Quantum Complexity in Neutrino Oscillations

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Motivation

- Quantum computational complexity: a problem of prime importance for quantum computation
  - Estimates the difficulty of constructing quantum states from elementary operations.
- It can also serve to study a completely different physical problem $\rightarrow$ Information processing inside black holes.
- Extends the connection between geometry and information. Growth of complexity $\equiv$ growth of black hole interiors. Susskind et al., 2014
- The highest rate of complexity growth has been observed for de Sitter space, most popular model for inflation, among expanding backgrounds. Bhattacharyya et al., 2020
- What characteristics complexity shows in other natural processes of evolution.
- Neutrinos have shown features such as entanglement and nonlocal correlations. Blasone et al., 2009, Formaggio et al., 2016
- How complex is an evolution of neutrino system and if complexity can also probe any open issue in the neutrino sector.
- Whether this maximization of complexity occurs in neutrino oscillations.

Neutrino Oscillations

- Flavor states $|\nu_\alpha\rangle$ are superposition of mass eigenstates $|\nu_i\rangle$ and vice-versa as

$$|\nu_\alpha\rangle = U^* |\nu_i\rangle$$

  where U is a unitary matrix.

- Time evolution of the flavor states is given by

$$i\frac{\partial}{\partial t}|\nu_\alpha(t)\rangle = H_f |\nu_\alpha(t)\rangle$$

  $H_f = UH_mU^{-1}$ - Hamiltonian in flavor basis;
  $H_m = diag(E_i)$ - Hamiltonian in mass basis.

- Time-evolved states from an initial state $|\nu_\alpha(0)\rangle$ at $t = 0$

$$|\nu_\alpha(t)\rangle = e^{-iH_f t} |\nu_\alpha(0)\rangle$$

- The Hamiltonian may describe propagation of neutrinos in vacuum and/or in a potential (matter effect).

Oscillation Probabilities

- 2-flavor mixing and propagation in vacuum

$$H_m = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \qquad U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

- Oscillation probabilities in vacuum for 2-flavor case

$$P_{\alpha\beta} = \sin^2 2\theta \sin^2\left(\frac{(E_2 - E_1)L}{2}\right) = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2_{21} L}{4E}\right)$$

$$P_{\alpha\alpha} = 1 - \sin^2 2\theta \sin^2\left(\frac{(E_2 - E_1)L}{2}\right) = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2_{21} L}{4E}\right)$$

- Note: $E_2 - E_1 \approx \Delta m^2_{21}/2E$ for common neutrino energy E.
- **Flavor oscillation requires non-degenerate neutrino masses.**
- 3-flavor mixing

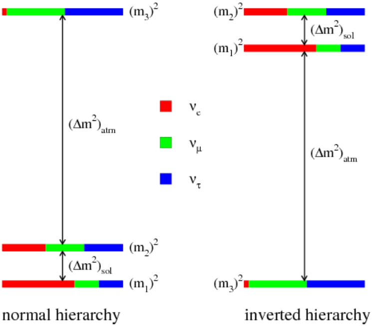
$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{13}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

Oscillation Parameters

Many parameters have been measured with good accuracy: θ_{12}, θ_{13}, Δm^2_{21} and $|\Delta m^2_{31}|$. Parameter θ_{23} is measured with relatively large uncertainties.

Open problems in 3-neutrino oscillation

- CP violation: $(\delta \neq 0) \Rightarrow P(\nu_\alpha \to \nu_\beta) \neq P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)$
- Absolute neutrino mass
- Neutrino mass hierarchy: whether $m_1 \leq m_2 \leq m_3$ or $m_3 \leq m_1 \leq m_2$.



Complexity

- How difficult is it to construct a desired target state with the elementary operations (gates) at your end?
- Or, the minimum number of unitaries required to construct a "target state" through a "reference state".
- For a system $|\phi(s)\rangle$, if

$$U_1 U_2 U_3 U_2 |\phi(s)\rangle = U_3 U_1 U_2 U_1 (U_1)^3 U_2 |\phi(s)\rangle \,,$$

  then the complexity $= 4$.
- Discrete notion of complexity is closely related to quantum computational setups.

Refs: Nielsen et al. (2006), Jefferson & Meyer (2017), ...

Complexity of spread of states

Balasubramanian et al. (2022), Caputa & Liu (2022)

- Spread complexity can be defined as the spread of the target state $|\psi(t)\rangle$ in the Hilbert space relative to the reference state $|\psi(0)\rangle$ through unitary transformations
- The complexity of the state can be defined by minimizing the spread of the wavefunction over all possible bases.
- This minimum is uniquely attained by an orthonormal basis produced by applying the Gram-Schmidt procedure.

Schrodinger equation for a system represented by $|\psi(t)\rangle$

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

Then, the time evolution of the state $|\psi(t)\rangle$ is obtained as

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle .$$

One can also write

$$|\psi(t)\rangle = \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}H^n|\psi(0)\rangle = \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}|\psi_n\rangle ,$$

where, $|\psi_n\rangle = H^n|\psi(0)\rangle$. Hence, we can see that the time evolved system-state $|\psi(t)\rangle$ is represented as superposition of infinite $|\psi_n\rangle$ states.

