Review of the BFSS conjecture
+
Three point amplitudes
+
Soft theorems in the BFSS matrix model

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Review of the BFSS conjecture

Bank, Fischler, Shenker, Susskind: ``M- theory as a matrix model, a conjecture'', hep-th/9610043

The matrix model can be used to calculate M-theory scattering amplitudes.

The matrix model

- Matrix quantum mechanics.
- Gauge group U(N) → restricted to U(N) invariant states (gauged).
- 9 adjoint scalars + 16 adjoint fermions, N x N matrices.
- 9 of SO(9) + 16 of SO(9).
- 16 supersymmetries.

$$L = \mathrm{Tr}\left[\frac{1}{2R}(\dot{X}^I)^2 + \frac{R}{4(2\pi)^2}\sum_{I,J}[X^I, X^J]^2 + \psi^t\left(\dot{\psi} + \frac{R}{2\pi}\Gamma_I[X^I,\psi]\right)\right]$$

X as positions. All in 11d Planck units. $X = X/l_p$, $R = R/l_p$.

A large N limit of a scattering problem in the matrix model

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scattering amplitude in M-theory.

This scattering problem involves very low energies, in the large N limit.

- Not the ‘t Hooft limit of the matrix model.
- Different energy regime from the one that describes the 10 dimensional black hole.

The potential has many flat directions. Matrices become diagonal with all different diagonal entries.

When some of the diagonal entries are the same → get bound states.

Simplest example:
U(N) ~ U(1) x SU(N)

The SU(N) part has a single bound state at threshold, at E=0.

Interpret it as a single graviton with momentum

$$-p_- = \frac{N}{R}$$

U(1), or center of mass, degrees of freedom →
description of massless superparticle in light-cone gauge.

bosons

$$L = \frac{N}{R}\frac{1}{2}\dot{\vec{x}}^{\,2} \longrightarrow H = \frac{1}{2}\frac{R}{N}\vec{p}^{\,2} = -p_+$$

Fermions → Fills out the supermultiplet, the different polarizations.

Expand around:

$$X \sim \begin{pmatrix} x_1 \mathbf{1}_{N_1} & 0 & 0 \\ 0 & x_2 \mathbf{1}_{N_2} & 0 \\ 0 & 0 & x_3 \mathbf{1}_{N_3} \end{pmatrix}$$

Three bound states with SU(N_i) → three gravitons, with momenta: $-p_-^i = \frac{N_i}{R}$

Each center of mass degree of freedom, each U(1)$_i$ gives superparticles.

Matrix block bound states to matrix block bound states scattering

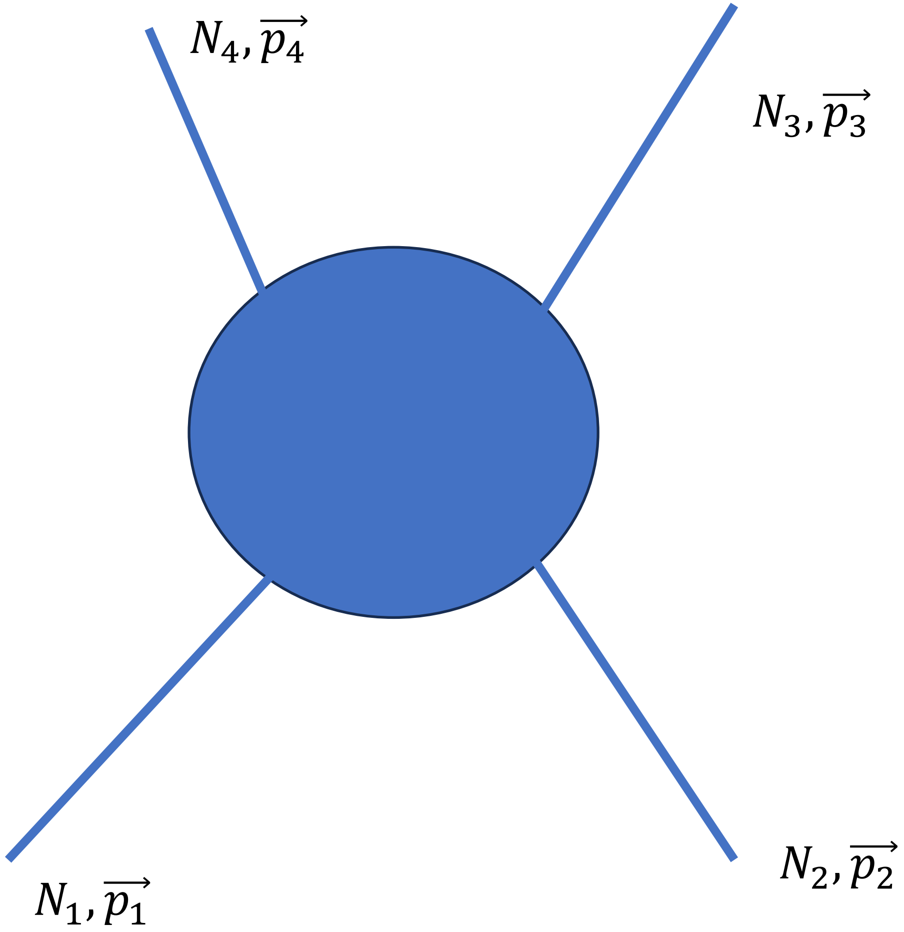
$$\begin{pmatrix} x_3 \mathbf{1}_{N_3} & 0 \\ 0 & x_4 \mathbf{1}_{N_4} \end{pmatrix}$$

