ even:

$$
\begin{aligned}
d\mathcal{J}_{1\ldots n} =& \epsilon \; d\log W_{1\ldots n} \; \mathcal{J}_{1\ldots n} \\
& + \epsilon \sum_{1\leq i\leq n} (-1)^{i+\left\lfloor \frac{n}{2} \right\rfloor} d\log W_{1\ldots(i)\ldots n} \; \mathcal{J}_{1\ldots\widehat{i}\ldots n} \\
& + \epsilon \sum_{1\leq i<j\leq n} (-1)^{i+j+\left\lfloor \frac{n}{2} \right\rfloor} d\log W_{1\ldots(i)\ldots(j)\ldots n} \; \mathcal{J}_{1\ldots\widehat{i}\ldots\widehat{j}\ldots n}.
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\end{aligned}
$$

Furthermore, compared to previous results for D_0 even based on

1. the diagrammatic coaction [Abreu,Britto,Duhr,Gardi'17]
2. the Baikov representation [Chen,Ma,Yang'22]

Agreement in form of CDE, as well as in letters for orientations presented in 2, see also. [Jiang,Yang'23]

Limits of generic to non-generic graphs

Proved that E_A has well-defined limit when any $m_i^2, p_j^2 \to 0$, namely it is unique regardless of the order with which we send them to zero.

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Limit of E_A when single parameter x takes value a may be defined as

$$\lim_{x\to a} E_A = \left.\frac{\partial^l \widetilde{E_A}}{\partial x^l}\right|_{x=a} \neq 0\,, \text{ with } \left.\frac{\partial^{l'} E_A}{\partial x^{l'}}\right|_{x=a} = 0 \text{ for } l' = 0, \ldots, l-1\,,$$

While multivariate generalization straightforward, highly nontrivial that limit does not depend on order. E.g. triangle Cayley in limit $p_i^2 \to 0$:

$$\det Y = 0 + 2\sum_{i=1}^{3} p_i^2 (m_i^2 - m_{i-1}^2)(m_{i+1}^2 - m_{i-1}^2) + \mathcal{O}(p_j^2 p_k^2)\,,$$

While limits of individual factors in E_A depend on limit order, E_A as a whole does not, since different orders produce factors it already contains.

Strong evidence that alphabet of non-generic FI correctly obtained as limit of generic one, in line with previous observations.

Mathematica Notebook

LandauAlphabetDE.nb

100%

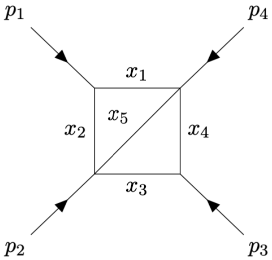
```
(* Symbol alphabets*)
(* Generic Box in even dimension *)
D0 = 4;
EvaluateLetter[AllLettersList[4]] // Short
(* Two-mass easy box limit *)
Factor[PLimit[%, ms[1] → 0, ms[2] → 0, ms[3] → 0, ms[4] → 0, s[1, 3] → 0, ps[2] → 0]];
(* In the limit the letters become multiplicatively dependent. Since all of them are rational, a basis may be found as follows *)
dlExpand[dl /@ %];
dlCollect /@ DeleteCases[RowReduce[CoefficientArrays[%, Variables[%]][[2]]].Variables[%], 0] /. dl[x_] → x
Length[%]
(* Product indeed yields corresponding limit of the principal A-determinant *)
LS2meBox - Times @@ %%

(* Differential equations *)
(*Box basis*)
basis @@ Range[4]
(*Box canonical differential equations*)
CDEs[%] // MatrixForm
```

```
Short=
ms[1] <<24>> (ms[3]^2 ps[1]^2 - 2 ms[3] × ms[4] ps[1]^2 + ms[4]^2 ps[1]^2 - 2 ms[1] × ms[3] × ps[1] × ps[2] +
   2 ms[1] × ms[4] × ps[1] × ps[2] + <<172>> + ms[3]^2 s[2, 3]^2 - 2 ms[1] × s[1, 2] s[2, 3]^2 - 2 ms[3] × s[1, 2] s[2, 3]^2 + s[1, 2]^2 s[2, 3]^2)

Out[180]= ps[1] × ps[3] (ps[1] - s[1, 2]) (ps[3] - s[1, 2]) s[1, 2] (ps[1] - s[2, 3]) (ps[3] - s[2, 3]) (ps[1] + ps[3] - s[1, 2] - s[2, 3]) s[2, 3] (ps[1] × ps[3] - s[1, 2] × s[2, 3])

Short=
{-1/(2 ms[1]), -1/(2 ms[2]), <<53>>, (-ms[1] × ms[3] × ps[1] + <<73>>)/(-ms[1] × ms[3] × ps[1] + <<71>> + Sqrt[<<1>>]),
 (-2 ms[1] × ms[2] × ps[1] + <<65>>)/(-2 ms[1] × ms[2] × ps[1] + <<63>> + Sqrt[(-ms[1]^2 + 2 ms[1] × ms[2] - <<1>>^2 + <<1>> + 2 ms[2] × ps[1] - ps[1]^2) (<<1>>)])}

Out[185]= {ps[1], ps[3], ps[1] - s[1, 2], ps[3] - s[1, 2], s[1, 2], ps[1] - s[2, 3], ps[3] - s[2, 3], ps[1] + ps[3] - s[1, 2] - s[2, 3], s[2, 3], ps[1] × ps[3] - s[1, 2] × s[2, 3]}

Out[186]= 10

Out[187]= 0

Out[188]= {IG[2][1], IG[2][2], IG[2][3], IG[2][4], IG[2][1, 2], IG[2][1, 3], IG[2][1, 4], IG[2][2, 3], IG[2][2, 4], IG[2][3, 4], IG[4][1, 2, 3], IG[4][1, 2, 4], IG[4][1, 3, 4], IG[4][2, 3, 4], IG[4][1, 2, 3, 4]}
```

