NLO EW corrections at a muon collider

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Why NLO EW corrections at muon colliders?

NLO EW corrections at muon colliders are typically as large as (or even more than) NLO QCD corrections at the LHC.

EW corrections should be considered not only for precision physics, since they give $\mathcal{O}(10-100\%)$ effects. This includes also BSM scenarios.

$\mu^+\mu^- \to X, \sqrt{s} = 3$ TeV	$\sigma^{\text{incl}}_{\text{LO}}$ [fb]	$\sigma^{\text{incl}}_{\text{NLO}}$ [fb]	δ_{EW} [%]
W^+W^-Z	$3.330(2)\cdot 10^{1}$	$2.568(8)\cdot 10^{1}$	$-22.9(2)$
W^+W^-H	$1.1253(5)\cdot 10^{0}$	$0.895(2)\cdot 10^{0}$	$-20.5(2)$
ZZZ	$3.598(2)\cdot 10^{-1}$	$2.68(1)\cdot 10^{-1}$	$-25.5(3)$
HZZ	$8.199(4)\cdot 10^{-2}$	$6.60(3)\cdot 10^{-2}$	$-19.6(3)$
HHZ	$3.277(1)\cdot 10^{-2}$	$2.451(5)\cdot 10^{-2}$	$-25.2(1)$
HHH	$2.9699(6)\cdot 10^{-8}$	$0.86(7)\cdot 10^{-8}$ *	
$W^+W^-W^+W^-$	$1.484(1)\cdot 10^{0}$	$0.993(6)\cdot 10^{0}$	$-33.1(4)$
W^+W^-ZZ	$1.209(1)\cdot 10^{0}$	$0.699(7)\cdot 10^{0}$	$-42.2(6)$
W^+W^-HZ	$8.754(8)\cdot 10^{-2}$	$6.05(4)\cdot 10^{-2}$	$-30.9(5)$
W^+W^-HH	$1.058(1)\cdot 10^{-2}$	$0.655(5)\cdot 10^{-2}$	$-38.1(4)$
$ZZZZ$	$3.114(2)\cdot 10^{-3}$	$1.799(7)\cdot 10^{-3}$	$-42.2(2)$
$HZZZ$	$2.693(2)\cdot 10^{-3}$	$1.766(6)\cdot 10^{-3}$	$-34.4(2)$
$HHZZ$	$9.828(7)\cdot 10^{-4}$	$6.24(2)\cdot 10^{-4}$	$-36.5(2)$
$HHHZ$	$1.568(1)\cdot 10^{-4}$	$1.165(4)\cdot 10^{-4}$	$-25.7(2)$

WHIZARD

Bredt, Kilian, Reuter, Stienemeier '22

NLO EW: some open questions/issues

Resummation?

When is it necessary to resum EW (Sudakov) corrections?

BSM?

What features of NLO EW corrections are universal and can be extended to the BSM case?

EW jets?

What should I do with Z,W radiation? Experimental set-up may impact the calculation result.

PDFs or VBF with matrix elements?

If PDF involves weak effects, weak counter terms in NLO EW corrections should be included. Resum logs or keep power corrections? Both?

MadGraph5_aMC@NLO: what can be done?

NLO EW hadron colliders: *Frederix, Frixione, Hirshi, DP, Shao, Zaro '18*

NLO EW e^+e^- colliders: *Bertone, Cacciari, Frixione, Stagnitto, Zaro, Zhao '22*

NLO EW Sudakov: *DP, Zaro '21*

The path is clear: extend NLO EW to muon collisions (muon PDFs), identify Sudakov corrections and therefore non-Sudakov effects.

one-loop EW virtual corrections $\mathcal{O}(\alpha)$

=

α **[Sudakov Logs $\mathcal{O}(-\log^k(s/m_W^2), k = 1{,}2)$ +**

constant term $\mathcal{O}(1)$ +

mass-suppressed terms $\mathcal{O}(m_W^2/s)$]

What are EW Sudakov logarithms?

QCD: virtual and real terms are separately IR divergent ($1/\epsilon$ poles). In physical cross sections the contributions are combined and poles cancel.

QED: same story, but I can also regularise IR divergencies via a photon-mass λ. So $1/\epsilon$ poles $\rightarrow \log(Q^2/\lambda^2)$, where Q is a generic scale.

EW: with weak interactions $\lambda \rightarrow m_W, m_Z$ and W and Z radiation are typically not taken into account, which is anyway IR-safe.

Therefore, at high energies EW loops induce corrections of order

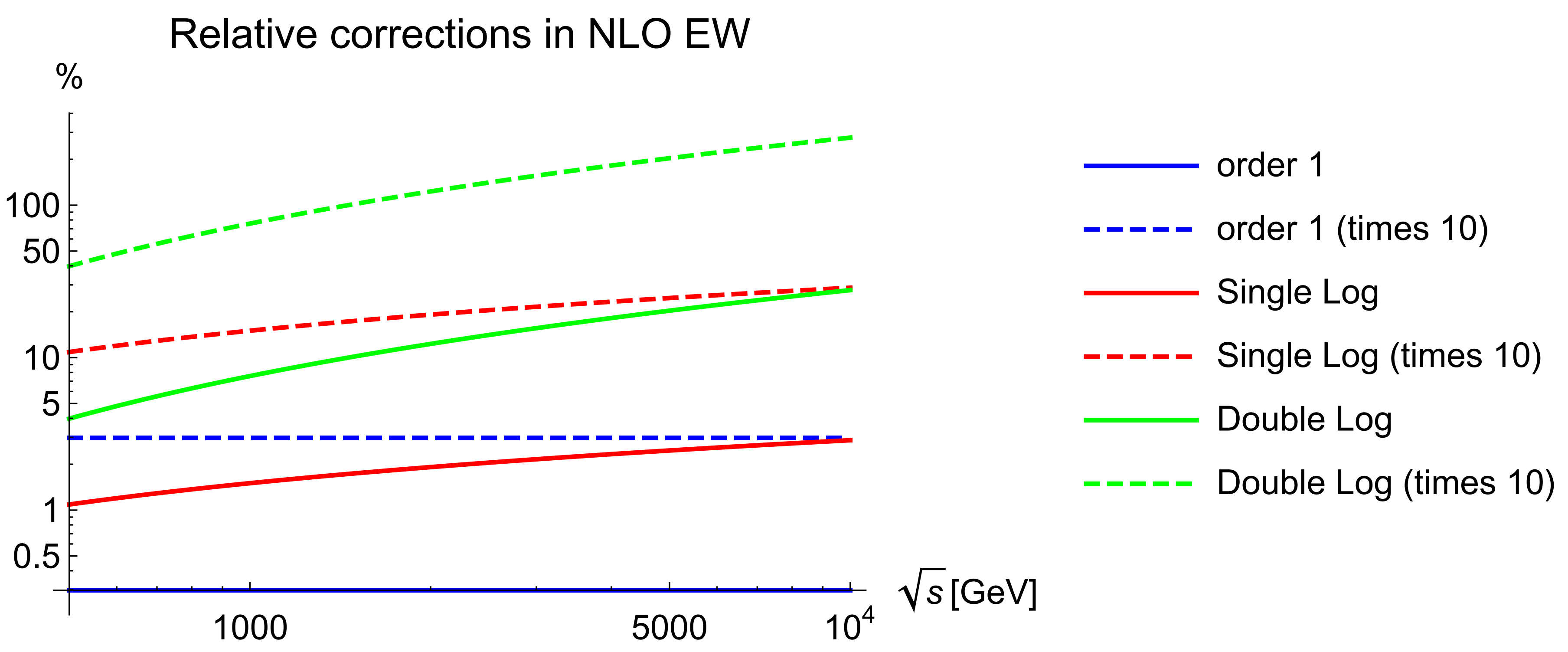
$$-\alpha^k \log^n(s/m_W^2)$$

where k is the number of loops and $n \leq 2k$. These logs are physical. Even including the real radiation of W and Z, there is not the full cancellation of this kind of logarithms.

What is the hierarchy?

$$\mathcal{O}(1) \to \frac{\alpha}{4\pi s_w^2} \sim 0.3\,\%\,,$$ **Single Log** $$\to \frac{\alpha}{4\pi s_w^2}\log(s/m_W^2),$$

Double Log $$\to \frac{\alpha}{4\pi s_w^2}\log^2(s/m_W^2)$$



The estimate done via the variation of a factor of 10 is actually conservative.

Just a representative example of a process

$$\delta^{\mathrm{RR,ew}}_{\mathrm{e^+e^-}\to\mu^+\mu^-} \boxed{= -2.58\,L(s)} \boxed{- 5.15\left(\log\frac{t}{u}\right) l(s) + 0.29\, l_{\mathrm{Z}} + 7.73\, l_{\mathrm{C}} + 8.80\, l_{\mathrm{PR}},}$$

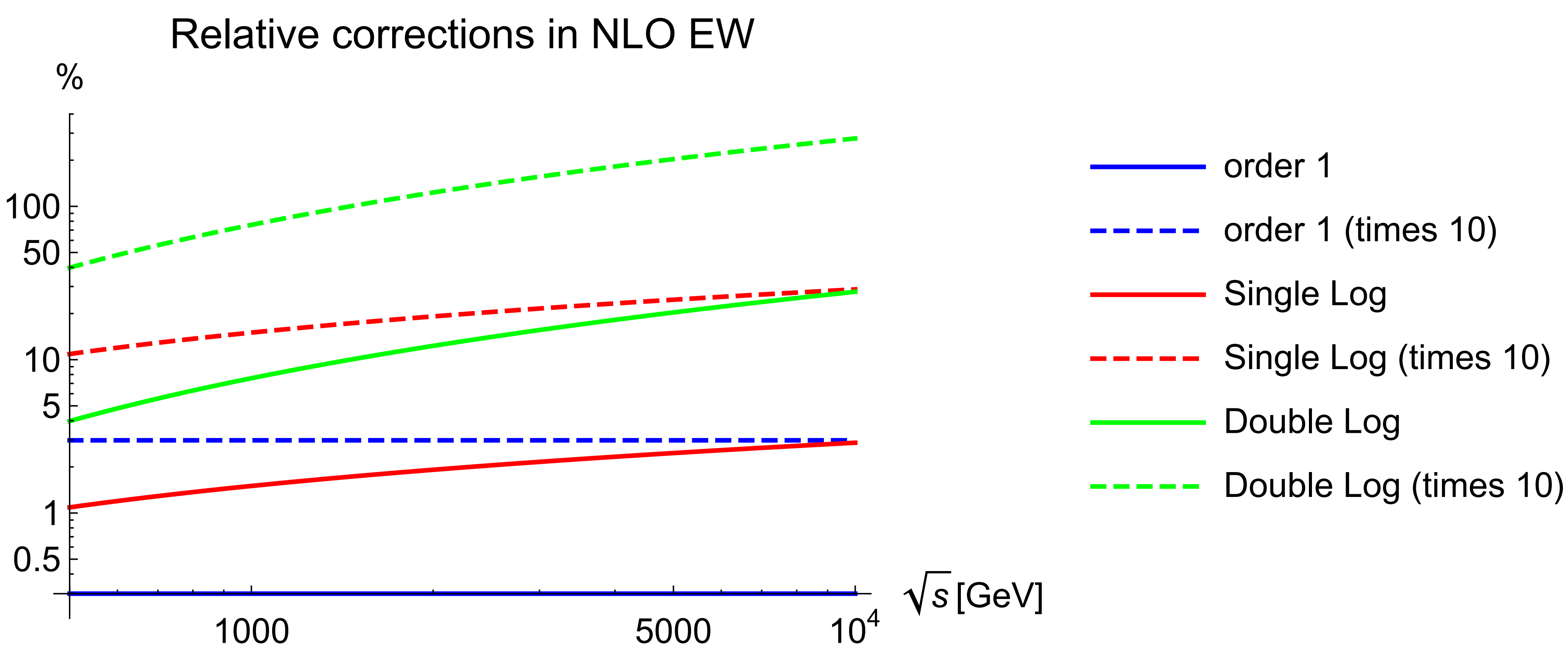
$$\delta^{\mathrm{RL,ew}}_{\mathrm{e^+e^-}\to\mu^+\mu^-} \boxed{= -4.96\,L(s)} \boxed{- 2.58\left(\log\frac{t}{u}\right) l(s) + 0.37\, l_{\mathrm{Z}} + 14.9\, l_{\mathrm{C}} + 8.80\, l_{\mathrm{PR}},}$$

Denner Pozzorini '01

What is the hierarchy?

$$\mathcal{O}(1) \rightarrow \frac{\alpha}{4\pi s_w^2} \sim 0.3\,\%\,, \quad \textbf{Single Log} \rightarrow \frac{\alpha}{4\pi s_w^2}\log(s/m_W^2),$$

$$\textbf{Double Log} \rightarrow \frac{\alpha}{4\pi s_w^2}\log^2(s/m_W^2)$$



The estimate done via the variation of a factor of 10 is actually conservative.

Definitely at 3 but also at 10 TeV both Single and Double logs should be taken into account, and obviously finite term for % acc.

Logs may often need to be resummed.