Krylov Basis and Cost Function

We have $|\psi_n\rangle = H^n |\psi(0)\rangle$.

- These states $\{|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle, \dots\}$ do not comprise a orthonormal set.
- Gram-Schmidt procedure can be used to obtain an ordered orthonormal basis set

$$\begin{aligned}
|K_0\rangle &= |\psi_0\rangle\,, \\
|K_1\rangle &= |\psi_1\rangle - \frac{\langle K_0|\psi_1\rangle}{\langle K_0|K_0\rangle}|K_0\rangle\,, \\
|K_2\rangle &= |\psi_2\rangle - \frac{\langle K_0|\psi_2\rangle}{\langle K_0|K_0\rangle}|K_0\rangle - \frac{\langle K_1|\psi_2\rangle}{\langle K_1|K_1\rangle}|K_1\rangle\,, \quad \text{and so on.}
\end{aligned}$$

$\mathcal{K} = \{|K_n\rangle\,, n = 0, 1, 2\dots\}$ ⇒ **Krylov basis** (Orthonormal ordered set)

Cost function to quantify the complexity

For a time evolved state $|\psi(t)\rangle$ and the Krylov basis defined as $\{|K_n\rangle\}$, the cost function is

$$\chi = \sum_{n=0}^{\infty} n|\langle K_n|\psi(t)\rangle|^2,$$

where $n = 0, 1, 2\dots$ For such Krylov basis the above defined cost function is minimised. *Balasubramanian et al. (2022)*

Complexity in 2-flavor neutrino oscillations

The evolution of flavor states can be represented by Schrodinger equation as

$$i\frac{\partial}{\partial t}\begin{pmatrix}|\nu_e(t)\rangle\\|\nu_\mu(t)\rangle\end{pmatrix} = H_f\begin{pmatrix}|\nu_e(t)\rangle\\|\nu_\mu(t)\rangle\end{pmatrix} \tag{1}$$

where $H_f = UH_mU^{-1}$, U being the mixing matrix and H_m is the Hamiltonian (diagonal) that governs the time evolution of neutrino mass eigenstate

$$H_m = \begin{pmatrix}E_1 & 0\\0 & E_2\end{pmatrix}, \qquad U = \begin{pmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{pmatrix}.$$

$$|\nu_e(0)\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |\nu_\mu(0)\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

We have

$$\{|\psi_n\rangle\} = \begin{cases}\{|\nu_e(0)\rangle, H_f|\nu_e(0)\rangle, H_f^2|\nu_e(0)\rangle\ldots\} & \textit{for initial } \nu_e \textit{ flavor}\\ \{|\nu_\mu(0)\rangle, H_f|\nu_\mu(0)\rangle, H_f^2|\nu_\mu(0)\rangle\ldots\} & \textit{for initial } \nu_\mu \textit{ flavor}\end{cases}$$

After applying Gram-Schmidt procedure we get $\{|K_n\rangle\} = \{|K_0\rangle, |K_1\rangle\}$, *i.e.*,

$$\{|K_n\rangle\} = \begin{cases}\{|K_0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |K_1\rangle = \begin{pmatrix}0\\1\end{pmatrix}\} = \{|\nu_e\rangle, |\nu_\mu\rangle\} & \textit{for initial } \nu_e\\ \{|K_0\rangle = \begin{pmatrix}0\\1\end{pmatrix}, |K_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}\} = \{|\nu_\mu\rangle, |\nu_e\rangle\} & \textit{for initial } \nu_\mu\end{cases}$$

Complexity and Probabilities

For a time evolved state $|\nu_e(t)\rangle = \begin{pmatrix} A_{ee}(t) \\ A_{e\mu}(t) \end{pmatrix} = \begin{pmatrix} \cos^2\theta e^{-iE_1 t} + \sin^2\theta e^{-iE_2 t} \\ \sin\theta\cos\theta(e^{-iE_2 t} - e^{-iE_1 t}) \end{pmatrix}$
(with $\{|K_n\rangle\} = \{|\nu_e(0)\rangle, |\nu_\mu(0)\rangle\}$)

$$\chi_e = \sum_{n=0}^{1} n|\langle K_n|\nu_e(t)\rangle|^2 = \mathbf{P_{e\mu}}$$

Similarly, for state $|\nu_\mu(t)\rangle = (A_{\mu e}(t), A_{\mu\mu}(t))^T$ (with $\{|K_n\rangle\} = \{|\nu_\mu(0)\rangle, |\nu_e(0)\rangle\}$)

$$\chi_\mu = \mathbf{P}_{\mu\mathbf{e}}$$

- The more the oscillation probability of neutrino flavor, the more complex the evolution of the neutrino flavor state.
- Since $P_{e\mu} = P_{\mu e}$ for standard vacuum oscillations, the complexity embedded in this system comes out to be same for both cases of initial flavor, *i.e.,* complexity of the system doesn't depend on the initial flavor of neutrino.

Complexity in 3-flavor neutrino oscillations

We have three types of initial states as $|\nu_e\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $|\nu_\mu\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $|\nu_\tau\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$ with Hamiltonian $H_f = UH_mU^{-1}$, $H_m = diag(0, \Delta m^2_{21}, \Delta m^2_{31})$ and U $\rightarrow 3 \times 3$ PMNS mixing matrix

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{13}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

Here, Krylov basis $\neq$ flavor basis.

