# Electroweak Factorization for Muon Colliders

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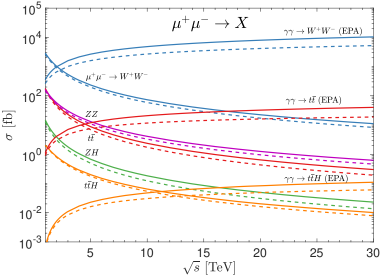
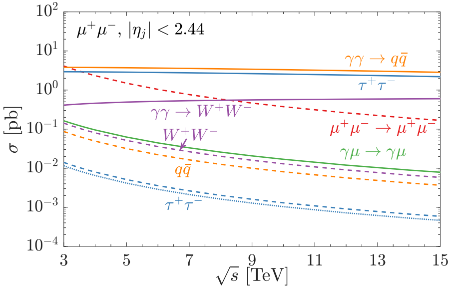
Based on work with T. Han and Y. Ma
2007.14300, 2103.09844, 2106.01393

# Vector boson fusions vs. annihilations




[Han, Ma, KX, 2007.14300]

[Han, Ma, KX, 2103.09844]

## General features:

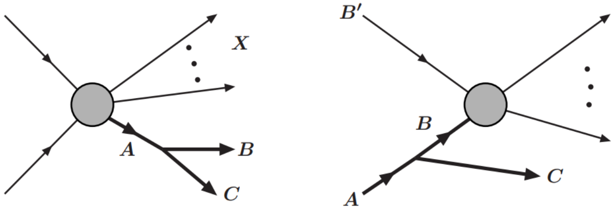
- The annihilations decrease as $1/s$.
- ISR needs to be considered, which can give over $10\%$ enhancement.
- The fusions increase as $\ln(s)$, which take over at high energies.
- The large collinear logarithm $\ln\left(Q^2/m_\ell^2\right)$ needs to be resummed.
- $W^+W^-$ as a reference to separate high-energy EW and low-energy QED/QCD

**Q: How to treat parton properly at high energies when $W/Z$ become active?**

# EW physics at high energies

- At high energies, every particle become massless
$$\frac{v}{E}:\frac{v}{100\ \mathrm{TeV}}\sim\frac{\Lambda_{\mathrm{QCD}}}{100\ \mathrm{GeV}},\ \frac{v}{E},\frac{m_t}{E},\frac{M_W}{E}\to 0!$$
- The splitting phenomena dominate due to large log enhancement
- The EW symmetry is restored: $SU(2)_L\times U(1)_Y$ unbroken
- Goldstone Boson Equivalence:
$$\varepsilon_L^\mu(k)=\frac{E}{M_W}(\beta_W,\hat{k})\simeq\frac{k^\mu}{M_W}+\mathscr{O}(\frac{M_W}{E})$$
The violation terms is power counted as $v/E\to$ QCD higher twist effects $\Lambda_{\mathrm{QCD}}/Q$ [Cuomo, Wulzer, 1703.08562; 1911.12366].
- We mainly focus on the splitting phenomena, which can be factorized and resummed as the EW PDFs in the ISR, and the Fragementaions/Parton Shower in the FRS.
- Other interesting aspects: the polarized EW boson scattering, top-Yukawa coupling effect

# Splitting phenomena




$$\mathrm{d}\sigma \simeq \mathrm{d}\sigma_X \times \mathrm{d}\mathscr{P}_{A\to B+C}\,, \quad E_B \approx zE_A, \quad E_C \approx \bar{z}E_A, \quad k_T \approx z\bar{z}E_A\theta_{BC}$$

$$\frac{\mathrm{d}\mathscr{P}_{A\to B+C}}{\mathrm{d}z\mathrm{d}k_T^2} \simeq \frac{1}{16\pi^2}\frac{z\bar{z}|\mathscr{M}^{(\mathrm{split})}|^2}{(k_T^2+\bar{z}m_B^2+zm_C^2-z\bar{z}m_A^2)^2}, \quad \bar{z}=1-z$$

- The dimensional counting: $|\mathscr{M}^{(\mathrm{split})}|^2 \sim k_T^2$ or $m^2$
- To validate the fractorization formalism
  - The observable $\sigma$ should be infra-red safe
  - Leading behavior comes from the collinear splitting

[Ciafaloni et al., hep-ph/0004071; 0007096; Bauer, Webber et al., 1703.08562; 1808.08831]

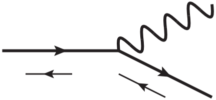
[Manohar et al., 1803.06347; Han, Chen, Tweedie, 1611.00788]

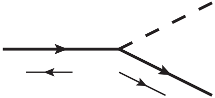
# EW Splitting functions

- Starting from the unbroken phase: all massless

$$\mathscr{L}_{SU(2)\times U(1)} = \mathscr{L}_{\text{gauge}} + \mathscr{L}_{\phi} + \mathscr{L}_{f} + \mathscr{L}_{\text{Yukawa}}$$

- Particle contents:
  - Chiral fermions $f_{L,R}$
  - Gauge bosons: $B$, $W^{0,\pm}$
  - Higgs $H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} \phi^+ \\ \frac{h - i\phi^0}{\sqrt{2}} \end{pmatrix}$
- Splitting functions [See Ciafaloni et al. hep-ph/0505047, Han et al. 1611.00788 for complete lists.]