$$\Big\uparrow$$

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$\Big\uparrow$$

$$\begin{pmatrix} x_1 \mathbf{1}_{N_1} & 0 \\ 0 & x_2 \mathbf{1}_{N_2} \end{pmatrix}$$



$$N_1 + N_2 = N_3 + N_4$$

Large N limit

$$N \to \infty \ , \qquad \frac{N}{R} = \text{fixed} \ , \qquad \vec{p} = \text{fixed} \ , \qquad \to \qquad -p_+ = \text{fixed}$$

This is a very low energy regime in the matrix model.

It corresponds to very strong coupling.

$$\frac{g_{YM}^2 N}{|p_+|^3} = R^3 N \propto N^4$$

This is also sometimes equivalently stated as

$$N \to \infty \ , \qquad R = \text{fixed} \ , \qquad \vec{p} = \text{fixed} \ , \qquad \to \qquad -p_+ \sim \frac{1}{N}\text{fixed}$$

The conjecture

$$\lim_{N\to\infty} \mathcal{A}_{MM}(\vec{p}_i, N_i) = \mathcal{A}(p_i^\mu)$$

Scattering in matrix model

11d M-theory scattering amplitudes.

<u>Subconjectures:</u>

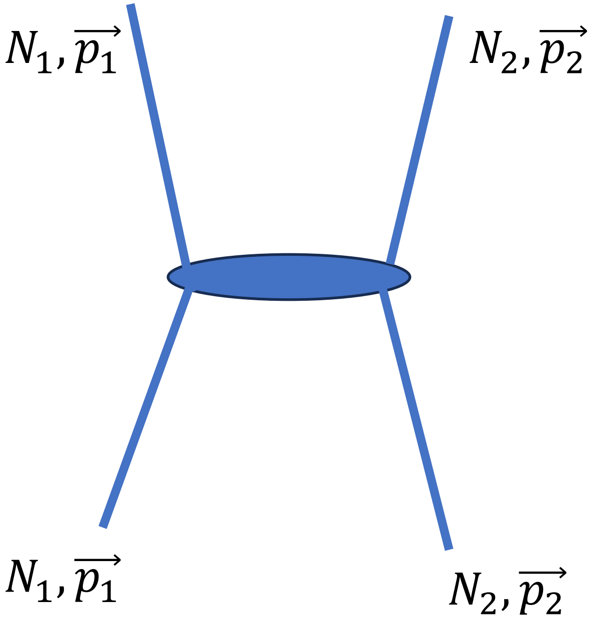
1) Limit exists
2) It is Lorentz invariant.
3) It is unitary (No probability for states that are not within this limit)

   (eg. Producing states with low N is suppressed).

4) It has all properties we expect from M theory.

   (e.g. reproduces gravity at low energies, incorporates black holes, etc)

What can be checked?



-No longitudinal momentum transfer.

-Very small transverse momentum transfer.

-Small velocity expansion.

-Leading interactions:

$$c_1 \frac{v^4}{r^7} + c_2 \frac{v^6}{r^{14}}$$

They are protected by non-renormalization theorems.

Paban, Sethi, Stern
Becker, Becker, Polchinski

Many nice properties under compactification

Compactifying one dimension

- $x^9 = x^9 + 2\pi R_9$
- Matrix QM → 1 + 1 dimensional field theory on circle of size $1/R_9$ (T-duality)
- For small R_9 → low energies in the 1+1 dimensional theory → moduli space = $(R^8)^N/S_N$ → viewed as strings in the light cone gauge.
- In this limit → interactions look like those of perturbative strings in light cone gauge.
- It looks like we are recovering string perturbation theory in this limit.

We get the right spectrum of BPS states for the compactified theories, up to M-theory on T^5

It is hard to compute anything…

It is a version of holography for flat space

We do not have a theory at the boundary, but we do have a procedure for determining the S-matrix.

Some questions

- Does the limit exist?
- Is there an argument similar to the t’ Hooft argument that tells us that we expect such a limit to exist?
- Can we show that objects that are outside this limit do are produced only with vanishingly small probability? (e.g. sub-blocks with N_i =1)

- Recovering gravity at low energies?

- Asymptotic symmetries, and other properties of flat space gravity ``celestial holography” ?

Now we describe our two recent papers with Aidan Herderschee:

ArXiv 2312.12592 Three point functions

ArXiv 2312.15111 Soft theorems

1) Using matrix theory, we computed the on shell three point amplitude for general (complex) momenta.

2) Using that the above 3pt amplitude agrees with gravity + some factorization properties of matrix theory $\rightarrow$ argue for the validity of the leading and subleading soft theorems $\rightarrow$ Lorentz invariance.