Master formula (Denner&Pozzorini)

Denner Pozzorini '01

Born amplitude: $\mathcal{M}_0^{i_1\dots i_n}(p_1,\dots,p_n)$

One-loop EW Sudakov corrections: $\delta\mathcal{M}^{i_1\dots i_n}(p_1,\dots,p_n) = \mathcal{M}_0^{i'_1\dots i'_n}(p_1,\dots,p_n)\delta_{i'_1 i_1\dots i'_n i_n}$

other tree-level amplitudes

the logs

eikonal approximation of soft EW boson exchange

$$\delta = \delta^{\mathrm{LSC}} + \delta^{\mathrm{SSC}} + \delta^{\mathrm{C}} + \delta^{\mathrm{PR}}$$

Leading Soft-Collinear: It depends only on s and it is the only term involving double logarithms.

Subleading Soft-Collinear: The only one involving ratios of s with other invariants and also angular dependences.

Collinear: In an on-shell scheme, the dependence on the UV regularisation scale cancels. No μ_r dependence is left.

Parameter renormalis.

The logs inside the δ^i have always the form:

$$L(|r_{kl}|, M^2) \equiv \frac{\alpha}{4\pi}\log^2\frac{|r_{kl}|}{M^2}$$

$$l(|r_{kl}|, M^2) \equiv \frac{\alpha}{4\pi}\log\frac{|r_{kl}|}{M^2}$$

$$M = M_W, M_Z, m_f, \lambda, \dots$$
$$r_{kl} \equiv (p_k + p_l)^2$$

ASSUMPTIONS:

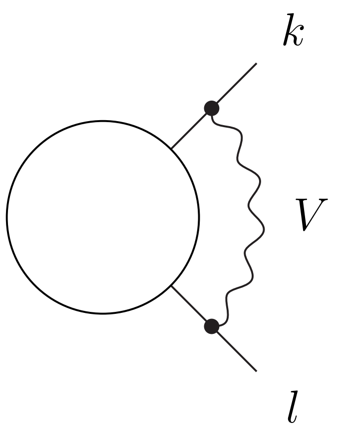
$$r_{kl} \equiv (p_k + p_l)^2 \simeq 2p_k p_l \gg M_W^2 \simeq M_H^2, m_t^2, M_W^2, M_Z^2$$

the high-energy limit

$$r_{kl}/r_{k'l'} \simeq 1$$

All invariants $\simeq s$. Reasonable, but $r_{kl} = s$ is impossible.

Derivation of LSC and SSC



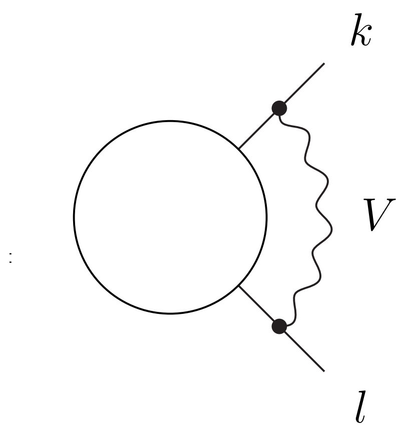
Denner&Pozzorini

$$
\begin{aligned}
L(|r_{kl}|, M^2) &= L(s, M^2) + 2l(s, M^2)\log\frac{|r_{kl}|}{s} + L(|r_{kl}|, s) \\
&= \underbrace{L(s) + 2l(s)\log\frac{M_W^2}{M^2}}_{\text{LSC}} + \underbrace{2l(s)\log\frac{|r_{kl}|}{s}}_{\text{SSC}} + \cdots
\end{aligned}
$$

$$L(s) \equiv L(s, M_W^2) \qquad \text{and} \qquad l(s) \equiv l(s, M_W^2)$$

The relation $\boxed{r_{kl} = r_{k'l'} = s}$ is used in all logs, unless they multiply $l(s)$.

Derivation of LSC and SSC



Denner&Pozzorini

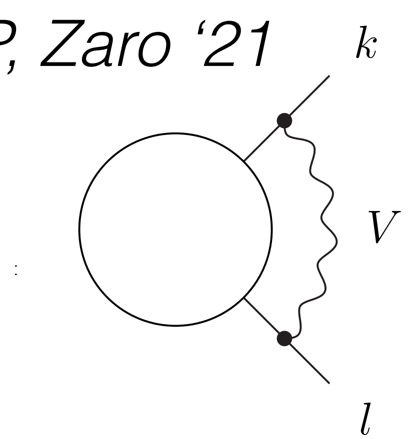
$$L(|r_{kl}|,M^2) = L(s,M^2) + 2l(s,M^2)\log\frac{|r_{kl}|}{s} + L(|r_{kl}|,s)$$

$$= \underbrace{L(s) + 2l(s)\log\frac{M_W^2}{M^2}}_{\textbf{LSC}} + \underbrace{2l(s)\log\frac{|r_{kl}|}{s}}_{\textbf{SSC}} + \cdots$$

$$L(s) \equiv L(s,M_W^2) \qquad \text{and} \qquad l(s) \equiv l(s,M_W^2)$$

The relation $r_{kl} = r_{k'l'} = s$ is used in all logs, unless they multiply $l(s)$.

DP, Zaro '21



Our approach: $\cancel{r_{kl} = r_{k'l'} = s}$ **in the expressions**

$$C_0(p_k, p_l, M, M_k, M_l) \quad \Longrightarrow \quad L(|r_{kl}|,M^2)\boxed{-\,2i\pi\Theta(r_{kl})l(|r_{kl}|,M^2)}$$

Previously omitted imaginary term

$$L(|r_{kl}|,M^2) - \boxed{2i\pi\Theta(r_{kl})l(|r_{kl}|,M^2)} =$$

$$= L(s,M^2) + 2l(s,M^2)\left(\log\frac{|r_{kl}|}{s}\boxed{-\,i\pi\Theta(r_{kl})}\right) + L(|r_{kl}|,s) - \boxed{2i\pi\Theta(r_{kl})l(|r_{kl}|,s)} =$$

$$= \underbrace{L(s) + 2l(s)\log\frac{M_W^2}{M^2}}_{\textbf{LSC}} + \underbrace{2l(s)\left(\log\frac{|r_{kl}|}{s} - \boxed{i\pi\Theta(r_{kl})}\right)}_{\textbf{SSC}} +$$

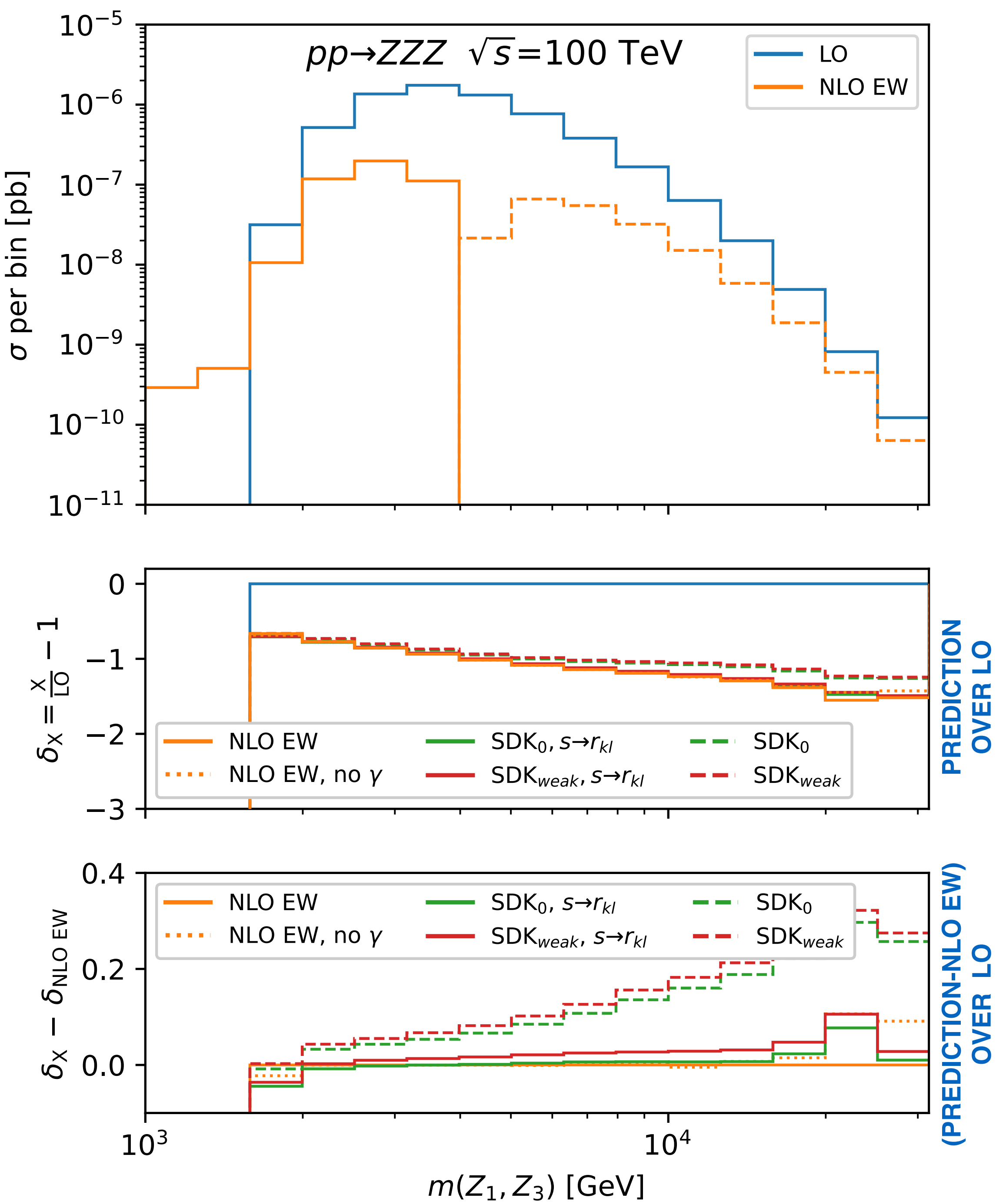
$$\underbrace{2l(M_W^2,M^2)\log\frac{|r_{kl}|}{s} + L(|r_{kl}|,s) - \boxed{2i\pi\Theta(r_{kl})l(|r_{kl}|,s)}}_{\mathrm{SSC}^{s\to r_{kl}}} + \cdots$$

New angular dependences via ratios among invariants

The conceptual derivation relies on the assumption $s = r_{kl}$, but is not actually used in the expressions.
Therefore, further angular dependencies are taken into account.

ZZZ production at 100 TeV **hadron**

$$p_T(Z_i) > 1\ \text{TeV}\,, \quad |\eta(Z_i)| < 2.5\,, \quad m(Z_i, Z_j) > 1\ \text{TeV}\,, \quad \Delta R(Z_i, Z_j) > 0.5\,.$$



Orange: NLO EW, (**dotted**: NLO EW no γ PDF)
Green = SDK_0, **Red** = $\mathrm{SDK}_{\mathrm{weak}}$
Dashed: standard approach for amplitudes.
Solid: our formulation (more angular information)

Reference Prediction: Red-solid line

$\mathrm{SDK}_{\mathrm{weak}}$ and SDK_0 explained afterwards (irrelevant for neutral final state).

Only the solid lines, having more angular information, correctly capture NLO EW.

Larger invariant -> larger correction

DP, Zaro '21

Cross-sections: our approach.

FOR WHAT EW SUDAKOV ARE USEFUL?
For providing a very **good approximation of NLO EW** in the **high-energy** limit.

HOW SHOULD ONE PERFORM THE CALCULATION IN THE HIGH-ENERGY LIMIT?
Photons have to be **always clustered with massless charged particle for IR**-safety reasons. But from an experimental point of view, **at high energy also clustering tops and W bosons with photons** is very reasonable, either if you imagine to tag heavy object directly or via their massless decay products.

