Applications to electroweak processes

Francesco Sanfilippo, INFN Roma Tre
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Extended Twisted Mass Collaboration

INFN

ICSC
Centro Nazionale di Ricerca in HPC, Big Data and Quantum Computing

Based on

Spectral-function determination of complex electroweak amplitudes with lattice QCD

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[Phys.Rev.D 108 (2023) 7, 074510]

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Bonus!, [gift from N.Tantalo]

“Updates on our calculation of inclusive τ decay into hadrons”

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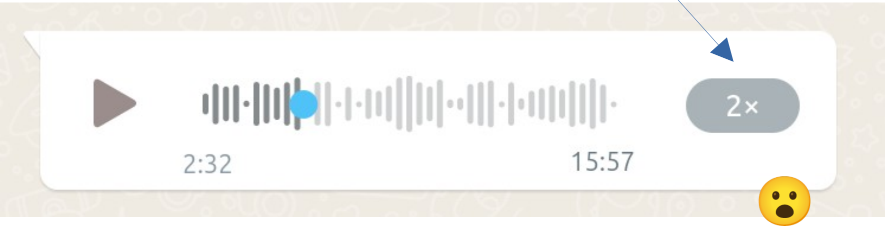
“Updates on our calculation of inclusive τ decay into hadrons”

Don’t worry, I have a solution



Hadronic amplitudes

Correlation function, e.g. J_A, J_B currents on state $|P\rangle$

$$C(t) \equiv \langle 0 | T\{J_A(t) J_B(0)\} | P \rangle \overset{t \geq 0}{=} \sum_{n=0}^{\infty} C_n \, e^{-iE_n t}$$

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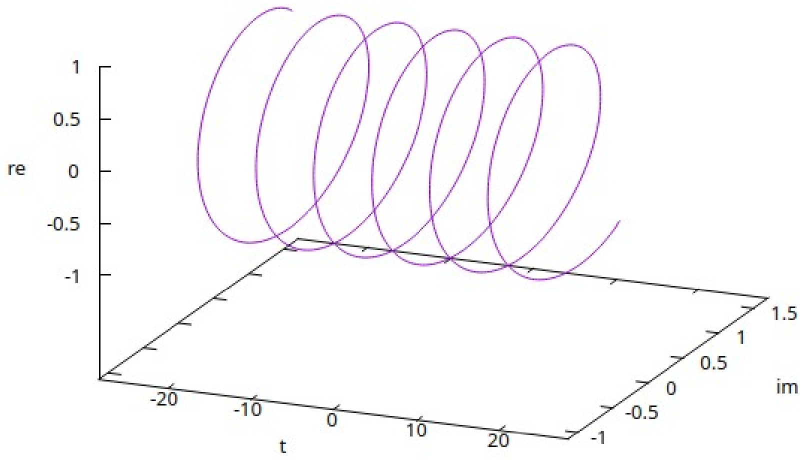

Extracting Hadronic Amplitude

in **Minkoskian** spacetime:

$$H(E) = i \int_0^{\infty} dt \, e^{iEt} \, C(t)$$

(inverse Fourier transform)

On the lattice

Analytic continuation to discrete Euclidean spacetime: $\tau = it$

$$C_e(\tau) \equiv C(-i\tau) = \sum_{n=0}^{\infty} C_n \, e^{-E_n \tau}$$

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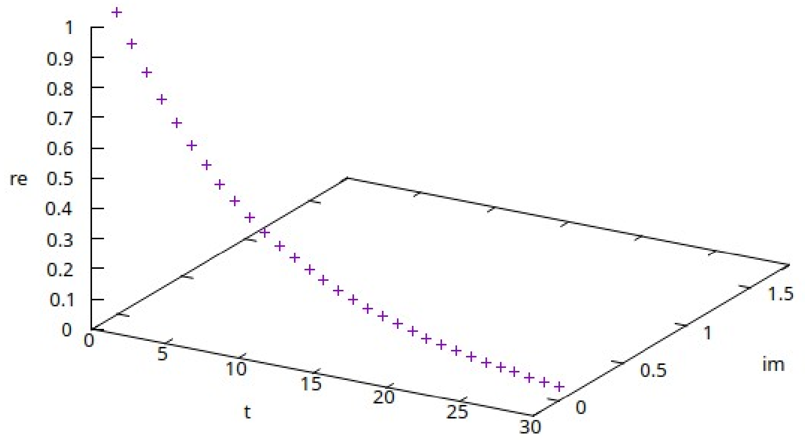

Extracting Hadronic Amplitude

in discrete **Euclidean** spacetime of length T:

$$H^T(E) = \int_0^T d\tau \, e^{E\tau} \, C_e(\tau)$$

A bit too naive...

Perils in the complex path...

We are actually taking different paths in the complex plane, do they agree?

$$\int_0^T d\tau\, e^{E\tau}\, C_e(\tau) = i\int_0^T dt\, e^{iEt}\, C(t) \quad ????$$

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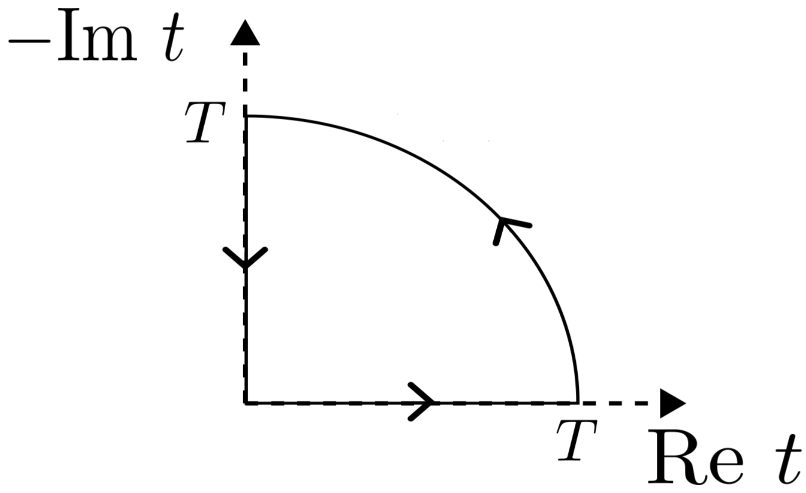

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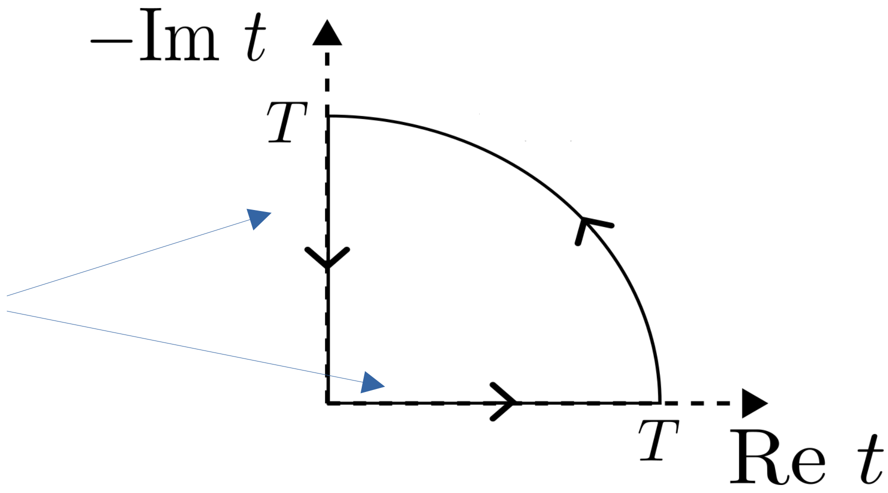
Do the two sides agree…?



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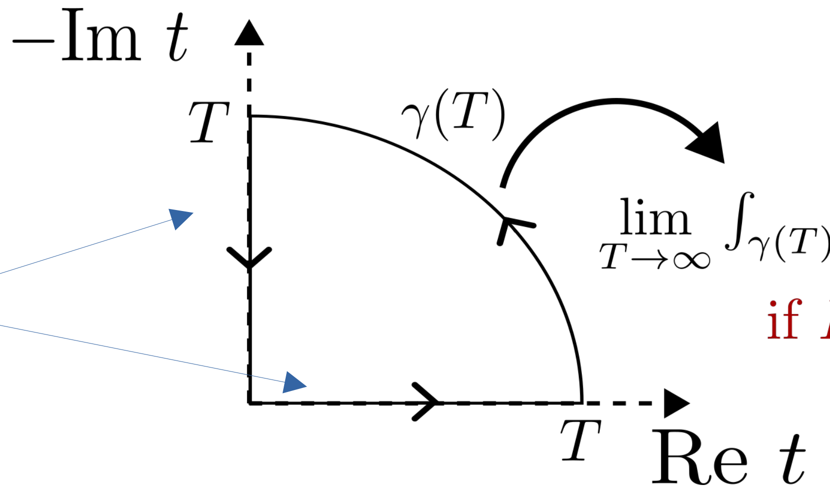

$$\lim_{T\to\infty} \int_{\gamma(T)} dt\, e^{Et} C(t) = 0$$

if $E < E_n$

Spectral representation

$$H^T(E) = \int_0^T dt\, e^{Et}\, C_E(t) = \int_0^T dt\, e^{Et} \sum_{n=0}^{\infty} C_n e^{-E_n t} = \sum_{n=0}^{\infty} C_n \frac{1 - e^{-(E_n - E)T}}{E_n - E}$$

Let us break down two energy regimes

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$$H^T(E) \underset{T \to \infty}{\to} \sum_{n=0}^{\infty} \frac{C_n}{E_n - E} + \sum_{n}^{E_n < E} C_n \frac{e^{(E - E_n)T}}{E - E_n}$$

BAD!!!

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BAD!!!

- How to subtract the divergent part?
- Needs all C_n to recover the imaginary part!

