J. Isaacson, S. Höche, D. Lopez Gutierrez, and N. Rocco, Phys. Rev. D 105, 096006 (2022)

Novel event generator for the automated simulation of neutrino scattering

Diego Lopez Gutierrez

Particle Physics in the Plains 2023

October 14, 2023

Washington University in St. Louis Fermilab

Outline

- Simulation Pipeline
- Current Setup
- Theory Overview
- Nuclear Physics
- Leptonic Current
- Phase Space Integral
- Results
- Summary and Outlook

Simulation Pipeline

Collider physics

$\mathscr{L}_{\text{BSM}}$ + Feynman rules

Neutrino physics

Simulation Pipeline

Collider physics

FeynRules, Universal FeynRules Output (UFO)

$\mathscr{L}_{\text{BSM}}$ + Feynman rules

Neutrino physics

Simulation Pipeline

Collider physics

FeynRules, Universal FeynRules Output (UFO)

$\mathscr{L}_{\text{BSM}}$ + Feynman rules

Neutrino physics

$$\sigma\,;\ \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$$

Simulation Pipeline

Collider physics

Neutrino physics

FeynRules, Universal FeynRules Output (UFO)

$\mathscr{L}_{\text{BSM}}$ + Feynman rules

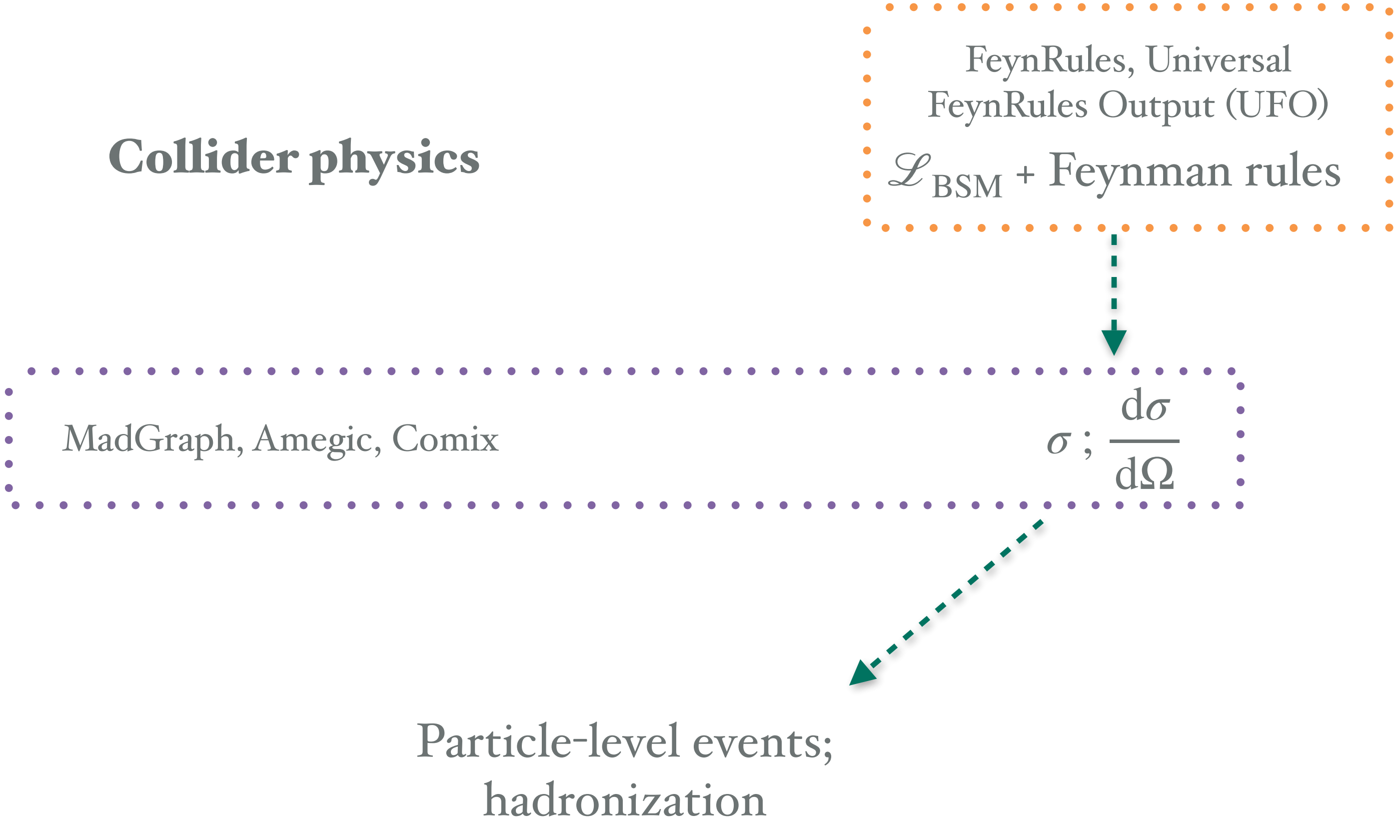
MadGraph, Amegic, Comix

$\sigma\,;\ \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$

Simulation Pipeline

Collider physics

Neutrino physics



FeynRules, Universal FeynRules Output (UFO)

$\mathscr{L}_{\mathrm{BSM}}$ + Feynman rules

MadGraph, Amegic, Comix

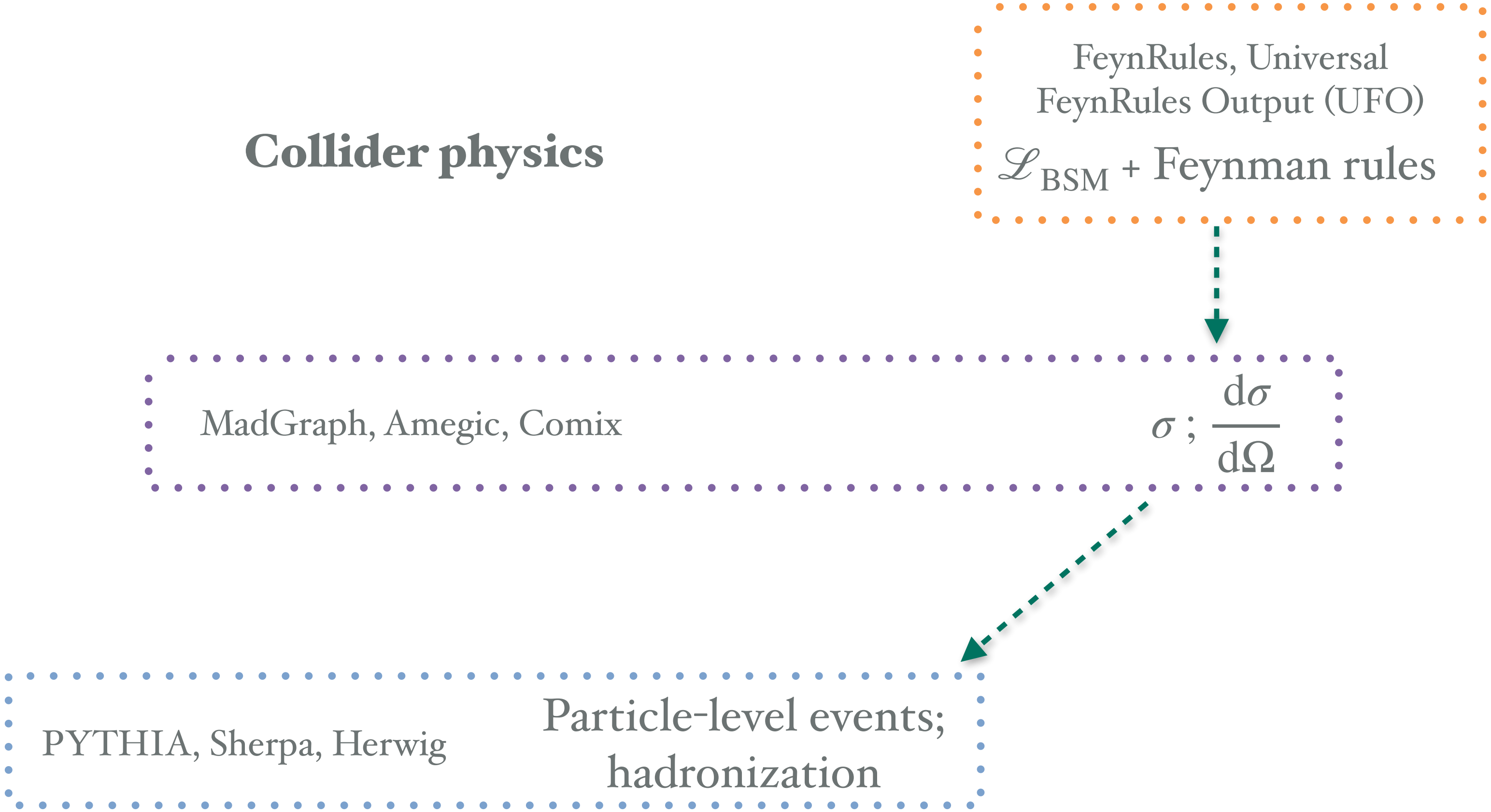
$\sigma \, ; \, \dfrac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$

Particle-level events;
hadronization

Simulation Pipeline

Collider physics

Neutrino physics



FeynRules, Universal FeynRules Output (UFO)