The **QED Logs,** involving s and λ^2 (or Q^2), **cancel against their real-emission counterparts and PDF counterterms**. The only one surviving are those from tops in vacuum polarisation for external (not tagged) photons, both in the initial and final state:

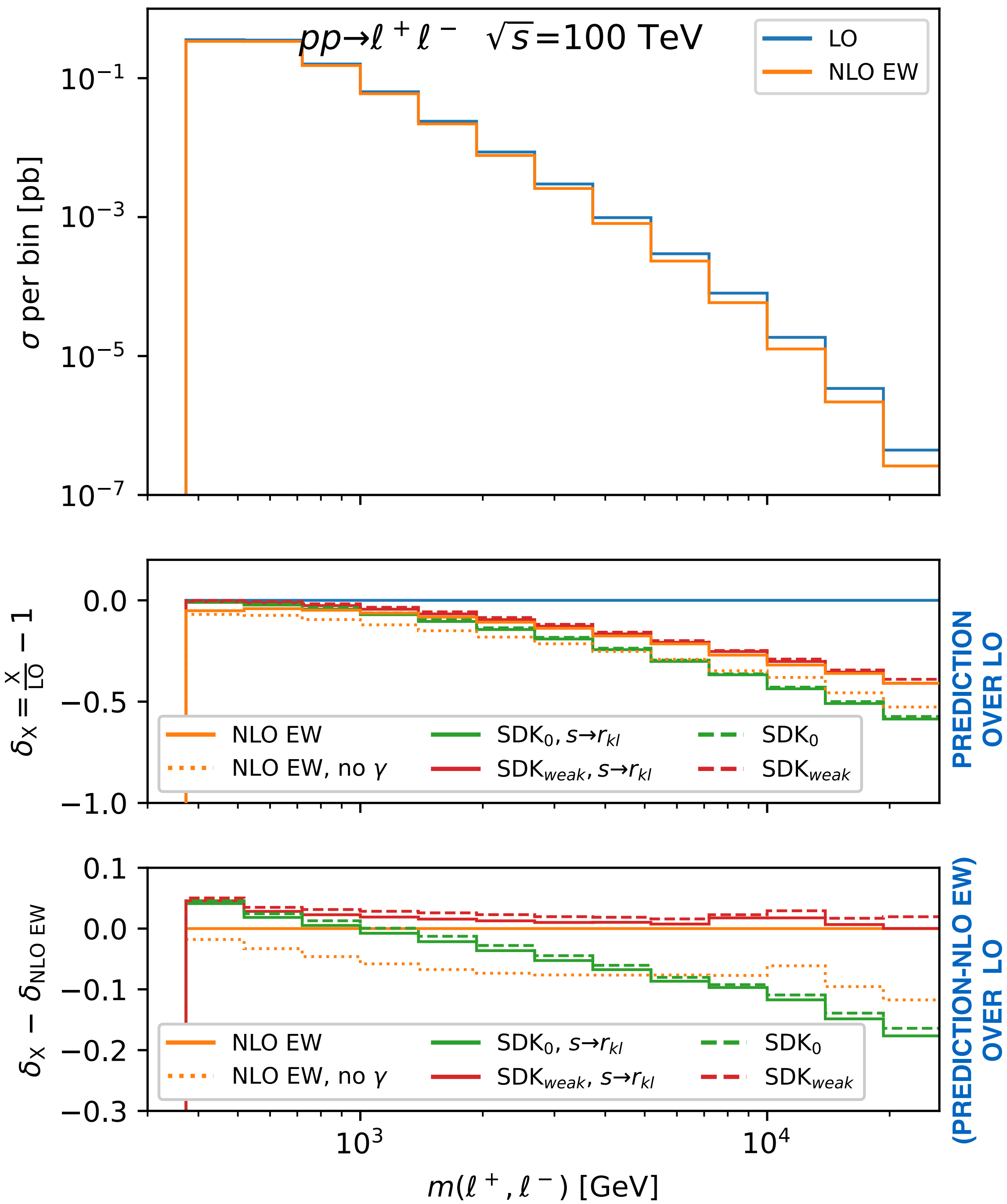
$$\mathrm{SDK}_{\mathrm{weak}}$$

Almost all the contributions of QED are removed
(e.g. $C_{\mathrm{EW}}(k) \to C_{\mathrm{EW}}(k) - Q_k^2$, $Q_k^2 = 0$**),**
but NOT in the parameter renormalisation δ^{PR}**.**

DP, Zaro '21

e^+e^- production at 100 TeV **hadron**

$$p_T(\ell^\pm) > 200 \ \text{GeV}, \quad |\eta(\ell^\pm)| < 2.5, \quad m(\ell^+,\ell^-) > 400 \ \text{GeV}, \quad \Delta R(\ell^+,\ell^-) > 0.5\,.$$



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Green = SDK_0, **Red** = $\mathrm{SDK}_{\mathrm{weak}}$

Dashed: standard approach for amplitudes.

Solid: our formulation (more angular information)

Reference Prediction: Red-solid line

Only the $\mathrm{SDK}_{\mathrm{weak}}$ approach correctly captures the NLO EW prediction.

Solid and dashed very similar.

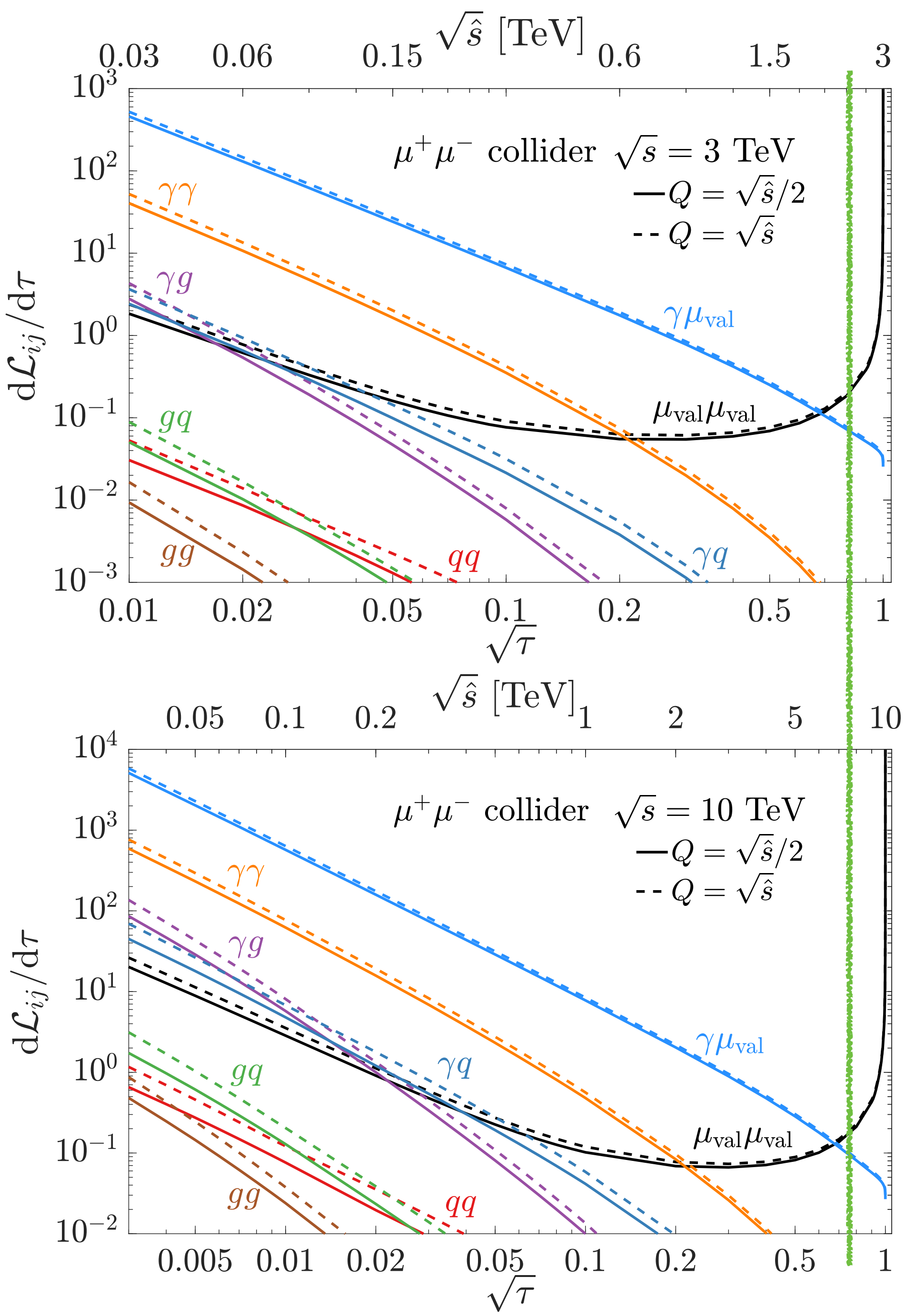
Photon PDF cannot be ignored.

Larger invariant -> larger correction

DP, Zaro '21

Calculation set up for showcasing some results

$\mu^+\mu^- \longrightarrow X$, where X is a generic final state involving W, Z, t, H, ℓ. Thus direct production, no VBF considered.



We require $m(X) > 0.8\sqrt{s}$, so that neither VBF nor PDFs other than μ are relevant.

We apply further experimentally motivated cuts for each i, j particle in X:

$$p_T(i) > 100 \text{ GeV},\ |\eta(i)| < 2.44,\ \Delta R(i,j) > 0.4$$

And we recombine photons with charged (also massive) particles.

Han, Ma, Xie '20, '21

Calculation set up for showcasing some results

$\mu^+\mu^- \longrightarrow X$, where X is a generic final state involving W, Z, t, H, ℓ. Thus direct production, no VBF considered.

ISR Treatment: we use the LL PDF for the muon only

$$\Gamma_{\text{LO}}(z) = \frac{\exp\left(3\beta_S/4 - \gamma_E\beta_E\right)}{\Gamma\left(1+\beta_E\right)}\,\beta_E(1-z)^{\beta_E-1} - \frac{1}{2}\beta_H(1+z) + \mathcal{O}(\alpha^2)$$

Bertone, Cacciari, Frixione, Stagnitto, Zaro, Zhao '22

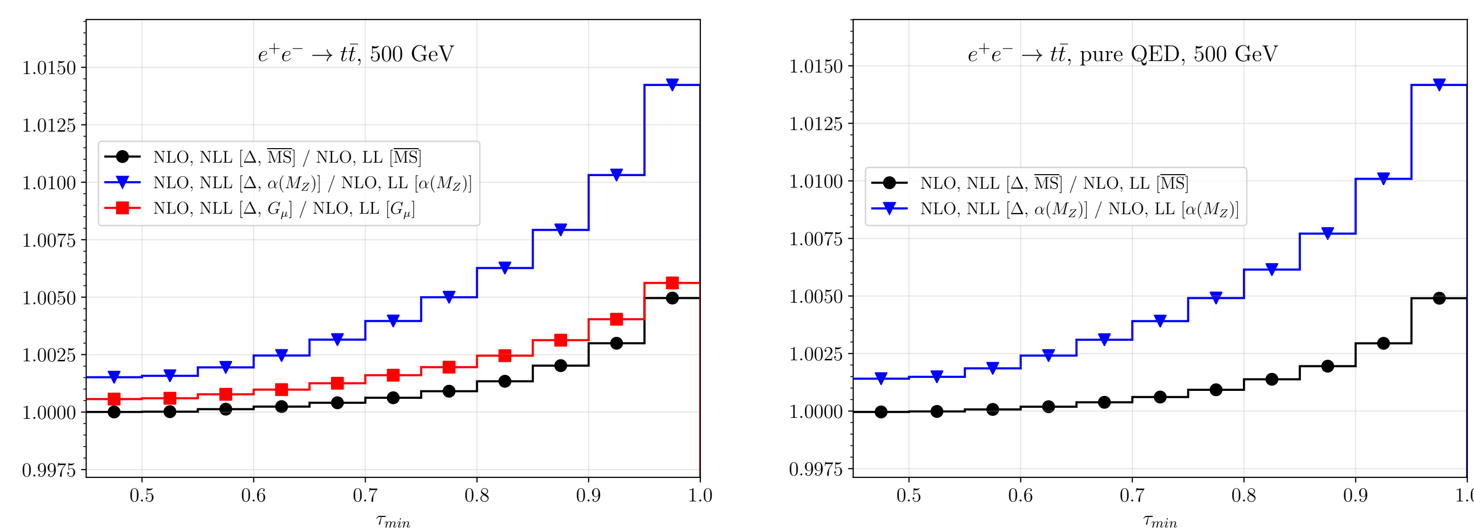
- Beta scheme:

$$\beta_E = \beta_S = \beta_H = e_e^2\beta\,.$$

- Eta scheme:

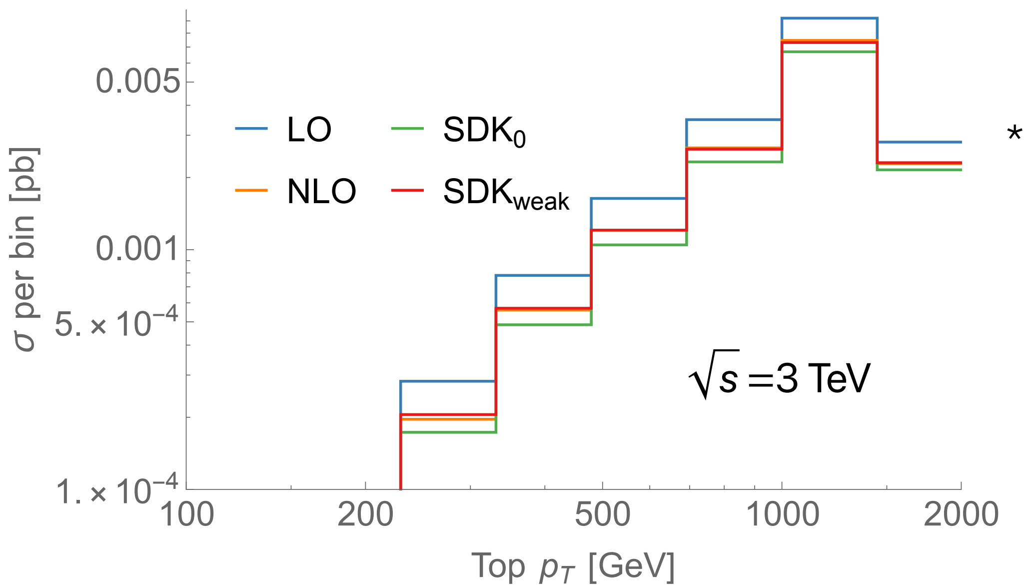
$$\beta_E = \beta_S = e_e^2\beta\,, \qquad \beta_H = e_e^2\eta\,.$$