General context

Problem well known since long time:

“*Electromagnetic pion form* $F_\pi(q^2)$ *factor in the time region* $q^2 > (2m_\pi)^2$ “

[Maiani L., Testa M., 1990]

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"*Particle scattering in Euclidean lattice field theories*" [Barata and Fredenhagen, 1991]

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Several applications, e.g. P_{ℓ_4} decays $P \to \bar{l}' l' \bar{l} \nu_l$ where P is e.g. a D_s meson

(see second part of the talk)

Introducing the spectral density

$$\rho(E') = 2\pi \langle 0 | J_A(0)\, \delta(\mathbb{H} - E')\, J_B(0) | P \rangle$$

$\mathbb{H}$ = hamiltonian, δ - function restricts the correlator to the states of energy E'

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Provides similar representations of the correlator in Minkowskian & Euclidean spacetime

$$C(t) \overset{t \geq 0}{=} \int_0^\infty \frac{dE'}{2\pi}\, \rho(E')\, e^{-iE' t}, \qquad C_E(t) \overset{t \geq 0}{=} \int_0^\infty \frac{dE'}{2\pi}\, \rho(E')\, e^{-E' t}$$

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In particular, in the Euclidean we represent the correlator $C_E(t)$ as the Laplace Transform of the spectral density $\rho(E)$

Hadronic amplitude via spectral density

Expressing the correlator in terms of the spectral density:

$$H(E) = \int_0^\infty dt\, e^{iEt}\, C(t) = \lim_{\epsilon \to 0^+} \int_0^\infty \frac{dE'}{2\pi}\, \rho(E') \int_0^\infty dt\, e^{-i(E'-E)t} e^{-\epsilon t}$$

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The term $e^{-\epsilon t}$ is introduced to guarantee the time integral convergence:

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Energy integral is always well-defined given that $\rho(E) = 0$ if $E < E_0$

Separating the real and imaginary parts

$$\text{Re}\,[H(E)] = \lim_{\epsilon\to 0^+}\int_0^\infty \frac{dE'}{2\pi}\,\rho(E')\,\frac{E'-E}{(E-E')^2+\epsilon^2} = \text{P.V.}\int_0^\infty \frac{dE'}{2\pi}\,\frac{\rho(E')}{E'-E}$$

$$\text{Im}\,[\mathrm{H(E)}] = \lim_{\epsilon\to 0^+}\int_0^\infty \frac{\mathrm{dE'}}{2\pi}\,\rho(\mathrm{E'})\,\frac{\epsilon}{(\mathrm{E}-\mathrm{E'})^2+\epsilon^2} = \frac{\rho(\mathrm{E})}{2}$$

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When studying energies $E < E_0$ one can rewrite: $\displaystyle \frac{1}{E'-E} = \int_0^\infty dt\, e^{-(E'-E)t}$ obtaining:

$$\mathrm{Re}\,[H(E)] = \int_{E_0}^\infty \frac{dE'}{2\pi}\rho(E') \int_0^\infty dt\, e^{-(E'-E)t} = \int_0^\infty dt\, e^{Et}\, C_E(t)$$

$\mathrm{Im}\,[H(E)] = 0$ by definition, since $\rho(E) = 0$ if $E < E_0$

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But these is **not possible** for $E > E_0$: the $\epsilon \to 0^+$ can be taken only **after integrating** in E

Convolution of the spectral density

Representing the hadronic matrix element in terms of the spectral density:

$$H(E) = \lim_{\epsilon \to 0^+} \int_{E_0}^{\infty} \frac{dE'}{2\pi} \frac{\rho(E')}{E' - E - i\epsilon} = \lim_{\epsilon \to 0^+} H(E, \epsilon)$$

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one can recognize the convolution of the spectral density with a kernel:

$$H(E, \epsilon) \equiv \int_{E_0}^{\infty} \frac{dE'}{2\pi} \rho(E') \, K(E' - E, \epsilon) \; , \qquad K(E' - E, \epsilon) \equiv \frac{1}{E' - E - i\epsilon}$$

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$H(E,\epsilon)$ can be interpreted as a **smeared amplitude**, let us see why...

Interpretation of the "smeared amplitude"

Using the identity:

$$\frac{1}{E' - E - i\epsilon} = \lim_{\eta \to 0^+} \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega \, \frac{\epsilon}{(E-\omega)^2 + \epsilon^2} \, \frac{1}{E' - \omega - i\eta}$$

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One can immediately see that:

$$H(E,\epsilon) = \int_{-\infty}^{+\infty} d\omega \, \frac{1}{\pi} \frac{1}{\left(\frac{E-\omega}{\epsilon}\right)^2 + 1} H(\omega)$$

Corresponding to a Lorentzian smearing of $H(E)$ over a region of width 2ϵ.

An example in a toy model

Let us take as an example a simple one-resonance model for $\rho(E)$

$$\rho(E') = \frac{A\,\Gamma}{(E-M)^2 + (\Gamma/2)^2}\,\theta(E) \quad \Longrightarrow \quad H(E) \simeq \frac{A}{M - E - i\,\Gamma/2}$$

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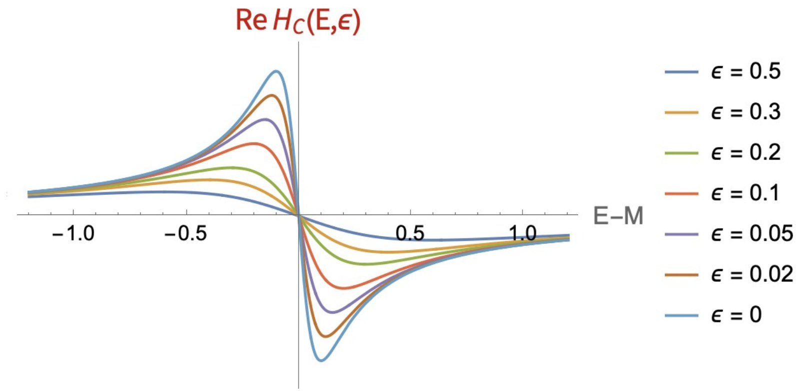
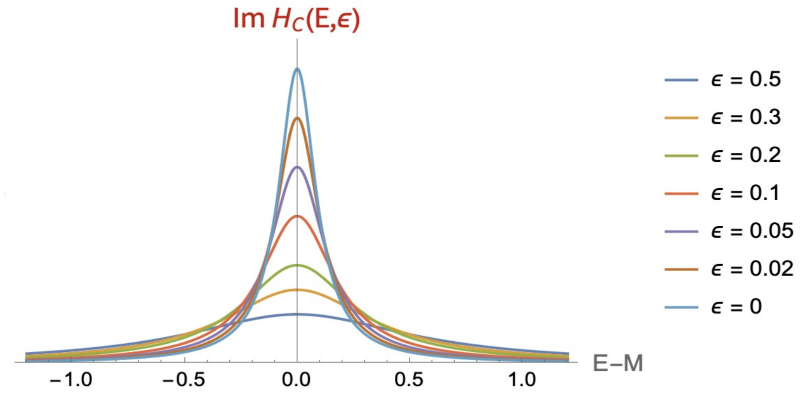
The smearing corresponds to a simple shift: $\Gamma \to \Gamma + 2i\epsilon$

$$H(E,\varepsilon) \simeq \frac{A}{M - E - i\,(\Gamma/2 + \epsilon)}$$





Taking the $\epsilon \to 0$ limit

Let us compare smeared and unsmeared amplitudes

$$H(E,\epsilon) \simeq \frac{A}{M - E - i\left(\frac{\Gamma}{2} + \epsilon\right)} \implies \frac{H(E,\varepsilon)}{H(E)} = \left[1 + \frac{i\epsilon}{\Delta(E)} e^{-i\phi(E)}\right]^{-1}$$

with $\Delta(E) = \sqrt{(E-M)^2 + (\Gamma/2)^2}$ and $\tan\phi(E) = \frac{\Gamma/2}{E-M}$

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Clearly $H(E,\varepsilon) = H(E) + \mathcal{O}(\epsilon)$ provided that $\epsilon = \sqrt{(E-M)^2 + (\Gamma/2)^2}$

- Far from the resonance peak, $M - E \gg \Gamma/2$ such that one needs just $\epsilon \ll |M - E|$
- Near the peak, $M \sim E$ such that one needs $\epsilon \ll \Gamma/2$

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This is in a toy model, but what in real life?

Taking the $\epsilon \to 0$ limit in a generic case

We need to give a general definition of the “width”. For sufficiently smooth amplitudes:

$$\frac{1}{\Delta(E)} \equiv \left| \frac{1}{H(E)} \frac{\partial H(E)}{\partial E} \right|$$

For each E the value of $\Delta(E)$ is a (linear estimate) of the region where $H(E)$ varies by 100%

Not always that simple...

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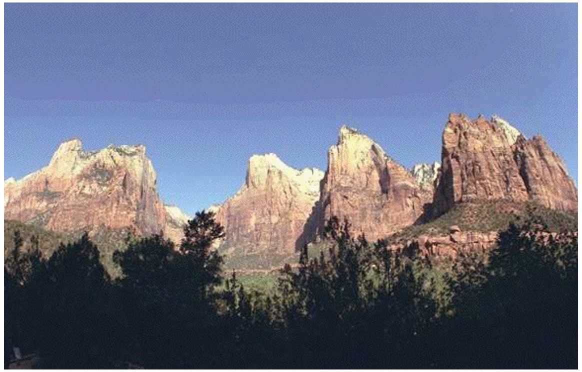
Not always that simple...

- In less trivial cases, <u>many features</u> might be present
- In this cases more care must be taken to take the extrapolation
- Some degree of modelling might be required
- Observing the asymptotic ϵ^n regime might require very small ϵ



Resonances in a finite volume

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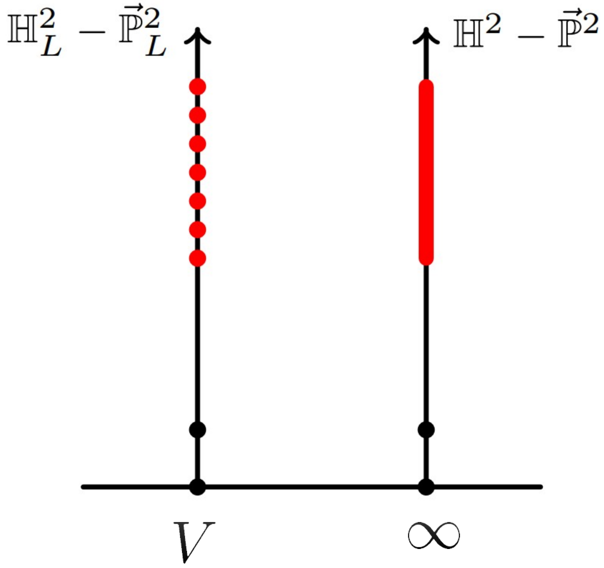
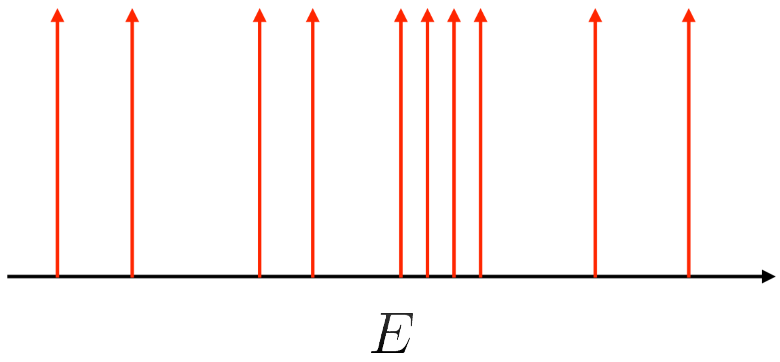
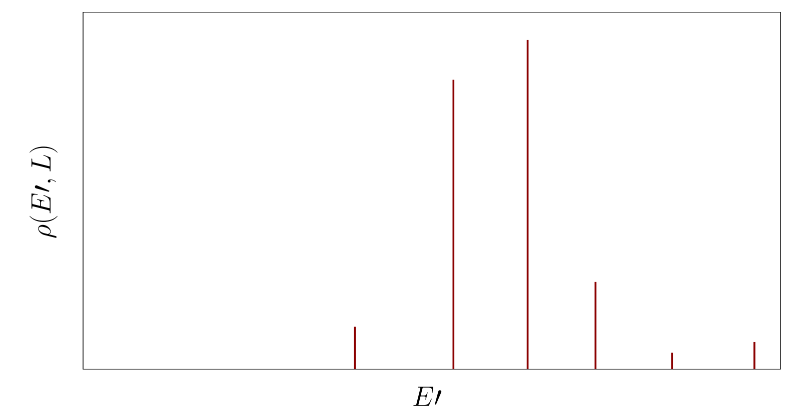


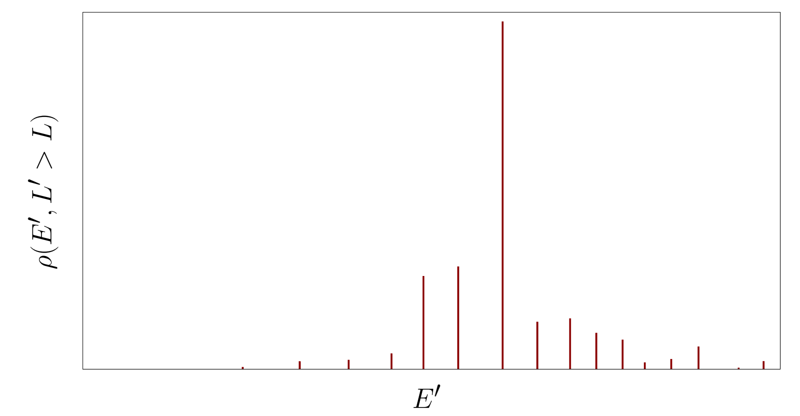
No direct connection to $\rho(E)$



Needs smearing, putting finite ϵ

Quantization conditions – 2 particles toy model





We must be sure not to undersmear!