$\mathscr{L}_{\text{BSM}}$ + Feynman rules

MadGraph, Amegic, Comix

$\sigma\,;\ \dfrac{d\sigma}{d\Omega}$

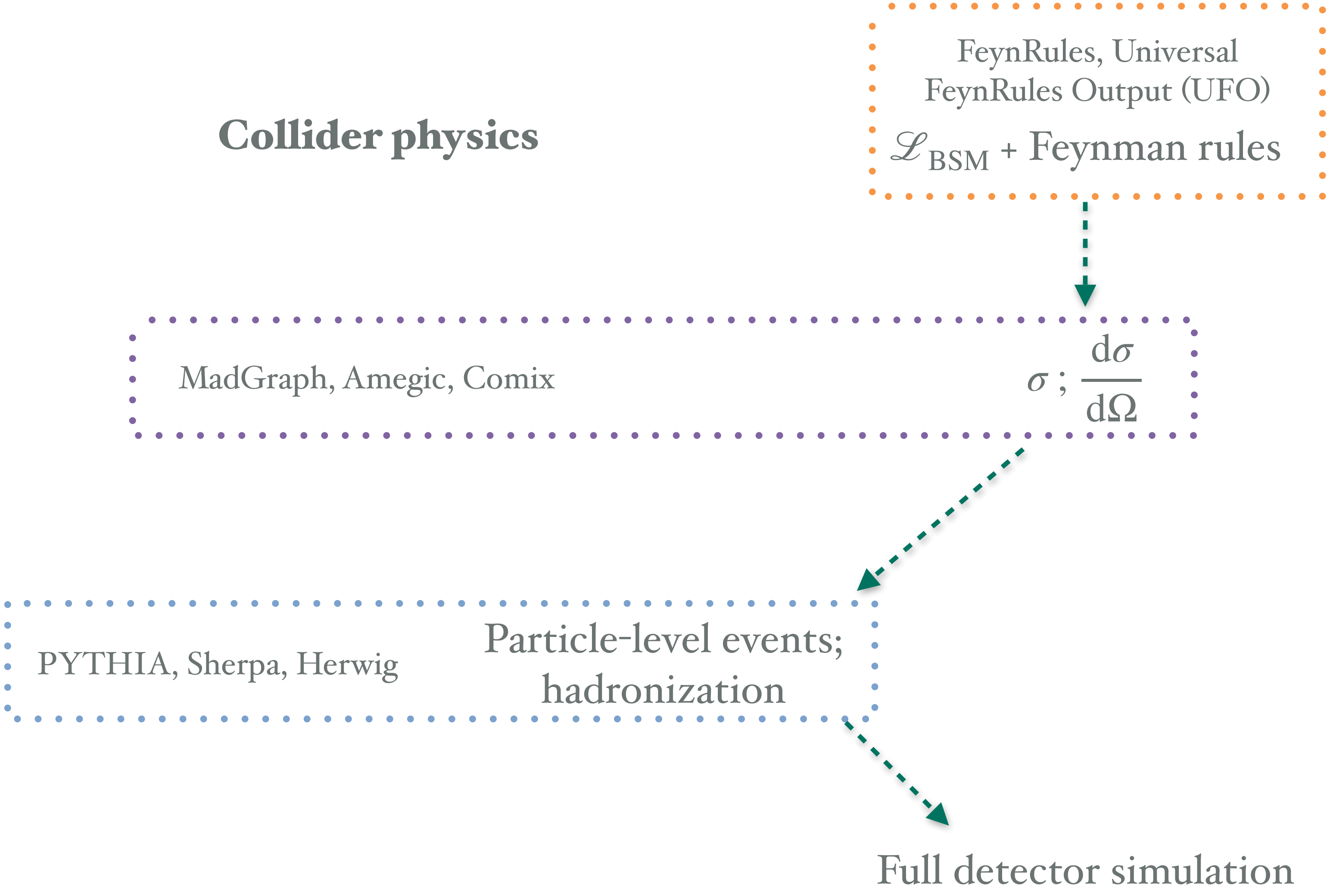
PYTHIA, Sherpa, Herwig

Particle-level events; hadronization

Simulation Pipeline

Collider physics

Neutrino physics



FeynRules, Universal FeynRules Output (UFO)

$\mathscr{L}_{\text{BSM}}$ + Feynman rules

MadGraph, Amegic, Comix

$\sigma\,;\ \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$

PYTHIA, Sherpa, Herwig

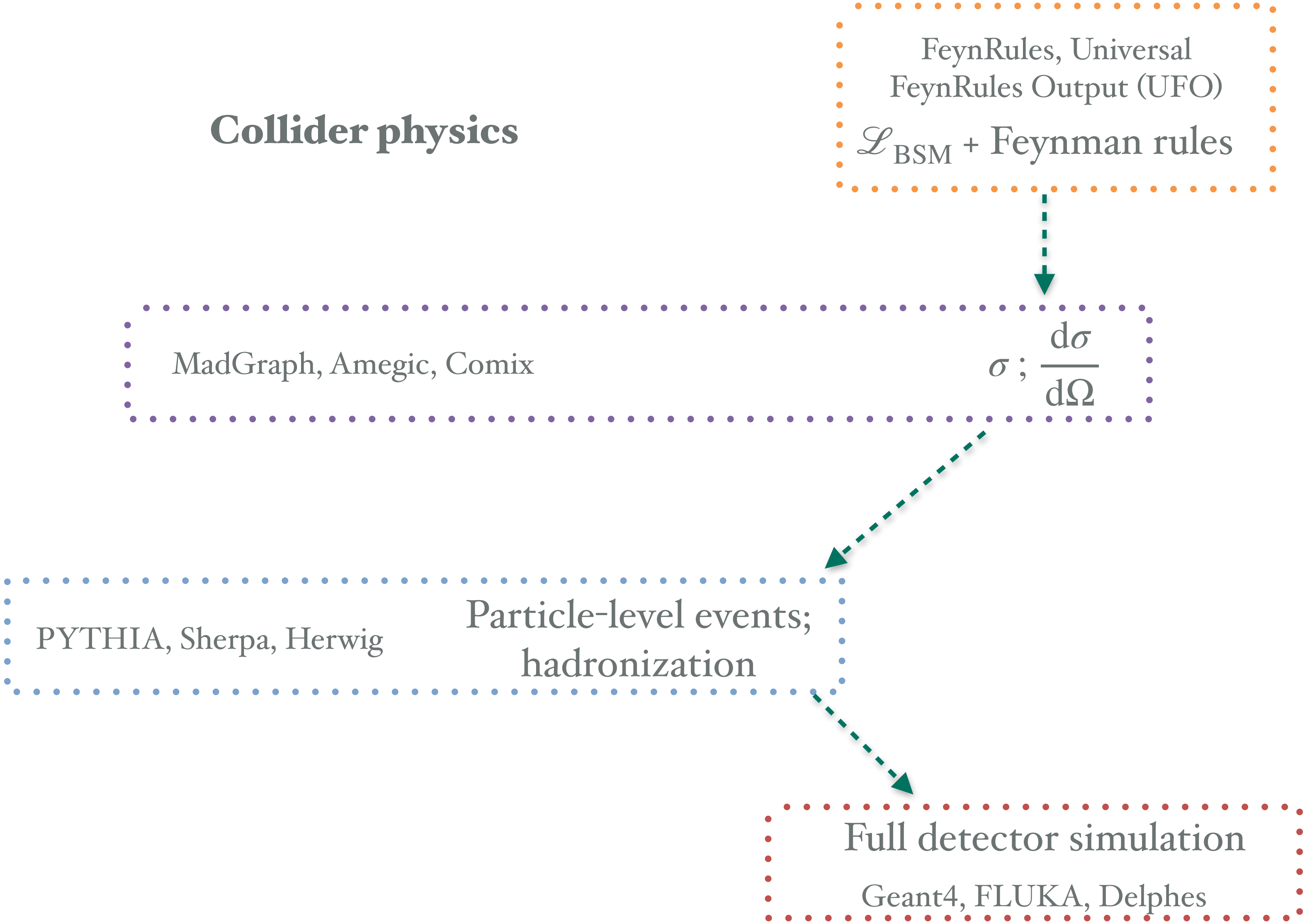
Particle-level events; hadronization

Full detector simulation

Simulation Pipeline

Collider physics

Neutrino physics



FeynRules, Universal FeynRules Output (UFO)

$\mathscr{L}_{\mathrm{BSM}}$ + Feynman rules

MadGraph, Amegic, Comix

$\sigma\,;\ \dfrac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$

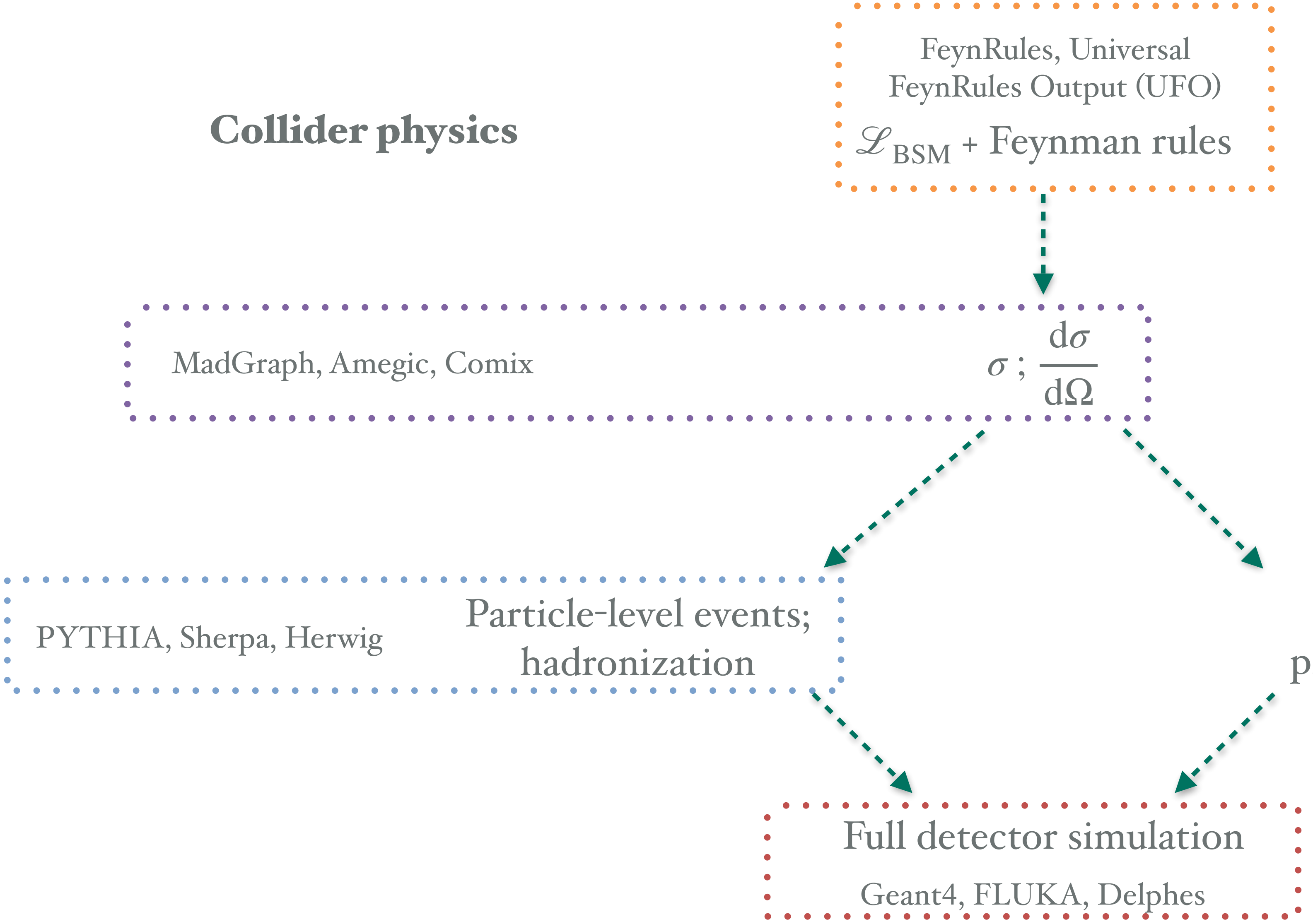
PYTHIA, Sherpa, Herwig

Particle-level events; hadronization

Full detector simulation

Geant4, FLUKA, Delphes

Simulation Pipeline



Collider physics

Neutrino physics

FeynRules, Universal FeynRules Output (UFO)