Let us discuss first the three point amplitudes

Three point amplitudes

- Massless three point amplitudes are non-zero for complex values of the momenta (or on (2,9) signature)

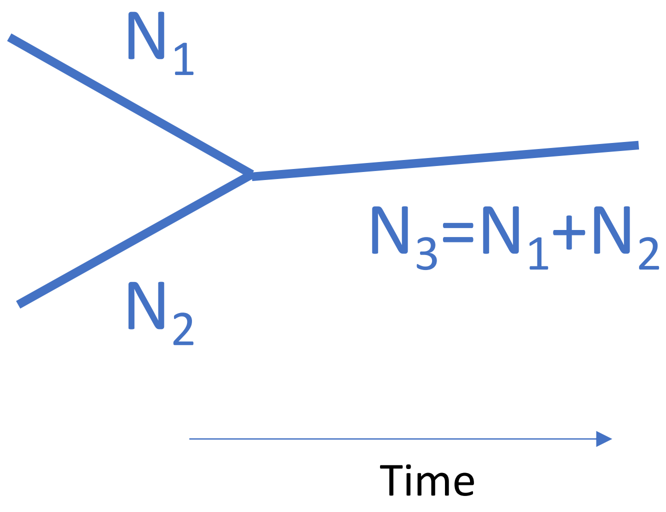
- Poincare symmetry (translations + Lorentz) fix these amplitudes up to a few coefficients.

- Maximal supersymmetry → fixes it completely up to an overall number, $\propto \sqrt{G_N}$. Only one structure.

Even though the three point amplitude is fixed by the symmetries, it is a non-trivial computation in the matrix model because it does <u>not</u> have all these symmetries (at finite N).

Suppose we have three momenta, with nonzero p_- , or non zero N.

In matrix theory the process involves two bound states with N_1, N_2 joining to make one with $N_3 = N_1 + N_2$



Our strategy is based on the following ideas:

- The kinematics of the on shell 3 point amplitudes preserves ¼ of SUSY.

- We first compactify one extra spatial dimension to transform the problem into a 1+1 gauge theory.

- We relate the amplitude to a supersymmetric index.

Kinematics of the three point amplitude

$$(p_+, p_-, p_z, p_{\bar{z}}) \ , \qquad p_z = p_9 + i p_8 \ , \qquad p_{\bar{z}} = p_9 - i p_8$$

$$p_1 = (0, -\frac{N_1}{R}, p_z, 0) \ , \qquad p_2 = (0, -\frac{N_2}{R}, -p_z, 0) \ , \qquad p_3 = (0, \frac{N_3}{R}, 0, 0)$$

With p_9 real, this implies that p_8 is purely imaginary.

(Only four of the components of the momenta are shown, the rest are zero)

The SUSY preserved by the three point amplitude

$$\not{p}_i \epsilon = 0$$

$$\Gamma^- \epsilon = \Gamma^z \epsilon = 0$$

These two conditions are compatible with each other.

Preserve ¼ of the supersymmetries.

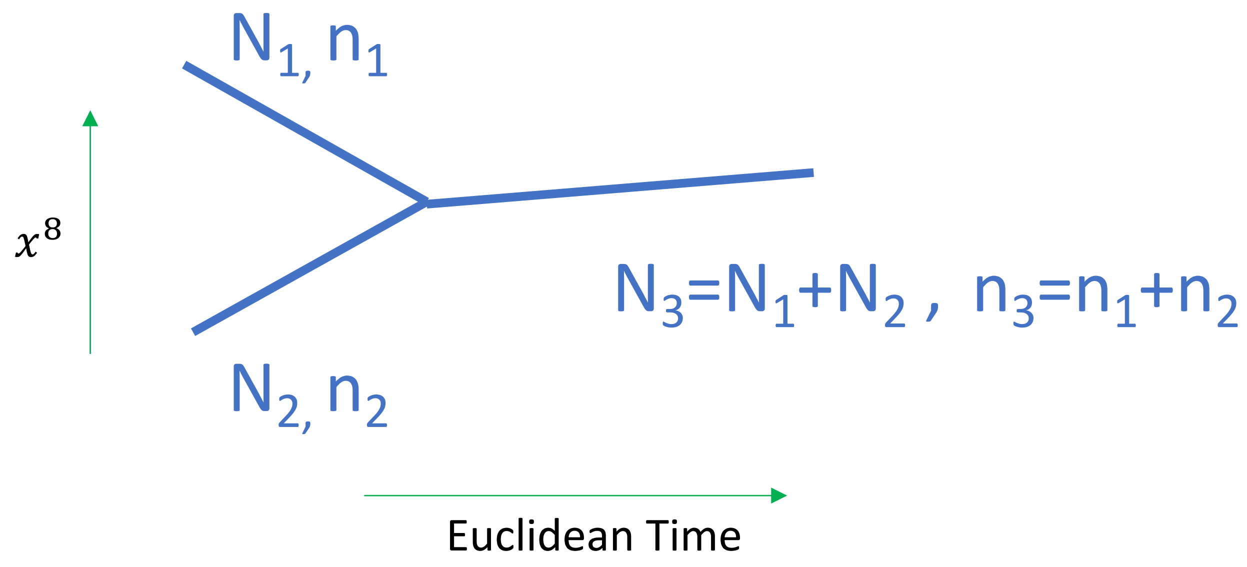
Compactifying the 9th dimension

Matrix QM to 1+1 dimensional matrix gauge theory.