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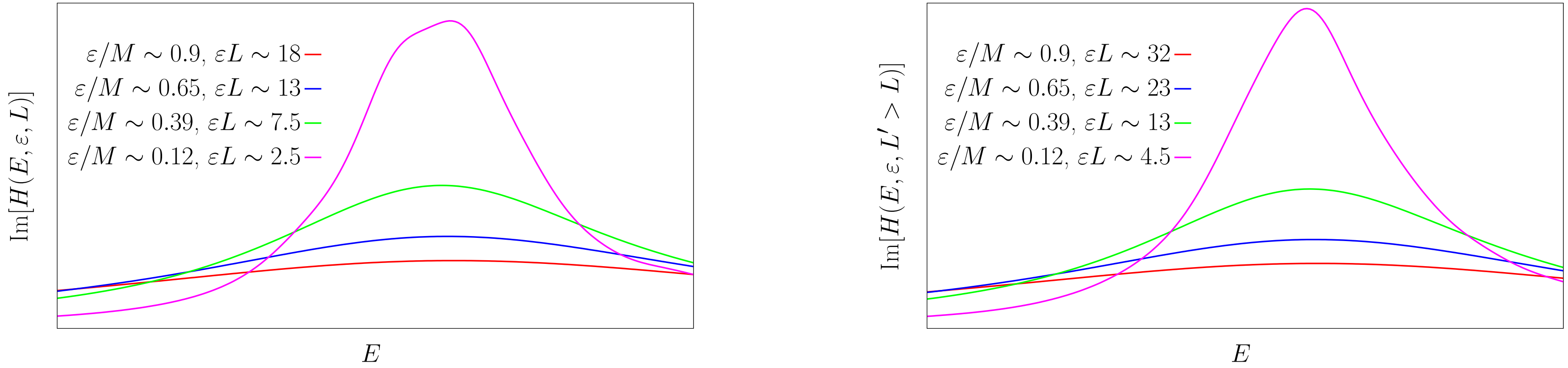

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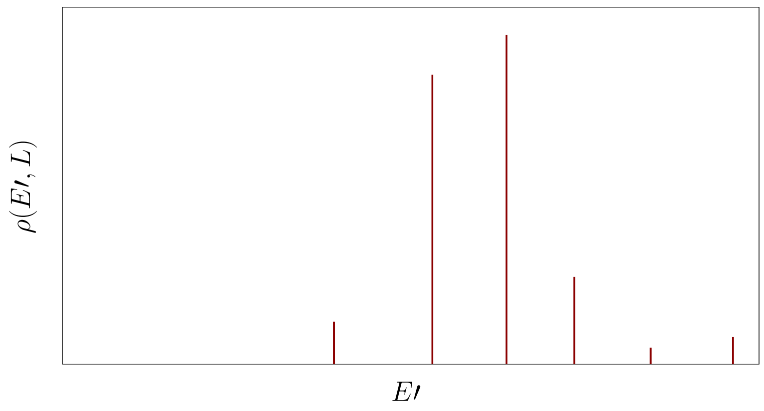

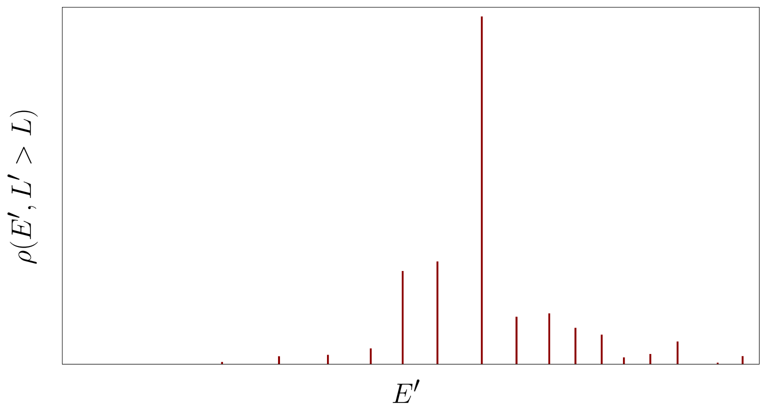

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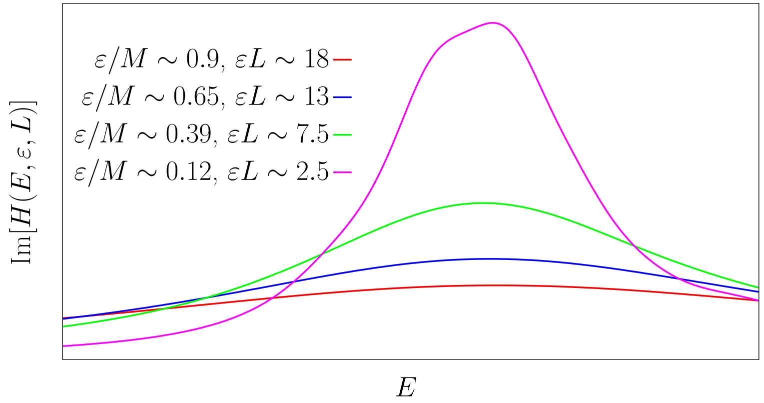

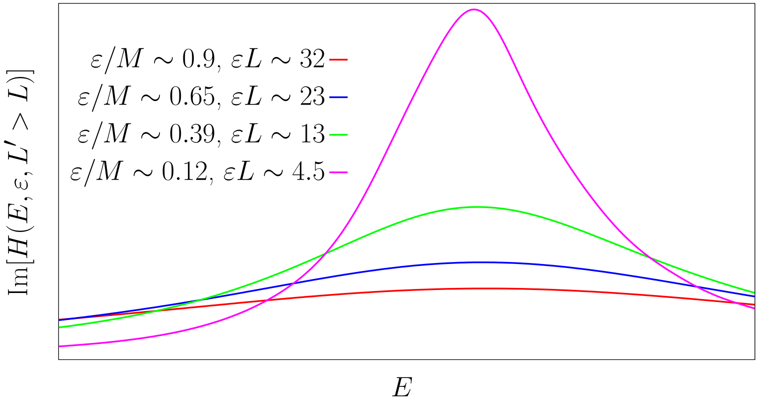

Small FVE guaranteed if $\epsilon L \gg 1$ [Bulava et al. 2021]

HLT method at work

We have a specific kernel to reconstruct, the HLT method offers the possibility to do it

$$K_{\mathrm{I}}(E' - E, \epsilon) \simeq \sum_{n=1}^{N} g_n^{\mathrm{I}}(E) e^{-aE'n}, \quad \mathrm{I} = \{\mathrm{Re}, \mathrm{Im}\}$$

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From the knowledge of the coefficients, one reconstructs the real and imaginary parts:

$$\sum_{n=1}^{N} g_n^{\mathrm{Re}}(E)\, C_E(na) = \int_0^{\infty} \frac{\mathrm{d}E'}{2\pi} \left(\sum_{n=1}^{N} g_n^{\mathrm{Re}}(E)\, e^{-naE'} \right) \rho(E') \simeq \mathrm{Re}[H(E,\epsilon)]$$

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No-regularization HLT

Minimize the L^2-distance between target and reconstructed function

$$A^{\mathrm{I}}[g] \equiv \int_0^{\infty} \frac{\mathrm{d}E'}{2\pi} \left| K_{\mathrm{I}}(E' - E, \varepsilon) - \sum_{n=1}^{N} g_n e^{-naE'} \right|^2$$

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Can be analytically minimized, leading to a “simple” closed form

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Unfortunately the Hilbert matrix is a textbook example of poorly conditioned matrix

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E.g. $\det H_{10} \simeq \mathcal{O}(10^{-53})$, $\det H_{20} \simeq \mathcal{O}(10^{-226})$

Two particle toy model

$$\rho(E) = \frac{1}{\pi} \sum_{n=1,2} \frac{\Gamma_n/2}{(E - E_n)^2 + (\Gamma_n/2)^2}\,,$$

$$E_1 = 0.10,\ \ \Gamma_1 = 10^{-2}$$

$$E_2 = 0.15,\ \ \Gamma_2 = 2 \cdot 10^{-2}$$

Let us construct the correlator and reconstruct amplitude via HLT...