| | $\frac{1}{8\pi^2}\frac{1}{k_T^2}\frac{1+\bar{z}^2}{z}$ | | $\frac{1}{8\pi^2}\frac{1}{k_T^2}\frac{z}{2}$ | |
|---|---|---|---|---|
| | $\to V_T f_s^{(\prime)}$ | $[BW]^0_T f_s$ | $H^{0(*)} f_{-s}$ | $\phi^\pm f'_{-s}$ |
| $f_{s=L,R}$ | $g_V^2 (Q^V_{f_s})^2$ | $g_1 g_2 Y_{f_s} T^3_{f_s}$ | $y^2_{f_R^{(\prime)}}$ | |
| | Soft & collinear singularites $(P_{gq})$ | | Collinear singularity chirality-flip, Yukawa | |

# Electroweak symmetry breaking

## Goldstone Boson Equivalence Theorem (GBET)

[Lee, Quigg, Thacker (1977); Chanowitz & Gailard (1984)]

- At high energies $E \gg M_W$, the longitudinally polarized gauge bosons $V_L$ behave like the corresponding Goldstone bosons
  They remember their origin!
- Scalarization of $V_L$

$$\varepsilon_L^{\mu}(k) = \frac{E}{M_W}(\beta_W, \hat{k}) \simeq \frac{k^{\mu}}{M_W} + \mathscr{O}(M_W/E)$$

- The GBET violation can be counted as power corrections $v/E$
  $\rightarrow$ Higher twist effects in QCD $(\Lambda_{\mathrm{QCD}}/Q)$

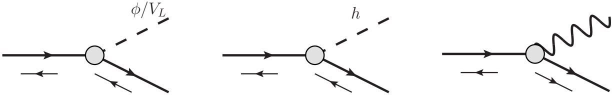
[Han et al. 1611.00788, Bauer et al. 1703.08562, Cuomo, Wulzer 1911.12366].

# New splitting in a broken gauge theory

- Fermion splitting into longitudinal gauge boson $f \to V_L$

$$P \sim \frac{v^2}{k_T^2}\frac{\mathrm{d}k_T^2}{k_T^2} \sim 1 - \frac{v^2}{Q^2}$$

- $V_L$ is of IR, $h$ has no IR [Han et al. 1611.00788]




| | | | |
|---|---|---|---|
| | $\frac{1}{16\pi^2}\frac{v^2}{k_T^4}\frac{1}{z}$ | $\frac{1}{16\pi^2}\frac{v^2}{k_T^4}$ | $\frac{1}{16\pi^2}\frac{v^2}{k_T^4}$ |
| $f_s \to$ | $V_L f_s^{(\prime)} (V \neq \gamma)$ | $h f_s$ | $V_T f_{-s}^{(\prime)}$ |
| | Chirality conserving non-zero for massless $f$ | | Chirality flipping $\sim m_f$ |

- The PDFs for $W_L/Z_L$ behaves as constants, which does not run at the leading log: "Bjorken scaling" restoration

$$f_{V_L/f}(x, Q^2) \sim \alpha \frac{1-x}{x}$$

Residuals of the EWSB, $v^2/E^2$, similar to higher-twist effects

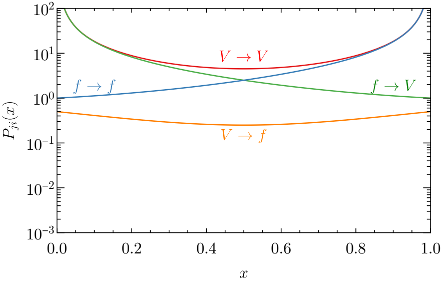
# Polarizations in the EW splittings

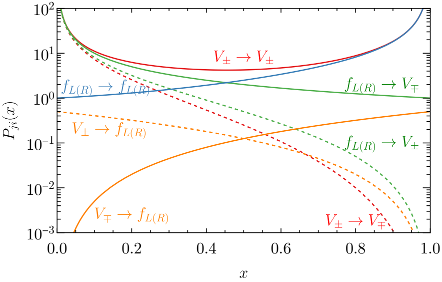
- The EW splittings must be polarized due to the chiral nature of the EW theory

$$f_{V_+/A_+} \neq f_{V_-/A_-}, \qquad f_{V_+/A_-} \neq f_{V_-/A_+},$$
$$\hat{\sigma}(V_+B_+) \neq \hat{\sigma}(V_-B_-), \quad \hat{\sigma}(V_+B_-) \neq \hat{\sigma}(V_-B_+)$$

We are not able to factorize the cross sections in an unporlarized form.

$$\sigma \neq f_{V/A}\hat{\sigma}(VB),\ f_{V/A} = \frac{1}{2}\sum_{\lambda,s_1} f_{V_\lambda/A_{s_1}},\ \hat{\sigma}(VB) = \frac{1}{4}\sum_{\lambda,s_2}\hat{\sigma}(V_\lambda B_{s_2})$$



# Definition of (QCD) PDFs

- Fast moving proton in the $z$ direction $p^\mu = (E, 0, 0, p)$

$$n^\mu = (1,0,0,1), \ \bar{n} = (1,0,0,-1)$$
$$n^2 = \bar{n}^2 = 0, n\cdot\bar{n} = 2$$

- Light-cone coordinates

$$p^\mu = \frac{1}{2}p^- n^\mu + \frac{1}{2}p^+ \bar{n}^\mu + p_\perp^\mu,$$

where

$$p^- = \bar{n}\cdot p = E + p_z \approx 2E, \ p^+ = n\cdot p = E - p_z \approx \frac{m_p^2}{2E}.$$

- Boost along $z$ axis,

$$p^+ \to \lambda p^+, \ p^- \to p^-/\lambda, p_\perp \to p_\perp.$$

- Quark PDFs: light-cone Fourier transforms [Collins & Soper, 1982]

$$f_q(x,\mu) = \langle p|O_q(r^-)|p\rangle, \ x = r^-/p^-$$
$$O_q(r^-) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \, e^{-i\xi r}[\bar{q}(\bar{n}\xi)\mathscr{W}(\bar{n}\xi)]\bar{\not{n}}[\mathscr{W}^\dagger(0)q(0)],$$

Similar expressions for antiquark PDFs and gluon PDFs.