Momentum along the 9th direction = n/R_9 $\rightarrow$

n = flux of the gauge field on the 1+1 dimensional worldvolume.

Similar to N D1 branes with n units of F1 string charge. (p,q) = (N,n) strings.
More precisely, U(N) gauge theory with n units of electric flux under the U(1).



Similar to (N,n) string junction.

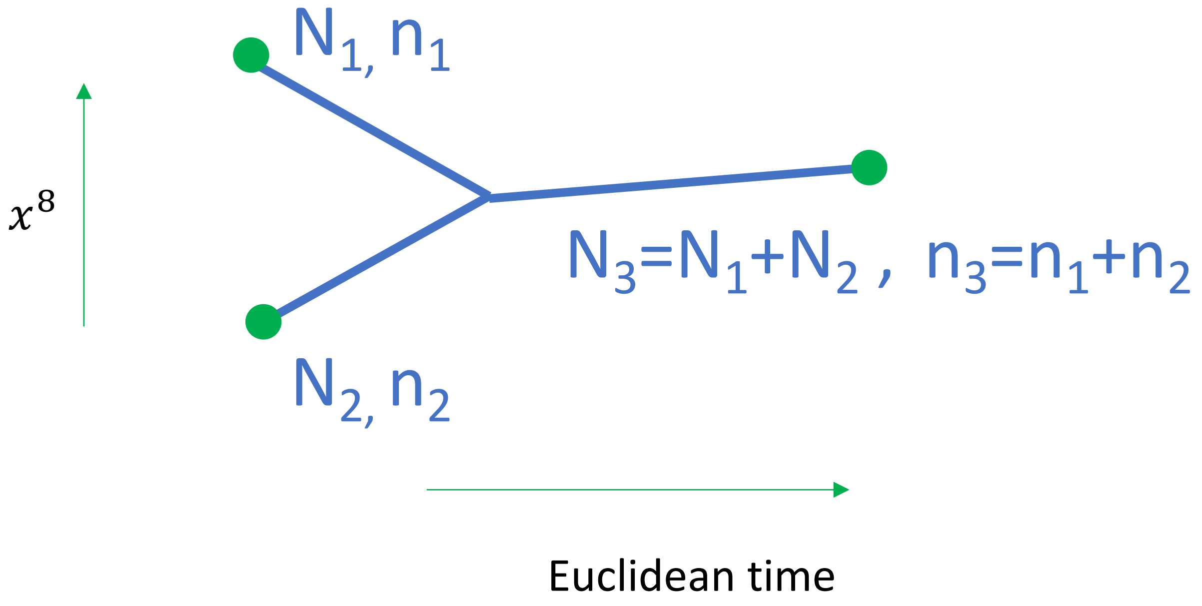
When N and n are coprime → subthreshold bound state.
More clearly separated from the continuum.

Witten

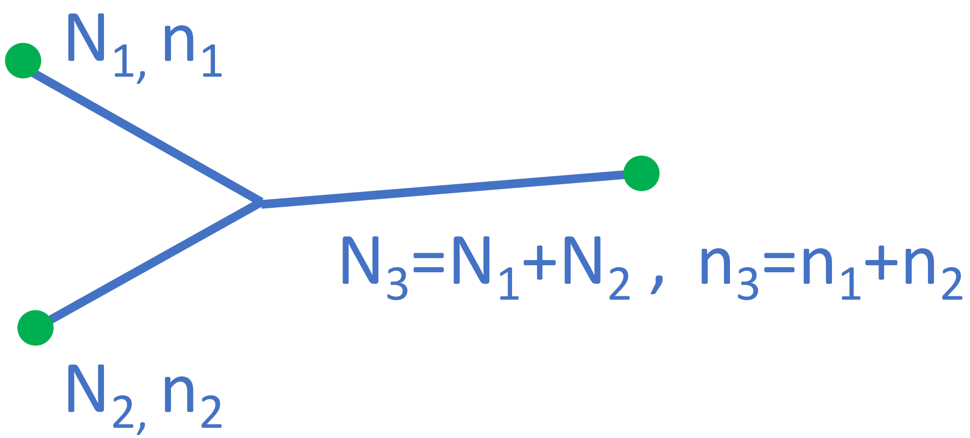
The are extended along the compact $\tilde{9}$ circle

Thinking of $\tilde{9}$ as the Euclidean time circle → index computation Tr[$(-1)^F$]

An extra step that helps the computation: Make them end on D3 branes. (still preserves SUSY).



We can now use a computation that Sen did for the index. (Related to dyons in SU(3) SYM).



The index is zero due to fermion zero modes coming from broken supersymmetries.

We can soak them up by inserting angular momentum generators.

$$Tr[J^6(-1)^F] = \pm(N_1 n_2 - N_2 n_1)$$ Sen

Sen computed it using the wall crossing formula.