$\mathscr{L}_{\mathrm{BSM}}$ + Feynman rules

MadGraph, Amegic, Comix

$\sigma\,;\ \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$

PYTHIA, Sherpa, Herwig

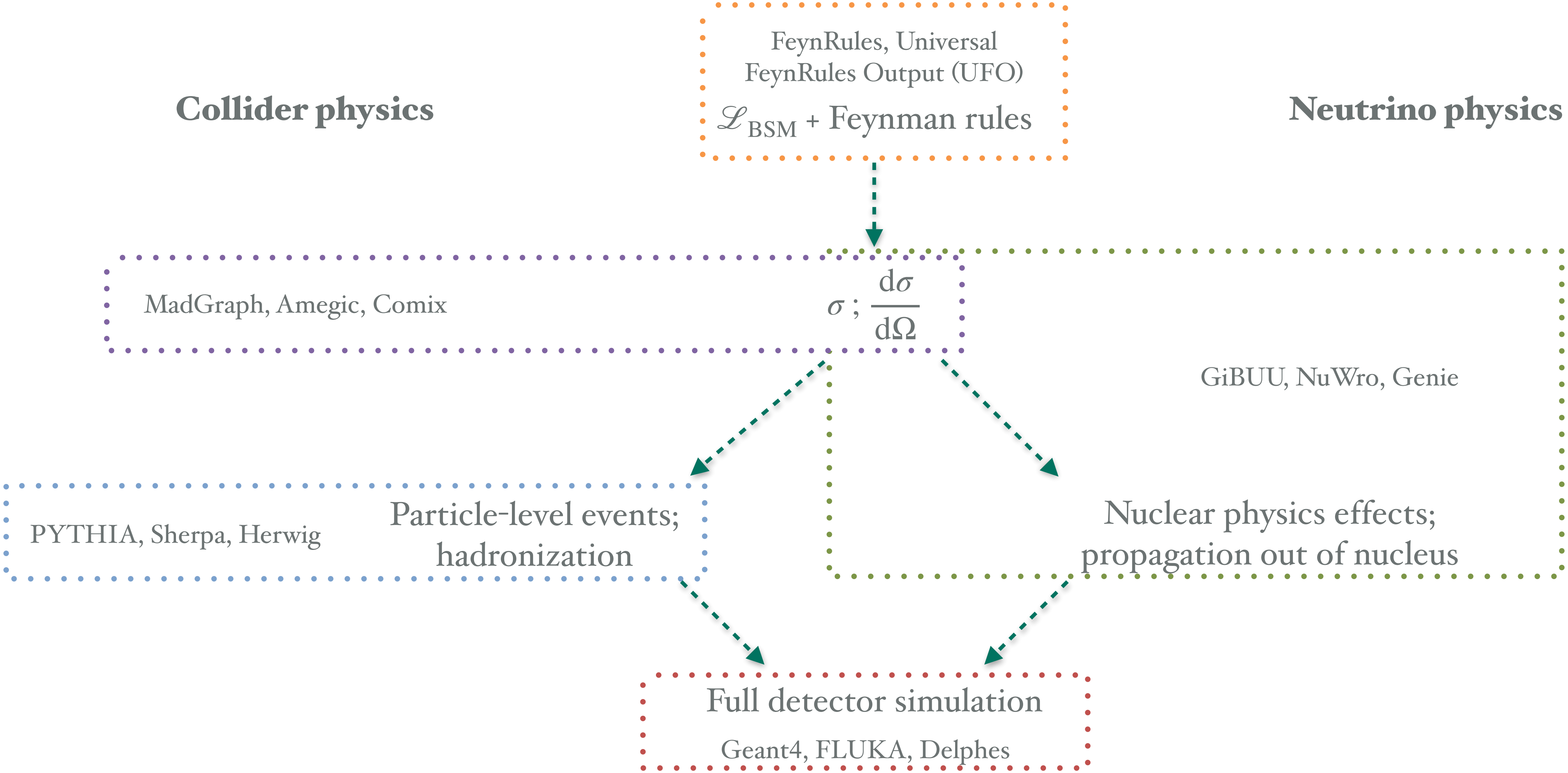
Particle-level events; hadronization

Nuclear physics effects; propagation out of nucleus

Full detector simulation

Geant4, FLUKA, Delphes

Simulation Pipeline



Collider physics

Neutrino physics

FeynRules, Universal FeynRules Output (UFO)

$\mathscr{L}_{\text{BSM}}$ + Feynman rules

MadGraph, Amegic, Comix

$\sigma\,;\ \dfrac{d\sigma}{d\Omega}$

GiBUU, NuWro, Genie

PYTHIA, Sherpa, Herwig

Particle-level events; hadronization

Nuclear physics effects; propagation out of nucleus

Full detector simulation

Geant4, FLUKA, Delphes

Current Setup

- BSM calculations implemented manually into different event generators

Current Setup

- BSM calculations implemented manually into different event generators
  - Prone to errors
  - Time-consuming
  - Infeasible

Current Setup

- BSM calculations implemented manually into different event generators
  - Prone to errors
  - Time-consuming
  - Infeasible
- Need for a flexible tool in neutrino physics community.

Current Setup

- BSM calculations implemented manually into different event generators
  - Prone to errors
  - Time-consuming
  - Infeasible
- Need for a flexible tool in neutrino physics community.
  - Particle-level events generated fully differentially in phase space

Current Setup

- BSM calculations implemented manually into different event generators
  - Prone to errors
  - Time-consuming
  - Infeasible
- Need for a flexible tool in neutrino physics community.
  - Particle-level events generated fully differentially in phase space
  - Easy to implement as part of experimental analysis frameworks

Current Setup

- BSM calculations implemented manually into different event generators
  - Prone to errors
  - Time-consuming
  - Infeasible
- Need for a flexible tool in neutrino physics community.
  - Particle-level events generated fully differentially in phase space
  - Easy to implement as part of experimental analysis frameworks
  - Focused on arbitrary BSM models while also including nuclear effects

Theory Overview

- Single gauge boson:

$$\frac{d\sigma}{d\Omega} \propto L_{\mu\nu} W^{\mu\nu}$$

BSM effects

Nuclear physics (model-dependent)

Theory Overview

- Single gauge boson:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto L_{\mu\nu} W^{\mu\nu}$$

BSM effects

Nuclear physics (model-dependent)

- Interference 1+ dominant boson. Easier to express:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left| \sum_i L_\mu^{(i)} W^{(i)\mu} \right|^2$$

allowed bosons

*Will calculate leptonic currents, but leptonic tensor is available

Nuclear Physics

- BSM effects independent of exact nuclear model.

Nuclear Physics

- BSM effects independent of exact nuclear model.
- Focus on quasielastic region for concreteness (impulse-approximation)

Nuclear Physics

- BSM effects independent of exact nuclear model.
- Focus on quasielastic region for concreteness (impulse-approximation)

$$\mathscr{J} \propto \sum_i \mathscr{J}_i^\mu \qquad ; \qquad |\psi_f^A\rangle \to |\psi_f^{A-1}\rangle \otimes |p_a\rangle$$

Nuclear Physics

- BSM effects independent of exact nuclear model.
- Focus on quasielastic region for concreteness (impulse-approximation)

$$\mathcal{J} \propto \sum_i \mathcal{J}_i^\mu \qquad ; \qquad |\psi_f^A\rangle \to |\psi_f^{A-1}\rangle \otimes |p_a\rangle$$

- $\mathcal{J}_i^\mu$ expressed in terms of process-dependent nuclear form factors:
  - UFO extended to interface with form factors used in neutrino event generators.
  - Express form factors of arbitrary models in terms of EM form factors.

Nuclear Physics

- BSM effects independent of exact nuclear model.
- Focus on quasielastic region for concreteness (impulse-approximation)

$$\mathcal{J} \propto \sum_i \mathcal{J}_i^\mu \qquad ; \qquad |\psi_f^A\rangle \rightarrow |\psi_f^{A-1}\rangle \otimes |p_a\rangle$$

- $\mathcal{J}_i^\mu$ expressed in terms of process-dependent nuclear form factors:
  - UFO extended to interface with form factors used in neutrino event generators.
  - Express form factors of arbitrary models in terms of EM form factors.
- $W^\mu / W^{\mu\nu}$ calculated using impulse-approximation in the spectral function formalism; $S(\vec{p}_a, E_r)$ probability of removing nucleon with $\vec{p}_a$, E_r.

Leptonic Current

- Berends-Giele recursive relations:
  - Full tree-level amplitude determined from off-shell currents:

Leptonic Current

- Berends-Giele recursive relations:
  - Full tree-level amplitude determined from off-shell currents:

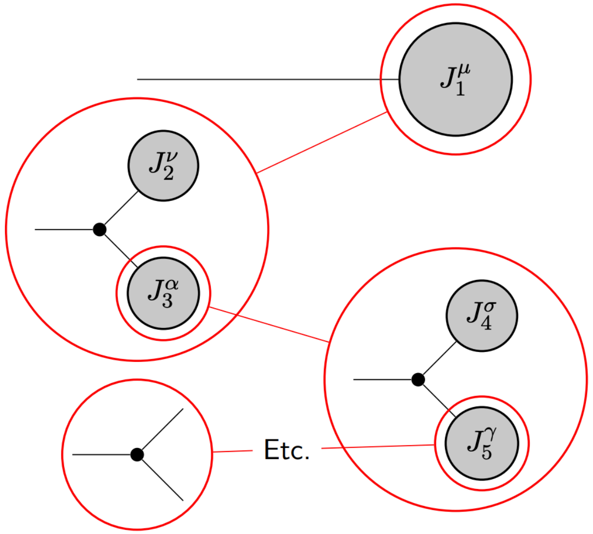
set of particles

$$\boxed{J_\alpha(\pi) = P_\alpha(\pi) \sum_{V_\alpha^{\alpha_1,\alpha_2}} \sum_{P_2(\pi)} S(\pi_1, \pi_2) V_\alpha^{\alpha_1,\alpha_2}(\pi_1, \pi_2) J_{\alpha_1}(\pi_1) J_{\alpha_2}(\pi_2)}$$

particle type

vertices

unordered partitions $\pi \rightarrow \pi_1, \pi_2$



Leptonic Current

- Berends-Giele recursive relations:
  - Full tree-level amplitude determined from off-shell currents:

set of particles

$$\boxed{J_\alpha(\pi) = P_\alpha(\pi) \sum_{V_\alpha^{\alpha_1,\alpha_2}} \sum_{P_2(\pi)} S(\pi_1, \pi_2) V_\alpha^{\alpha_1,\alpha_2}(\pi_1, \pi_2) J_{\alpha_1}(\pi_1) J_{\alpha_2}(\pi_2)}$$

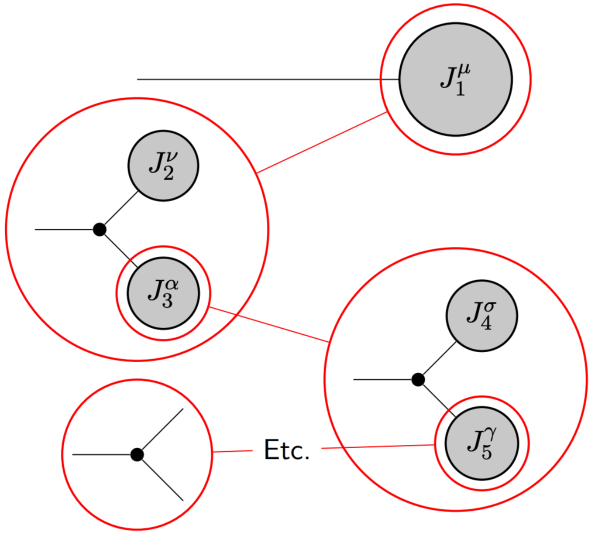
particle type

vertices

unordered partitions $\pi \rightarrow \pi_1, \pi_2$

  - Initial (on-shell, external) currents:

$$J_{\alpha_i}(p_i) = 1 \qquad J_{\alpha_i}(p_i) = u(p_i)\ ,\ v(p_i) \qquad J_{\alpha_i}(p_i) = \varepsilon_\alpha(p_i, k)$$



Leptonic Current

- Berends-Giele recursive relations:
  - Full tree-level amplitude determined from off-shell currents:

set of particles

particle type

$$\boxed{J_\alpha(\pi) = P_\alpha(\pi) \sum_{V_\alpha^{\alpha_1,\alpha_2}} \sum_{P_2(\pi)} S(\pi_1, \pi_2) V_\alpha^{\alpha_1,\alpha_2}(\pi_1, \pi_2) J_{\alpha_1}(\pi_1) J_{\alpha_2}(\pi_2)}$$