- Collinear PDFs are defined at $x^- = 0$ and $x_\perp = 0$, which are boost invariant.

# EW PDFs (different from the QCD ones)

- Due to confinement, QCD observables are color invariant.

$$O_q(r^-) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi\, e^{-i\xi r}[\bar{q}(\bar{n}\xi)\mathscr{W}(\bar{n}\xi)]\bar{\not{n}}\begin{bmatrix}1\\T^a\end{bmatrix}[\mathscr{W}^\dagger(0)q(0)],$$

$$\langle p|\bar{q}\cdots q|p\rangle = f_q(x,\mu),\ \langle p|\bar{q}\cdots T^a\cdots|p\rangle = 0.$$

  Equal probabilities to find the different colors, $q_1, q_2, q_3$

- EW symmetry is broken

$$\langle p|\bar{q}\cdots t^a\cdots|p\rangle \neq 0$$

  That is, isospin charge is not invariant in a physical observable.

- Non-singlet PDFs ($I \neq 0$)

$$\langle p|\bar{q}_L\cdots t^3\cdots q_L|p\rangle = \frac{1}{2}\left[f_{u_L} - f_{d_L}\right] \neq 0,\ f_{u_L} \neq f_{d_L},$$

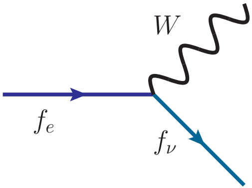
  which gives non-zero non-singlet PDFs.

[Bauer et al., 1703.08562, 1712.07147; Manohar, Waalewijn, 1802.08687; Han, Ma, KX, 2007,14300, 2103.09844]

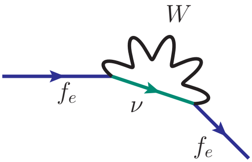
# Factorization violation

- Recall the QCD collinear factorization [CSS, 80s']
  - One-side QCD radiation (Drell-Yan, SIA, and DIS)
  - Sufficiently inclusive, *i.e.*, $pp \to V + X$
    Unitary condition $\sum_X |X\rangle\langle X| = \mathbb{1} \implies$ Ward identity $\implies$ factorization

- In the EW case, the factorization violation is everywhere [Rothstein et al., 1811.04120]: when $SU(2)$ quantum numbers are not summed/averaged (non-singlet case), collinear factorization formalism may NOT hold solely




- A natural consequence of the infrared divergence is not canceled in $f_{I\neq 0}$, (individually in $f_e$ or $f_\nu$)
- Can we rescue it? $\implies$ **Partly!**
  Deep $\leftrightarrow$ Shallow factorization [Sterman, 2207.06507]
- We need sufficiently inclusive observables, *e.g.*, EW jets.




- New operators and formalism are needed, *e.g.* bared charges, cutoff and matching

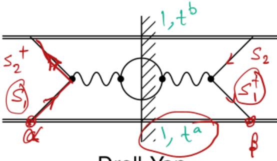
# Shallow EW factorization

- A inclusive cross section can be factorized into hard, collinear (PDFs and/or FFs) and soft functions [Manohar, 1802.08687]

$$\sigma(AB \to X) = \sum_{a,b} \mathscr{C}_{a/A} \mathscr{C}_{b/B} \mathscr{S}_{ab} \mathscr{H}_{ab},$$

where the soft function

$$\mathscr{S}_{ab} = \langle 0 | S_1^\dagger t^a S_1 S_2^\dagger t^b S_2 \cdots | 0 \rangle.$$



Drell-Yan

- In the QCD case, $t^a \to T^a$ vanish unless $T^A = 1$. $S^\dagger S = 1$ leaves a trivial soft function $\mathscr{S}_{ab} = 1$.
- In the EW theory, $\mathscr{S}$ is not trivially identity, leads a angular dependence $\to$Rapidity divergence $\Longrightarrow$ Collins-Soper Equation [1981] $\Leftrightarrow$ rapidity RGE in SCET [Chiu et al. 1104.0881, 1202.0814]
- EW PDFs/FFs involve both singlet and non-singlet components $(I = 0, 1, 2)$.
- DGLAP equation in $I \neq 0$ sector will give double-log evolution.

# PDFs and Fragmentations (parton showers)

- Initial state radiation (ISR): PDFs [Bauer et al., 1703.08562; 1808.08831, Manohar et al., 1808.08831, Han, Ma, KX, 2007.14300],

$$f_B(z,\mu^2,\nu^2) = \sum_A \int_z^1 \frac{\mathrm{d}\xi}{\xi} f_A(\xi,\mu^2,\nu^2) \int_{m^2}^{\mu^2} \mathrm{d}k_T^2 \mathscr{P}_{A\to B+C}(z/\xi, k_T^2, \nu^2)$$

$$\frac{\mathrm{d}f_B(z,\mu^2,\nu^2)}{\mathrm{d}\ln\mu^2} = \sum_A \int_z^1 \frac{\mathrm{d}\xi}{\xi} \frac{\mathrm{d}\mathscr{P}_{A\to B+C}(z/\xi,\mu^2,\nu^2)}{\mathrm{d}z\mathrm{d}k_T^2} f_A(\xi,\mu^2,\nu^2)$$

$$\frac{\mathrm{d}f_B^{(I\neq 0)}(z,\mu^2,\nu^2)}{\mathrm{d}\ln\nu^2} = \widehat{\gamma}_\nu f_B^{(I\neq 0)}(z,\mu^2,\nu^2)$$