The expected three point amplitude

To describe the multiplet it is convenient to use four dimensional notation.

$$\mathcal{A}_3 \sim \sqrt{G_{N,11}}\,\delta\left(\sum p_-^i\right)\delta\left(\sum p_+^i\right)\delta^9\left(\sum \vec{p}^{\,i}\right)\delta^{16}\left(\lambda_\alpha^i \bar{\eta}_I^i\right)\frac{1}{\left(\langle 1,2\rangle\langle 2,3\rangle\langle 1,3\rangle\right)^2}$$

We now do three things:

1) Express the amplitude starting from the polarization state created by the D3 branes.

2) Set the momenta equal to the N, n values.

After a bunch of redefinitions of the η variables we get the two and three point amplitudes

$$\hat{\mathcal{A}}_2^c \sim \delta^8(\vec{p}+\vec{p}')\delta^4(\bar{\eta}_J+\bar{\eta}'_J)\delta^4(\eta^K-\eta'^K)\ , \qquad N=N'\ , \qquad n=n' \qquad \text{(normalization)}$$

$$\begin{aligned}\hat{\mathcal{A}}_3^c &\sim \delta^8(\sum \vec{p}^i)\delta^4(\bar{\eta}_J^{cm}+\bar{\eta}_J^3)\delta^4(\eta^{cmK}-\eta^{3K})\times \\ &\times l_p^{9/2}\frac{R_-}{R_9^{5/2}}\frac{(n_1N_2-n_2N_1)^2N_3^2}{(N_1N_2)^2\sqrt{N_1N_2N_3}}\delta(\sum p_+^i)\delta^4(\bar{\eta}_J^r)\ .\end{aligned}$$

First line comes from the CM degrees of freedom, or U(1) in U(N_3).

The second from SU(N_3). $N_3=N_1+N_2$

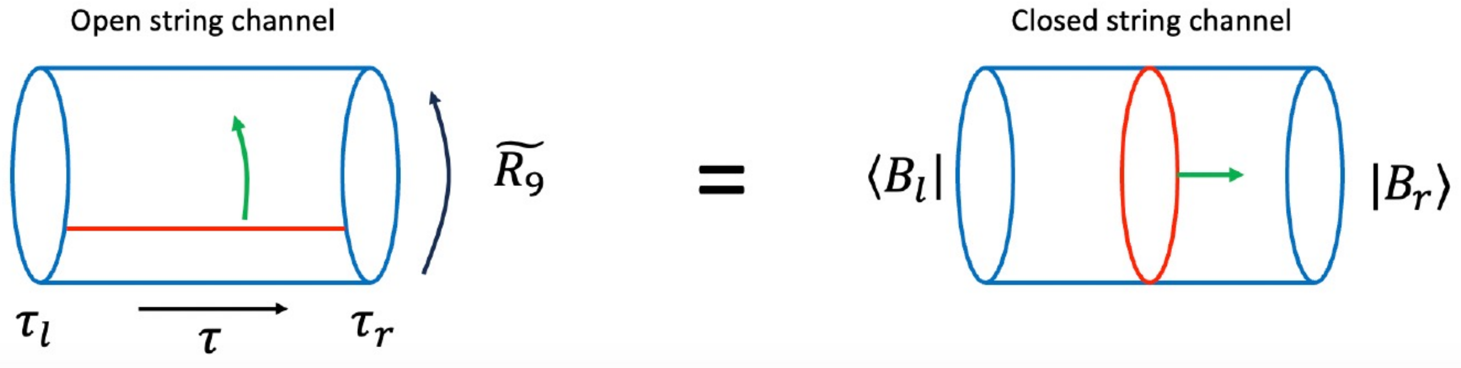
$$
\begin{aligned}
\hat{\mathcal{A}}_2^c &\sim \delta^8(\vec{p}+\vec{p}')\delta^4(\bar{\eta}_J+\bar{\eta}'_J)\delta^4(\eta^K-{\eta'}^K)\ , \qquad N=N'\ , \qquad n=n' \\
\hat{\mathcal{A}}_3^c &\sim \delta^8\Big(\sum \vec{p}^i\Big)\delta^4(\bar{\eta}_J^{cm}+\bar{\eta}_J^3)\delta^4(\eta^{cmK}-\eta^{3K})\times \\
& \boxed{\times l_p^{9/2}\frac{R_-}{R_9^{5/2}}\frac{(n_1N_2-n_2N_1)^2N_3^2}{(N_1N_2)^2\sqrt{N_1N_2N_3}}\delta\Big(\sum p_+^i\Big)\delta^4(\bar{\eta}_J^r)}\ .
\end{aligned}
$$

The fermionic delta functions are related to the insertions of the angular momentum (which inserts fermion zero modes)

We expected the second line to match the index for the $SU(N_3)$ theory.