- For initial $|\nu_e\rangle$ state $|K_0\rangle \equiv |\nu_e\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$, other states spanning the Krylov basis take the form

$$|K_1\rangle = N_1 \begin{pmatrix}0\\a_1\\a_2\end{pmatrix} = N_1 \begin{pmatrix} 0 \\ \left(\frac{\Delta m^2_{21}}{2E}\right) U^*_{e2}U_{\mu 2} + \left(\frac{\Delta m^2_{31}}{2E}\right) U^*_{e3}U_{\mu 3} \\ \left(\frac{\Delta m^2_{21}}{2E}\right) U^*_{e2}U_{\tau 2} + \left(\frac{\Delta m^2_{31}}{2E}\right) U^*_{e3}U_{\tau 3} \end{pmatrix},$$

$$|K_2\rangle = N_2 \begin{pmatrix}0\\b_1\\b_2\end{pmatrix} = N_2 \begin{pmatrix} 0 \\ \left(\frac{\Delta m^2_{21}}{2E}\right)\left(\frac{\Delta m^2_{21}}{2E} - A\right) U^*_{e2}U_{\mu 2} + \left(\frac{\Delta m^2_{31}}{2E}\right)\left(\frac{\Delta m^2_{31}}{2E} - A\right) U^*_{e3}U_{\mu 3} \\ \left(\frac{\Delta m^2_{21}}{2E}\right)\left(\frac{\Delta m^2_{21}}{2E} - A\right) U^*_{e2}U_{\tau 2} + \left(\frac{\Delta m^2_{31}}{2E}\right)\left(\frac{\Delta m^2_{31}}{2E} - A\right) U^*_{e3}U_{\tau 3} \end{pmatrix}$$

Complexity in 3-flavor neutrino oscillations

$$\chi_e = P_{e\mu}(t)(N_1^2|a_1|^2 + 2N_2^2|b_1|^2) + P_{e\tau}(t)(N_1^2|a_2|^2 + 2N_2^2|b_2|^2) + 2\Re(N_1^2 a_1^* a_2 A_{e\mu}(t) A_{e\tau}(t)^*) + 4\Re(N_2^2 b_1^* b_2 A_{e\mu}(t) A_{e\tau}(t)^*)$$

with

$$A = \frac{\left(\begin{array}{c} \left(\Delta m_{21}^2\right)^3 |U_{\alpha 2}|^2(1-|U_{\alpha 2}|^2) + \left(\Delta m_{31}^2\right)^3 |U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2) \\ -\left(\Delta m_{21}^2\right)\left(\Delta m_{31}^2\right)|U_{\alpha 2}|^2|U_{\alpha 3}|^2\left(\Delta m_{21}^2 + \Delta m_{31}^2\right) \end{array} \right)}{\left(\Delta m_{21}^2\right)^2 |U_{\alpha 2}|^2(1-|U_{\alpha 2}|^2) + \left(\Delta m_{31}^2\right)^2 |U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2) - 2\left(\Delta m_{21}^2\right)\left(\Delta m_{31}^2\right)|U_{\alpha 2}|^2|U_{\alpha 3}|^2},$$

Effects of different oscillation parameters

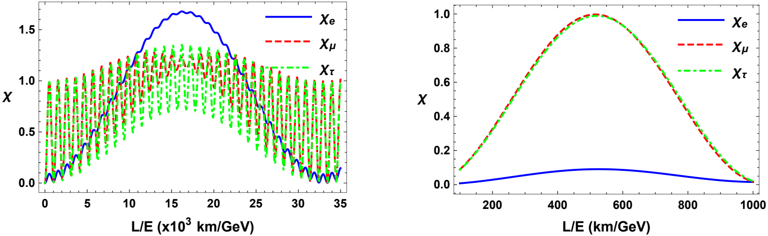


Figure: Complexity χ_α Vs. L/E in vacuum in case of initial flavor ν_e (blue solid line), ν_μ (red dashed line) and ν_τ (green dot-dashed line) for CP-violating phase $\delta = 0^o$. Here, mixing parameters $\theta_{12} = 33.64^o$, $\theta_{13} = 8.53^o$, $\theta_{23} = 47.63^o$, $\Delta m^2_{21} = 7.53 \times 10^{-5}\ eV^2$ and $\Delta m^2_{31} = 2.45 \times 10^{-3}\ eV^2$ are considered.

- Rapid oscillation pattern seen in the left panel (zoomed-in in the right panel) is due to Δm^2_{31} mass-squared difference in the oscillation phase, while the longer oscillation pattern is due to Δm^2_{21} in the oscillation phase. The oscillation length is $\sim 10^3$ km at $E = 1$ GeV for Δm^2_{31} and $\sim 3 \times 10^4$ km at $E = 1$ GeV for Δm^2_{21}.
- In the general case the complexity is maximum if the neutrino is produced initially as ν_e, however, this happens only at a very large L/E value of $\sim 1.6 \times 10^4$ km/GeV.
- In current experimental setups (right panel), which covers roughly one oscillation length for Δm^2_{31}, the initial ν_e flavor provides the least complexity among all neutrino flavors.