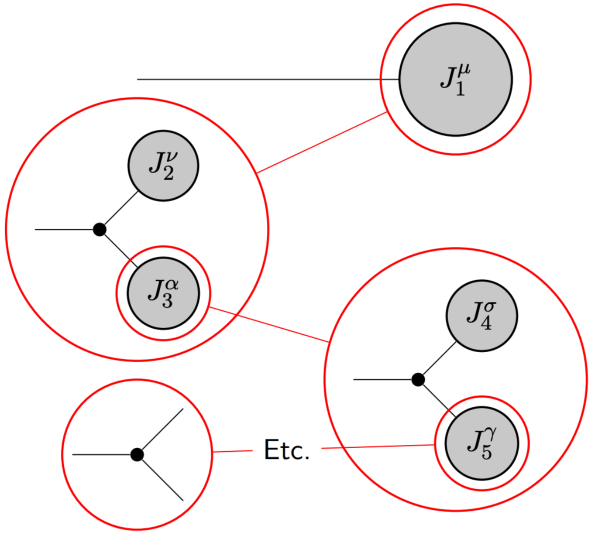
vertices

unordered partitions $\pi \rightarrow \pi_1, \pi_2$

  - Initial (on-shell, external) currents:

$$J_{\alpha_i}(p_i) = 1 \qquad J_{\alpha_i}(p_i) = u(p_i)\ ,\ v(p_i) \qquad J_{\alpha_i}(p_i) = \varepsilon_\alpha(p_i, k)$$

  - For n particles, $\mathcal{O}(3^n)$ instead of $\mathcal{O}(n!)$



Leptonic Current

- Interface to Comix matrix element generator
  - Handles N–point vertices, interfaces to UFO files.

Leptonic Current

- Interface to Comix matrix element generator
  - Handles N–point vertices, interfaces to UFO files.
  - Comix must calculate full matrix element. Point-like nucleons for bookkeeping purposes.

$$L_\mu^{(i)}(1,\ldots,m) = J_\mu^{(i)}(1,\ldots,m)$$

$$L_{\mu\nu}^{(ij)}(1,\ldots,m) = J_\mu^{(i)}(1,\ldots,m) J_\nu^{(j)\dagger}(1,\ldots,m)$$

Leptonic Current

- Interface to Comix matrix element generator
  - Handles $N-$point vertices, interfaces to UFO files.
  - Comix must calculate full matrix element. Point-like nucleons for bookkeeping purposes.

$$L_\mu^{(i)}(1,\ldots,m) = J_\mu^{(i)}(1,\ldots,m)$$

$$L_{\mu\nu}^{(ij)}(1,\ldots,m) = J_\mu^{(i)}(1,\ldots,m)J_\nu^{(j)\dagger}(1,\ldots,m)$$

- Limitations:
  - Handle spin ≤ 1 particles → Implement necessary external states and propagators

Leptonic Current

- Interface to Comix matrix element generator
  - Handles $N-$point vertices, interfaces to UFO files.
  - Comix must calculate full matrix element. Point-like nucleons for bookkeeping purposes.

$$L_\mu^{(i)}(1,\ldots,m) = J_\mu^{(i)}(1,\ldots,m)$$

$$L_{\mu\nu}^{(ij)}(1,\ldots,m) = J_\mu^{(i)}(1,\ldots,m)J_\nu^{(j)\dagger}(1,\ldots,m)$$

- Limitations:
  - Handle spin ≤ 1 particles → Implement necessary external states and propagators
  - Vector particles probe nucleus → Update nuclear physics code to include appropriate form factors for different spin

Leptonic Current

- Interface to Comix matrix element generator
  - Handles $N-$point vertices, interfaces to UFO files.
  - Comix must calculate full matrix element. Point-like nucleons for bookkeeping purposes.

$$L_\mu^{(i)}(1,\ldots,m) = J_\mu^{(i)}(1,\ldots,m)$$

$$L_{\mu\nu}^{(ij)}(1,\ldots,m) = J_\mu^{(i)}(1,\ldots,m)J_\nu^{(j)\dagger}(1,\ldots,m)$$

- Limitations:
  - Handle spin ≤ 1 particles → Implement necessary external states and propagators
  - Vector particles probe nucleus → Update nuclear physics code to include appropriate form factors for different spin
  - Tree-level diagrams → Automation of one-loop diagrams discussed elsewhere

Leptonic Current

- Interface to Comix matrix element generator
  - Handles $N-$point vertices, interfaces to UFO files.
  - Comix must calculate full matrix element. Point-like nucleons for bookkeeping purposes.

$$L_\mu^{(i)}(1,\ldots,m) = J_\mu^{(i)}(1,\ldots,m)$$

$$L_{\mu\nu}^{(ij)}(1,\ldots,m) = J_\mu^{(i)}(1,\ldots,m)J_\nu^{(j)\dagger}(1,\ldots,m)$$

- Limitations:
  - Handle spin ≤ 1 particles → Implement necessary external states and propagators
  - Vector particles probe nucleus → Update nuclear physics code to include appropriate form factors for different spin
  - Tree-level diagrams → Automation of one-loop diagrams discussed elsewhere
  - Only colorless particles → Unsolvable. Assumption of QCD d.o.f. as protons, neutrons

Phase Space Integral

- Recursive phase space:
  - In 2 → n, full phase space can be written:

$$\mathrm{d}\Phi_{\mathrm{n}}(\mathrm{a},\mathrm{b};1,\ldots,\mathrm{n}) = \mathrm{d}\Phi_{\mathrm{n-m+1}}(\mathrm{a},\mathrm{b};\pi,\mathrm{m}+1,\ldots,\mathrm{n})\frac{\mathrm{d}s_{\pi}}{2\pi}\mathrm{d}\Phi_{\mathrm{m}}(\pi,1,\ldots,\mathrm{m})$$

Phase Space Integral

- Recursive phase space:
  - In 2 → n, full phase space can be written:

$$\mathrm{d}\Phi_{\mathrm{n}}(\mathrm{a},\mathrm{b};1,\ldots,\mathrm{n})=\mathrm{d}\Phi_{\mathrm{n-m+1}}(\mathrm{a},\mathrm{b};\pi,\mathrm{m}+1,\ldots,\mathrm{n})\frac{\mathrm{d}s_{\pi}}{2\pi}\mathrm{d}\Phi_{\mathrm{m}}(\pi,1,\ldots,\mathrm{m})$$

  - Each dΦ can be written in a way that optimally samples $s-$ or $t-$ channel processes

Phase Space Integral

- Recursive phase space:
  - In 2 → n, full phase space can be written:

$$\mathrm{d}\Phi_{\mathrm{n}}(\mathrm{a},\mathrm{b};1,\ldots,\mathrm{n}) = \mathrm{d}\Phi_{\mathrm{n-m+1}}(\mathrm{a},\mathrm{b};\pi,\mathrm{m}+1,\ldots,\mathrm{n})\frac{\mathrm{d}s_{\pi}}{2\pi}\mathrm{d}\Phi_{\mathrm{m}}(\pi,1,\ldots,\mathrm{m})$$

  - Each dΦ can be written in a way that optimally samples $s-$ or $t-$ channel processes.

- Optimal integrator:
  - Recursive phase space
  - Multichannel technique ($s-$ and $t-$channels)
  - Adaptive multidimensional Vegas algorithm

Results

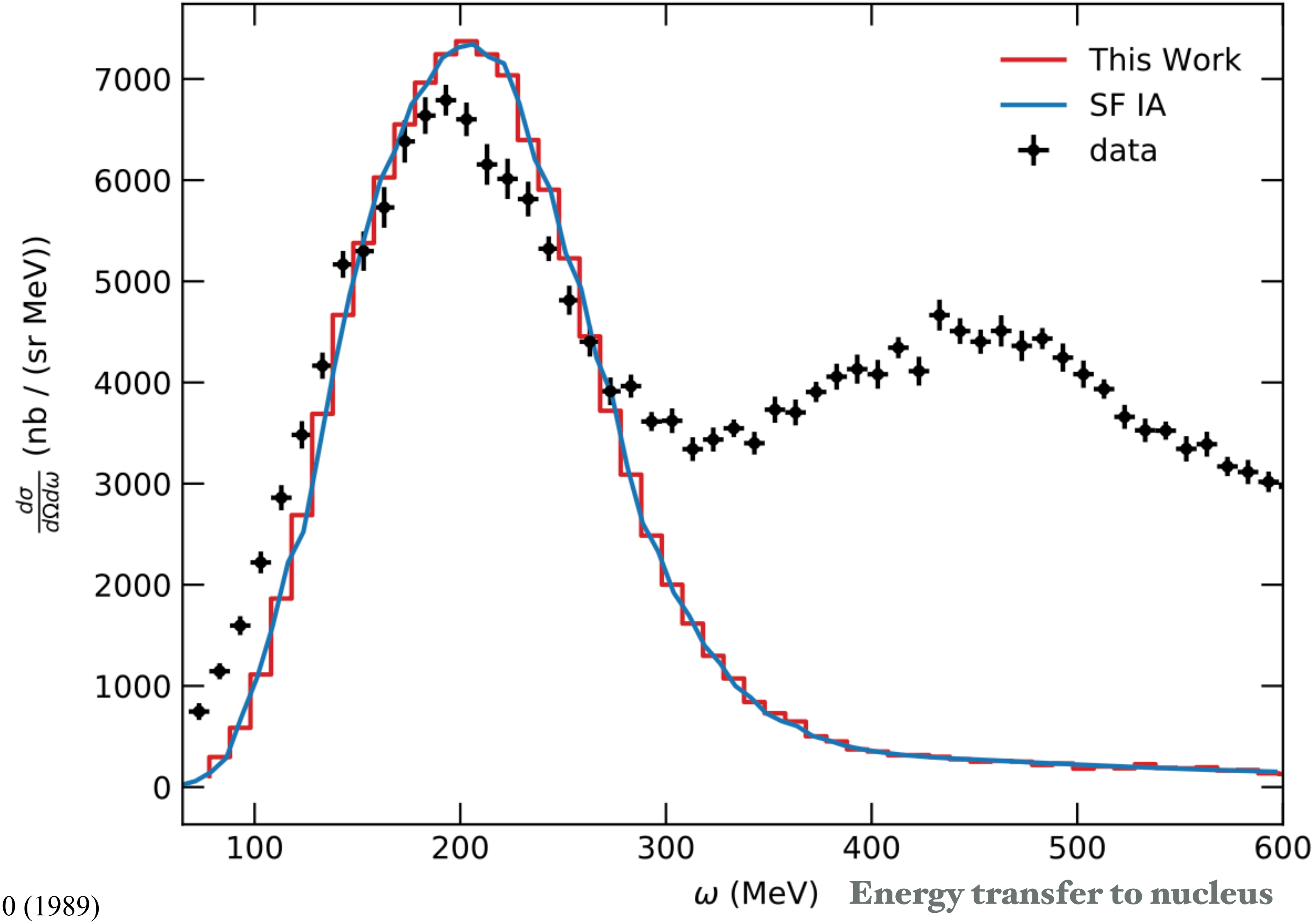
Some choices:

- Choose monochromatic lepton beam, but straightforward to include fluxes.
- No final-state interactions.
- Leptons considered massless.
- Full propagator for $W^{\pm}, Z$ bosons.

$$\begin{pmatrix} \alpha \\ G_F \\ M_Z \end{pmatrix} = \begin{pmatrix} 1/137 \\ 1.16637 \times 10^5\,\mathrm{GeV} \\ 91.1876\,\mathrm{GeV} \end{pmatrix}$$

e^-C scattering

- 961 MeV e⁻ scattering of carbon-12 at an angle of 37.5 deg.