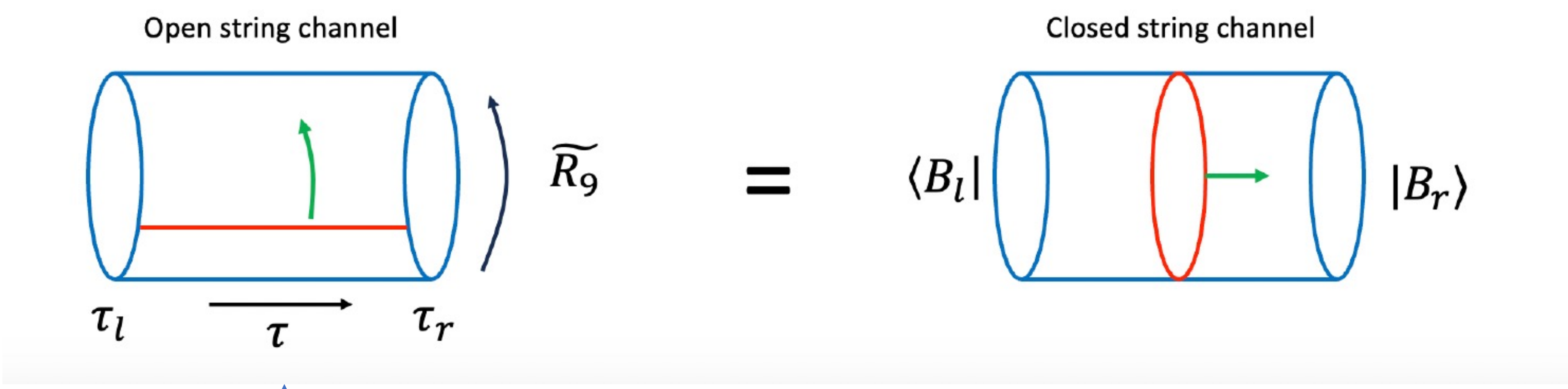
The computation has an ``open string'' channel and a ``closed string'' channel.

We have a 2d theory, not a string worldsheet.

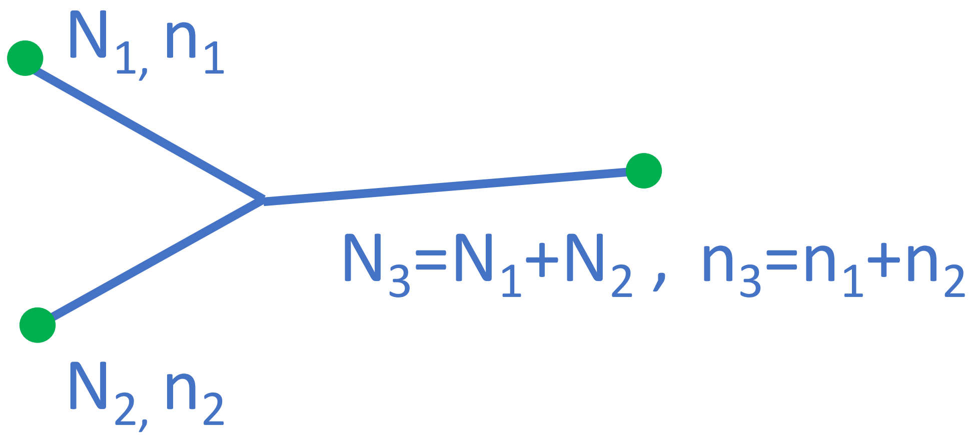


Index

The boundary states creates the states
we are scattering with some amplitudes which are non-trivial functions.



Index



$$\langle B_l|p_1, p_2\rangle\mathcal{A}_3\langle p_3|B_r\rangle = \langle B_1|p_1\rangle\langle B_2|p_2\rangle\mathcal{A}_3\langle p_3|B_r\rangle$$

In addition, in the scattering amplitude, we need to integrate over the interaction time.

Fix $\langle B|p\rangle$ by looking at the two point function.

N, n

$$Tr[J^4(-1)^F] = 1$$

This involves again the relation between an index and an amplitude.

The fermion insertions in the open and closed string channels are normalized differently.

This leads to some $\tilde{R}_9$ dependence for the overlaps.

Fix $\langle B|p\rangle$

After working this out explicitly (I spare you details) we get precise agreement with the full functional form of the amplitude.

We did not compute the overall numerical coefficient, but it should also work. In other words, our method can in principle also determine the coefficient. (We did argue that we get the right coefficient when $N_2 = 0$, but this is a very special case).

We could now take the large N and n limits and recover the amplitudes in completely un-compacified eleven dimensions.

Conclusions

- We obtained precise agreement with the expected amplitude, even at finite N, n.
- (We did not compute the numerical coefficient, but it should be computable using this method).