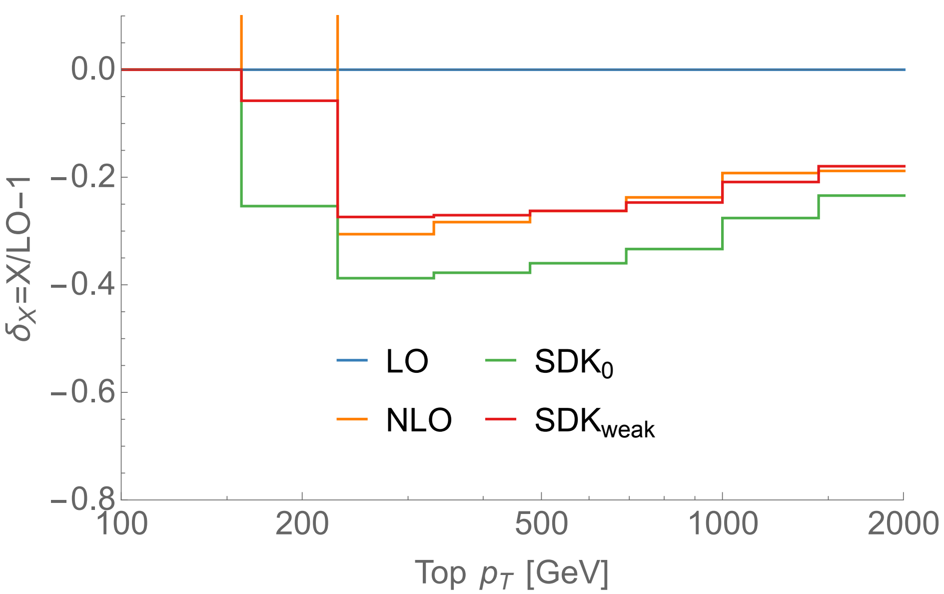
$$\eta = \frac{\alpha}{\pi}\log\frac{\mu^2}{m^2}\,, \qquad \beta = \frac{\alpha}{\pi}\left(\log\frac{\mu^2}{m^2} - 1\right)$$

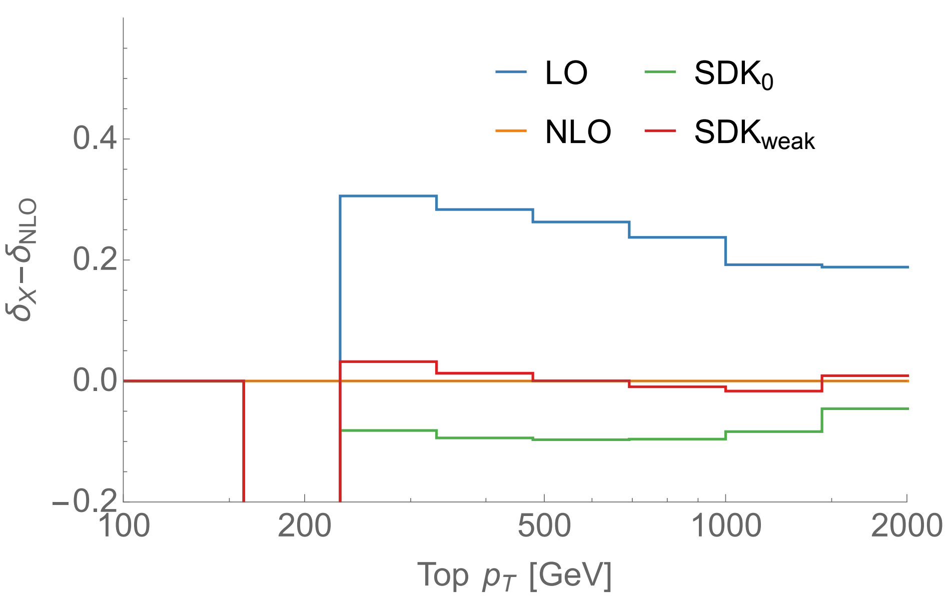


For precision physics the scheme adopted and the NLL accuracy (*Frixione, Stagnitto '23*) are mandatory. But it is not the focus of this talk.

$$t\bar{t}$$







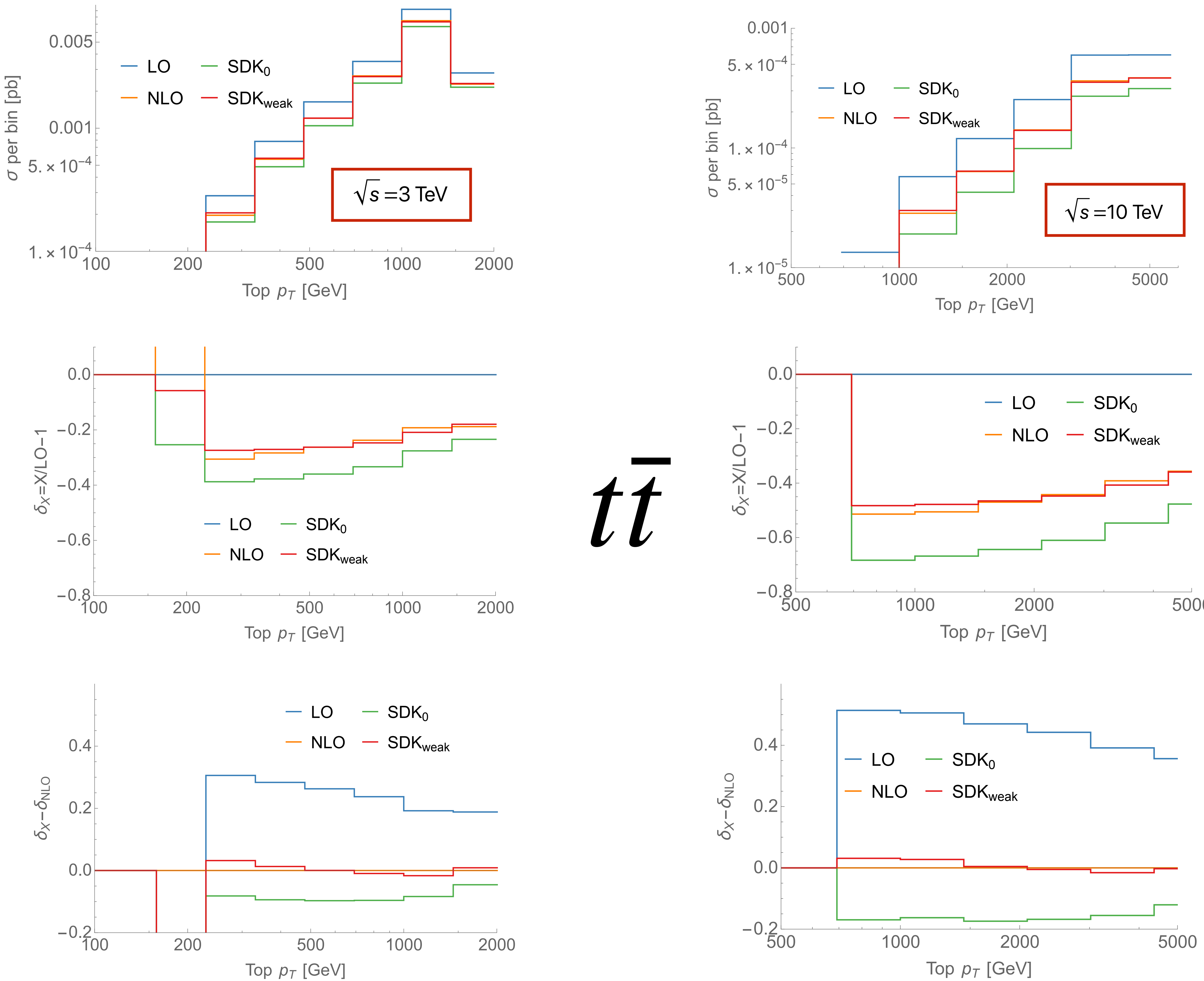
For smaller p_T , larger corrections

Sudakov (in the $\mathrm{SDK_{weak}}$ scheme) **capture NLO EW corrections** up to the % level.

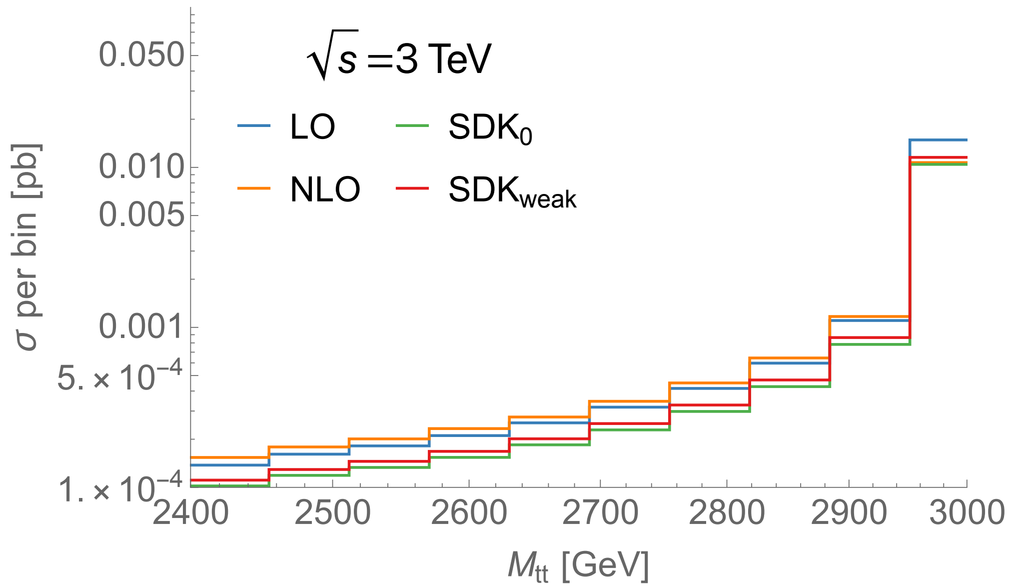
If double logs are expressed as $\log^2(s/m_W^2)$, the shapes observed here are all arising from **single logs**.

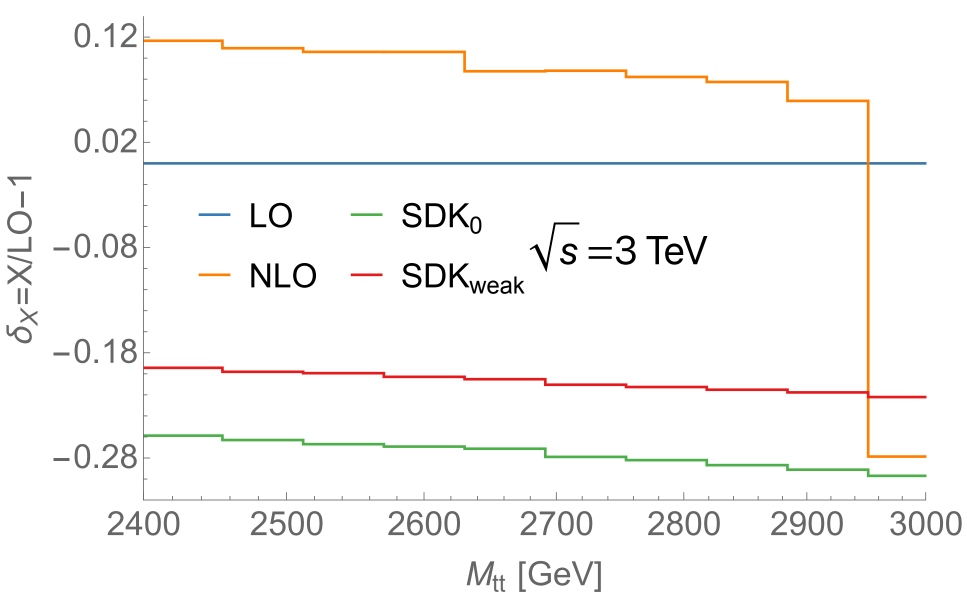
$$t\bar{t}$$

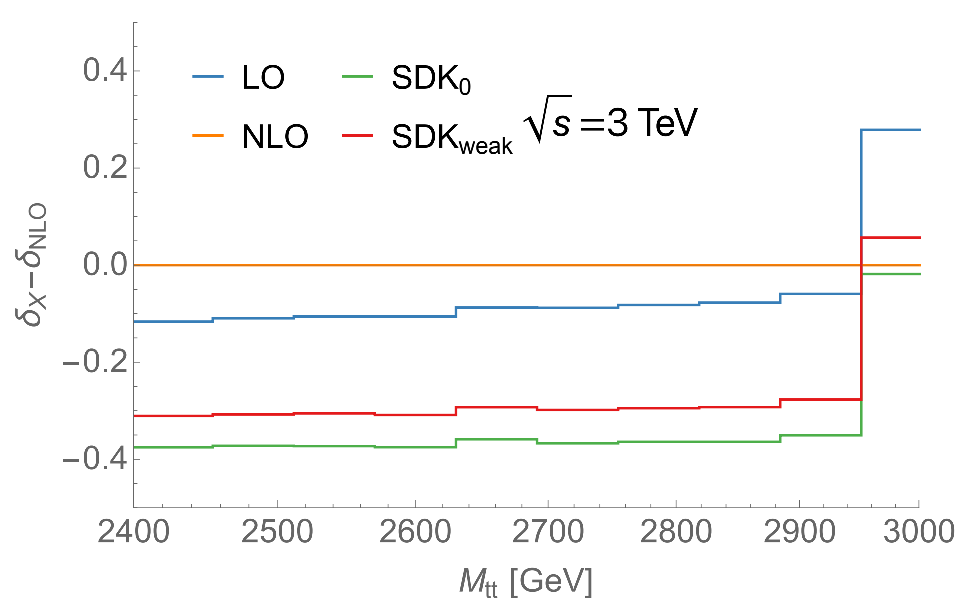
*Obviously, the last bin on the right is filled only up to 1500 GeV at LO, that's why there is a drop in the σ per bin.