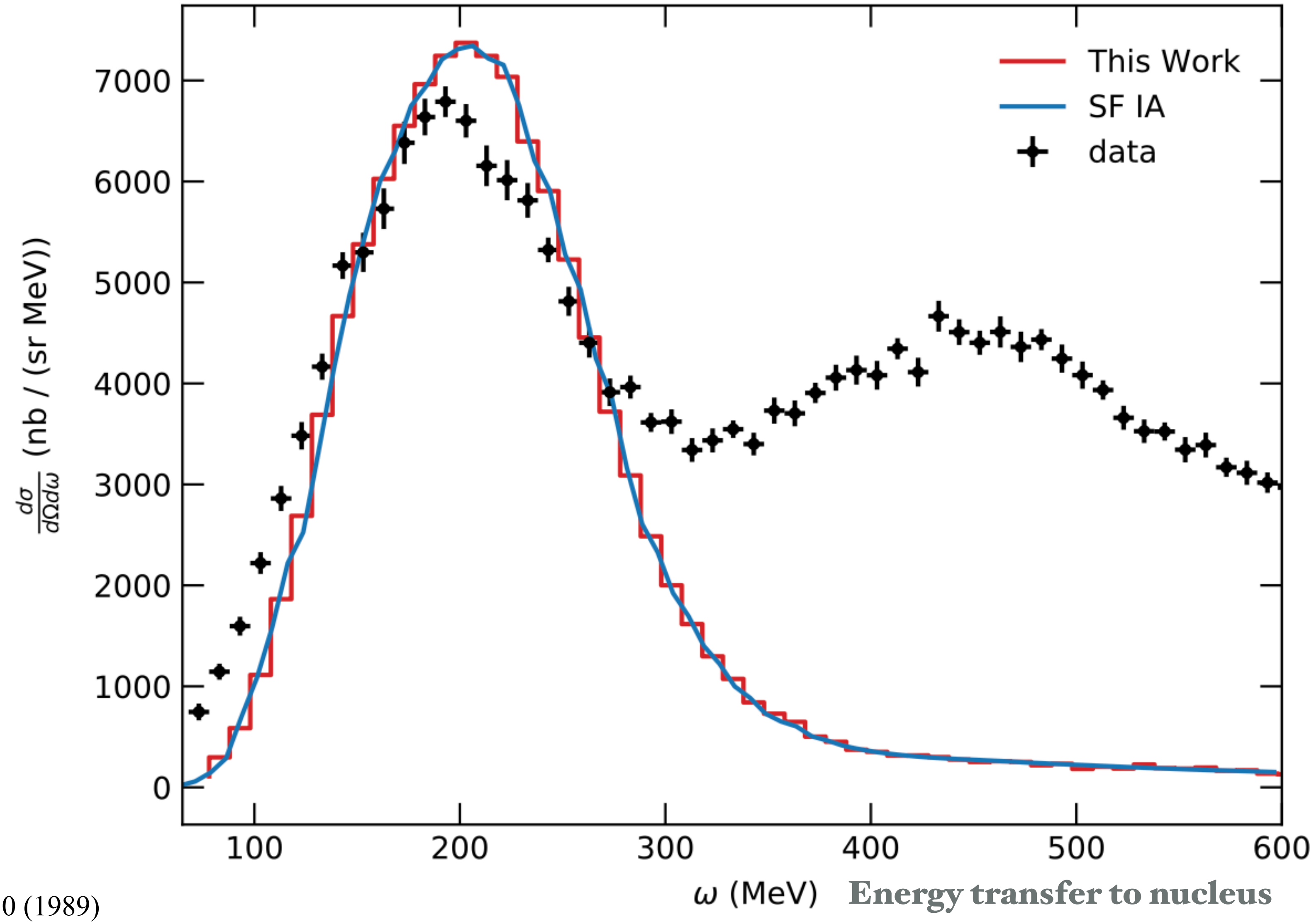


Energy transfer to nucleus

Data: R. M. Sealock et al., Phys. Rev. Lett. 62, 1350 (1989)
SF IA: N. Rocco, S. X. Nakamura, T. S. H. Lee, and A. Lovato, Phys. Rev. C 100, 045503 (2019)

e^-C scattering

- 961 MeV e^- scattering of carbon-12 at an angle of 37.5 deg.
- Difference in QE region peak due to final-state interaction effects.

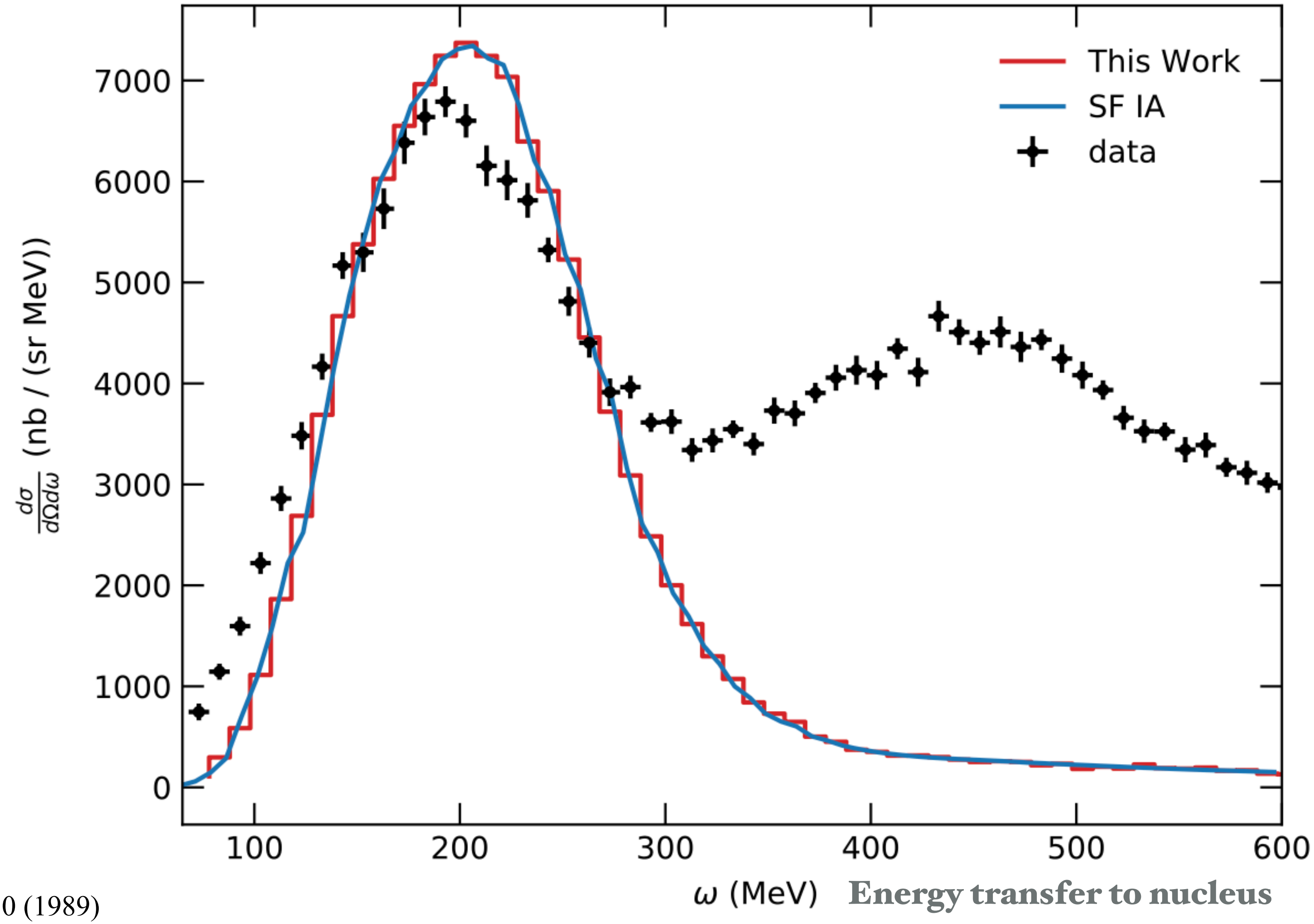


Energy transfer to nucleus

Data: R. M. Sealock et al., Phys. Rev. Lett. 62, 1350 (1989)
SF IA: N. Rocco, S. X. Nakamura, T. S. H. Lee, and A. Lovato, Phys. Rev. C 100, 045503 (2019)

$e^{-}C$ scattering

- 961 MeV e⁻ scattering of carbon-12 at an angle of 37.5 deg.
- Difference in QE region peak due to final-state interaction effects.
- High-energy tail due to two-body currents, resonance production, DIS.