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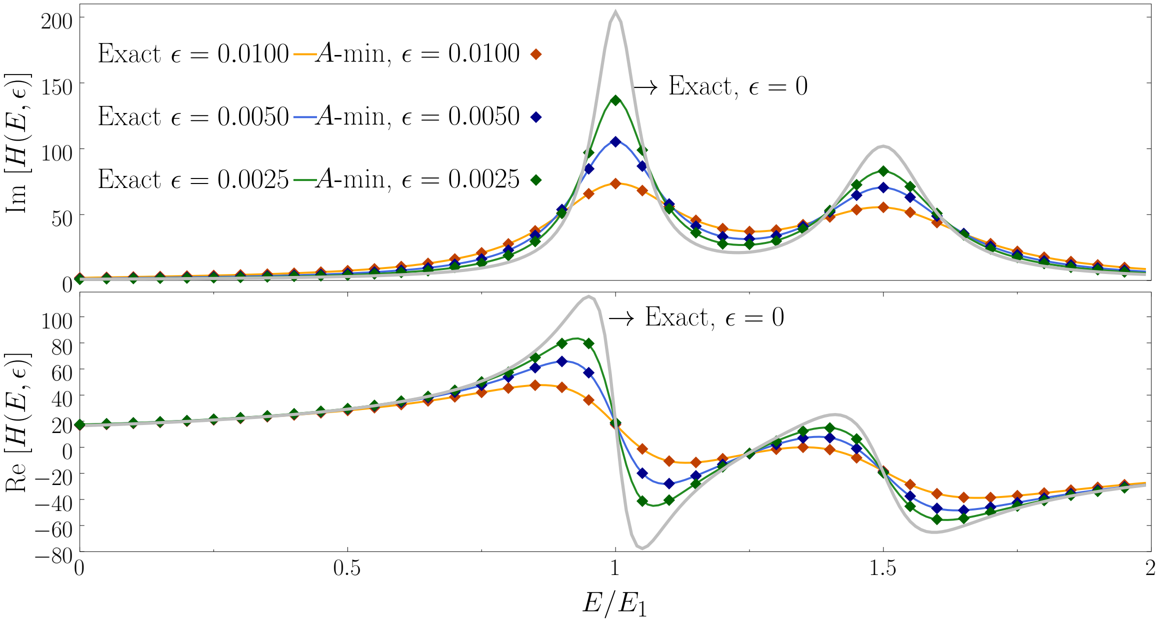
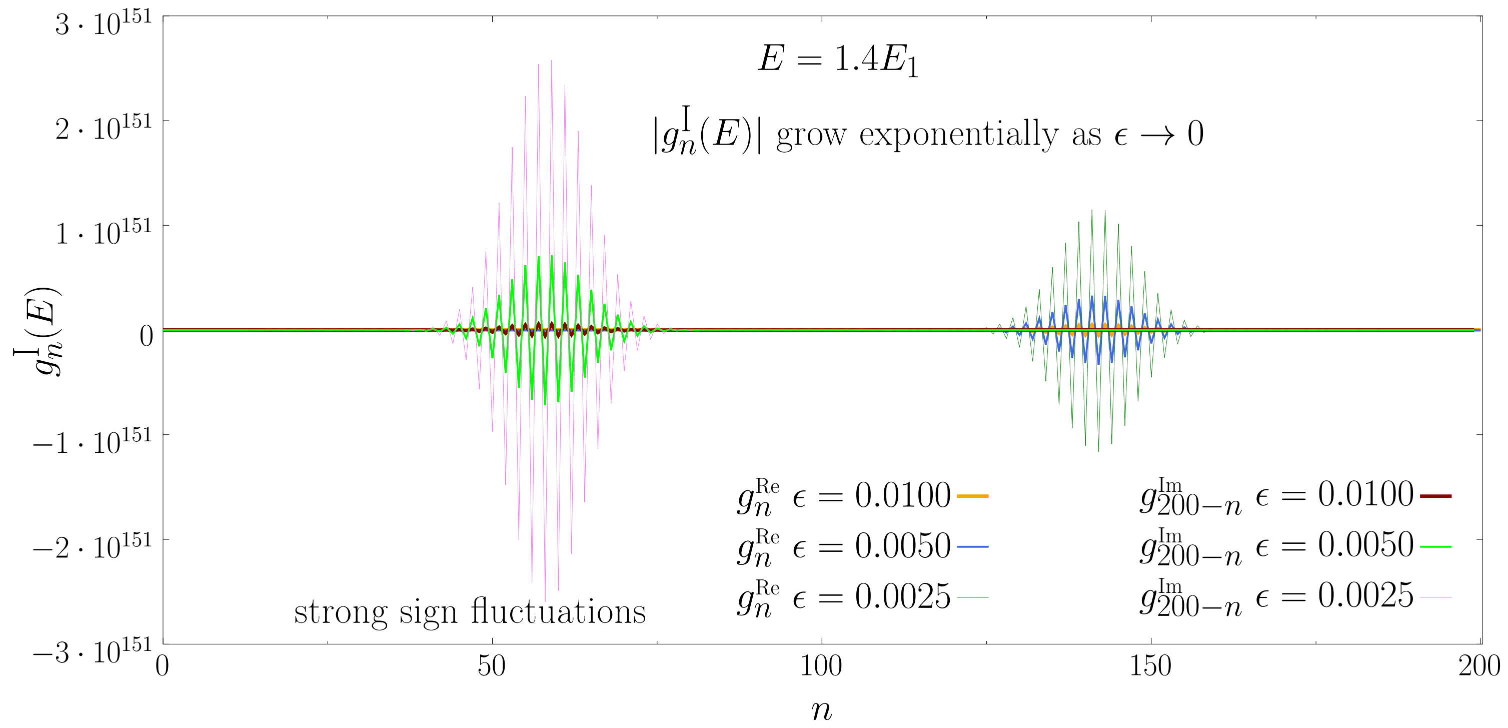


The problem with the coefficients



This giant coefficients will make the error on the amplitude EXPLODE once used on real data

$$\Delta H(E,\epsilon) = \sqrt{\sum_n \left[g_n^{\mathrm{Re}}(E)\,\delta C_E(na)\right]^2} + i\sqrt{\sum_n \left[g_n^{\mathrm{Im}}(E)\,\delta C_E(na)\right]^2}$$

Cure the problem with the HLT

Include the error weight in the functional to be minimized

$$W^{\mathrm{I}}[g] = \frac{A^{\mathrm{I}}[g]}{A^{\mathrm{I}}[0]} + \lambda B[g]$$

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Systematic error due to reconstruction

$$B[g] \propto \sum_{n_1, n_2 = 1}^{N} g_{n_1} g_{n_2} \, \mathrm{Cov}\left(C_E(an_1), C_E(an_2)\right)$$

Statistical error due to correlator fluctuations

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Statistical error
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λ balances between the two error contributions

A practical test: the $D_s \to \bar{\ell}'\ell'\ell\nu_\ell$ decay

Rare electroweak decay of a flavoured, charged meson into a dilepton $\bar{\ell}'\ell'$ and a lepton pair $\nu_\ell \ell$

Decay rates suppressed by $\mathcal{O}(G_F^2\,\alpha_{\rm em}^2)$, interesting probe of NP beyond the SM

Mediated by virtual photon emission, two diagrams at LO in $\alpha_{\rm em}$:

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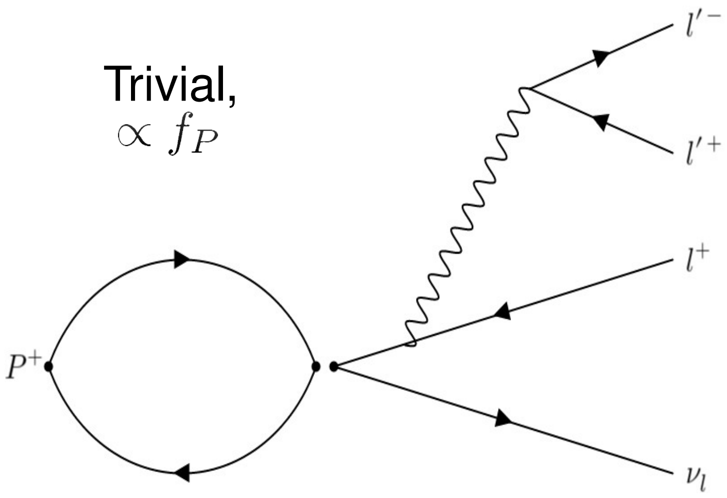
Bremsstrahlung contribution,

Trivial,
$\propto f_P$



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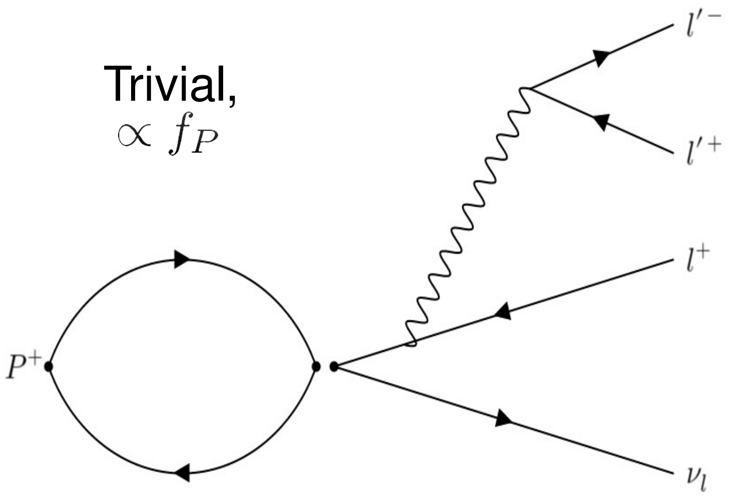
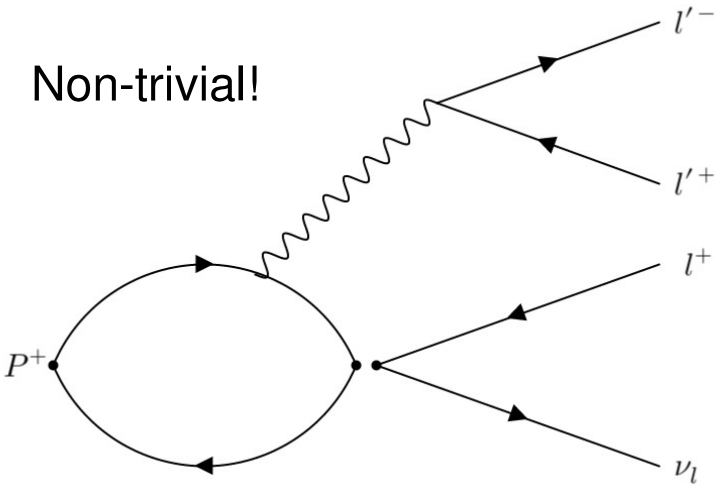
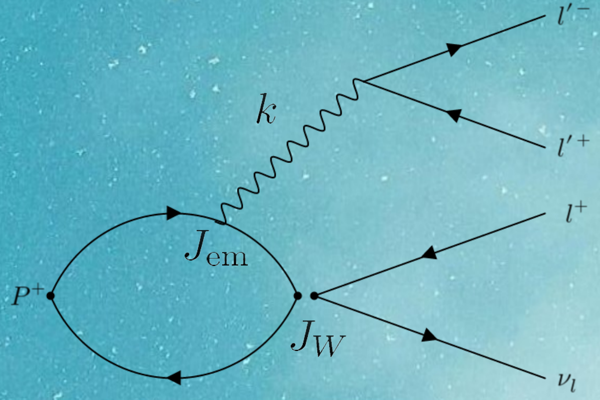


Structure-dependent contribution

Non-trivial!



Hadronic tensor

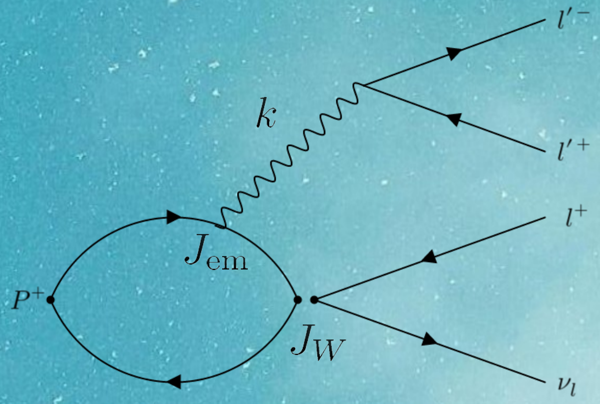


Both quarks can emit the photon

$$H_W^{\mu\nu}(k) = i \int dt\, e^{iEt}\, \mathrm{T} \langle 0 | J_{\rm em}^{\mu}(t, k) J_W^{\nu}(0) | P \rangle$$

Vector and Axial part of the Weak current take part

Hadronic tensor



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let us analyse separately the positive and negative times

$$= \underbrace{i\int_{-\infty}^{0} dt\, e^{iEt}\langle 0|J^{\nu}_{W}(0) J^{\mu}_{\rm em}(t,k)|P\rangle}_{H^{\mu\nu}_{W,1}(k)} + \underbrace{i\int_{0}^{\infty} dt\, e^{iEt}\langle 0|J^{\mu}_{\rm em}(t,k) J^{\nu}_{W}(0)|P\rangle}_{H^{\mu\nu}_{W,2}(k)}$$

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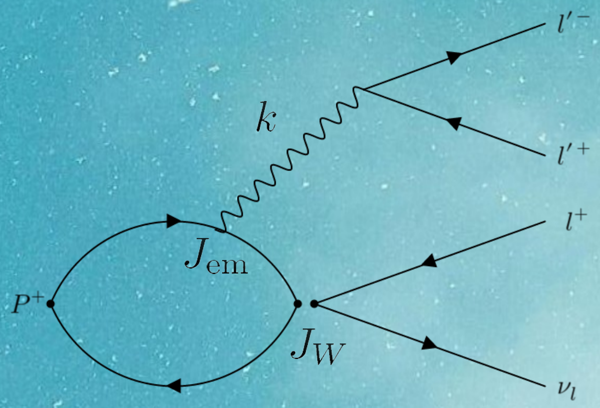

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Evolving the e.m current to zero time $J^\mu_{\rm em}(t,k) = e^{i(\mathbb{H}-i\varepsilon)t}\, J^\mu_{\rm em}(0,k)\, e^{-i(\mathbb{H}-i\varepsilon)t}$ and integrating...