$\sqrt{s}$ =3 TeV
$\sqrt{s}$ =10 TeV
σ per bin [pb]
Top p_T [GeV]
LO
NLO
SDK$_0$
SDK$_{weak}$
δ_X=X/LO−1
$\delta_X - \delta_{NLO}$
$t\bar{t}$







Some distributions are pathological at Fixed Order, like $M_{t\bar{t}}$ for $t\bar{t}$ production. At LO all bins but the last one are due to PDFs effects. At NLO all bins receive contributions even with no PDFs.

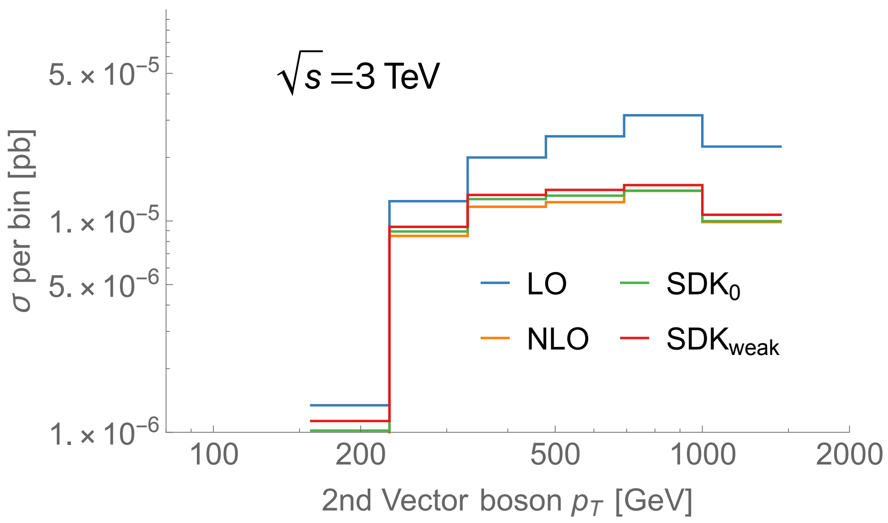
Sudakov logs cannot catch the NLO EW corrections for this distribution.
If we considered photon shower effects from the beginning, it may be a completely different story.

$$t\bar{t}$$

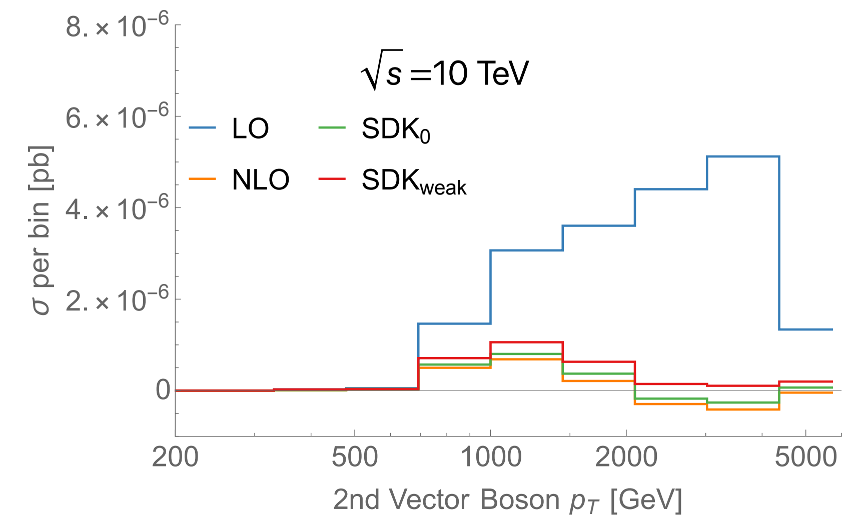
ZZZ

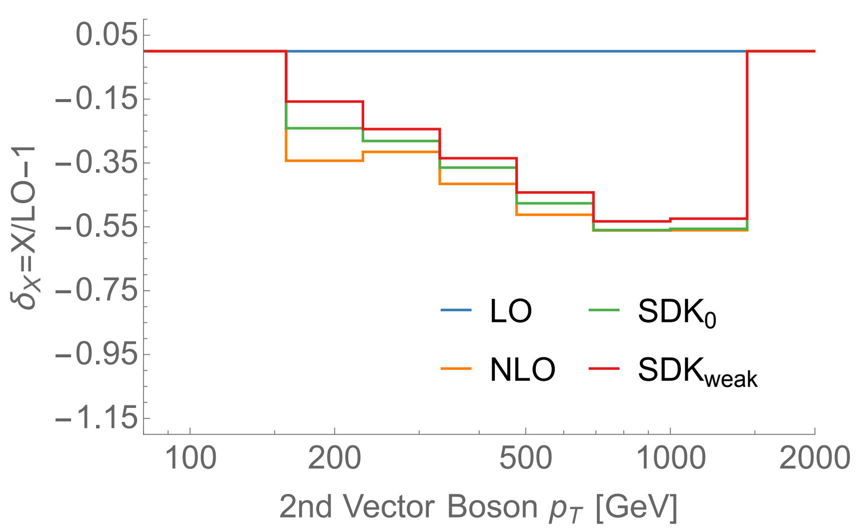
One should keep in mind that BSM effects may increase total rates. Most of the Sudakov effects would be unchanged with no new resonant particles.

ZZZ

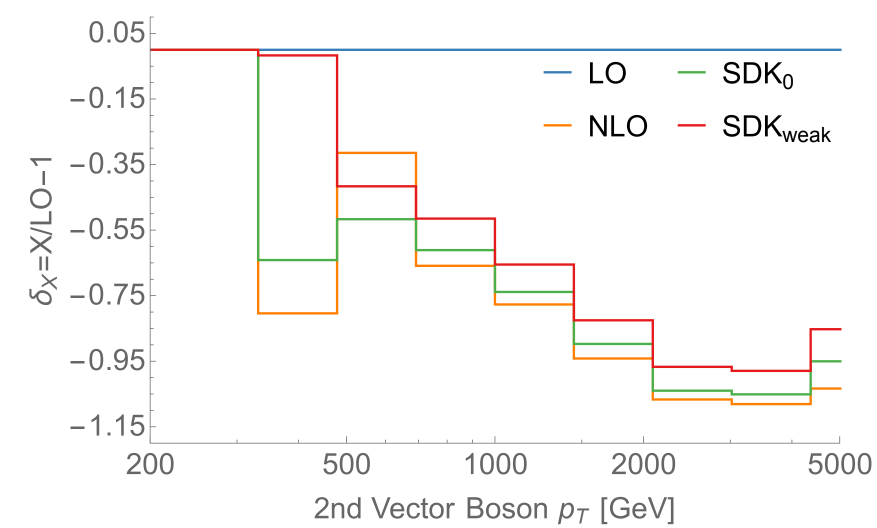


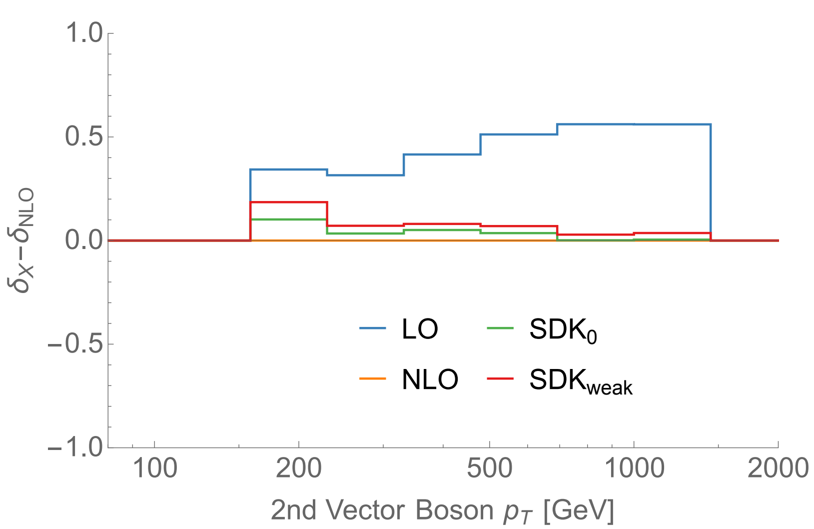
Spectrum dominated by large momenta.



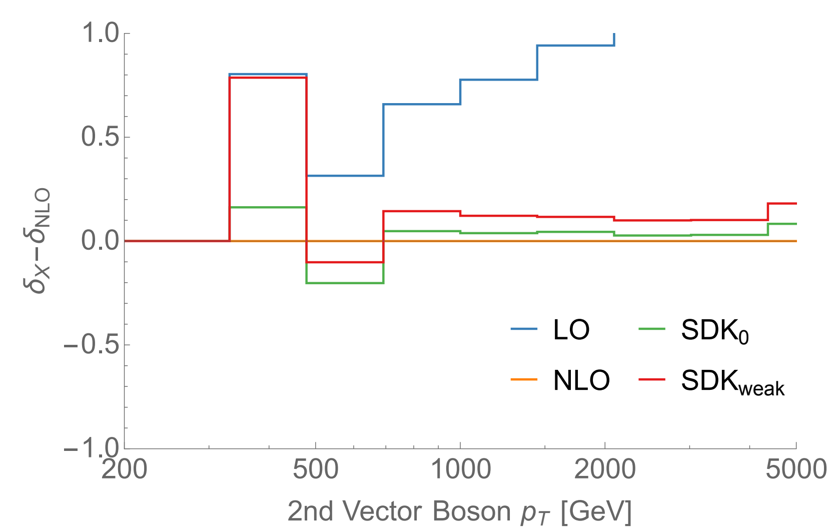


Resummation is clearly necessary.



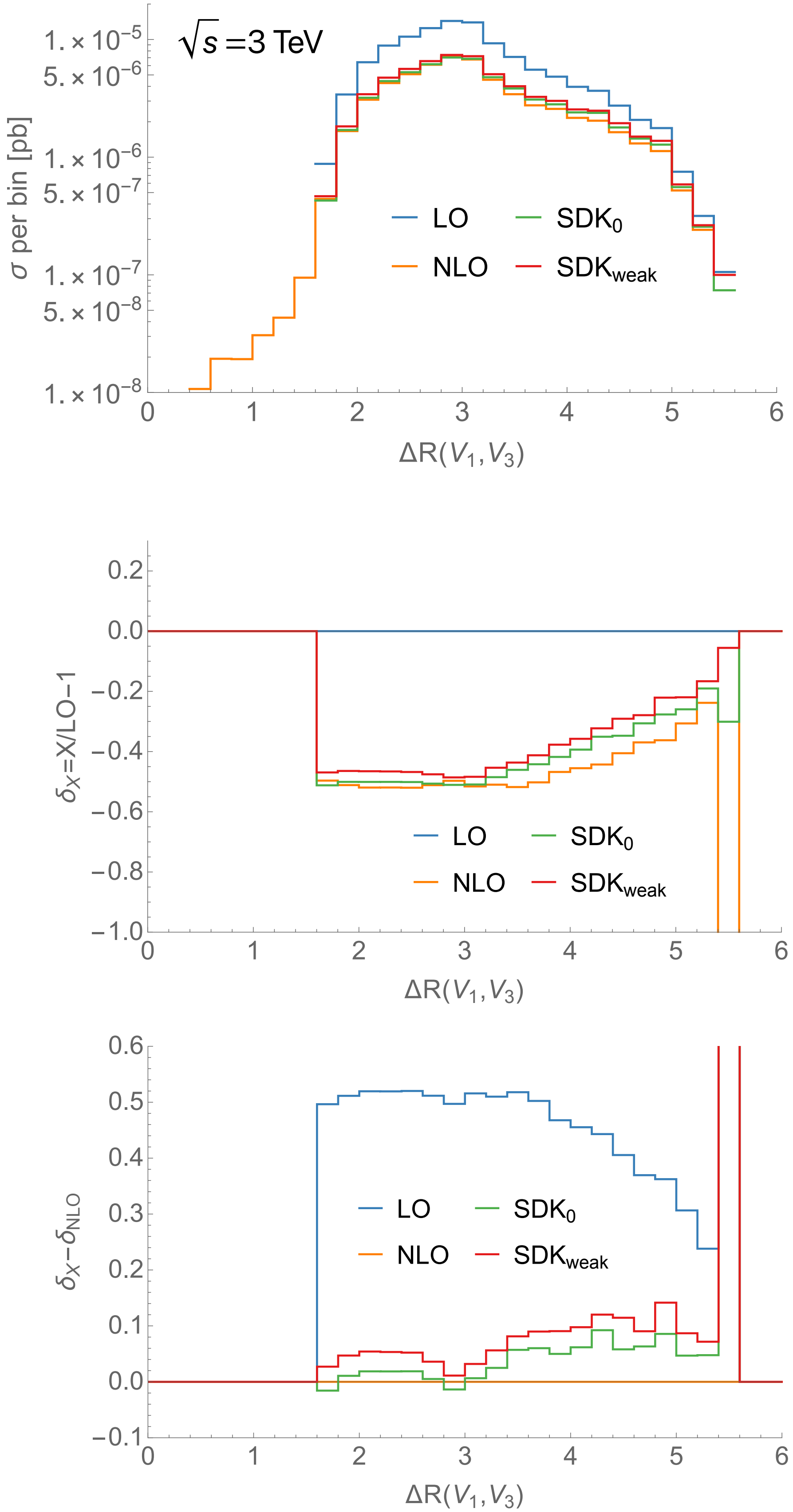


Finite flat corrections of just a few percents from non-Sudakov effects.



