M(embrane) Theory

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As derived 45* years ago, M-dimensional objects moving in such a way that their world volume swept out in $\mathbb{R}^{1,D-1}$ is extremal can be described by a light-cone (~~[illegible]~~ w.r.t. $\varphi^0 = \frac{x^0 \pm x^{D-1}}{2} =: \tau$)

Hamiltonian $P_- = \frac{1}{2\eta}\int \frac{\vec{p}^{\,2}+g}{\rho}\, d^M\varphi = H[\vec{x},\vec{p};\eta,\zeta_0]$,

polynomially depending on $d := D-2$ transversal coordinate fields $\vec{x} = \sum_{\alpha=0}^{\infty} \vec{x}_\alpha(\tau)\, Y_\alpha(\varphi^1 \ldots \varphi^M)$,

canonically conjugate momenta $\vec{p} = \sum_\beta \rho\, \vec{p}_\beta(\tau) Y_\beta$,

ρ a τ-independent density of weight 1 $(= \int \rho\, d^M\varphi)$, $\int Y_\alpha Y_\beta \rho\, d^M\varphi = \delta_{\alpha\beta}$,

$\{x_{i\alpha}, p_{j\beta}\} = \delta_{ij}\delta_{\alpha\beta}$, as well as two discrete degrees of freedom, $\eta = P_+$ and its canonical conjugate $\zeta_0 = \int (x^0 - x^{D-1})\rho\, d^M\varphi$, $\{\eta, \zeta_0\} = 1$,

$$g = \det\left(\partial_a \vec{x}\, \partial_b \vec{x}\right)_{a,b=1\ldots M}.$$

* Sorry for, during my talk on Jan. 8, having thought that $2024 - 1979 = 55$

The second important step, for $M=2$ leading to the $SU(N)$-invariant matrix model

$$H_N = \mathrm{Tr}\left(\vec{P}^2 - \sum_{i<j}[X_i, X_j]^2\right),$$

derived in my 1982 MIT Ph.D.-thesis, was to replace basisfunctions $Y_\alpha(\varphi^1, \varphi^{M=2})$ by $N\times N$ matrices $T_\alpha^{(N)}$. For spherical membranes,

$$T_{lm}^{(N)} = \sqrt{4\pi N}\sqrt{\frac{(N^2-1)^l (N-1-l)!}{(N+l)!}}\sum C^{(m)}_{a_1\dots a_l} X^{(N)}_{a_1}\dots X^{(N)}_{a_l},$$

with the 3 hermitean matrices $X_a^{(N)}$ being an irreducible representation of $SU(2)$, normalized such that $X_1^2+X_2^2+X_3^2 = \mathbb{1}$,

$$[X_a^{(N)}, X_b^{(N)}] = i\epsilon_{abc}\frac{2}{\sqrt{N^2-1}}X_c^{(N)}, \quad \frac{1}{N}\mathrm{Tr}\, T^{(N)\dagger}_{lm} T^{(N)}_{l'm'} = \delta_{ll'}\delta_{mm'}$$

for $l=1,2,\dots,N-1$ ($m=-l\dots+l$; note that $T^{(N)}_{l\geq N, m}\equiv 0$) are N^2-1 linear independent $N\times N$ matrices for the (complex, traceless, symmetric) tensors $C^{(m)}_{a_1\dots a_l}$ appearing in the similar expression for r^l times the ordinary spherical harmonics $Y_{lm}(\theta\varphi)$ as harmonic homogeneous polynomials in

$x_1 = r\sin\theta\cos\varphi$, $x_2 = r\sin\theta\sin\varphi$, $x_3 = r\cos\theta$.

(cf. arXiv 0206192 (hep-th), 1101.4403
CMP 195 (1998) 67 Nucl. Phys. B 849
Int. J. Mod. Phys. A 4 (1989) #19, 5235 2011, 628) (2

For general M the (purely internal)
$\mathbb{M}^2 = 2P_+P_- - \vec{P}^2$ can be written as

$$\sum_{\substack{\alpha=1 \\ i=1\ldots d}}^{\infty} p_{i\alpha}p_{i\alpha} + \frac{1}{M!} g_{\alpha\alpha_1\ldots\alpha_M} g_{\alpha\beta_1\ldots\beta_M} x_{i_1\alpha_1} x_{i_1\beta_1} \cdots x_{i_M\alpha_M} x_{i_M\beta_M}$$

$$g_{\alpha\alpha_1\ldots\alpha_M} := \int Y_\alpha \epsilon^{r_1\ldots r_M} \partial_{r_1} Y_{\alpha_1} \cdots \partial_{r_M} Y_{\alpha_M} d^M\varphi$$

(arXiv hep-th 9602020, considering as well
$[[X^{\mu_1}, \ldots, X^{\mu_{M+1}}], X_{\mu_2}, \ldots, X_{\mu_{M+1}}] = 0$)

Lorentzinvariance of the classical theory was proven by Goldstone around 1985 (see the attached 7-page note)

Dynamical Symmetry and Reconstruction Algebra : JH, PLB 695
resp. arXiv 1003.5189, 1004.0266, 1006.4714,
1007.5505, 1101.4334

general review of relativistic membranes:
J. Phys. A 46 (2013)
(~~unfortunately~~ but with many typos kindly typed by the Journal from my handwritten notes)

$*$-product membranes: Phys. Lett. B 250 (1990) 44

Lax-pair formulation
(of general supersymmetrizable bosonic systems)
arXiv 2101.01803, 2101.04495, 2101.11510
(PoS CORFU 2021 (2022) 258, Recent Progress in Membrane Theory)

Integrability in the dynamics of axially symmetric membranes / exact solutions*:

arXiv 2107.00569, 2201.02524, 2202.06955, 2303.03920, 2211.03887 (Asian J. Math 26 (2022) 253, PLB 822 (2021) 136658)

* in 4 space-time dimensions

only very few are known explicitly

$$(t^2+x^2+y^2-z^2)(t+z)^2 = C \ (\leq 0)$$

$$(t^2-x^2-y^2-z^2) = \underset{>0}{c}(t+z)^6 \ \cancel{(\leq 0)}$$

$$(t^2+x^2+y^2-z^2) - 6\underset{<0}{C}\sqrt{x^2+y^2}(t+z)^2 + 3C^2(t+z)^4 = 0,$$

the latter leading to a very interesting class of exact solutions of the classical matrix model equations,

$$\ddot{X}_i = -\sum_j [[X_i, X_j], X_j] \qquad \sum_i [X_i, \dot{X}_i] = 0$$

The fast non-commutative sharp drop

JH, Phys. Lett. B 2024