- This is a simple example of an amplitude with generic values of the N_i

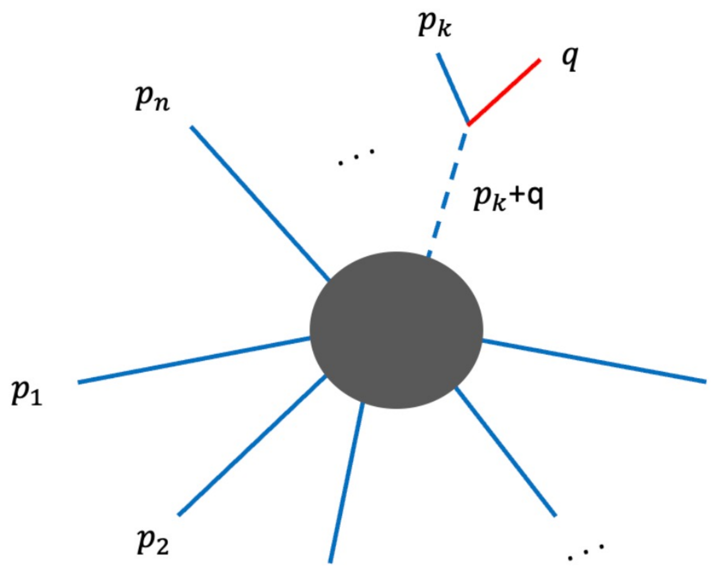
Soft factors from the matrix model

Previous discussions: Miller, Strominger, Tropper, Wang ;   Tropper, Wang.

Soft factors in gravity amplitudes

$$\mathcal{A}_{n+1}(p_1,\cdots p_n,q)\sim(\underset{\substack{\uparrow\\ \frac{1}{q}}}{S_0}+\underset{\substack{\uparrow\\ q^0}}{S_1})\mathcal{A}_n(p_1,\cdots,p_n)+\cdots\ ,\qquad \text{as}\quad q\to 0$$

They involve the three point amplitude



Look at the theory at long distances. The n point amplitude is like an n-point vertex.

The soft factors is putting this vertex in a curved background. → covariantizing the interaction.

We should be able to consistently couple to a slowly changing background gravitational field.

Consistency of the soft factors implies Poincare symmetry: Translations + Lorentz

General argument for soft factors

- <u>Assumptions</u>:
  - 1) Amplitudes are suitably analytic.
  - 2) When the soft q and one external line add up to zero the amplitude factorizes

$$\mathcal{A}_{n+1} \sim \mathcal{A}_3 \frac{1}{(p_i+q)^2} \mathcal{A}_n \ , \qquad \text{for} \quad (p_i+q)^2 \to 0$$

  - 3) All other singularities, branch cuts, etc, are subleading as a function of q. (they give higher powers of q as q → 0).
  - 4) The three point function is the standard relativistic one.

We do not assume that the amplitude is Lorentz invariant.

We will argue that these properties are true in the matrix model.

1) Is an assumption.
2) Follows from scattering properties in the matrix model and the existence of the onshell intermediate state
3) Follows from phase space volume considerations for multiparticle states.
4) Previous argument.

When two sub-blocks merge or give a singularity → we expect that it should come from some on shell intermediate states, a vanishing denominator in perturbation theory.

$$\sum_I \frac{1}{E_I - E_{ext}} \mathcal{M}$$

Energy of the intermediate state
In $U(N_I)$ theory.

Energy of the external states , $E_k + E_s$

$$E_I = \frac{1}{2}\frac{\vec{p}_I^2 R_-}{N_I} + \epsilon_I$$

$$N_I = N_k + N_s$$

$SU(N_I)$

Centre of mass $U(1)$

We use momentum conservation to write

Combine to the full Lorentz invariant in 11d.

$$E_I - E_{ext} = \frac{p^s . p^k}{N_I} + \epsilon_I$$

Two types of intermediate states:

-- Bound state with $\epsilon_I = 0$. → gives us the desired on shell pole.

-- Continuum of states, multiparticle states. Integral over the relative momenta of subblocks.

$$\int d^9 p_r \frac{1}{p_r^2 + \zeta} \propto \zeta^{7/2} \ , \qquad \text{as} \ \ \zeta \to 0 \ , \qquad \zeta \propto p^k . p^s$$

Leads to branch cuts, but it is ``small'' when the soft momentum goes to zero.

We have now argued that the matrix model obeys the assumptions mentioned above.

We can now use a contour deformation argument similar to the one used by BCFW to find the expression for the soft factor.