Effects of CP-violating parameter δ

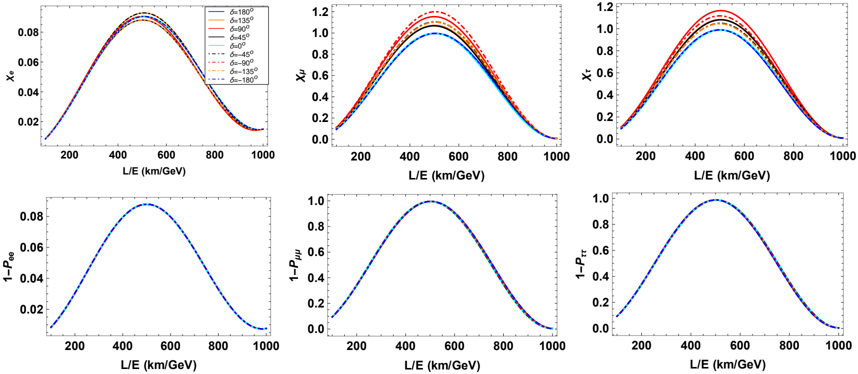


Figure: Complexities and 1-$P_{\alpha\alpha}$ with respect to L/E.

- Complexity mimics the features of the total oscillation probability $1 - P_{\alpha\alpha}$.
- However, it is visible that χ_α for all three flavors provide more information regarding the CP-violating phase δ.

Matter Effect on Complexity

- For any initial flavor ν_α : $|K_0\rangle_\alpha^{matter} = |K_0\rangle_\alpha^{vacuum}$, $|K_1\rangle_\alpha^{matter} = |K_1\rangle_\alpha^{vacuum}$
- $|K_2\rangle$ contains the effects of constant matter density
- For the initial ν_μ flavor: $|K_2\rangle_e = N_{2e}^m(0, b_1^m, b_2^m)^T$
  where,

$$b_1^m = \left(\frac{\Delta m_{21}^2}{2E}\right)\left(\frac{\Delta m_{21}^2}{2E} + V - B_e\right) U_{e2}^* U_{\mu 2} + \left(\frac{\Delta m_{31}^2}{2E}\right)\left(\frac{\Delta m_{31}^2}{2E} + V - B_e\right) U_{e3}^* U_{\mu 3},$$
$$b_2^m = \left(\frac{\Delta m_{21}^2}{2E}\right)\left(\frac{\Delta m_{21}^2}{2E} + V - B_e\right) U_{e2}^* U_{\tau 2} + \left(\frac{\Delta m_{31}^2}{2E}\right)\left(\frac{\Delta m_{31}^2}{2E} + V - B_e\right) U_{e3}^* U_{\tau 3}.$$

- Similarly, for the initial ν_μ flavor: $|K_2\rangle_\mu = N_{2\mu}^m(d_1^m, 0, d_2^m)^T$
  where,

$$d_1^m = \left(\frac{\Delta m_{21}^2}{2E}\right)\left(\frac{\Delta m_{21}^2}{2E} + V - B_\mu\right) U_{e2} U_{\mu 2}^* + \left(\frac{\Delta m_{31}^2}{2E}\right)\left(\frac{\Delta m_{31}^2}{2E} + V - B_\mu\right) U_{e3} U_{\mu 3}^*$$
$$d_2^m = \left(\frac{\Delta m_{21}^2}{2E}\right)\left(\frac{\Delta m_{21}^2}{2E} - B_\mu\right) U_{\mu 2}^* U_{\tau 2} + \left(\frac{\Delta m_{31}^2}{2E}\right)\left(\frac{\Delta m_{31}^2}{2E} - B_\mu\right) U_{\mu 3}^* U_{\tau 3},$$

- Similar approach can be followed for the initial ν_τ flavor.

Matter effects on complexity

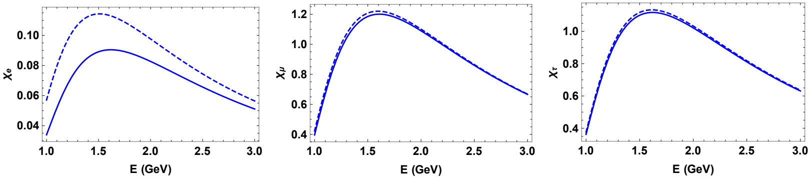


Figure: Cost function χ_e (left), χ_μ (middle) and χ_τ (right) w. r. t. neutrino-energy E is shown. Here, $L = 810$ km, $\delta = -90^o$ and higher octant of θ_{23} is considered. Solid and dashed curves represent the case of vacuum and matter oscillations, respectively. $V = 1.01 \times 10^{-13}$ eV.

- Matter effect increases complexity of the system in all cases of initial flavors of the neutrino, most significantly for ν_e as expected.