...after integrating (in Minkoswy!)...

$$H^{\mu\nu}_{W,1}(k) = \langle 0|J^{\nu}_{W}(0)\frac{1}{\mathbb{H}+E-M_P-i\epsilon}J^{\mu}_{\mathrm{em}}(0,k)|P\rangle$$

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Inserting a complete set of states, one gets

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$|r\rangle$ states have same flavor and are not lighter than $|P\rangle$, **no issue** with analytic continuation

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$$H^{\mu\nu}_{W,2}(k) = \sum_n \frac{\langle 0|J^\mu_{\rm em}(0,k)|n\rangle\langle n|J^\nu_W(0)|P(0)\rangle}{E_n-E-i\epsilon}, \quad p_n=+k$$

$|n\rangle$ states are unflavored and vectorial: $|\phi\rangle$, **issue** with analytic continuation above $|k^2|=M_\phi$

Proof-of-principle calculation

We computed the needed three-points correlator:

$$C_W^{\mu\nu}(t,k) \equiv T\langle 0|J^{\mu}_{\text{em}}(t,k)\, J^{\nu}_{W}(0)|D_s(0)\rangle$$

...and extracted the hadronic matrix elements.

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Single N_f=2+1+1 Wilson-clover twisted-mass ETMC gauge ensemble at the physical point

Ensemble	a [fm]	L/a	T/a	$N_{\rm confs}$	$N_{\rm sources}$
cB211.072.64	0.079	64	128	302	4

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Let's study the problematic time ordering, for V channel, and to $\vec{k} = k\hat{z}$, for $C_E(t) = C_V^{12}(t,k)$

Extracting $H(E,\epsilon) = \int_0^\infty \frac{dE'}{2\pi}\,\rho(E')\,K(E'-E,\epsilon)$ from $C_E(t) = \int_0^\infty \frac{dE'}{2\pi} e^{-E't}\,\rho(E')$

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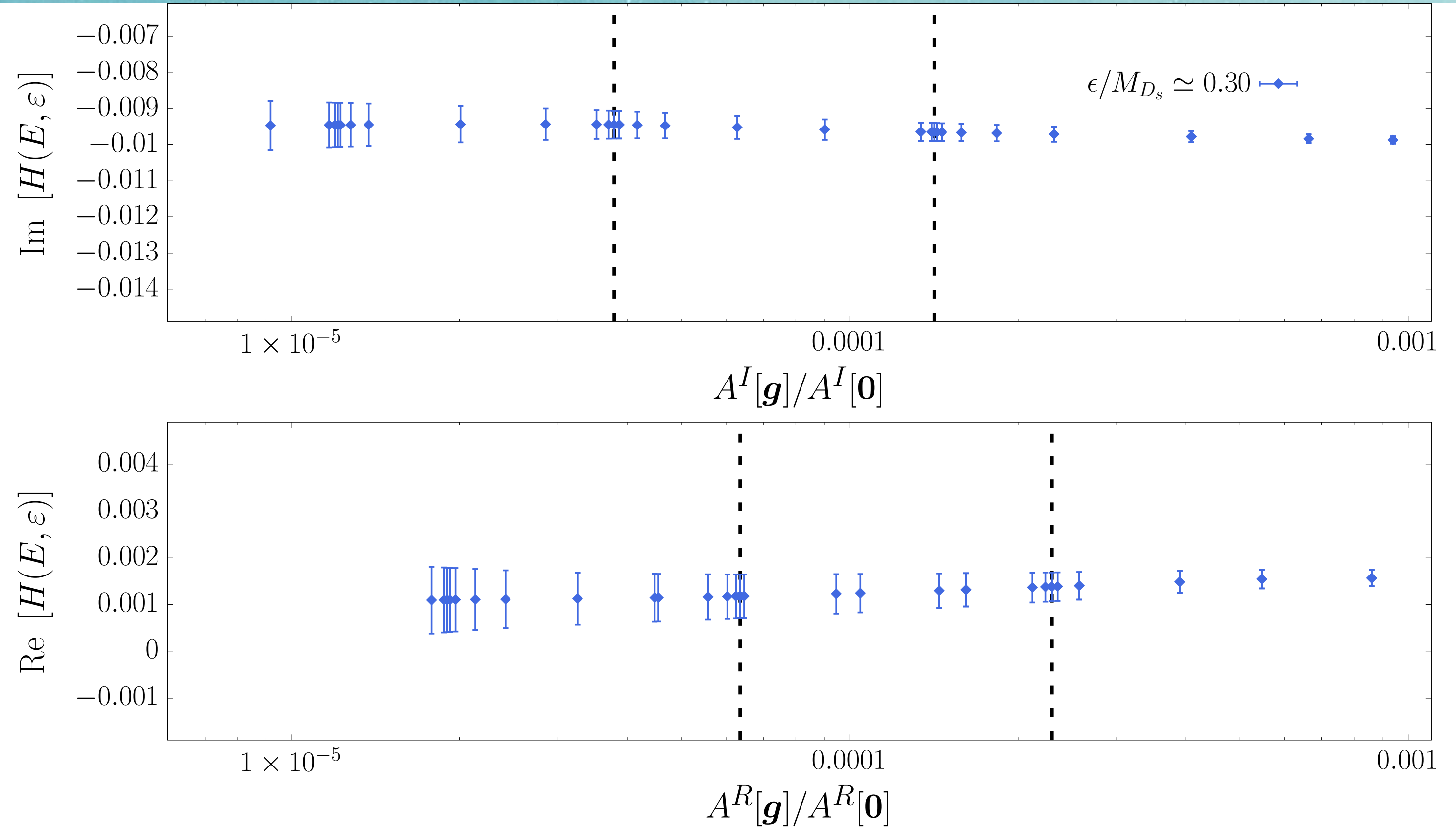
Let's go...

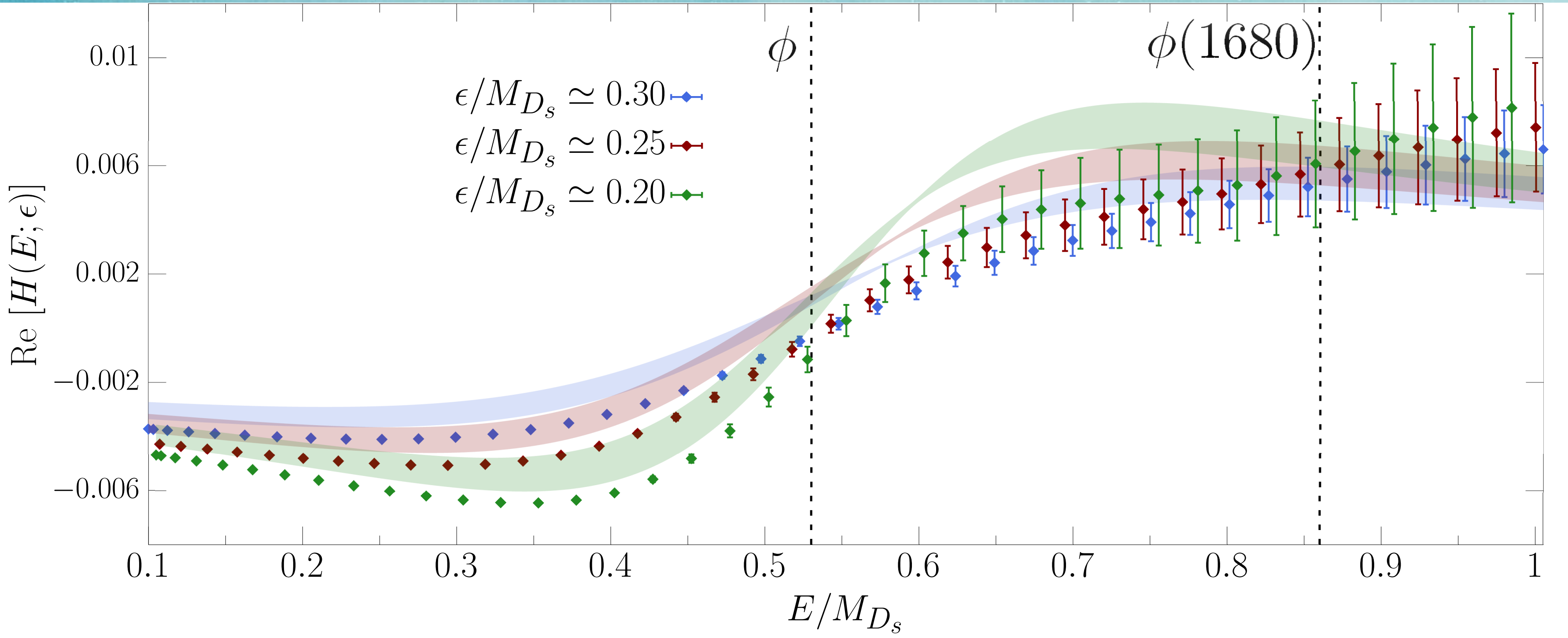
Stability plot analysis



Extract at small $A[g]/A[0]$ (used as a proxy for λ) where stability is seen, before error explodes

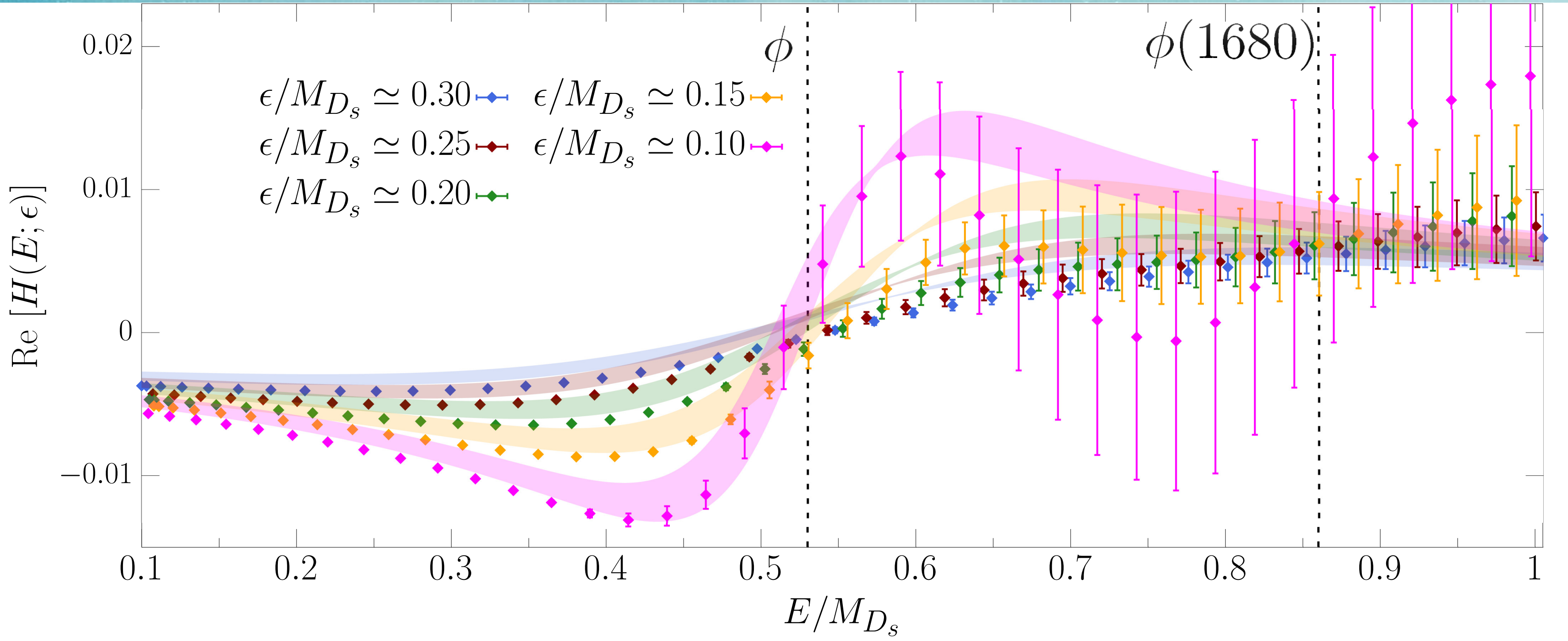
Systematic error of the choice estimated by varying the chosen point of $A[g]/A[0]$