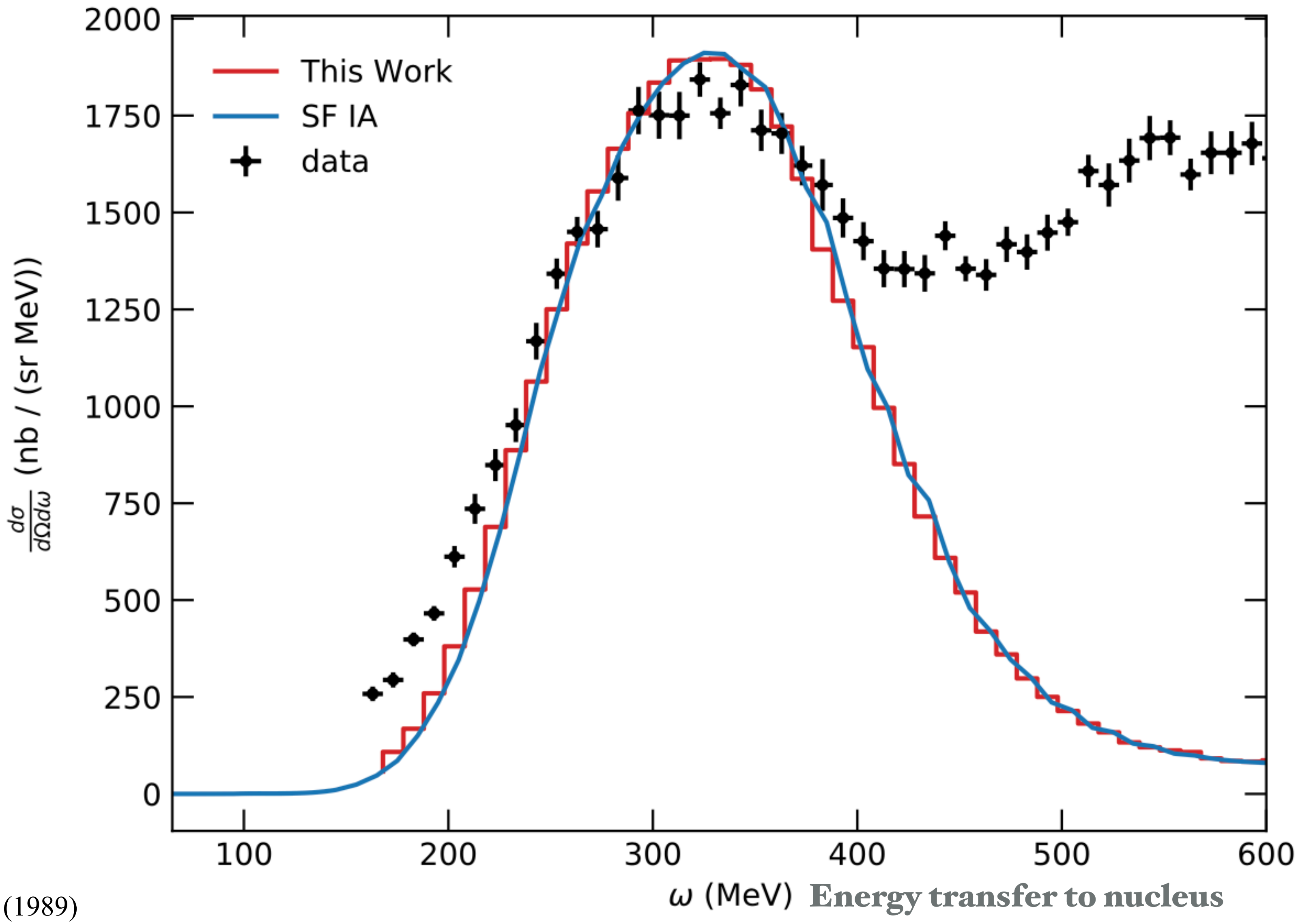


Energy transfer to nucleus

Data: R. M. Sealock et al., Phys. Rev. Lett. 62, 1350 (1989)
SF IA: N. Rocco, S. X. Nakamura, T. S. H. Lee, and A. Lovato, Phys. Rev. C 100, 045503 (2019)

$e^{-}C$ scattering

- 1300 MeV e⁻ scattering of carbon-12 at an angle of 37.5 deg.
- Difference in QE region peak due to final-state interaction effects.
- High-energy tail due to two-body currents, resonance production, DIS.

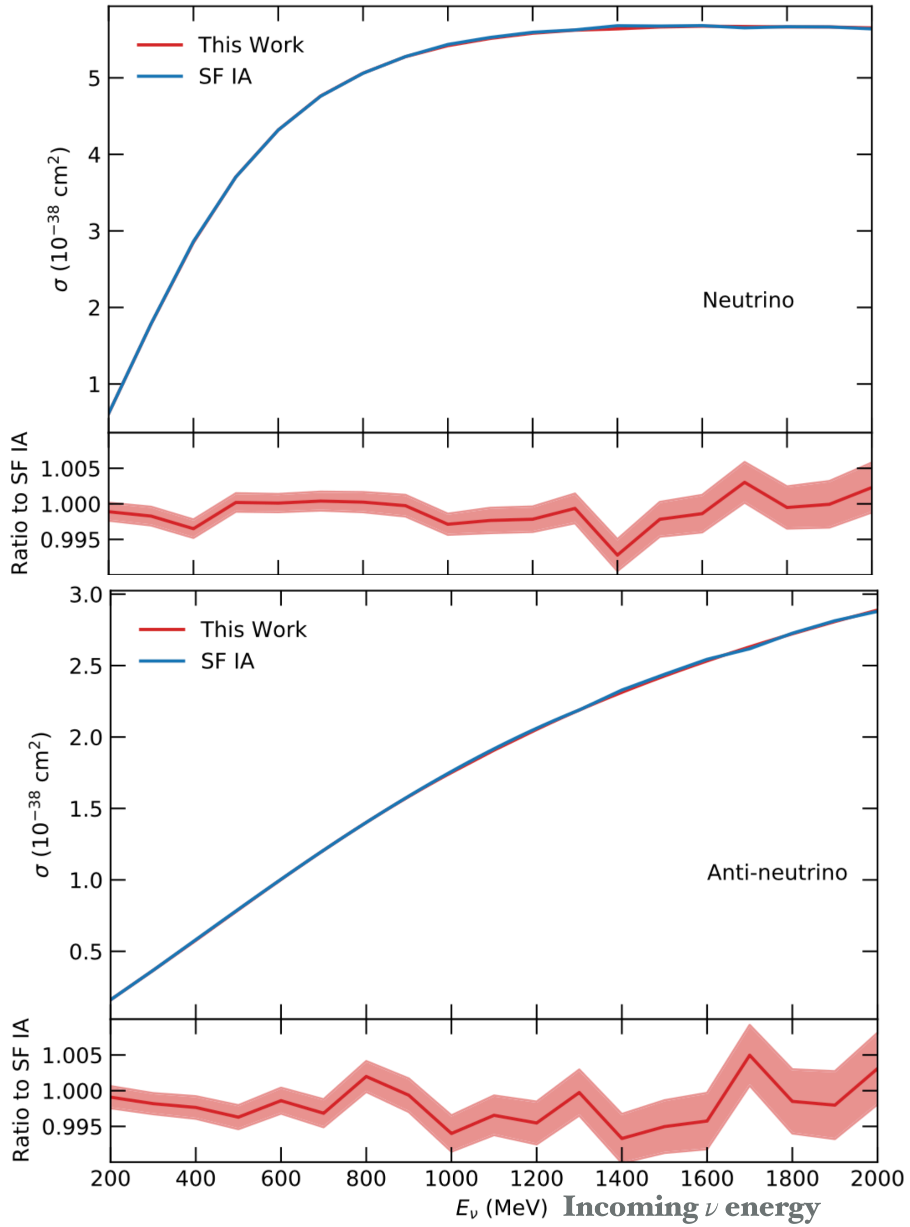


Energy transfer to nucleus

Data: R. M. Sealock et al., Phys. Rev. Lett. 62, 1350 (1989)
SF IA: N. Rocco, S. X. Nakamura, T. S. H. Lee, and A. Lovato, Phys. Rev. C 100, 045503 (2019)

νC scattering

- Only compare to theory calculations due to lack of high-energy monochromatic neutrino beam.
- Only CC interactions.
- Outgoing nucleon momentum greater than 225 MeV for fair comparison with theory calculation.

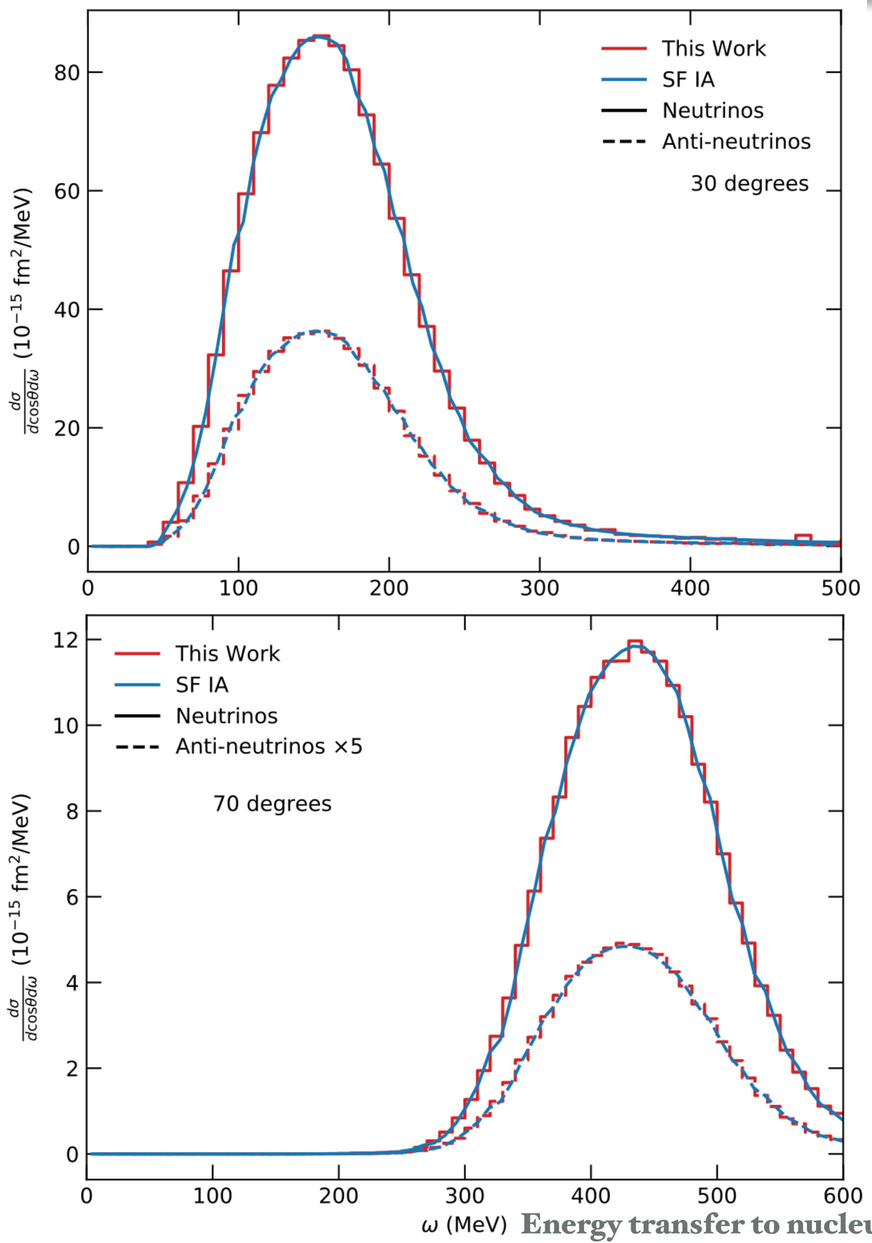


SF IA: N. Rocco, C. Barbieri, O. Benhar, A. De Pace, and A. Lovato, Phys. Rev. C 99, 025502 (2019)

νC scattering

- Only compare to theory calculations due to lack of high-energy monochromatic neutrino beam.
- Only CC interactions.
- Outgoing nucleon momentum greater than 225 MeV for fair comparison with theory calculation.
- Incoming E_ν set to 1 GeV.
  - Fixed outgoing lepton angle: 30 deg (top), 70 deg (bottom)

SF IA: N. Rocco, C. Barbieri, O. Benhar, A. De Pace, and A. Lovato, Phys. Rev. C 99, 025502 (2019)