  - The leading order splitting gives the effective $W$ approximation (EWA) [Kane, Repko, Rolnick, PLB1984, Dawson, NPB1985, Chanowitz, Gaillard, NPB1985]
  - $W_L(Z_L)$ capture the remnants of EWSB, governed by power correction $\mathscr{O}(M_Z^2/Q^2)$ to the Goldstone Equivalence.

- Final state radiation (FSR): Fragmentations [Bauer et al., 1806.10157; Han, Ma, KX, 2203.11129] or parton showers [Han et al., 1611.00788]

$$\Delta_A(t) = \exp\left[-\sum_B \int_{t_0}^t \mathrm{d}t' \int \mathrm{d}z \frac{\mathrm{d}\mathscr{P}_{A\to B+C}(z,t')}{\mathrm{d}z\mathrm{d}t'}\right]$$

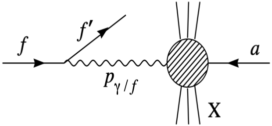
# Parton inside of a lepton

## Equivalent photon approximation (EPA) [Fermi, Z. Phys. 29, 315 (1924), von Weizsacker, Z. Phys. 88, 612 (1934)]

Treat photon as a parton constituent in the lepton [Williams, Phys. Rev. 45, 729 (1934)]

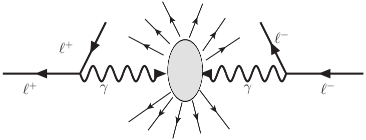
$$\sigma(\ell^- + a \to \ell^- + X) = \int \mathrm{d}x\, f_{\gamma/\ell}\, \hat{\sigma}(\gamma a \to X)$$

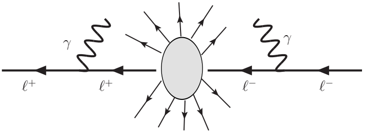
$$f_{\gamma/\ell,\mathrm{EPA}}(x_\gamma, Q^2) = \frac{\alpha}{2\pi}\frac{1+(1-x_\gamma)^2}{x_\gamma}\ln\frac{Q^2}{m_\ell^2}$$



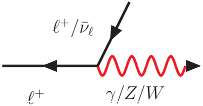

Extra terms to Improve: [Budnev, Ginzburg, Meledin, Serbo, Phys. Rept. (1975)], [Frixione, Mangano, Nason, Ridolfi, 9310350]

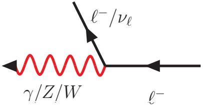
Photon fusions and annihilations with initial state radiations




## Effective $W$ approximation (EWA) [Kane, Repko, Rolnick, PLB1984, Dawson, NPB1985, Chanowitz, Gaillard, NPB1985]



# The novel features of the EWA

- The EW PDFs must be polarized due to the chiral nature of the EW theory

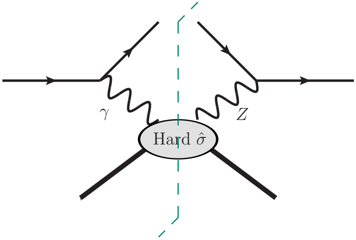
$$f_{V_+/A_+} \neq f_{V_-/A_-}, \qquad f_{V_+/A_-} \neq f_{V_-/A_+},$$
$$\hat{\sigma}(V_+B_+) \neq \hat{\sigma}(V_-B_-), \quad \hat{\sigma}(V_+B_-) \neq \hat{\sigma}(V_-B_+)$$

  We are not able to factorize the cross sections in an unporlarized form.

$$\sigma \neq f_{V/A}\hat{\sigma}(VB), \; f_{V/A} = \frac{1}{2}\sum_{\lambda,s_1} f_{V_\lambda/A_{s_1}}, \; \hat{\sigma}(VB) = \frac{1}{4}\sum_{\lambda,s_2}\hat{\sigma}(V_\lambda B_{s_2})$$

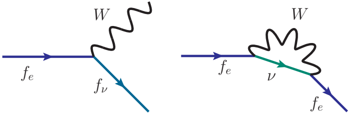
- The interference gives the mixed PDFs

$$f_{\gamma Z} \sim A^{\mu\nu} Z_{\mu\nu} + \text{h.c.}, \; f_{hZ_L} \sim hZ_L$$




- Bloch-Nordsieck theorem violation due to the non-cancelled divergence in $f \to f'V$: fully inclusive observables [Manohar, 1802.08687]

$$f_1 = \frac{1}{2}(f_\nu + f_e) \sim \frac{\alpha_W}{2\pi}\log,$$
$$f_3 = \frac{1}{2}(f_\nu - f_e) \sim \frac{\alpha_W}{2\pi}\log^2$$



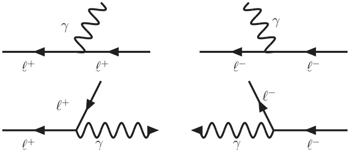

- Numerical small $\Longrightarrow$ cutoff $M_V/Q$ [Bauer et al., 1703.08562, Han, Ma, KX, 2007.14300]