Complexity in Long Baseline Neutrino Experiments: T2K

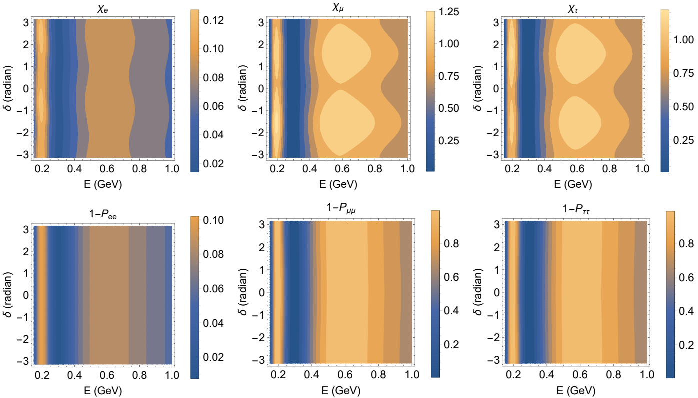


Figure: T2K: Cost function (upper panel) and 1-$P_{\alpha\alpha}$ (lower panel) in the plane of $E - \delta$ in case of initial flavor ν_e (left), ν_μ (middle) and ν_τ (right). Here, $L = 295$ km and mixing parameters $\theta_{12} = 33.64^o$, $\theta_{13} = 8.53^o$, $\theta_{23} = 47.63^o$, $\Delta m^2_{21} = 7.53 \times 10^{-5}\ eV^2$ and $\Delta m^2_{31} = 2.45 \times 10^{-3}\ eV^2$ are considered.

Complexity in Long Baseline Neutrino Experiments: NOvA

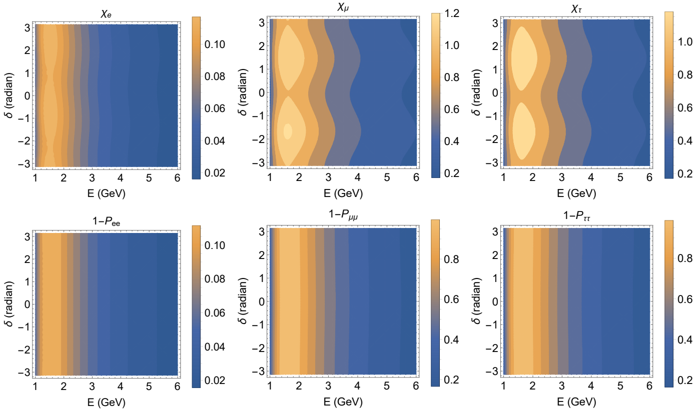


Figure: NOνA: Cost function (upper panel) and 1-$P_{\alpha\alpha}$ (lower panel) in the plane of $E-\delta$ in case of initial flavor ν_e (left), ν_μ (middle) and ν_τ (right). Here, $L = 810$ km, and higher octant of θ_{23} (47.63^o) is considered.

- For both the experiments, the maxima of χ_μ and χ_τ are found at $\delta \approx -\pi/2$ and $\delta = \pi/2$, respectively.
- This means that the matter effect just enhances the magnitude of complexities, however, the characteristics of χ_α with respect to δ are almost similar for both T2K and NOvA experiments.

What complexity tells us about neutrino oscillation

- In the T2K and NOvA experimental setups, where only ν_μ beams are produced, the only relevant complexity is χ_μ.
- For both the T2K and NOvA χ_μ is maximized at $\delta \approx -1.5$ radian at the relevant experimental energies. The T2K best-fit value of $\delta = -2.14^{+0.90}_{-0.69}$ radian is consistent with this expectation.
- The NOvA best-fit, however, is at $\delta \approx 2.58$ radian which is far away from the maximum χ_μ in the lower-half plane of δ but is still within a region of high χ_μ value in the upper-half plane of δ.
- $P_{\mu e}$, which is the only oscillation probability accessible to the T2K and NOvA setups, it becomes maximum at $\delta \approx -1.5$ radian. This is compatible with T2K best-fit but is in odd with the NOvA best-fit.
- Complexity provides correct prediction for the δ in experimental setups.

Summary & Conclusions

- We examined the spread complexity of neutrino states in two- and three-flavor oscillation scenarios.
- In the two-flavor scenario, complexity and transition probabilities yield equivalent information.
- In case of three-flavor oscillation, initial flavor state evolves into two mixed final states. Hence, the complexity contains additional information regarding open issues related to neutrinos, compared to the total oscillation probability.
- Remarkably, we found that the complexity is maximized for a value of the phase angle for which CP is also maximally violated. T2K experimental data also favors this phase angle, which is obtained from flavor transition.
- Quantum spread complexity emerges as a potent and novel quantity for investigating neutrino oscillations. It successfully reproduces existing results, also demonstrates the potential to serve as a theoretical tool for predicting new outcomes in future experiments.

Thank you for your attention!

BACKUP SLIDES

Comparing the effects of neutrino mass ordering for neutrinos & antineutrinos

For antineutrino $\rightarrow \{V \rightarrow -V, \delta \rightarrow -\delta\}$

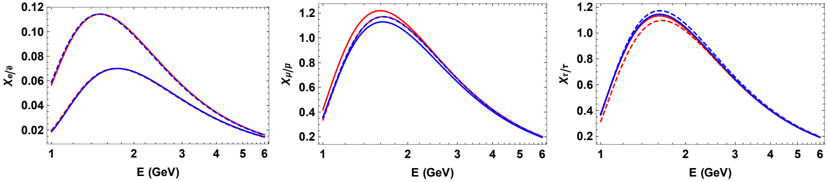


Figure: NOvA: Complexities and $P_{\mu e}$ with respect to neutrino-energy E where red and blue curves represent neutrino and antineutrino case, respectively, with solid (normal ordering) and dashed (inverted ordering) lines. Here $L = 810$ km and $\delta = -90^o$ are considered.