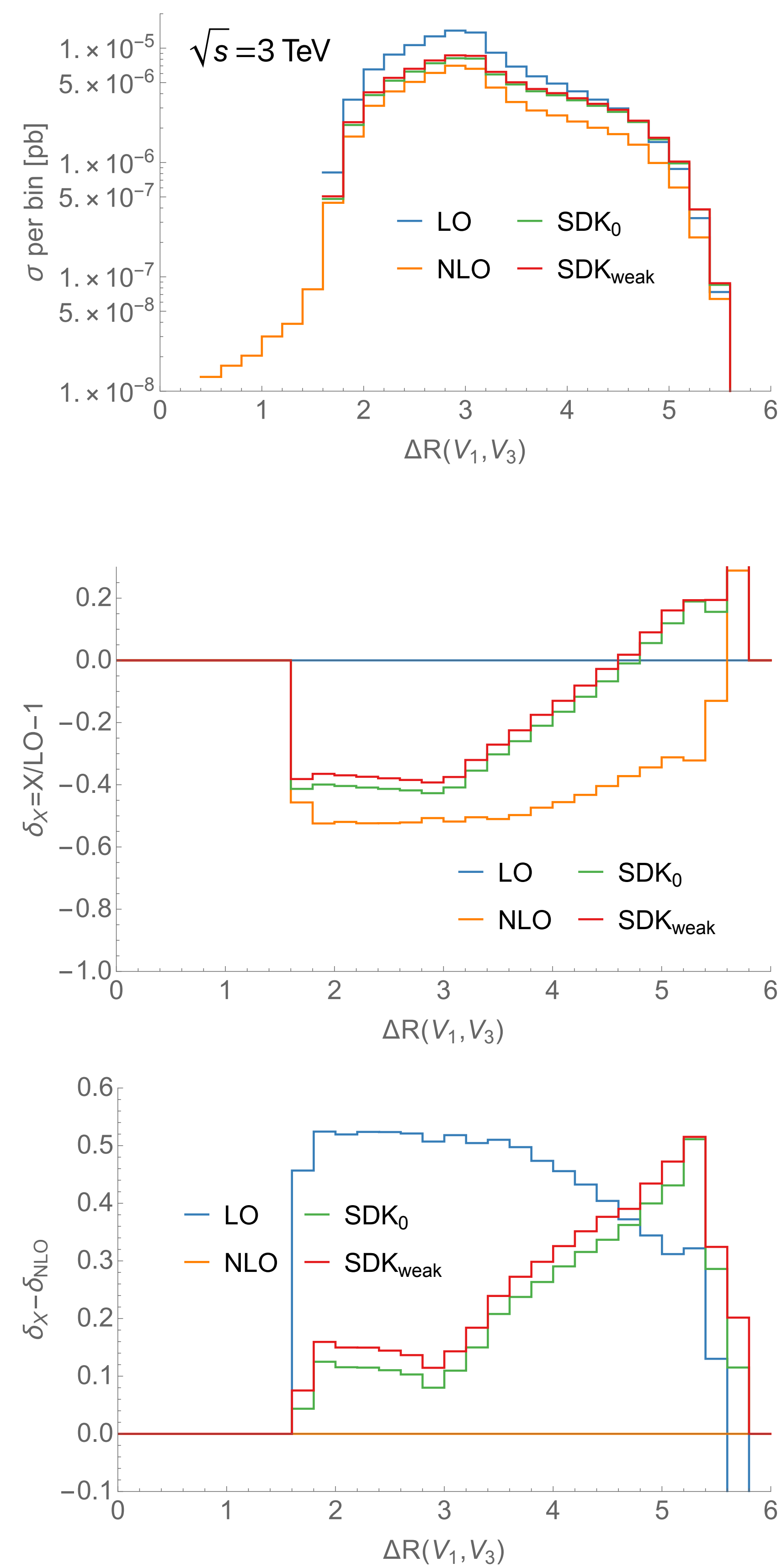
ZZZ

NO $\boxed{2l(M_W^2, M^2)\log\frac{|r_{kl}|}{s} + L(|r_{kl}|, s)}$ term



Sudakov logs can approximate very well the NLO EW but not only the logs of the form $\frac{\alpha}{4\pi s_w^2}\log^k(s/m_W^2)$ with $k = 1{,}2$ are relevant.

Also double and single logs among invariants are not negligible.



ZHH

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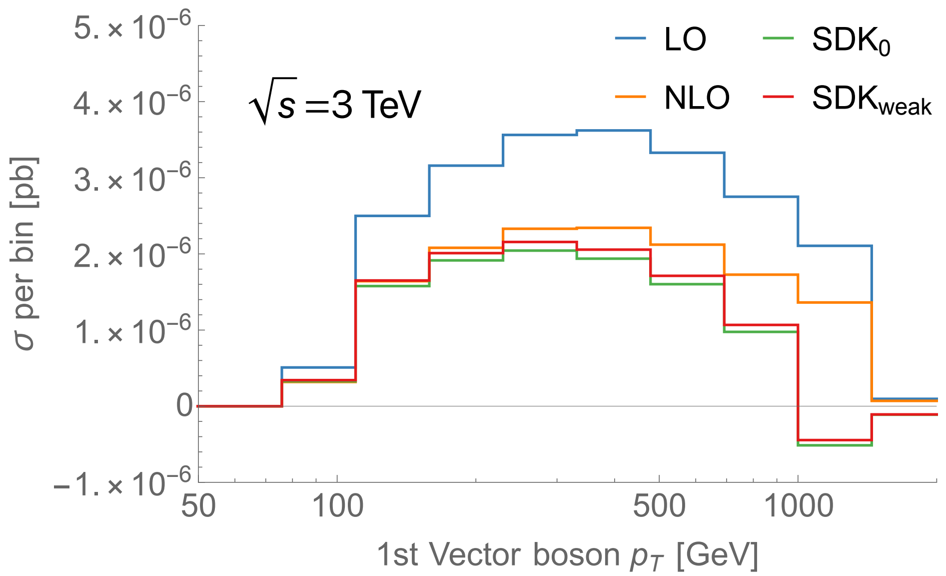
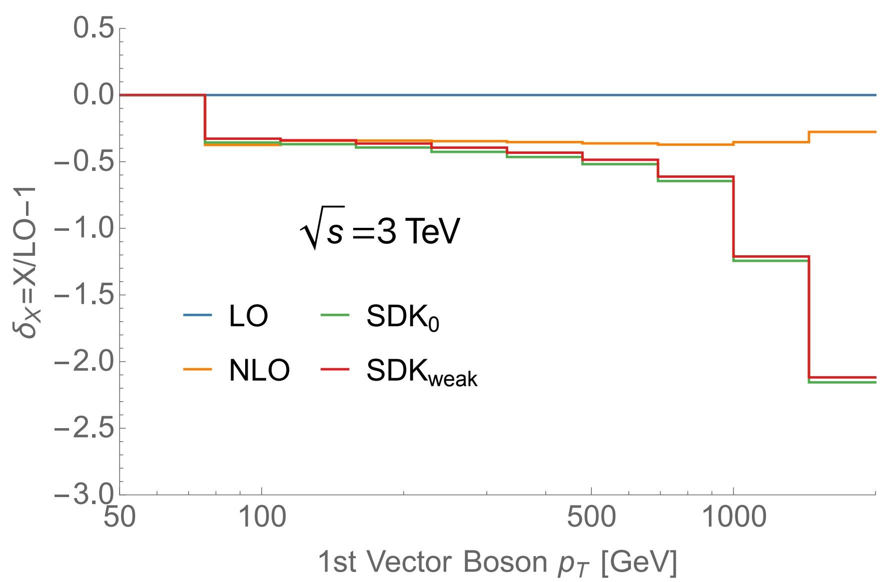
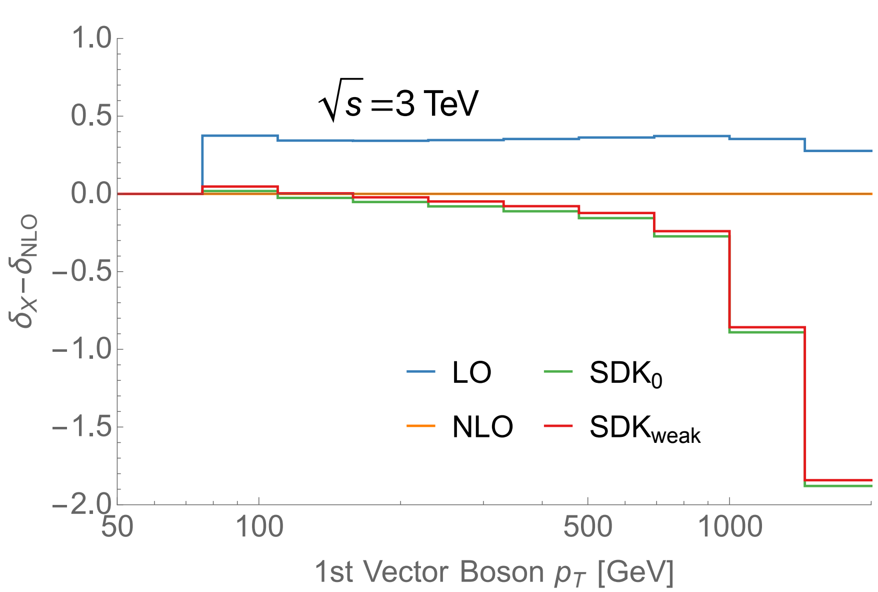