# Go beyond the EPA/EWA

**We have been doing:**

- $\ell^+\ell^-$ annihilation

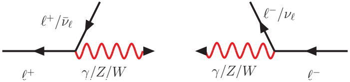

- EPA and ISR




- "Effective W Approx." (EWA)

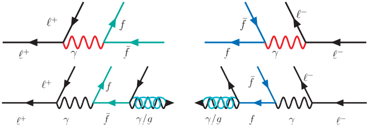
  [Kane, Repko, Rolnick, PLB 148 (1984) 367]

  [Dawson, NPB 249 (1985) 42]

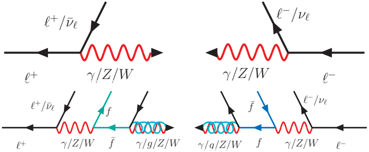



**We complete:**

- Above $\mu_{\rm QCD}$: QED$\otimes$QCD
  $q, g$ become active [Han, Ma, KX, 2103.09844]




- Above $\mu_{\rm EW} = M_Z$: EW$\otimes$QCD
  EW partons emerge [Han, Ma, KX, 2007.14300]



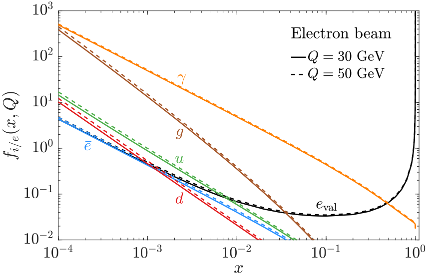

**In the end, every content is a parton, i.e. the full SM PDFs.**

# The QED⊗QCD PDFs for lepton colliders

## Electron beam:

- Scale unc. $10\%$ for $f_{g/e}$ [2103.09844]
- $\mu_{\rm QCD}$ unc. 15%
- The averaged momentum fractions $\langle x_i \rangle = \int x f_i(x) \mathrm{d}x$ [%]

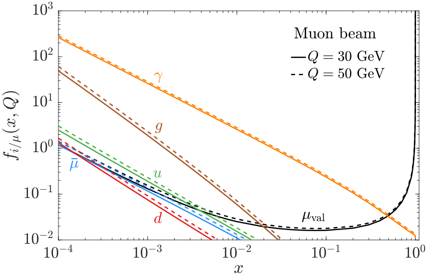
| $Q(e^{\pm})$ | $e_{\rm val}$ | $\gamma$ | $\ell$sea | $q$ | $g$ |
|---|---|---|---|---|---|
| 30 GeV | 96.6 | 3.20 | 0.069 | 0.080 | 0.023 |
| 50 GeV | 96.5 | 3.34 | 0.077 | 0.087 | 0.026 |
| $M_Z$ | 96.3 | 3.51 | 0.085 | 0.097 | 0.028 |




## Muon beam:

- Scale unc. $20\%$ for $f_{g/\mu}$ [2103.09844]
- $\mu_{\rm QCD}$ unc. 5% [2106.01393]

| $Q(\mu^{\pm})$ | $\mu_{\rm val}$ | $\gamma$ | $\ell$sea | $q$ | $g$ |
|---|---|---|---|---|---|
| 30 GeV | 98.2 | 1.72 | 0.019 | 0.024 | 0.0043 |
| 50 GeV | 98.0 | 1.87 | 0.023 | 0.029 | 0.0051 |
| $M_Z$ | 97.9 | 2.06 | 0.028 | 0.035 | 0.0062 |

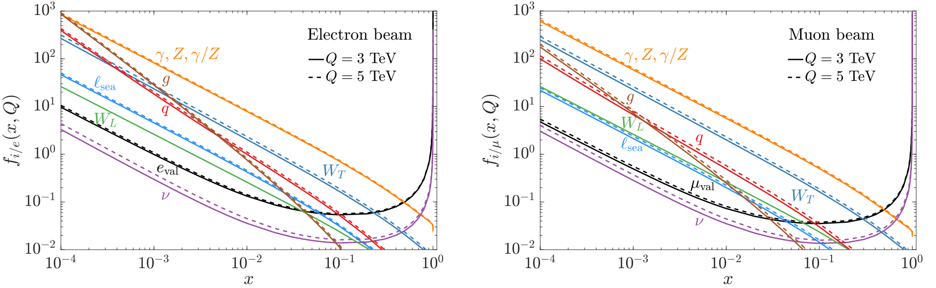


# EWPDFs of a lepton

- The sea leptonic and quark PDFs

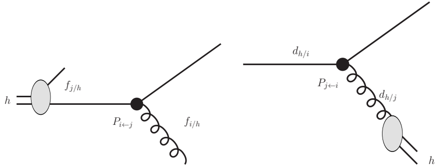
$$\nu = \sum_i (\nu_i + \bar{\nu}_i), \;\; \ell\text{sea} = \bar{\ell}_{\text{val}} + \sum_{i \neq \ell_{\text{val}}} (\ell_i + \bar{\ell}_i), \;\; q = \sum_{i=d}^{t} (q_i + \bar{q}_i)$$

Even neutrino becomes active.