- For both neutrino and antineutrino, the effects of NH and IH are significantly distinguishable for all three flavors.
- In case of χ_e, red-solid line (neutrinos for NH) and blue-dashed line (antineutrinos for IH) exhibit more complexity, *i.e.*, complete swap between the NH (IH) hierarchy and ν ($\bar{\nu}$).
- For χ_μ and χ_τ the maximum is achieved in case of neutrinos with NH and Antineutrinos with IH, respectively.

Spread complexity in three flavor (vacuum) neutrino oscillations

- Similarly, for initial $|\nu_\mu\rangle$, $|K_0\rangle \equiv |\nu_\mu\rangle = (0,1,0)^T$, then we get

$$|K_1\rangle = N_{1\mu}\begin{pmatrix} c_1 \\ 0 \\ c_2 \end{pmatrix} = N_{1\mu}\begin{pmatrix} \left(\frac{\Delta m^2_{21}}{2E}\right) U^*_{\mu 2}U_{e2} + \left(\frac{\Delta m^2_{31}}{2E}\right) U^*_{\mu 3}U_{e3} \\ 0 \\ \left(\frac{\Delta m^2_{21}}{2E}\right) U^*_{\mu 2}U_{\tau 2} + \left(\frac{\Delta m^2_{31}}{2E}\right) U^*_{\mu 3}U_{\tau 3} \end{pmatrix},$$

$$|K_2\rangle = N_{2\mu}\begin{pmatrix} d_1 \\ 0 \\ d_2 \end{pmatrix} = N_{2\mu}\begin{pmatrix} \left(\frac{\Delta m^2_{21}}{2E}\right)\left(\frac{\Delta m^2_{21}}{2E} - A\right) U^*_{\mu 2}U_{e2} + \left(\frac{\Delta m^2_{31}}{2E}\right)\left(\frac{\Delta m^2_{31}}{2E} - A\right) U^*_{\mu 3}U_{e3} \\ 0 \\ \left(\frac{\Delta m^2_{21}}{2E}\right)\left(\frac{\Delta m^2_{21}}{2E} - A\right) U^*_{\mu 2}U_{\tau 2} + \left(\frac{\Delta m^2_{31}}{2E}\right)\left(\frac{\Delta m^2_{31}}{2E} - A\right) U^*_{\mu 3}U_{\tau 3} \end{pmatrix}$$

$$\chi_\mu = P_{\mu e}(t)(N^2_{1\mu}|c_1|^2 + 2N^2_{2\mu}|d_1|^2) + P_{\mu\tau}(t)(N^2_{1\mu}|c_2|^2 + 2N^2_{2\mu}|d_2|^2) + 2\Re(N^2_{1\mu}c_1^* c_2 A_{\mu e}(t)A_{\mu\tau}(t)^*) + 4\Re(N^2_{2\mu}d_1^* d_2 A_{\mu e}(t)A_{\mu\tau}(t)^*).$$

Spread complexity in three flavor neutrino oscillations

- In case of $|K_0\rangle \equiv |\nu_\tau\rangle = (0,0,1)^T$,

$$|K_1\rangle = N_{1\tau}(e_1, e_2, 0)^T = N_{1\tau}\begin{pmatrix} \left(\frac{\Delta m_{21}^2}{2E}\right) U_{\tau 2}^* U_{e2} + \left(\frac{\Delta m_{31}^2}{2E}\right) U_{\tau 3}^* U_{e3} \\ \left(\frac{\Delta m_{21}^2}{2E}\right) U_{\tau 2}^* U_{\mu 2} + \left(\frac{\Delta m_{31}^2}{2E}\right) U_{\tau 3}^* U_{\mu 3} \\ 0 \end{pmatrix},$$

$$|K_2\rangle = N_{2\tau}(f_1, f_2, 0)^T = N_{2\tau}\begin{pmatrix} \left(\frac{\Delta m_{21}^2}{2E}\right)\left(\frac{\Delta m_{21}^2}{2E} - A\right) U_{\tau 2}^* U_{e2} + \left(\frac{\Delta m_{31}^2}{2E}\right)\left(\frac{\Delta m_{31}^2}{2E} - A\right) U_{\tau 3}^* U_{e3} \\ \left(\frac{\Delta m_{21}^2}{2E}\right)\left(\frac{\Delta m_{21}^2}{2E} - A\right) U_{\tau 2}^* U_{\mu 2} + \left(\frac{\Delta m_{31}^2}{2E}\right)\left(\frac{\Delta m_{31}^2}{2E} - A\right) U_{\tau 3}^* U_{\mu 3} \\ 0 \end{pmatrix}.$$

$$\chi_\tau = P_{\tau e}(t)(N_1^2|e_1|^2 + 2N_2^2|f_1|^2) + P_{\tau\mu}(t)(N_1^2|e_2|^2 + 2N_2^2|f_2|^2) + 2\Re(N_1^2 e_1^* e_2 A_{\tau e}(t) A_{\tau\mu}(t)^*) + 4\Re(N_2^2 f_1^* f_2 A_{\tau e}(t) A_{\tau\mu}(t)^*).$$