Real part of the amplitude



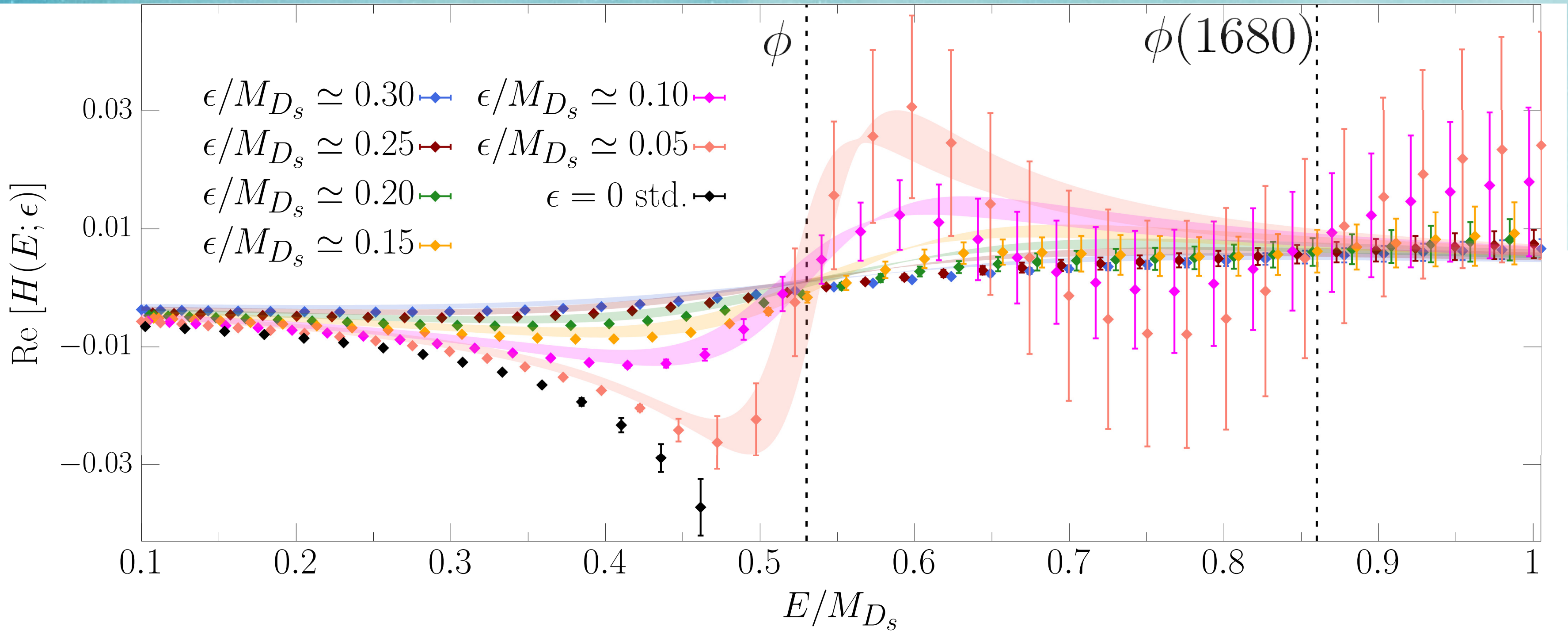
Band: comparison with **Vector Meson Dominance** model tuned on the correlator

Decreasing ϵ



For $E > E_\phi = \sqrt{M_\phi^2 + |k|^2}$ the error increases when decreasing ϵ

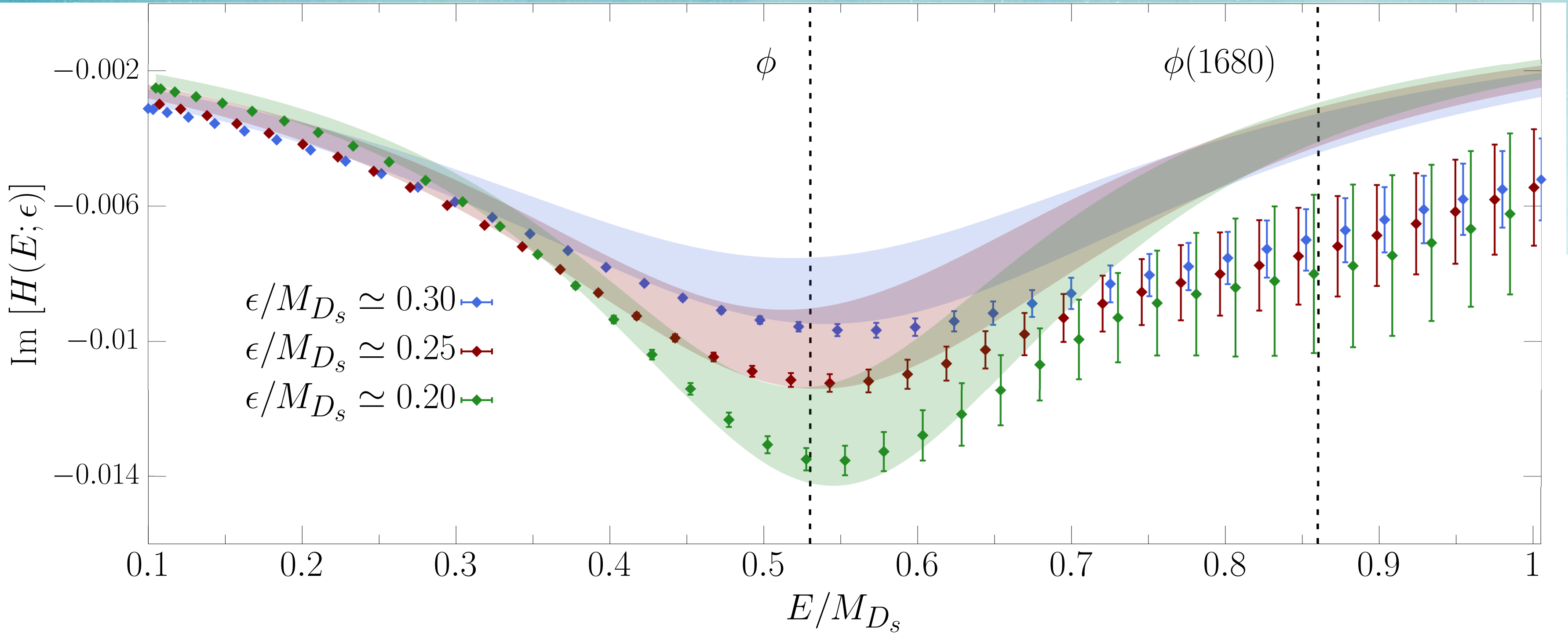
Decreasing ϵ



Below threshold one can use **standard methods** (no analytic continuation issue)

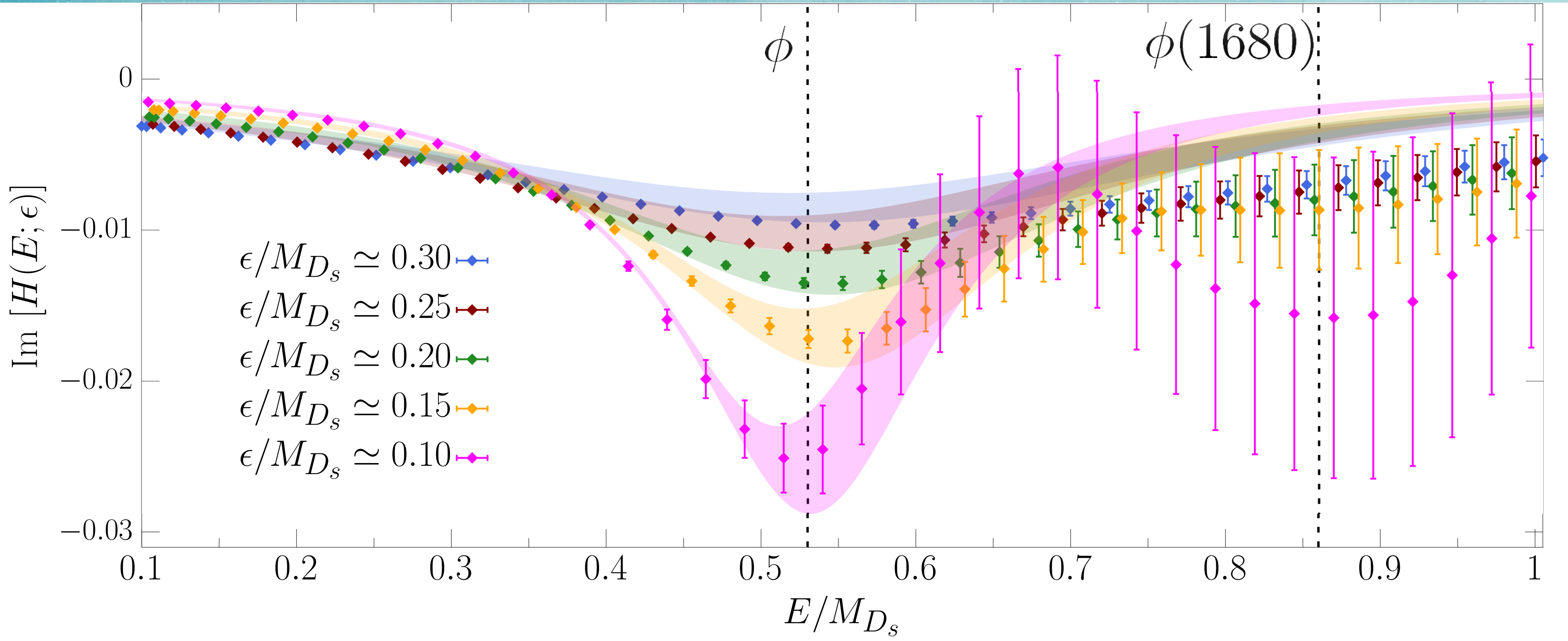
Divergence at $E = M_\phi$ damped only by finite width of the ϕ meson, but $\Gamma_\phi \sim 4$ MeV!