- **All SM particles are partons** [Han, Ma, KX, 2007.14300]
- $W_L(Z_L)$ does not evolve: **Bjorken-scaling restoration**: $f_{W_L}(x) = \frac{\alpha_2}{4\pi}\frac{1-x}{x}$.
- The EW correction can be large: $\sim 50\%$ $(100\%)$ for $f_{d/e}$ $(f_{d/\mu})$ due to the relatively **large SU(2) gauge coupling**. [Han, Ma, KX et. al, 2106.01393]
- Scale uncertainty: $\sim 15\%$ $(20\%)$ between $Q = 3$ TeV and $Q = 5$ TeV

# EW Fragmentations




- Backward evolution

$$\frac{\mathrm{d}d_i}{\mathrm{d}\log Q^2} = \sum_I \frac{\alpha_I}{2\pi} \sum_j d_j \otimes P^I_{ji}$$

- Initial conditions for fermions [Bauer et al. 1806.10157, Han, Ma, KX, 2203.11129]

$$d^f_{f_L}(x, Q_0^2) = d^f_{f_R}(x, Q_0^2) = \delta(1-x),$$
$$d^\nu_{\nu_L}(x, Q_0^2) = \delta(1-x),\ d_i^f = 0 \text{ for } i \neq f.$$

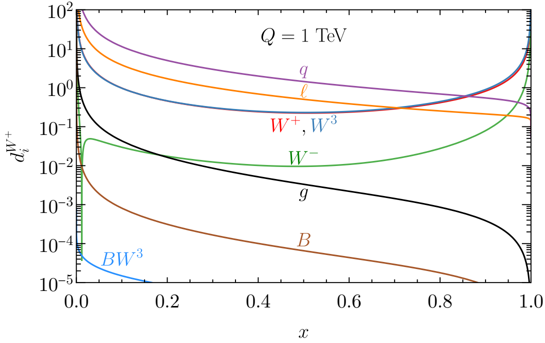
  vector bosons

$$d^V_{V_+}(x, Q_0^2) = d^V_{V_-}(x, Q_0^2) = \delta(1-x),\ d_i^V(x, Q_0^2) = 0 \text{ for } i \neq V.$$

- We take the same techniques developed for PDF evolution. See backup slides for details.
- The DGLAP evolution resumms the EW logarithms, which is equivalent to the Sudakov in showering.

# An example: a final-state $W^+$

- At future high-energy colliders, collinear splittings also happen to energetic final state particles $\Rightarrow$ **EW jets**
- Electroweak fragmentation function (EW FF) $d_i^{W^+}$, defined as the probability of finding a $W^+$ in the mother particle $i$ (i.e., $i \to W$) [Han, Ma, KX, 2203.11129]




The parton shower approach can be found in [Han et al. 1611.00788, Herwig 2108.10817]

# Summary and prospects

- A high-energy muon collider is a dream machine for new physics search, both for energy and precision frontiers
- **The parton picture play an important role**
  - At very high energies, the collinear splittings dominate. **All SM particles should be treated as partons that described by EW PDFs and FFs/PS.**
  - The large collinear logarithm needs to be resummed via solving the DGLAP equations, so the **QCD partons (quarks and gluons) emerge**.
  - When $Q > \mu_{\mathrm{EW}}$, the EW splittings are activated: the EW partons appear, and the existing QED$\otimes$QCD PDFs may receive big corrections.
- **A high-energy muon collider is an EW gauge boson collider**
  - Two classes of processes: $\mu^+\mu^-$ annihilation v.s. VBF [Han, Ma, KX, 2007.14300]
  - Quark and gluon initiated jet production dominates [Han, Ma, KX, 2103.09844]
  - EW PDFs are essential for high-energy muon colliders [Han, Ma, KX, 2007.14300, 2106.01393]
- Final-state EW logarithms can be resummed with parton showers [Chen, Han, Tweedie, arXiv:1611.00788] or fragementations [Han, Ma, KX, 2203.11129].
- The non-trivial soft functions are essential to complete the electroweak factorization.

# The PDF evolution: DGLAP

- The DGLAP equations

$$\frac{\mathrm{d}f_i}{\mathrm{d}\log Q^2} = \sum_I \frac{\alpha_I}{2\pi} \sum_j P^I_{ij} \otimes f_j$$

- The initial conditions

$$f_{\ell/\ell}(x, m_\ell^2) = \delta(1-x)$$

- Three regions and two matchings
  - $m_\ell < Q < \mu_{\rm QCD}$: QED
  - $Q = \mu_{\rm QCD} \lesssim 1$ GeV: $f_q \propto P_{q\gamma} \otimes f_\gamma, f_g = 0$ [Simplified Non-pert. parameterization.]
  - $\mu_{\rm QCD} < Q < \mu_{\rm EW}$: QED⊗QCD
  - $Q = \mu_{\rm EW} = M_Z$: $f_\nu = f_t = f_W = f_Z = f_{\gamma Z} = 0$
  - $Q > \mu_{\rm EW}$: EW⊗QCD.

$$\begin{pmatrix} f_B \\ f_{W^3} \\ f_{BW^3} \end{pmatrix} = \begin{pmatrix} c_W^2 & s_W^2 & -2c_W s_W \\ s_W^2 & c_W^2 & 2c_W s_W \\ c_W s_W & -c_W s_W & c_W^2 - s_W^2 \end{pmatrix} \begin{pmatrix} f_\gamma \\ f_Z \\ f_{\gamma Z} \end{pmatrix}$$

- We work in the $(B, W)$ basis [See backup for details.]
- Double logs are retained through $M_V/Q$ cutoff [Bauer, Ferland, Webber, 1703.08562].

$$f_3 = \frac{\alpha_W}{2\pi} \log \int_x^{1-M_V/Q} \mathrm{d}z P_{ff} \otimes \frac{f_\nu - f_e}{2} \sim \frac{\alpha_W}{2\pi} \log^2$$

Same physics as the Rapidity RGE [Manohar, Waalewijn 1802.08687]