Here,

$$A = \frac{\left[\begin{matrix} \left(\Delta m_{21}^2\right)^3 |U_{\alpha 2}|^2(1-|U_{\alpha 2}|^2) + \left(\Delta m_{31}^2\right)^3 |U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2) \\ - \left(\Delta m_{21}^2\right)\left(\Delta m_{31}^2\right)|U_{\alpha 2}|^2|U_{\alpha 3}|^2\left(\Delta m_{21}^2 + \Delta m_{31}^2\right) \end{matrix}\right]}{\left(\Delta m_{21}^2\right)^2 |U_{\alpha 2}|^2(1-|U_{\alpha 2}|^2) + \left(\Delta m_{31}^2\right)^2 |U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2) - 2\left(\Delta m_{21}^2\right)\left(\Delta m_{31}^2\right)|U_{\alpha 2}|^2|U_{\alpha 3}|^2},$$

Spread complexity in three flavor neutrino oscillations

$$N_{1\alpha} = \left(\left(\frac{\Delta m^2_{21}}{2E}\right)^2 |U_{\alpha 2}|^2 (1 - |U_{\alpha 2}|^2) + \left(\frac{\Delta m^2_{31}}{2E}\right)^2 |U_{\alpha 3}|^2 (1 - |U_{\alpha 3}|^2) \right.$$
$$\left. -2 \left(\frac{\Delta m^2_{21}}{2E}\right)\left(\frac{\Delta m^2_{31}}{2E}\right) |U_{\alpha 2}|^2 |U_{\alpha 3}|^2 \right)^{-1/2},$$

$$N_{2\alpha} = \left(\left(\frac{\Delta m^2_{21}}{2E}\right)^2 \left(\frac{\Delta m^2_{21}}{2E} - A\right)^2 |U_{\alpha 2}|^2 (1 - |U_{\alpha 2}|^2) \right.$$
$$+ \left(\frac{\Delta m^2_{31}}{2E}\right)^2 \left(\frac{\Delta m^2_{31}}{2E} - A\right)^2 |U_{\alpha 3}|^2 (1 - |U_{\alpha 3}|^2)$$
$$\left. -2 \left(\frac{\Delta m^2_{21}}{2E}\right)\left(\frac{\Delta m^2_{31}}{2E}\right)\left(\frac{\Delta m^2_{21}}{2E} - A\right)\left(\frac{\Delta m^2_{31}}{2E} - A\right) |U_{\alpha 2}|^2 |U_{\alpha 3}|^2 \right)^{-1/2}$$

Matter effects on complexity

$$B_e = \Bigg[\left(\Delta m^2_{21}\right)^2\left(\Delta m^2_{21}+2EV\right)|U_{e2}|^2(1-|U_{e2}|^2)+\left(\Delta m^2_{31}\right)^2\left(\Delta m^2_{31}+2EV\right)|U_{e3}|^2$$
$$(1-|U_{e3}|^2)-\left(\Delta m^2_{21}\right)\left(\Delta m^2_{31}\right)|U_{e2}|^2|U_{e3}|^2\left((\Delta m^2_{21}+2EV)+(\Delta m^2_{31}+2EV)\right)\Bigg]$$
$$\Bigg[2E\Bigg[\left(\Delta m^2_{21}\right)^2|U_{e2}|^2(1-|U_{e2}|^2)+\left(\Delta m^2_{31}\right)^2|U_{e3}|^2(1-|U_{e3}|^2)$$
$$-2\left(\Delta m^2_{21}\right)\left(\Delta m^2_{31}\right)|U_{e2}|^2|U_{e3}|^2\Bigg]\Bigg]^{-1}.$$

For initial ν_μ and ν_τ state the constant B_α is

$$B_\alpha = \Bigg[\left(\Delta m^2_{21}\right)^3|U_{\alpha 2}|^2(1-|U_{\alpha 2}|^2)+\left(\Delta m^2_{31}\right)^3|U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2)-\left(\Delta m^2_{21}\right)\left(\Delta m^2_{31}\right)$$
$$|U_{\alpha 2}|^2|U_{\alpha 3}|^2\left(\Delta m^2_{21}+\Delta m^2_{31}\right)+2EV\Bigg(\left(\Delta m^2_{21}\right)^2|U_{e2}|^2|U_{\alpha 2}|^2+\left(\Delta m^2_{31}\right)^2|U_{e3}|^2|U_{\alpha 3}|^2$$
$$+2\left(\Delta m^2_{21}\right)\left(\Delta m^2_{31}\right)\Re(U^*_{e2}U_{\alpha 2}U_{e3}U^*_{\alpha 3})\Bigg)\Bigg]\Bigg[2E\Bigg[\left(\Delta m^2_{21}\right)^2|U_{\alpha 2}|^2(1-|U_{\alpha 2}|^2)+\left(\Delta m^2_{31}\right)^2$$
$$|U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2)-2\left(\Delta m^2_{21}\right)\left(\Delta m^2_{31}\right)|U_{\alpha 2}|^2|U_{\alpha 3}|^2\Bigg]\Bigg]^{-1},$$