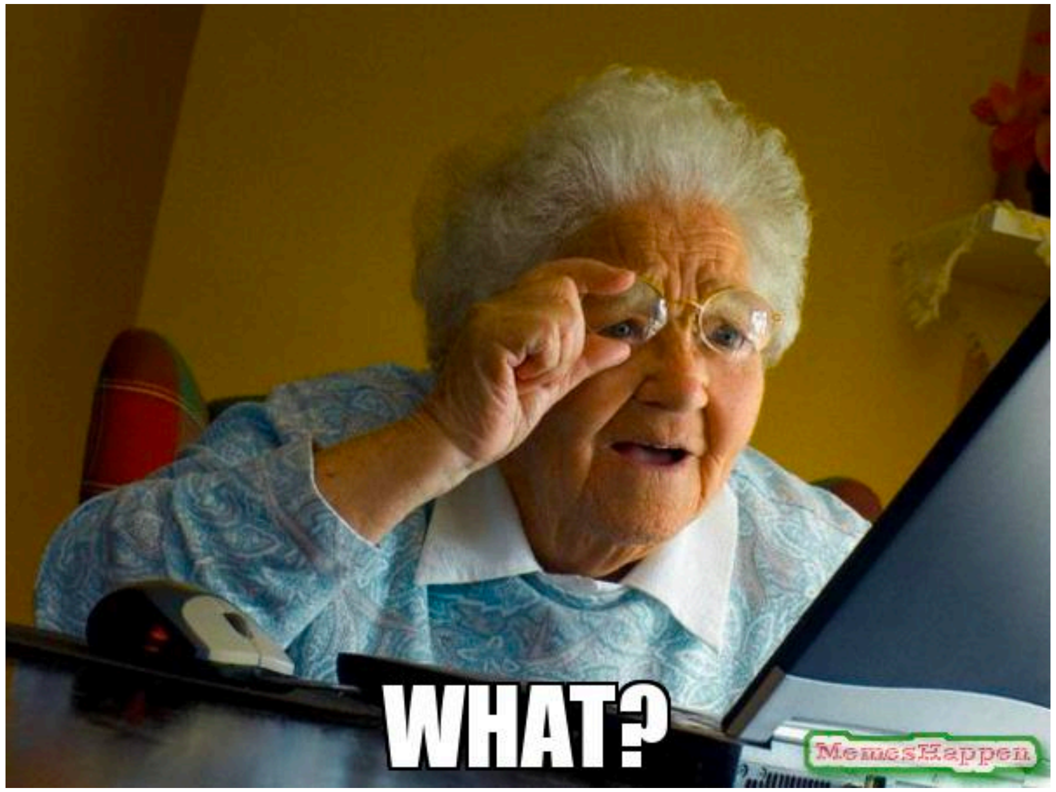




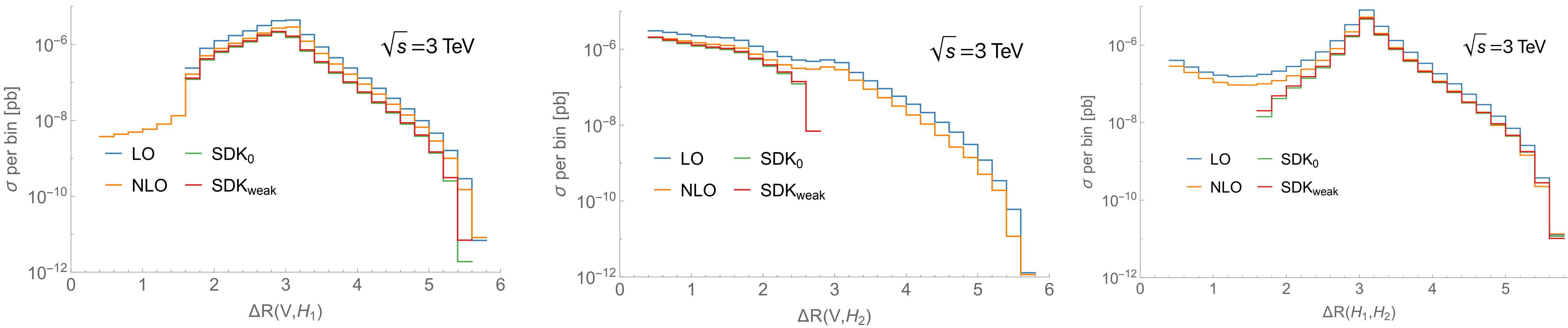
ZHH

NLO EW corrections are flat.

Sudakov logarithms work **very well at low pt and very bad at high pt.**

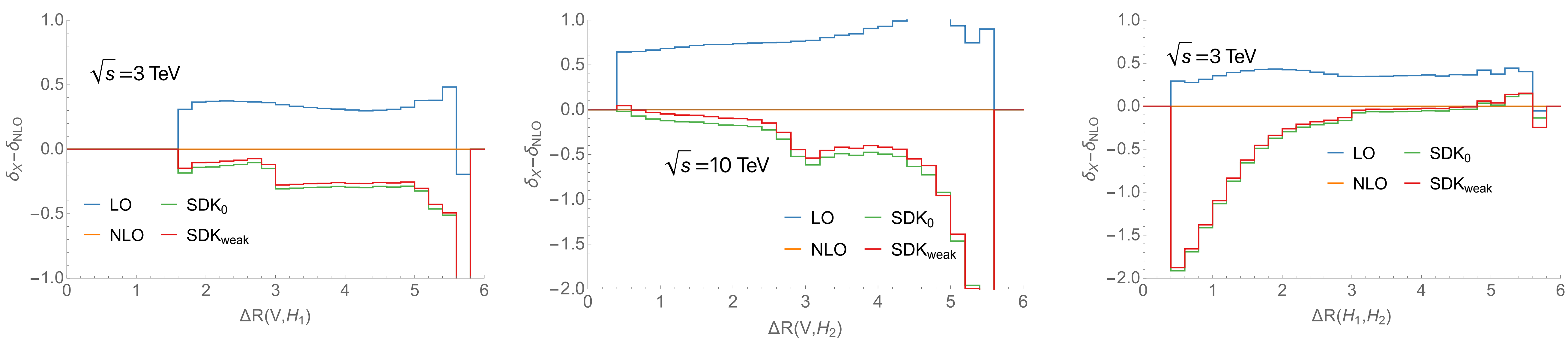


ZHH



For High pt of the Z boson, the two Higgs can have very small ΔR and so small $m_{H_1H_2}$, recoiling against the Z.

In that configuration, formally mass suppressed terms $\sim \dfrac{v}{m(H_1H_2)}$ can become numerically sizeable, and the DP algorithm fails. **As it fails also for VBF single Higgs production.**



What about extra radiation of Z (and H)?

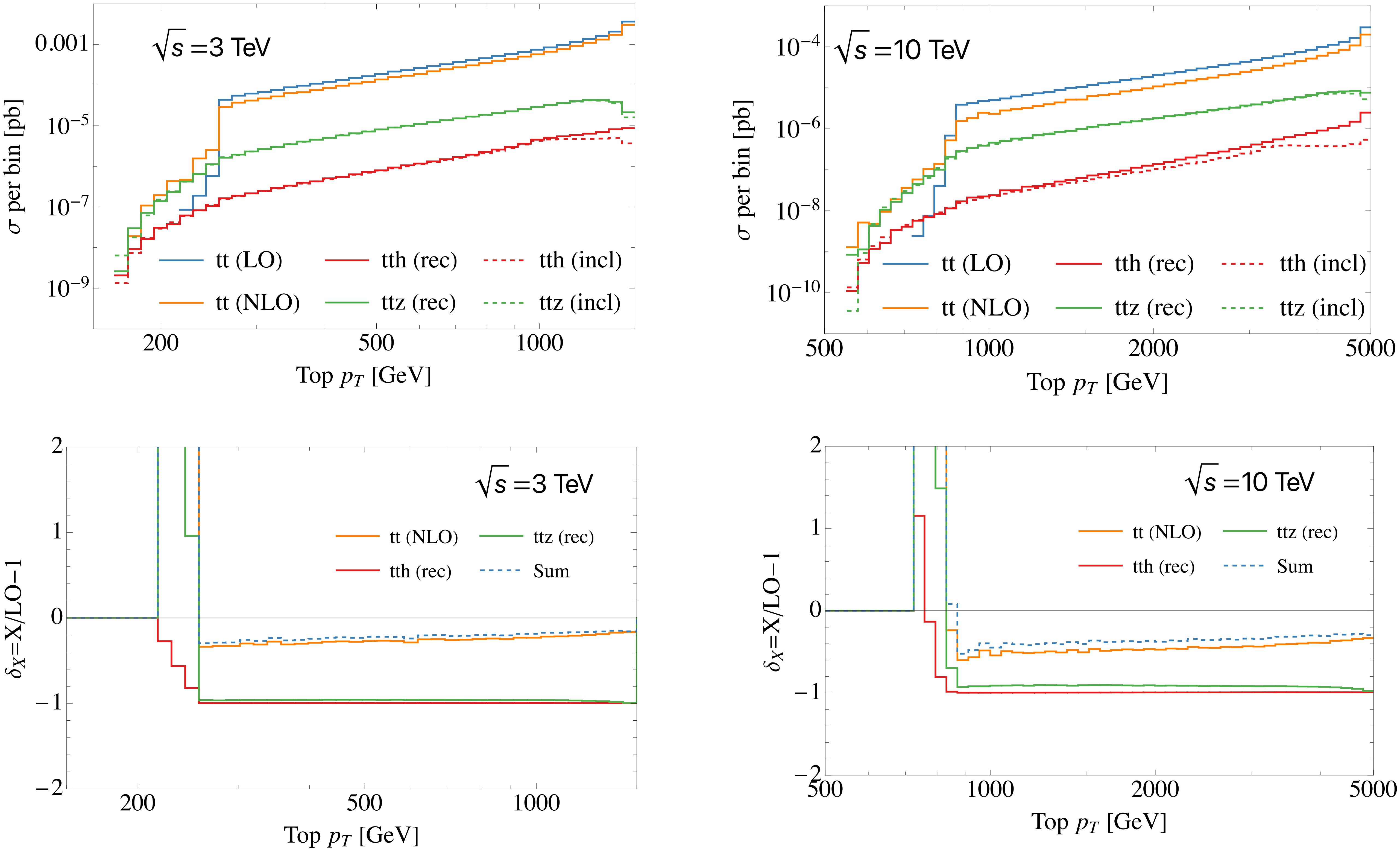
We know that unlike QCD in virtual+real there is not the exact cancellation of logarithms.
But a cancellation is still present, how much large?

back to

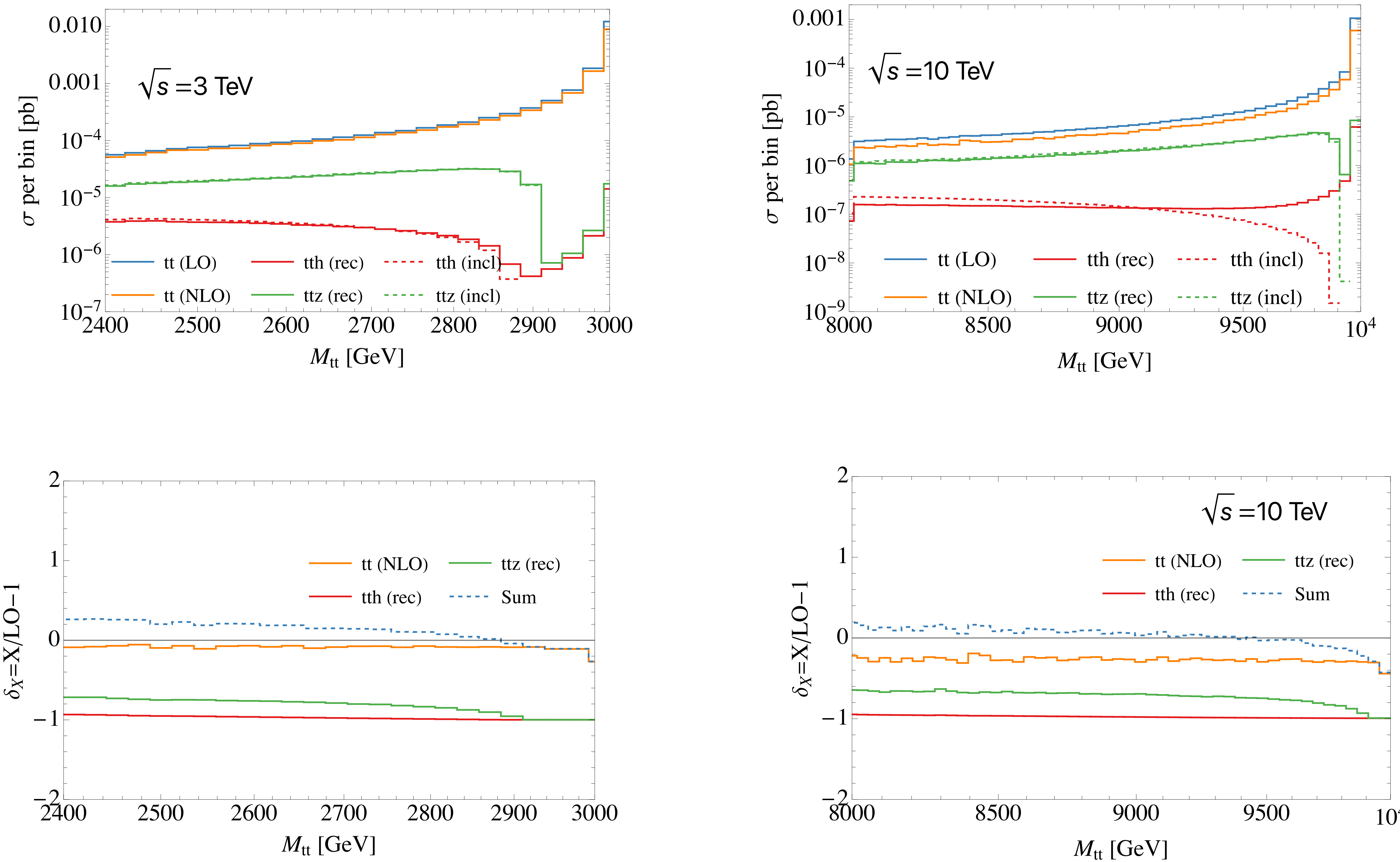
$t\bar{t}$

We consider the idea that not only the photon, but also the Z or the Higgs are recombined together with the top if they are within a $\Delta R < 0.4$.

.. very small effects from Z and H radiation



sum= NLO EW + H and Z radiation



sum= NLO EW + H and Z radiation

OUTLOOK rather than CONCLUSION

Why did we automate Sudakov in aMG5?

- EW corrections are mandatory for phenomenology at muon colliders.
- There are many interesting open questions/issues on this subject.
- We discussed the Sudakov approximation vs the exact NLO EW for direct production.
- Sudakov logs are the bulk of the NLO EW contribution and they are a good approximation under some conditions: single logs present, logs among invariants present, correct scheme $\mathrm{SDK_{weak}}$ adopted, mass-suppressed contributions negligible.
- Heavy-Boson Radiation has an impact, how much it is still to be studied in detail.
- Resummation is mandatory for sensible results in many configurations.