# The QED⊗QCD DGLAP evolution

- The singlets and gauge bosons

$$\frac{\mathrm{d}}{\mathrm{d}\log Q^2}\begin{pmatrix} f_L \\ f_U \\ f_D \\ f_\gamma \\ f_g \end{pmatrix} = \begin{pmatrix} P_{\ell\ell} & 0 & 0 & 2N_\ell P_{\ell\gamma} & 0 \\ 0 & P_{uu} & 0 & 2N_u P_{u\gamma} & 2N_u P_{ug} \\ 0 & 0 & P_{dd} & 2N_d P_{d\gamma} & 2N_d P_{dg} \\ P_{\gamma\ell} & P_{\gamma u} & P_{\gamma d} & P_{\gamma\gamma} & 0 \\ 0 & P_{gu} & P_{gd} & 0 & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} f_L \\ f_U \\ f_D \\ f_\gamma \\ f_g \end{pmatrix}$$

- The non-singlets

$$\frac{\mathrm{d}}{\mathrm{d}\log Q^2} f_{NS} = P_{ff} \otimes f_{NS}.$$

- The averaged momentum fractions of the PDFs: $f_{\ell_{\rm val}}, f_\gamma, f_{\ell_{\rm sea}}, f_q, f_g$

$$\langle x_i \rangle = \int x f_i(x)\mathrm{d}x, \ \sum_i \langle x_i \rangle = 1$$

$$\frac{\langle x_q \rangle}{\langle x_{\ell{\rm sea}} \rangle} \lesssim \frac{N_c\left[\sum_i (e_{u_i}^2 + e_{\bar{u}_i}^2) + \sum_i (e_{d_i}^2 + e_{\bar{d}_i}^2)\right]}{e_{\bar{\ell}_{\rm val}}^2 + \sum_{i\neq\ell{\rm val}} (e_{\ell_i}^2 + e_{\bar{\ell}_i}^2)} = \frac{22/3}{5}$$

# The EW isospin (T) and charge-parity (CP) basis

- The leptonic doublet and singlet in the (T,CP) basis

$$f_\ell^{0\pm} = \frac{1}{4}\left[(f_{\nu_L} + f_{\ell_L}) \pm (f_{\bar{\nu}_L} + f_{\bar{e}_L})\right], \quad f_\ell^{1\pm} = \frac{1}{4}\left[(f_{\nu_L} - f_{\ell_L}) \pm (f_{\bar{\nu}_L} - f_{\bar{e}_L})\right].$$

$$f_e^{0\pm} = \frac{1}{2}[f_{e_R} \pm f_{\bar{e}_R}]$$

- Similar for the quark doublet and singlets.
- The bosonic

$$f_B^{0\pm} = f_{B_+} \pm f_{B_-},\; f_{BW}^{1\pm} = f_{BW_+} \pm f_{BW_-},$$

$$f_W^{0\pm} = \frac{1}{3}\left[\left(f_{W_+^+} + f_{W_+^-} + f_{W_+^3}\right) \pm \left(f_{W_-^+} + f_{W_-^-} + f_{W_-^3}\right)\right],$$

$$f_W^{1\pm} = \frac{1}{2}\left[\left(f_{W_+^+} - f_{W_+^-}\right) \mp \left(f_{W_-^+} - f_{W_-^-}\right)\right],$$

$$f_W^{2\pm} = \frac{1}{6}\left[\left(f_{W_+^+} + f_{W_+^-} - 2f_{W_+^3}\right) \pm \left(f_{W_-^+} + f_{W_-^-} - 2f_{W_-^3}\right)\right].$$

# The EW PDFs in the singlet/non-singlet basis

Construct the singlets and non-singlets

- Singlets

$$f_L^{0,1\pm} = \sum_i^{N_g} f_\ell^{0,1\pm}, \quad f_E^{0\pm} = \sum_i^{N_g} f_e^{0\pm},$$

- Non-singlets

$$f_{L,NS}^{0,1\pm} = f_{\ell_1}^{0,1\pm} - f_{\ell_2}^{0,1\pm}, \quad f_{E,NS}^{0\pm} = f_{e_1}^{0\pm} - f_{e_2}^{0\pm}$$

- The trivial non-singlets

$$f_{L,23}^{0,1\pm} = f_{E,23}^{0\pm} = 0$$

Reconstruct the PDFs for each flavors

- The leptonic PDFs

$$f_{\ell_1}^{0,1\pm} = \frac{f_L^{0,1\pm} + (N_g - 1) f_{L,NS}^{0,1\pm}}{N_g}, \quad f_{\ell_2}^{0,1\pm} = f_{\ell_3}^{0,1\pm} = \frac{f_L^{0,1\pm} - f_{L,NS}^{0,1\pm}}{N_g},$$

$$f_{e_1}^{0\pm} = \frac{f_E^{0\pm} + (N_g - 1) f_{E,NS}^{0\pm}}{N_g}, \quad f_{e_2}^{0\pm} = f_{e_3}^{0\pm} = \frac{f_E^{0\pm} - f_{E,NS}^{0\pm}}{N_g}.$$

- The quark components can be constructed as singlets/non-singlets, and reconstructed correspondingly as well.