Matter effects on complexity

$$N^m_{2e} = \left(\left(\frac{\Delta m^2_{21}}{2E}\right)^2 |U_{e2}|^2(1-|U_{e2}|^2) \left[\left(\frac{\Delta m^2_{21}}{2E}+V-B_e\right)^2\right] + \left(\frac{\Delta m^2_{31}}{2E}\right)^2 |U_{e3}|^2(1-|U_{e3}|^2) \left[\left(\frac{\Delta m^2_{31}}{2E}+V-B_e\right)^2\right] -2\left(\frac{\Delta m^2_{21}}{2E}\right)\left(\frac{\Delta m^2_{31}}{2E}\right)\left(\frac{\Delta m^2_{21}}{2E}+V-B_e\right)\left(\frac{\Delta m^2_{31}}{2E}+V-B_e\right)|U_{e2}|^2|U_{e3}|^2 \right)^{-1/2},$$

$$N^m_{2\mu} = \left(\left(\frac{\Delta m^2_{21}}{2E}\right)^2 |U_{\mu 2}|^2 \left[\left(\frac{\Delta m^2_{21}}{2E}+V-B_\mu\right)^2 |U_{e2}|^2 + \left(\frac{\Delta m^2_{21}}{2E}-B_\mu\right)^2 |U_{\tau 2}|^2\right] + \left(\frac{\Delta m^2_{31}}{2E}\right)^2 |U_{\mu 3}|^2 \left[\left(\frac{\Delta m^2_{31}}{2E}+V-B_\mu\right)^2 |U_{e3}|^2 + \left(\frac{\Delta m^2_{31}}{2E}-B_\mu\right)^2 |U_{\tau 3}|^2\right] +2\left(\frac{\Delta m^2_{21}}{2E}\right)\left(\frac{\Delta m^2_{31}}{2E}\right)\left[\left(\frac{\Delta m^2_{21}}{2E}+V-B_\mu\right)\left(\frac{\Delta m^2_{31}}{2E}+V-B_\mu\right)\Re(U^*_{\mu 2}U_{e2}U_{\mu 3}U^*_{e3}) + \left(\frac{\Delta m^2_{21}}{2E}-B_\mu\right)\left(\frac{\Delta m^2_{31}}{2E}-B_\mu\right)\Re(U^*_{\mu 2}U_{\tau 2}U_{\mu 3}U^*_{\tau 3})\right]\right)^{-1/2},$$

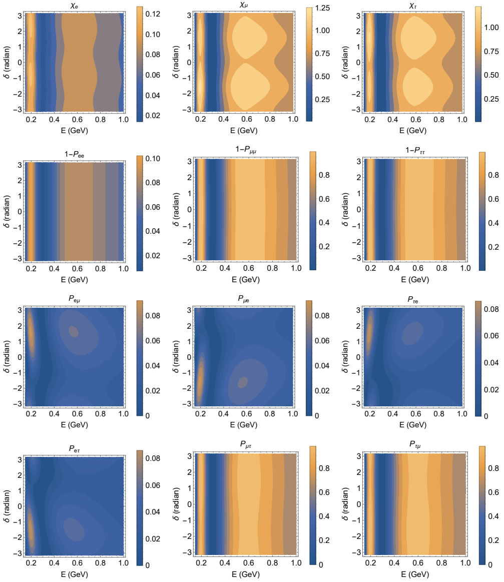


Figure: T2K: Cost function (upper panel) and 1-$P_{\alpha\alpha}$ (lower panel) in the plane of $E - \delta$ in case of initial flavor ν_e (left), ν_μ (middle) and ν_τ (right). Here, $L = 295$ km is considered.

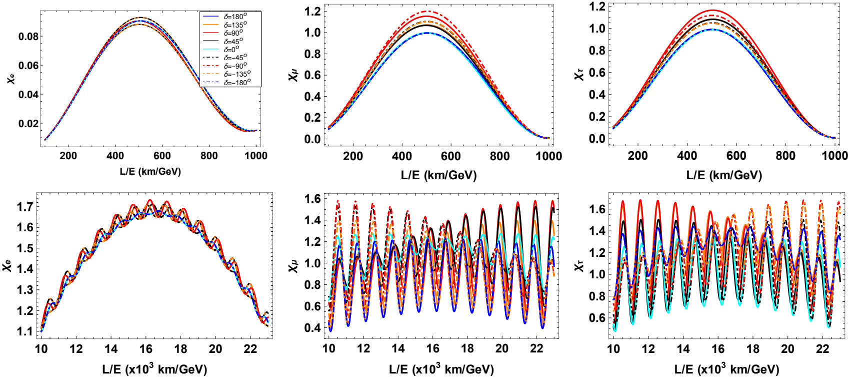


Figure: Complexity for small L/E range (upper panels), large L/E range (lower panels) with respect to L/E for initial flavor is ν_e (left), ν_μ (middle) and ν_τ (right) for different values of the CP-phase δ depicted by different colors.

- For large L/E range the complexities are maximized and the corresponding $\delta = +90^o$ or -90^o for χ_μ and χ_τ, and at $\delta = \pm 90^o$ for χ_e where CP is maximally violated.
- In the limited L/E range χ_μ and χ_τ are maximized at $\delta = -90^o$ (red-dashed line) and at $\delta = +90^o$ (red-solid line), respectively. However, χ_e is maximized at $\delta = +135^o$ and at -45^o.