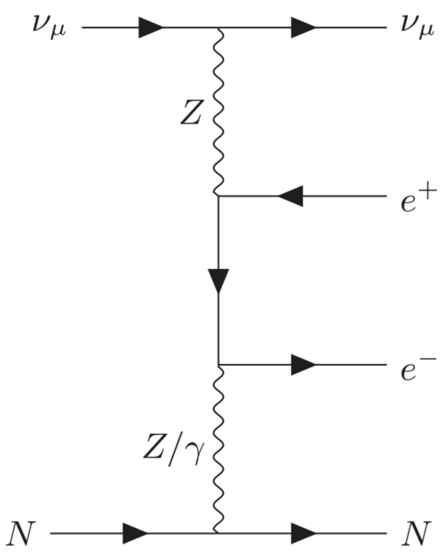


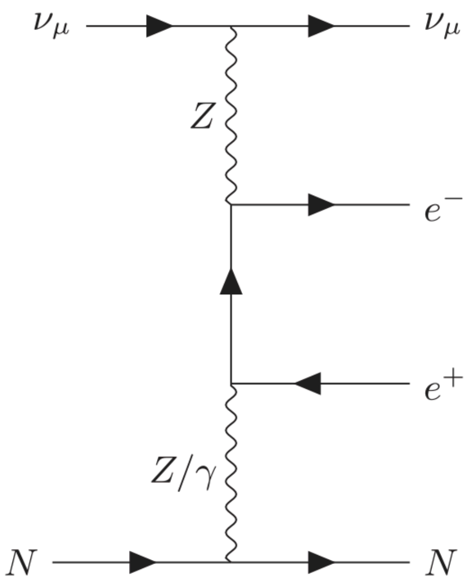
Energy transfer to nucleus

Neutrino Trident

Neutrino Trident

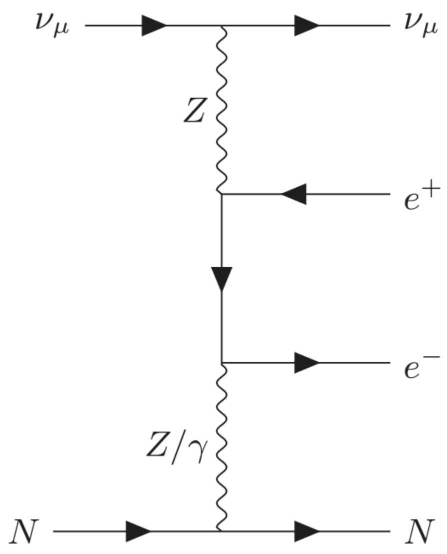
- $\nu_\mu C \rightarrow \nu_\mu e^+ e^- X$
- Interference terms (γ, Z) and $2 \rightarrow 4$

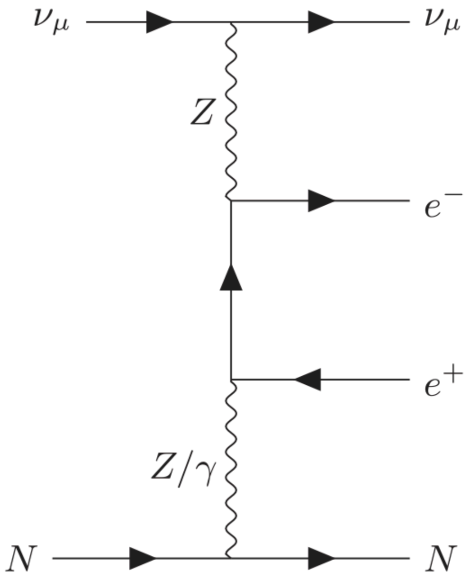




Neutrino Trident

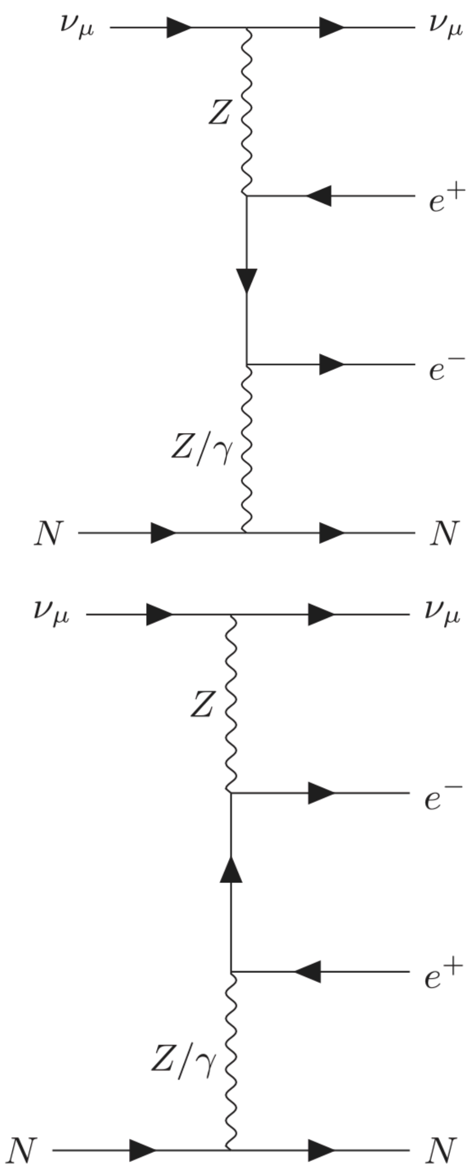
- $\nu_\mu C \to \nu_\mu e^+ e^- X$
- Interference terms (γ, Z) and $2 \to 4$
- Assumptions:
  - Fixed E_ν to 1 GeV
  - $\Delta\theta_{ee} > 5$ deg
  - $E_e > 30$ MeV
  - θ with beam axis > 10 deg.





Neutrino Trident

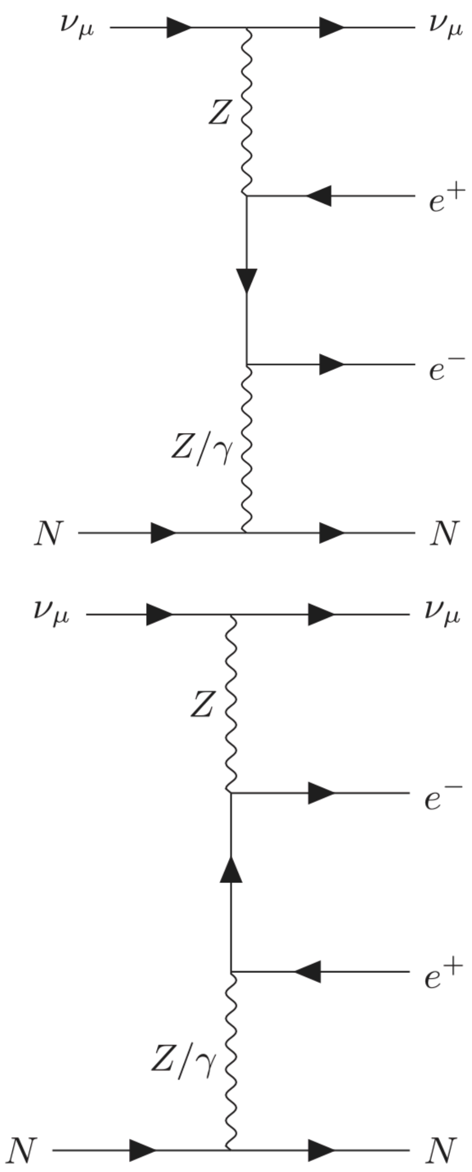
- $\nu_\mu C \rightarrow \nu_\mu e^+ e^- X$
- Interference terms (γ, Z) and $2 \rightarrow 4$
- Assumptions:
  - Fixed E_ν to 1 GeV
  - $\Delta\theta_{ee} > 5$ deg
  - $E_e > 30$ MeV
  - θ with beam axis > 10 deg.
- Important background for multiple lepton final state BSM explanations of MiniBooNE excess.



Neutrino Trident

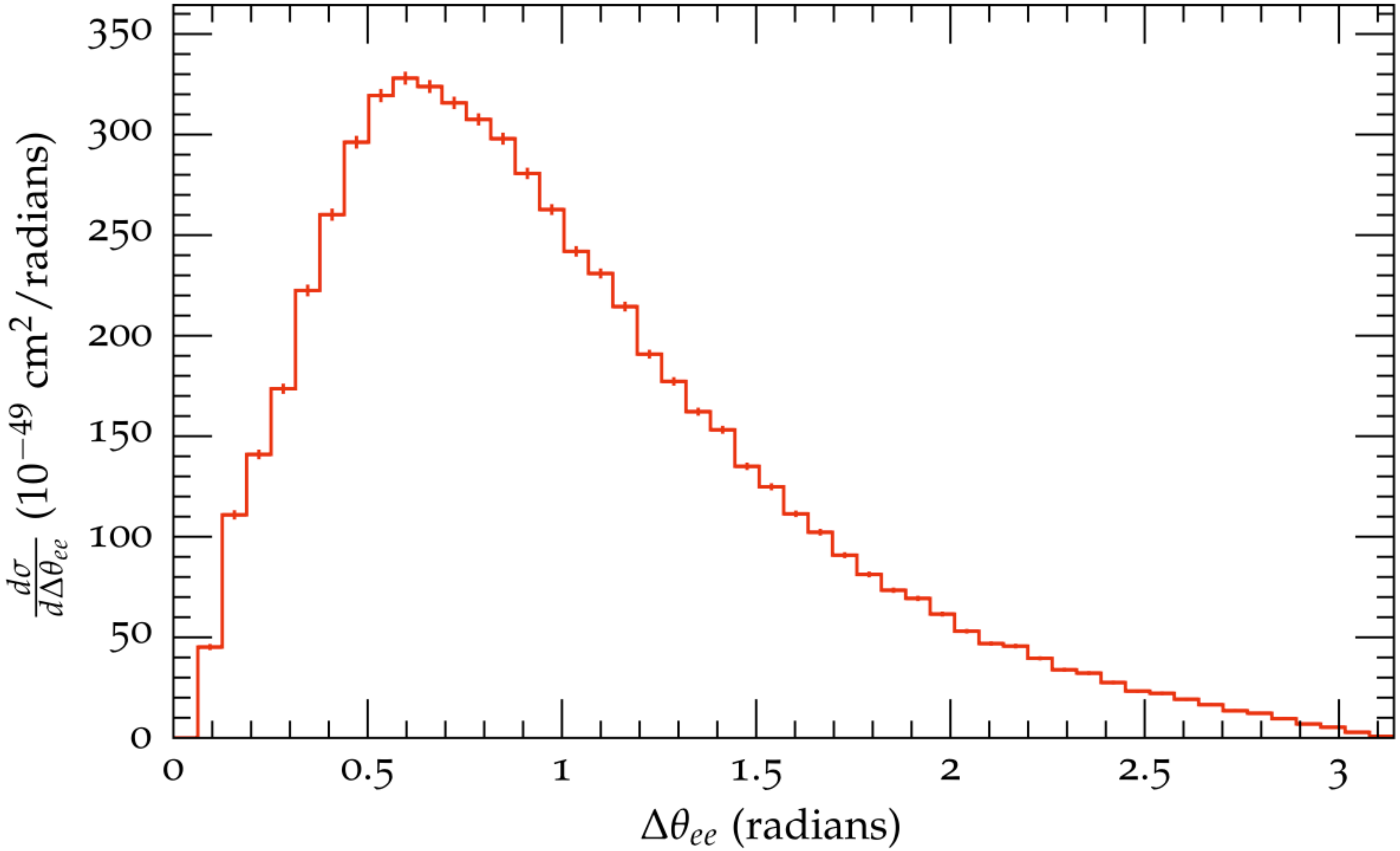
- $\nu_\mu C \rightarrow \nu_\mu e^+ e^- X$
- Interference terms (γ, Z) and $2 \rightarrow 4$
- Assumptions:
  - Fixed E_ν to 1 GeV
  - $\Delta\theta_{ee} > 5$ deg
  - $E_e > 30$ MeV
  - θ with beam axis > 10 deg.
- Important background for multiple lepton final state BSM explanations of MiniBooNE excess.
- Total $\sigma = 3.973 \times 10^{-11} \pm 2.764 \times 10^{-14}$ pb

(Consistent with P. Ballett, M. Hostert, S. Pascoli, Y. F. Perez-Gonzalez, Z. Tabrizi, and R. Zukanovich Funchal, J. High Energy Phys. 01 (2019) 119.)