MatrixForm=

w[1]	0	0	0	0	0	0	0	0	0
0	w[2]	0	0	0	0	0	0	0	0
0	0	w[3]	0	0	0	0	0	0	0
0	0	0	w[4]	0	0	0	0	0	0
-w[1, {2}]	w[{1}, 2]	0	0	w[1, 2]	0	0	0	0	0
-w[1, {3}]	0	w[{1}, 3]	0	0	w[1, 3]	0	0	0	0
-w[1, {4}]	0	0	w[{1}, 4]	0	0	w[1, 4]	0	0	0
0	-w[2, {3}]	w[{2}, 3]	0	0	0	0	w[2, 3]	0	0
0	-w[2, {4}]	0	w[{2}, 4]	0	0	0	0	w[2, 4]	0
0	0	-w[3, {4}]	w[{3}, 4]	0	0	0	0	0	w[3, 4]
-w[1, {2}, {3}]	w[1, {2}, {3}] + w[{1}, {2}, 3]	-w[{1}, {2}, 3]	0	-w[1, 2, {3}]	w[1, {2}, 3]	0	-w[{1}, 2, 3]	0	0

Two-loop example of principal A-determinant-alphabet relation



1-mass slashed box,

$$p_1^2 \neq 0, p_2^2 = p_3^2 = p_4^2 = 0$$

$$E_A(\mathcal{G}) = (p_1^2 - t)(p_1^2 - s)(p_1^2 - s - t)(s+t)stp_1^2.$$

Agrees precisely with (2dHPL) alphabet known to describe 2-loop master integrals with these kinematics! [Gehrmann,Remiddi'00]

Conclusions and Outlook

Evidence that rational letters of polylogarithmic FI captured by polynomial form of Landau equations in terms of *principal A-determinant* E_A!

- Through 2 loops
- 1 loop: Also obtain square-root letters from Jacobi identities + CDE
- Strong evidence for well-defined limits to non-generic kinematics
- Easy-to-use Mathematica file with our results

Next Stage

1. More efficient evaluation of E_A + more 2-loop checks
2. New predictions for pheno, e.g. letters for $2 \to 3$ with 2 massive legs
   [Les Houches Standard Model Precision Wishlist'21]
3. Explore implications for beyond-polylogarithmic case

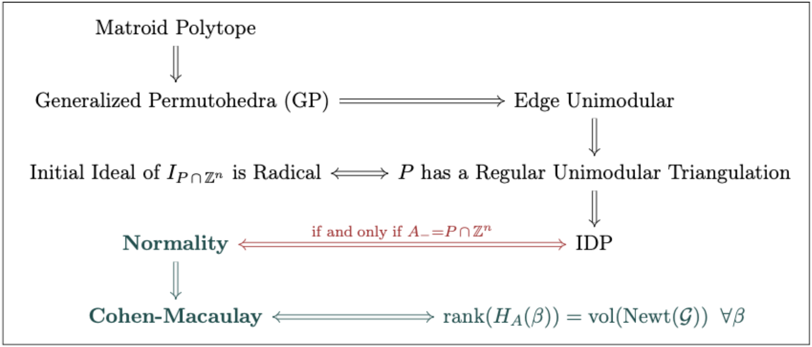
Further mathematical properties of Feynman integrals:Cohen-Macauley

Guarantees that

$$\#\text{ master integrals} = \text{volume of Newt}(\mathcal{G})$$

Proved it for currently largest known class of 1-loop integrals, including completely on-shell/massless. For earlier work, see [Tellander,Helmer'21] [Walther'22]

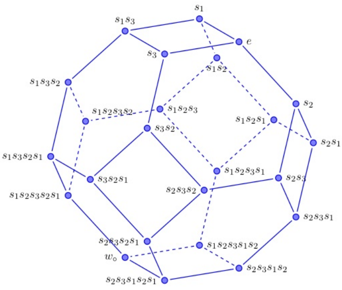
Relation to other properties:

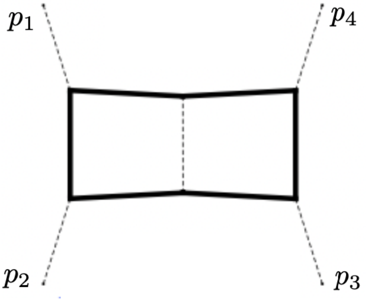


Further mathematical properties of Feynman integrals

:Generalized permutohedron (GP) property

A polytope $P \subset \mathbb{R}^n$ is GP if and only if every edge is parallel to $\mathbf{e}_i - \mathbf{e}_j$, where $\mathbf{e}_i$ is unit vector on coordinate axis, for some $i, j \in \{1, \ldots, n\}$. E.g.





Practical utility: This property facilitates new methods for fast Monte Carlo evaluation of Feynman integrals. [Borinsky'20] [Borinsky,Munch,Tellander'23]

Previously proven for generic kinematics. [Schultka'18] Here: Generalized to any graph where all external vertices joined by massive path.