Britto, Cachazo, Feng, Witten

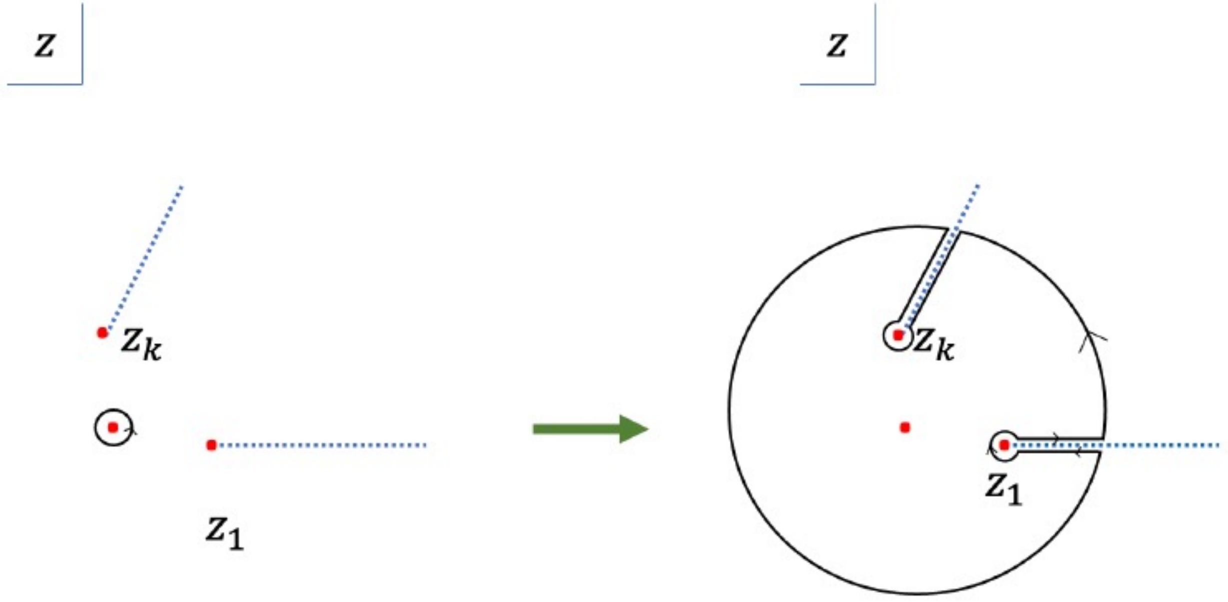
Arkani-Hamed, Cachazo, Kaplan

Let’s first give the general idea

We introduce a complex deformation of the kinematics of the amplitude, z=0 is the original amplitude.

$$\mathcal{A}_{n+1} = \frac{1}{2\pi i}\oint \frac{dz}{z}\mathcal{A}_{n+1}(z)$$

We now deform the contour



(a) (b)

We only pick up poles near the origin, which appear when q → 0.
The contour is shifted to a finite position, not to infinity.
Except for the poles, the rest of the contour has a subleading behavior in the limit q → 0.

To describe the deformation, we used four dimensional kinematics. (It would be nice to extend the argument to general kinematics)(It is good enough for the four point amplitude).

$$\mathcal{A}_{n+1}(s, 1, \cdots, n)$$

All momenta are in four of the dimensions.

Use the usual spinor helicity variables $\lambda_\alpha = |\lambda\rangle$, $\bar{\lambda}_{\dot{\beta}} = |\lambda]$

$$|s\rangle \to |s(z)\rangle = |s\rangle + z|n\rangle \ , \qquad |n] \to |n(z)] = |n] - z|s]$$

Soft particle

Another particle

Picking up the residues at all poles we obtain

$$\mathcal{A}_{\text{poles}} \propto \sum_{k \neq n} \frac{\langle n, k \rangle^2}{\langle n, s \rangle^2} \frac{[s, k]}{\langle s, k \rangle} A_n(z_k)$$

Expanding we find

$$\mathcal{A}_{n+1} = S_0 \mathcal{A}_n + S_1 \mathcal{A}_n + \text{subleading}$$

With

$$S_0 \mathcal{A}_n \propto \sum_k \frac{\langle n, k\rangle^2}{\langle n, s\rangle^2} \frac{[s, k]}{\langle s, k\rangle} A_n(0)$$

$$S_1 \mathcal{A}_n = \frac{1}{2} \sum_k \frac{\langle n, k\rangle}{\langle n, s\rangle} \frac{[s, k]}{\langle s, k\rangle} \bar{\lambda}^s_{\dot\beta} \frac{\partial \mathcal{A}_n}{\partial \bar{\lambda}^k_{\dot\beta}}$$

There is only one aspect of this that is important for us.

These expressions are selecting one of the particles, the particle n in this case. We could repeat this argument with other particles, say m, or m'.

$$S_0 \mathcal{A}_n \propto \sum_k \frac{\langle n,k\rangle^2}{\langle n,s\rangle^2} \frac{[s,k]}{\langle s,k\rangle} A_n(0)$$

$$S_1 \mathcal{A}_n = \frac{1}{2} \sum_k \frac{\langle n,k\rangle}{\langle n,s\rangle} \frac{[s,k]}{\langle s,k\rangle} \bar{\lambda}^s_{\dot{\beta}} \frac{\partial \mathcal{A}_n}{\partial \bar{\lambda}^k_{\dot{\beta}}}$$

Demanding that the answer is independent of the ``special'' particle we get a constraint

Leading soft factor → energy momentum conservation for the n-point amplitude.

Subleading soft factor → Lorentz invariance of the n-point amplitude.

Tropper, Wang

So we see that we get a consistent answer only when the amplitude is Lorentz invariant.

This is interesting because this is not a property that we have put in.
We have used other reasonable properties of the matrix model amplitude.

Comment:

It is important for this argument that the soft momentum has a non-zero N_{soft} , and that we take the large N limit first, and then we take $q_- \to 0$.

It is important because the non-trivial Lorentz generators involve a derivative with respect to p_-.

Conclusion

- We have argued for the soft limits using the matrix model.
- We found that consistency implies Lorentz invariance.

Future

- Perhaps a similar BCFW-like argument would work for showing that the tree level gravity amplitudes are reproduced… (Recall that BCFW showed that the three point amplitudes determines all other amplitudes by a recursion relation). (Issues: Large z, other branch cuts, etc.)