# The DGLAP in the singlet and non-singlet basis

$$\frac{\mathrm{d}}{\mathrm{d}L}\begin{pmatrix} f_L^{0\pm} \\ f_Q^{0\pm} \\ f_E^{0\pm} \\ f_U^{0\pm} \\ f_D^{0\pm} \\ f_B^{0\pm} \\ f_W^{0\pm} \\ f_g^{0\pm} \end{pmatrix} = \begin{pmatrix} P_{LL}^{0\pm} & 0 & 0 & 0 & 0 & P_{LB}^{0\pm} & P_{LW}^{0\pm} & 0 \\ 0 & P_{QQ}^{0\pm} & 0 & 0 & 0 & P_{QB}^{0\pm} & P_{QW}^{0\pm} & P_{Qg}^{0\pm} \\ 0 & 0 & P_{EE}^{0\pm} & 0 & 0 & P_{EB}^{0\pm} & 0 & 0 \\ 0 & 0 & 0 & P_{UU}^{0\pm} & 0 & P_{UB}^{0\pm} & 0 & P_{Ug}^{0\pm} \\ 0 & 0 & 0 & 0 & P_{DD}^{0\pm} & P_{DB}^{0\pm} & 0 & P_{Dg}^{0\pm} \\ P_{BL}^{0\pm} & P_{BQ}^{0\pm} & P_{BE}^{0\pm} & P_{BU}^{0\pm} & P_{BD}^{0\pm} & P_{BB}^{0\pm} & 0 & 0 \\ P_{WL}^{0\pm} & P_{WQ}^{0\pm} & 0 & 0 & 0 & 0 & P_{WW}^{0\pm} & 0 \\ 0 & P_{gQ}^{0\pm} & 0 & P_{gU}^{0\pm} & P_{gD}^{0\pm} & 0 & 0 & P_{gg}^{0\pm} \end{pmatrix} \otimes \begin{pmatrix} f_L^{0\pm} \\ f_Q^{0\pm} \\ f_E^{0\pm} \\ f_U^{0\pm} \\ f_D^{0\pm} \\ f_B^{0\pm} \\ f_W^{0\pm} \\ f_g^{0\pm} \end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}L}\begin{pmatrix} f_L^{1\pm} \\ f_Q^{1\pm} \\ f_W^{1\pm} \\ f_{BW}^{1\pm} \end{pmatrix} = \begin{pmatrix} P_{LL}^{1\pm} & 0 & P_{LW}^{1\pm} & P_{LM}^{1\pm} \\ 0 & P_{QQ}^{1\pm} & P_{QW}^{1\pm} & P_{QM}^{1\pm} \\ P_{WL}^{1\pm} & P_{WQ}^{1\pm} & P_{WW}^{1\pm} & 0 \\ P_{ML}^{1\pm} & P_{MQ}^{1\pm} & 0 & P_{MM}^{1\pm} \end{pmatrix} \otimes \begin{pmatrix} f_L^{1\pm} \\ f_Q^{1\pm} \\ f_W^{1\pm} \\ f_{BW}^{1\pm} \end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}L} f_W^{2\pm} = P_{WW}^{2\pm} \otimes f_{WW}^{2\pm}$$

The splitting functions can be constructed in terms of Refs. [Han et al. 1611.00788, Bauer et al. 1703.08562,1808.08831]

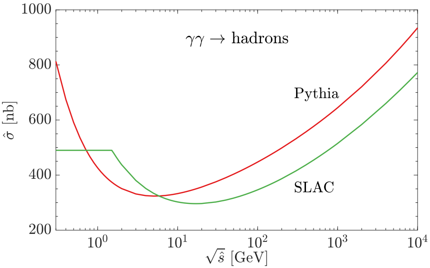
# $\gamma\gamma \to$ hadrons at lepton colliders

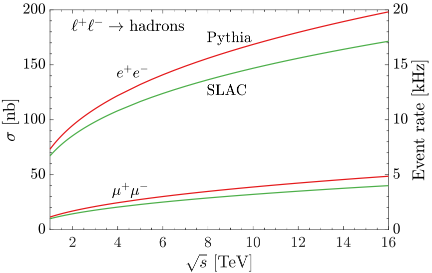
- Large photon induced non-perturbative hadronic production

  [Drees and Godbole, PRL 67 (1991) 1189, hep-ph/9203219]

  [Chen, Barklow, and Peskin, hep-ph/9305247; Godbole et al., Nuovo Cim. C 034S1 (2011)]

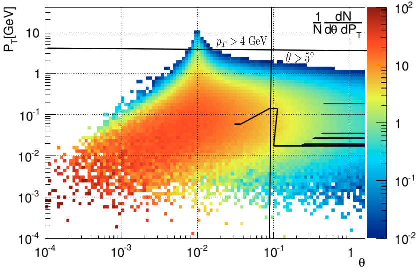
  - $\sigma_{\gamma\gamma\to\ \mathrm{hadrons}}$ may reach micro-barns level at TeV c.m. energies
  - $\sigma_{\ell\ell\to\ \mathrm{hadrons}}$ may reach nano-barns, after folding in the $\gamma\gamma$ luminosity



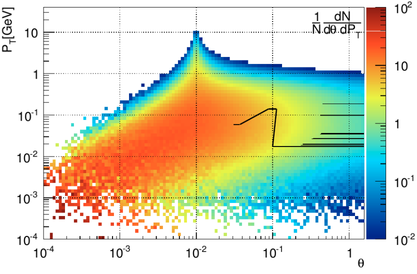



# Main hadronic events (background)

- The events populate at low $p_T$ regime
  So we can separate from this non-perturbative range via a $p_T$ cut.




(a) Pythia sample




(b) SLAC sample

[T. Barklow, D. Dannheim, M. O. Sahin, and D. Schulte, LCD-2011-020]