Final advertisement

Considering only the dominant contribution from NLO EW:
Sudakov and QED FSR.

Matching of NLO QCD + PS + EWSL + QED FSR at the LHC:

Improving NLO QCD event generators with high-energy EW corrections

Davide Pagani,[a] Timea Vitos,[b] Marco Zaro[c]

This method is based on the $\mathrm{SDK_{weak}}$ approach discussed in the talk.

Both Born, QCD virtual and separately real corrections are consistently reweighted via NLO EW Sudakov logarithms.

EXTRA SLIDES

Our revisitation and automation: Amplitude level

We have **revisited** and **automated** in aMG5 the **Denner&Pozzorini algorithm** for the evaluation of one-loop EW Sudakov corrections to amplitudes (*Denner, Pozzorini '01*). In particular we have introduced the **following novelties**.

- **IR QED** divergencies are dealt with **via D**imensional **R**egularisation, with strictly massless photons and light fermions.

- **Additional logarithms** that involve ratios between invariants, and therefore **angular** dependences, are taken into account.

- We correctly take into account an **imaginary term** that was **previously omitted** in the literature. Relevant for $2 \to n$ processes with $n > 2$

- Moving to the level of interferences of tree and one-loop amplitudes, we take into account NLO EW contributions originating from **QCD loops on top of subleading LO terms**.

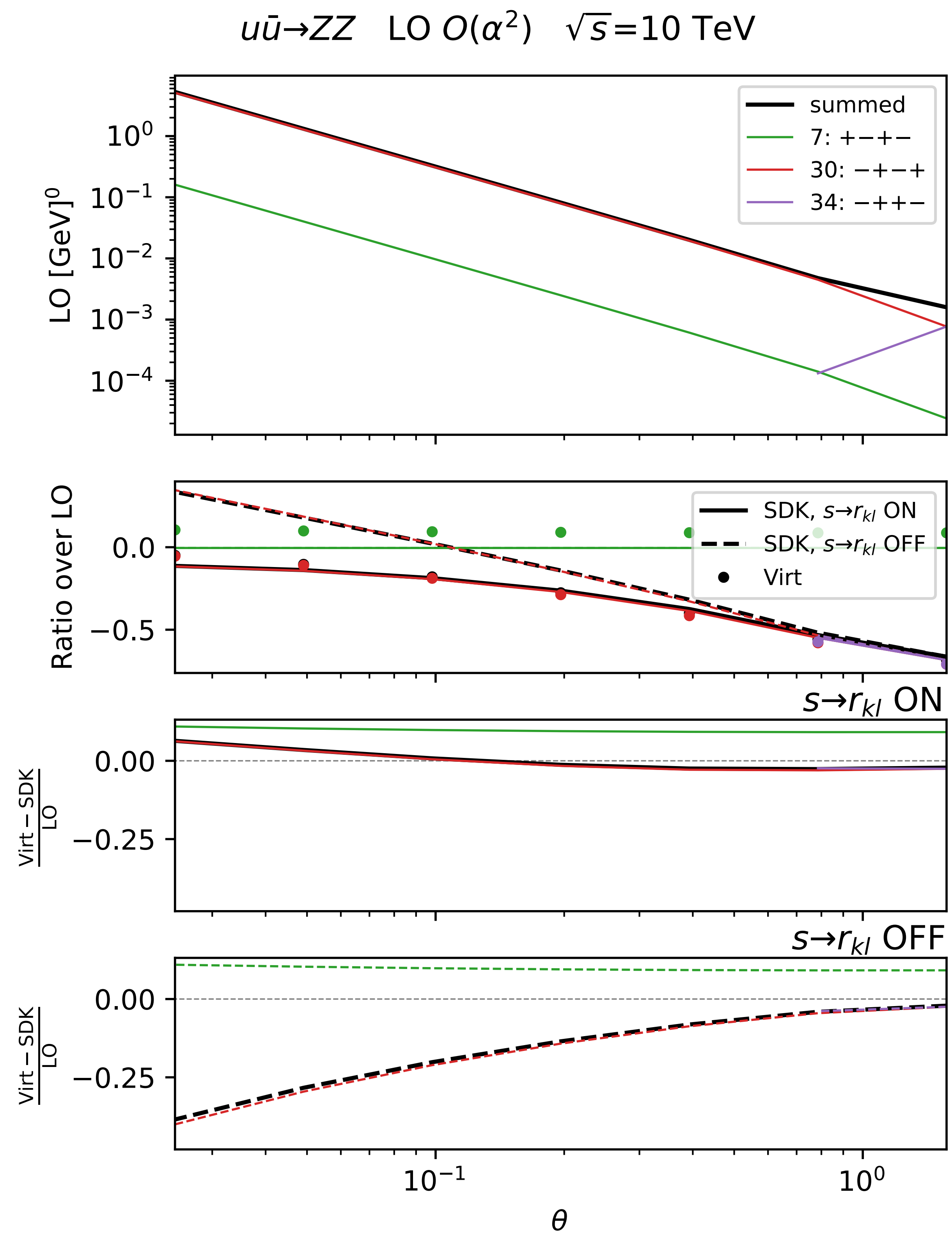
Example ($2 \to 2$): $u\bar{u} \to ZZ$ scan in θ

Denner&Pozzorini algorithm works only with non mass-suppressed LO processes: we select only helicity configurations > 10^(-3) of the dominant one.

Dots: NLO EW (MadLoop). **Lines =** Sudakov.
Dashed: standard approach.
Solid: our formulation (more angular information)

Dots-Solid/LO: quite horizontal, the correct Log dependence is very-well approximated.

Dots-Dashed/LO: not horizontal, the correct Log dependence is lost.



Implementation

Born amplitude: $\mathcal{M}_0^{i_1\dots i_n}(p_1,\dots,p_n)$

One-loop EW Sudakov corrections: $\delta\mathcal{M}^{i_1\dots i_n}(p_1,\dots,p_n) = \mathcal{M}_0^{i'_1\dots i'_n}(p_1,\dots,p_n)\delta_{i'_1 i_1\dots i'_n i_n}$

other tree-level amplitudes

the logs

Born process: $\boxed{\varphi_{i_1}(p_1)\dots\varphi_{i_n}(p_n)\to 0}$

$$Z \longrightarrow \chi\,,$$
$$W^\pm \longrightarrow \phi^\pm\,,$$

GBE theorem for longitudinal W and Z bosons.

$\varphi_{i_1}(p_1)\dots\varphi_{i'_k}\dots\varphi_{i_n}(p_n)\to 0\,,$

$$Z \longleftrightarrow A\,,$$
$$H \longleftrightarrow \chi\,.$$

Relevant for LSC and C contributions.

Amplitudes with one or 2 different external particles w.r.t. the Born have to be generated.

$\varphi_{i_1}(p_1)\dots\varphi_{i'_k}\dots\varphi_{i'_l}\dots\varphi_{i_n}(p_n)\to 0$

$$f_\sigma \longleftrightarrow f_{-\sigma}\,,$$
$$H \longleftrightarrow \phi^\pm\,,$$
$$\chi \longleftrightarrow \phi^\pm\,,$$
$$A \longleftrightarrow W^\pm\,,$$
$$Z \longleftrightarrow W^\pm\,.$$

Relevant for SSC charged contributions.

Organisation of the logs in the algorithm

Two examples: LSC and C for fermions

$$\delta^{\mathrm{LSC}}_{i'_k i_k}(k) = -\frac{1}{2}\left[C^{\mathrm{ew}}_{i'_k i_k}(k)\boxed{L(s)} - 2(I^Z(k))^2_{i'_k i_k}\log\frac{M_\mathrm{Z}^2}{M_\mathrm{W}^2}\boxed{l(s)} + \delta_{i'_k i_k} Q_k^2 \boxed{L^{\mathrm{em}}(s,\lambda^2,m_k^2)}\right]$$

Casimir for the entire $SU(2)_L \times U(1)_B$

Charge for $U(1)_{QED}$

$$\delta^{\mathrm{C}}_{f_\sigma f_{\sigma'}}(f^\kappa) = \delta_{\sigma\sigma'}\left\{\left[\frac{3}{2}C^{\mathrm{ew}}_{f^\kappa} - \frac{1}{8s_\mathrm{w}^2}\left((1+\delta_{\kappa\mathrm{R}})\frac{m_{f_\sigma}^2}{M_\mathrm{W}^2} + \delta_{\kappa\mathrm{L}}\frac{m_{f_{-\sigma}}^2}{M_\mathrm{W}^2}\right)\right]\boxed{l(s)} + Q_{f_\sigma}^2 \boxed{l^{\mathrm{em}}(m_{f_\sigma}^2)}\right\},$$

$$\boxed{L(s)} \equiv L(s, M_W^2) \quad \text{and} \quad \boxed{l(s)} \equiv l(s, M_W^2)$$

$$\boxed{l^{\mathrm{em}}(m_f^2)} := \frac{1}{2} l(M_\mathrm{W}^2, m_f^2) + l(M_\mathrm{W}^2, \lambda^2) \qquad \boxed{L^{\mathrm{em}}(s,\lambda^2,m_k^2)} := 2l(s)\log\left(\frac{M_\mathrm{W}^2}{\lambda^2}\right) + L(M_\mathrm{W}^2,\lambda^2) - L(m_k^2,\lambda^2)$$

The full EW is present between s and M_W^2, while only QED is present between M_W^2 and λ^2.

So the QED contribution is split between the intervals $(s, M_W^2) + (M_W^2, \lambda^2)$. But the division at M_W^2 is simply determined by convenience, in parallel with the weak case. In this case M_W^2 is just a technical parameter and not a physical quantity.

Cross-sections: standard approach in the literature

SDK_0

Two examples: LSC and C for fermions

$$\delta^{\mathrm{LSC}}_{i'_k i_k}(k) = -\frac{1}{2}\left[C^{\mathrm{ew}}_{i'_k i_k}(k) \boxed{L(s)} - 2(I^Z(k))^2_{i'_k i_k} \log\frac{M_Z^2}{M_W^2} \boxed{l(s)} + \cancel{\delta_{i'_k i_k} Q_k^2 L^{\mathrm{em}}(s, \lambda^2, m_k^2)} \right]$$

Casimir for the entire
$SU(2)_L \times U(1)_B$

$$\delta^{\mathrm{C}}_{f_\sigma f_{\sigma'}}(f^\kappa) = \delta_{\sigma\sigma'} \left\{ \left[\frac{3}{2} C^{\mathrm{ew}}_{f^\kappa} - \frac{1}{8 s_{\mathrm{w}}^2} \left((1+\delta_{\kappa \mathrm{R}}) \frac{m_{f_\sigma}^2}{M_W^2} + \delta_{\kappa \mathrm{L}} \frac{m_{f_{-\sigma}}^2}{M_W^2} \right) \right] \boxed{l(s)} + \cancel{Q_{f_\sigma}^2 l^{\mathrm{em}}(s, m_{f_\sigma}^2)} \right\}$$

$\boxed{L(s)} \equiv L(s, M_W^2)$ and $\boxed{l(s)} \equiv l(s, M_W^2)$

The logarithms between M_W^2 and the infrared scale are simply removed. Equivalently in the case of DR, logarithms involving M_W^2 and the IR regulator Q^2.

Easy, but not very well motivated.

We will denote in the following this approach as SDK_0.

Purely Weak

1. Calculate the δ^{PR} in eq. (2.12) as in the standard SDK approach.

2. For each external particle φ_{i_k} in (2.9), set
$$Q_k = I^A(k) = 0\,. \tag{4.1}$$
This step alone has the effect of eliminating all the terms tagged as "em", with the exception of $\delta Z^{\mathrm{em}}_{AA}$. It also eliminates all the SSC terms and C terms that lead to SL originating from photons, with the exception of those related to transverse W bosons.

3. For each external particle φ_{i_k} in (2.9), perform the replacement
$$C^{\mathrm{ew}}_{i'_k i_k}(k) \longrightarrow C^{\mathrm{ew}}_{i'_k i_k}(k) - Q_k^2\,, \tag{4.2}$$
with the value of Q_k^2 before enforcing eq. (4.1). This, in combination with eq. (4.1), has the effect of eliminating the DL due to photons.

4. Perform the replacement
$$b^{\mathrm{ew}}_W \longrightarrow b^{\mathrm{ew}}_W - 11/3\,. \tag{4.3}$$
This has the effect of eliminating for the transverse W bosons the C terms that lead to SL originating from photons.

5. Set
$$\delta Z^{\mathrm{em}}_{AA} = 0\,, \tag{4.4}$$
and perform the replacement
$$b^{\mathrm{ew}}_{AA} \longrightarrow b^{\mathrm{ew}}_{AA} + \frac{4}{3}\sum_{f,i,\sigma\neq t} N_{\mathrm{C}}^f Q_{f_\sigma}^2 = b^{\mathrm{ew}}_{AA} + 80/9\,. \tag{4.5}$$
This has the effect of eliminating, for the photons, the C terms that lead to SL originating from light fermions.

6. Calculate the remaining terms in eq. (2.12) with the new redefinitions of steps 2–5.