Imaginary part of the amplitude



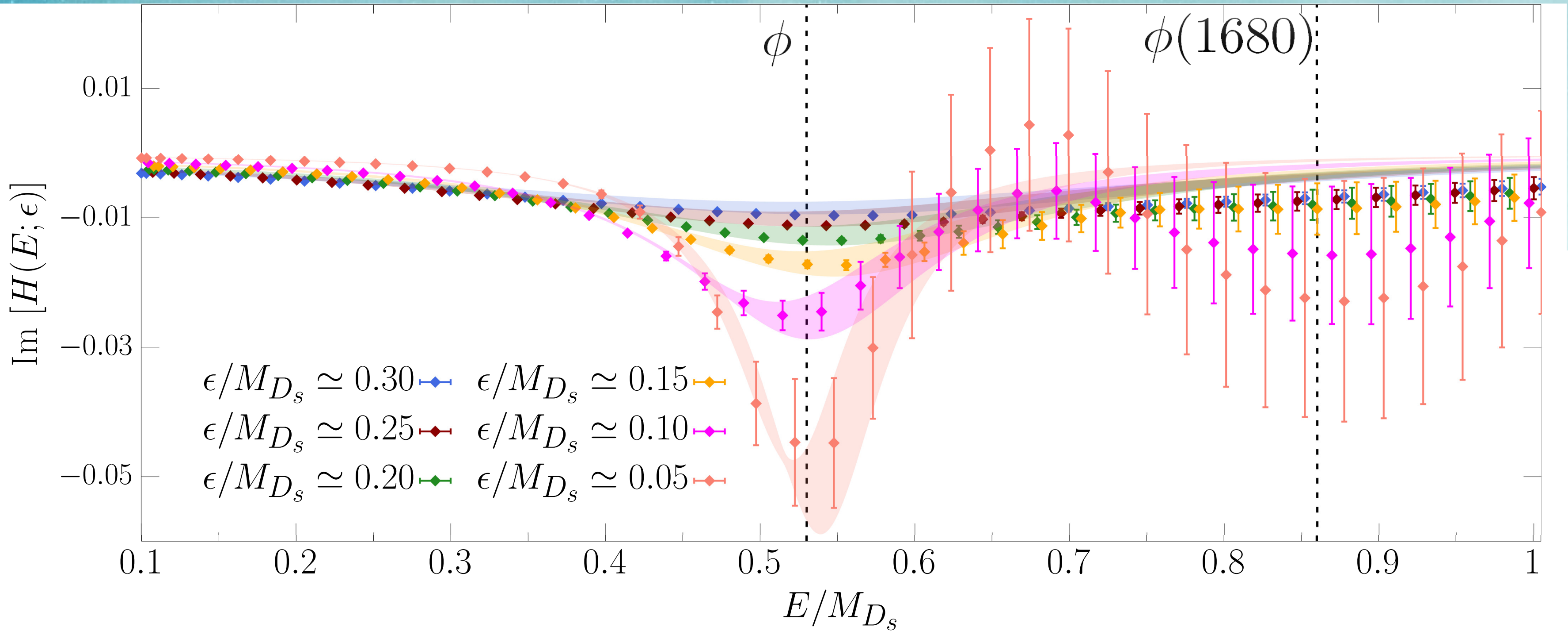
The imaginary part of the amplitude is the (smeared) spectral density itself

Decreasing ϵ



Decreasing the smearing size, one should see the very narrow resonance

Can we see the resonance...?

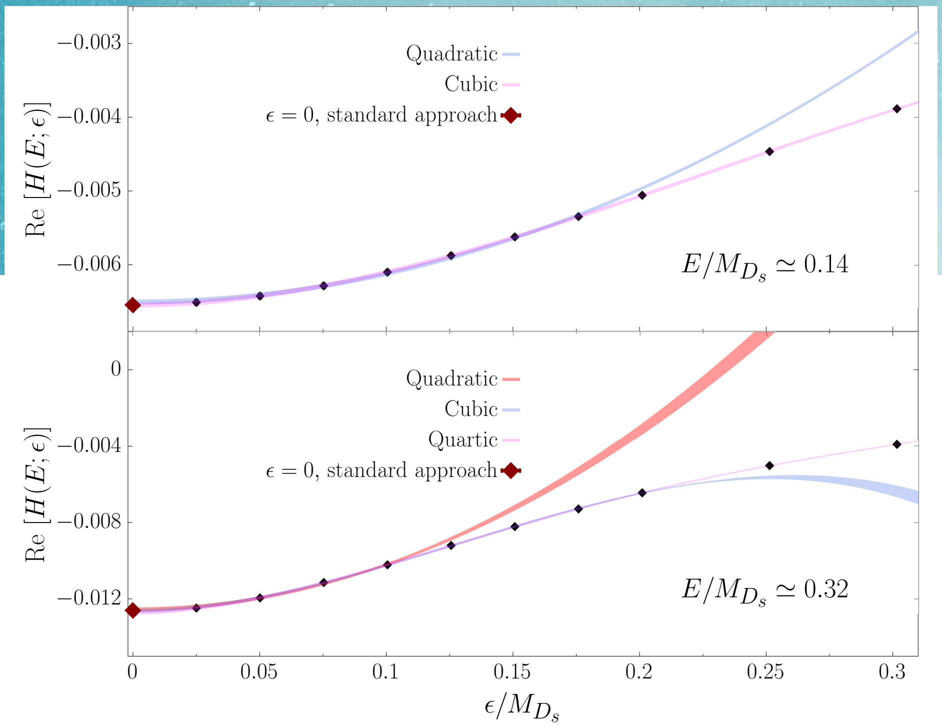


...not really… indeed, the width of the ϕ meson has just $\Gamma_\phi \sim 4$ MeV !

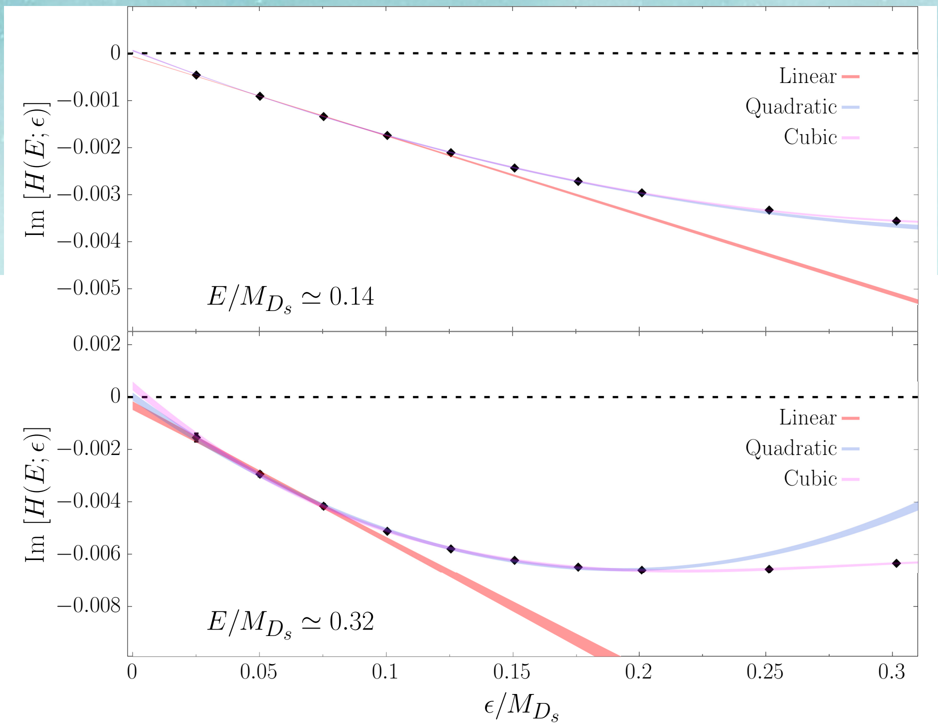
This is a <u>though</u> case, but illustrates well the various regimes

Extrapolating to $\epsilon = 0$ at $E < E_\phi$

Expected LO for real part: ϵ^2



Expected LO for imaginary part: ϵ^1



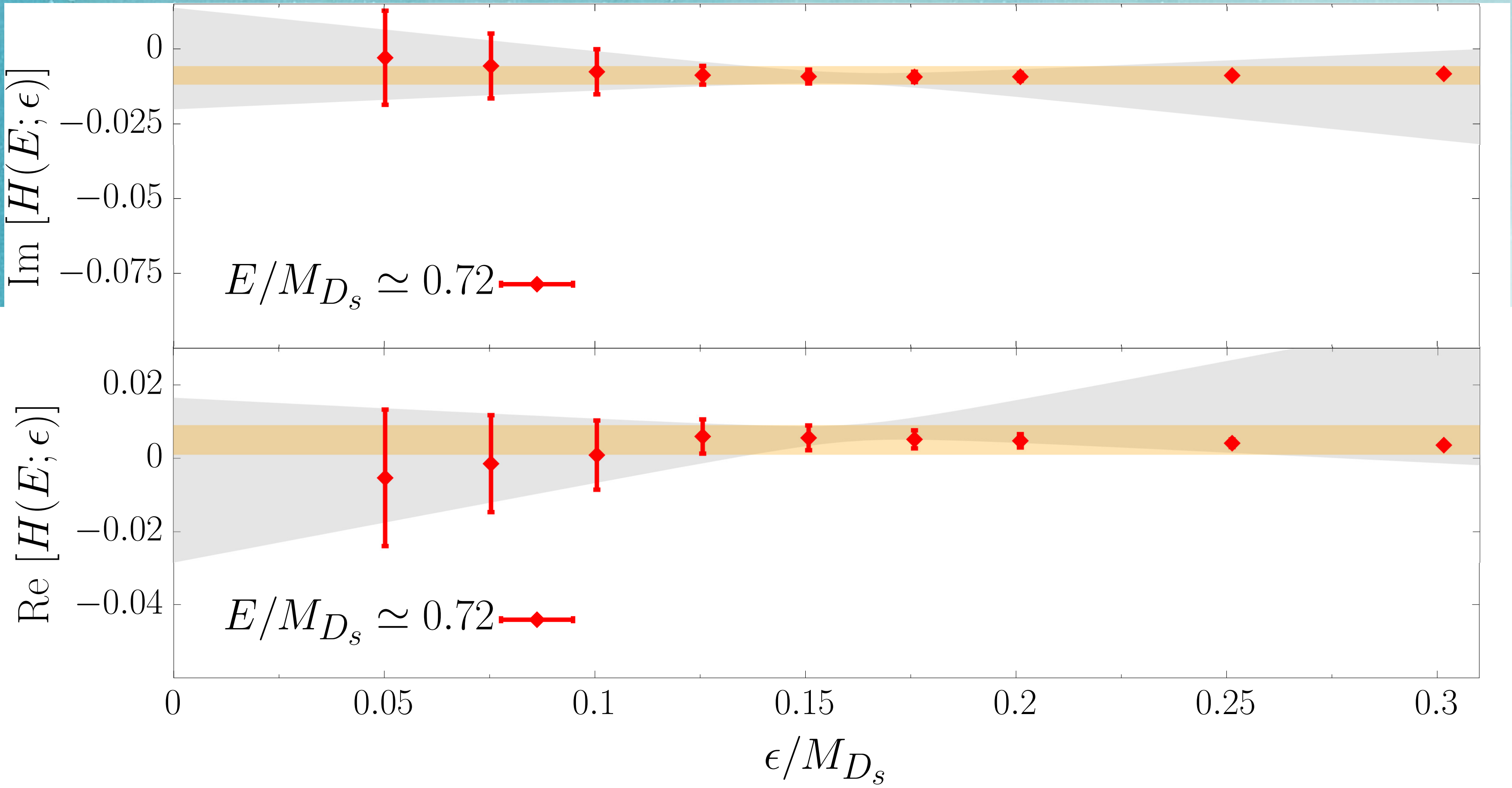
Standard approach: $H(E) = \int_0^\infty dt\, e^{Et}\, C_E(t)\,,$

is validating the reconstruction

At the same time: $\lim_{\varepsilon \to 0^+} \mathrm{Im}[H(E,\varepsilon)] = 0\,,$