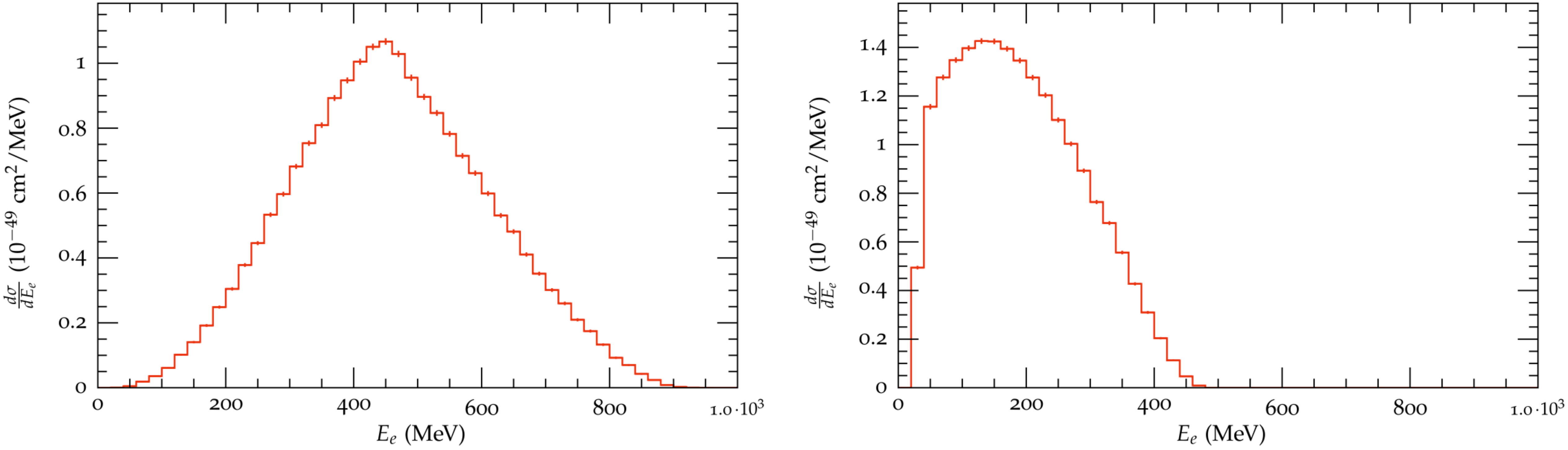
Neutrino Trident

- Angular separation $\Delta\theta_{ee}$ of both electrons. Ability of next-generation experiments to observe this process.



Neutrino Trident

- Leading (left) and subleading (right) electron energies.

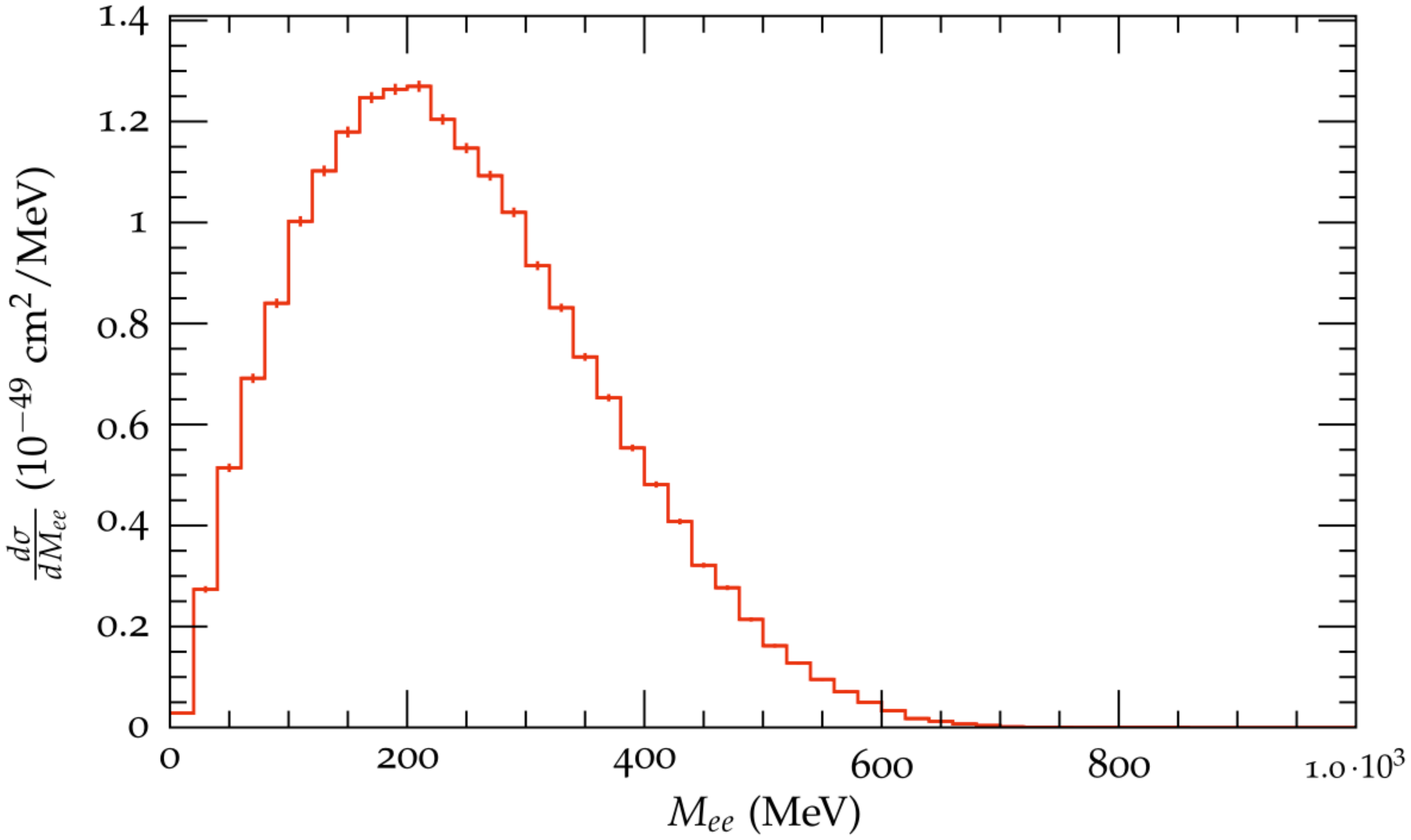


Leading e^-

Subleading e^-

Neutrino Trident

- Invariant mass of electron pair. Potentially distinguish trident processes from BSM scenarios with an electron pair in the final state.



Summary and Outlook

- We have developed an event generator framework for the automated simulation of neutrino scattering at next-generation experiments.

Summary and Outlook

- We have developed an event generator framework for the automated simulation of neutrino scattering at next-generation experiments.
- Main output is in the form of **leptonic currents/tensors of arbitrary BSM Lagrangians**.

Summary and Outlook

- We have developed an event generator framework for the automated simulation of neutrino scattering at next-generation experiments.
- Main output is in the form of **leptonic currents/tensors of arbitrary BSM Lagrangians**.
- BSM effects independent of nuclear model; chose to focus on QE region (impulse-approximation) using the spectral function formalism for concreteness.

Summary and Outlook

- We have developed an event generator framework for the automated simulation of neutrino scattering at next-generation experiments.
- Main output is in the form of **leptonic currents/tensors of arbitrary BSM Lagrangians**.
- BSM effects independent of nuclear model; chose to focus on QE region (impulse-approximation) using the spectral function formalism for concreteness.
- Performed phase space integral to get cross sections to validate against carbon-12 electron and neutrino scattering data. Obtained first fully differential neutrino trident production in the QE region using SF formalism.

Summary and Outlook

- We have developed an event generator framework for the automated simulation of neutrino scattering at next-generation experiments.
- Main output is in the form of **leptonic currents/tensors of arbitrary BSM Lagrangians**.
- BSM effects independent of nuclear model; chose to focus on QE region (impulse-approximation) using the spectral function formalism for concreteness.
- Performed phase space integral to get cross sections to validate against carbon-12 electron and neutrino scattering data. Obtained first fully differential neutrino trident production in the QE region using SF formalism.
- By design, generator is easily interfaced with other neutrino event generators and allows the user choice over the nuclear model to use.

Summary and Outlook

- Context: Our generator, though fully independent, is a module of the ACHILLES (A CHIcago Land Lepton Event Simulator) theory-driven lepton event generator.
- ACHILLES encompasses the full simulation pipeline from beams, hard interactions, intranuclear cascades, and decays of unstable particles.
- Our generator extends ACHILLES ability by enabling BSM calculations on $2 \rightarrow n$ processes.

ACHILLES: J. Isaacson, W. Jay, A. Lovato, P. A. N. Machado, and N. Rocco, Phys. Rev. D 107, 033007 (2023)

```
+=====================================================================+
|                                                                     |
|    .d8b.   .o88b. db   db d888888b db      db      d88888b .d8888.  |
|   d8' `8b d8P  Y8 88   88   `88'   88      88      88'     88'  YP  |
|   88ooo88 8P      88ooo88    88    88      88      88ooooo `8bo.    |
|   88~~~88 8b      88~~~88    88    88      88      88~~~~~   `Y8b.  |
|   88   88 Y8b  d8 88   88   .88.   88booo. 88booo. 88.     db   8D  |
|   YP   YP  `Y88P' YP   YP Y888888P Y88888P Y88888P Y88888P `8888Y'  |
|                                                                     |
+---------------------------------------------------------------------+
|                                                                     |
|                 .d88888888888888888888888888b.                      |
|            .d8888888888888888888888888888888888b.                   |
|         .d88888888888888888888888888888888888888888b.               |
|      88888888888888888888888888888888888888888888888888b.           |
|       `Y8888888888888888888888888888888888888888888888888b.         |
|            `Y88888888888888888888888888888888888888888888888        |
|              `Y88888888888888888888888888888888888888888888888      |
|                 `Y888888888888888888888888888P  Y88888888888888     |
|                    `Y88  8888  8888  88Y'        Y8888888888888     |
|                   .d888888888888888888888b.       Y888888888888.    |
|                .d88888888888888888888888888b.      8888888888888    |
|              .d8888888888888888888888888888888b   88888888888888    |
|            .d8888888888888888888888888888888888   8888888888888D    |
|            d88888888888888888888888888888888888   8888888888888     |
|           d888888888888888888888888888888888888   Y888888888888     |
|          d888888P'   d888888888888888888888888P       88888888      |
|          8888P     .d888888888888888888888888P         888888       |
|          888P    .d8888888888888888888888888P          8888Y        |
|          888b  .d888888888888888888888888888           8888P        |
|          Y88  d888888888888888888888888888888           888         |
|          `8' d8888888888888888888888888888888            88         |
|              8888888888888888888888888888888P             8         |
|              8888888888888888P     Y8888888P                        |
|             d8888888888888888       Y88888P                         |
|             8888888888888888P        Y888P                          |
|            d8888888888888P            Y8P                           |
|            888888888P                  8                            |
|           d888P                                                     |
|                                                                     |
+---------------------------------------------------------------------+
|                                                                     |
|    Version: {:56}|
|    Authors: Joshua Isaacson, William Jay, Alessandro Lovato,        |
|        Pedro A. Machado, Noemi Rocco                                |
|                                                                     |
|    Undergraduate Student Contributions:                             |
|        Diego Lopez Gutierrez, Sherry Wang, Russell Farnsworth       |
|                                                                     |
+=====================================================================+
```

Summary and Outlook

Thank you!