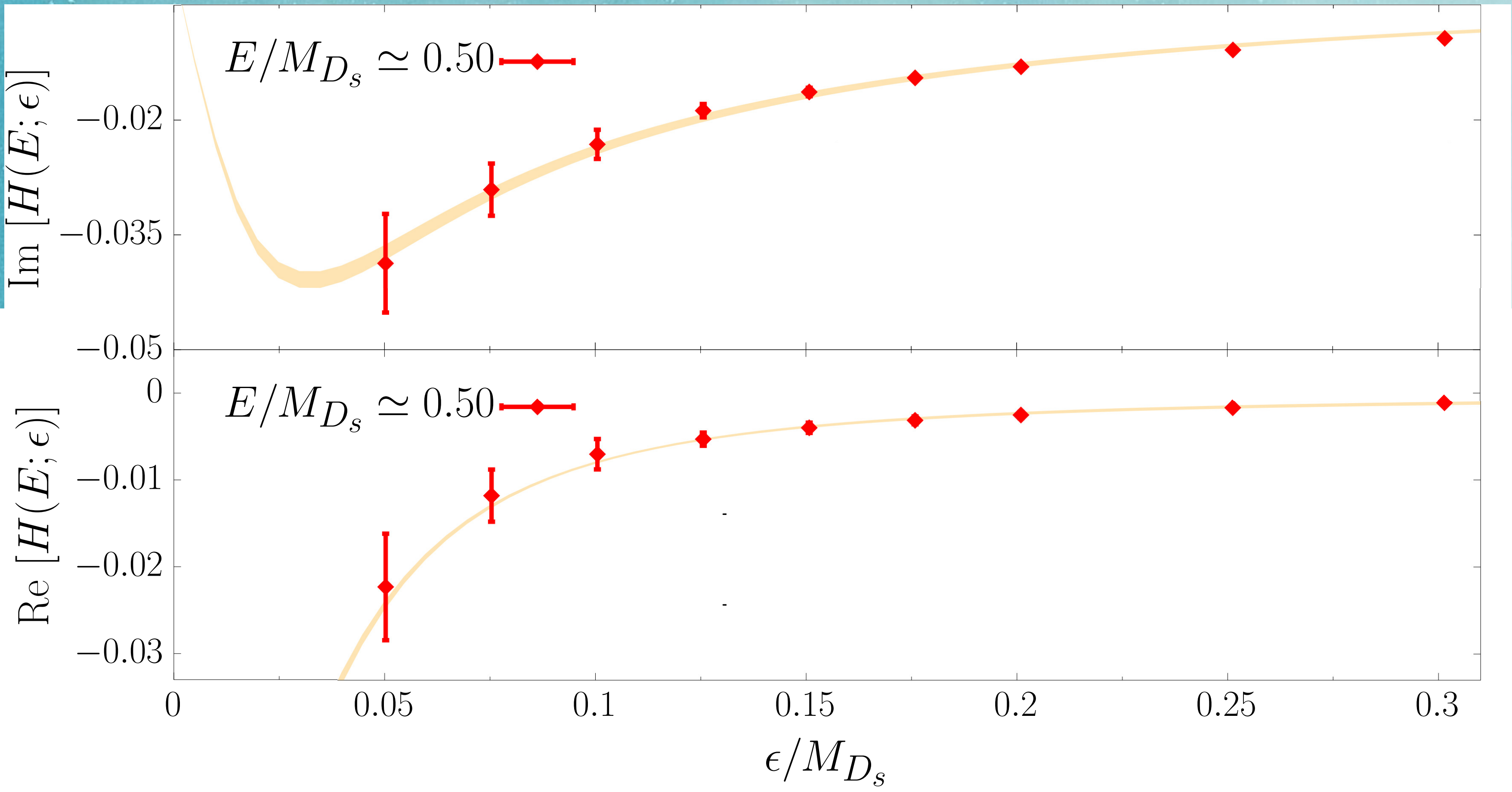
as expected below threshold

Extrapolating to $\epsilon = 0$ at $E > E_\phi$



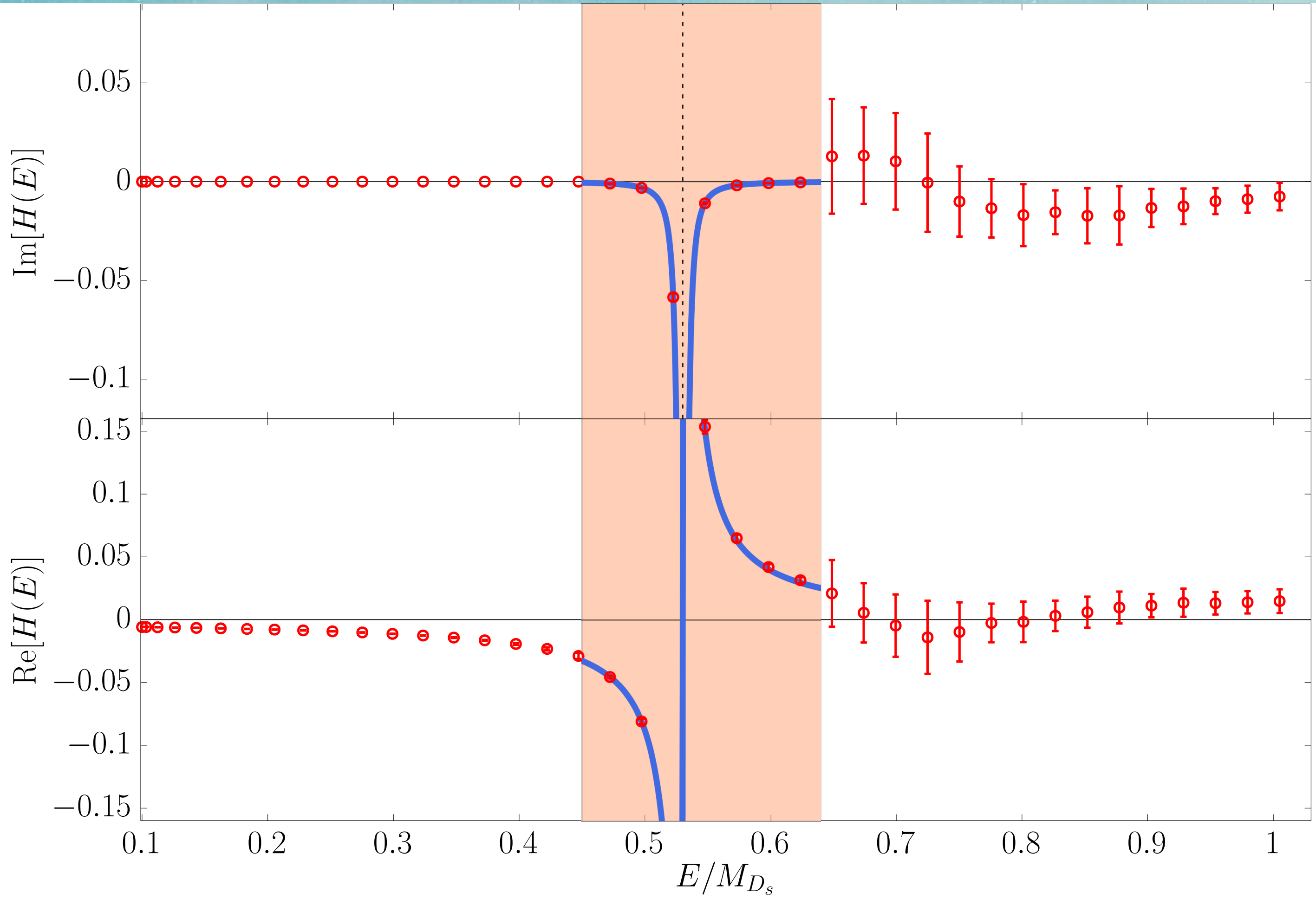
Basically independent on the smearing parameter, different ansatze used to test systematics

Extrapolating to $\epsilon = 0$ at $E \sim E_\phi$



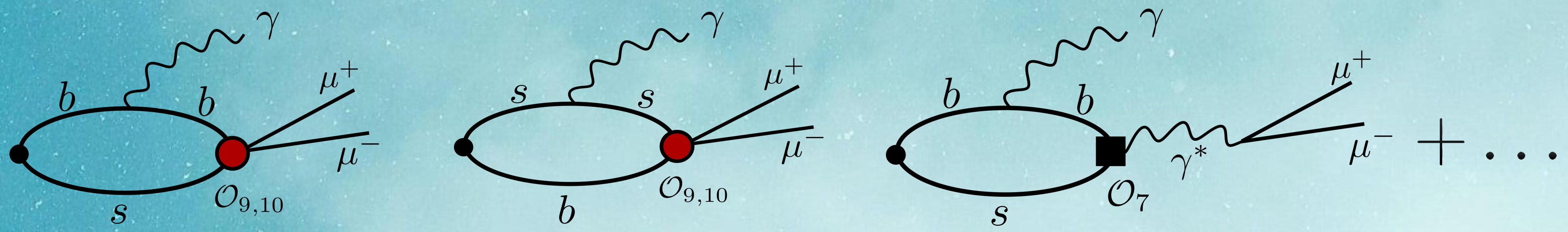
Needs to use a Breit Wegner model to extrapolate...

Putting all energies together



Flashing on the $B_s \to \mu^+\mu^-\gamma$ at large q^2

[Phys.Rev.D 109 (2024) 11, 114506]



FCNC process useful for NP searches, like $B_s \to \mu^+\mu^-$, suppressed by α_{em} but helicity enhanced

A really hard job, combining many contributions, extrapolating from $m_c \to m_b$ on 4 lattice spacings

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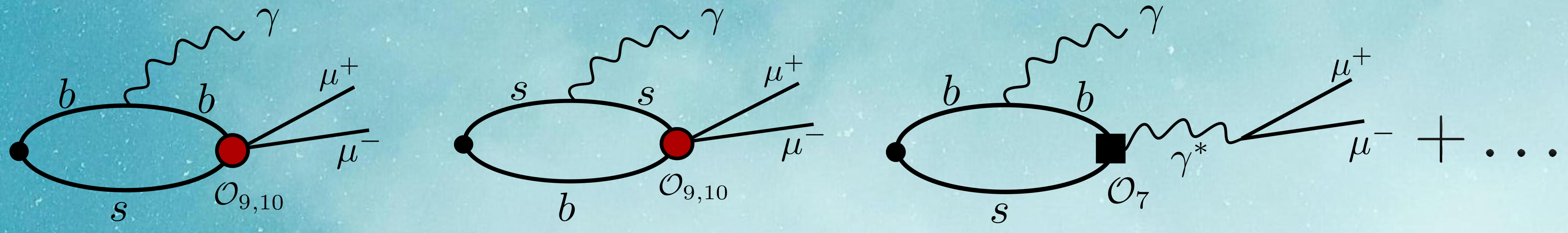

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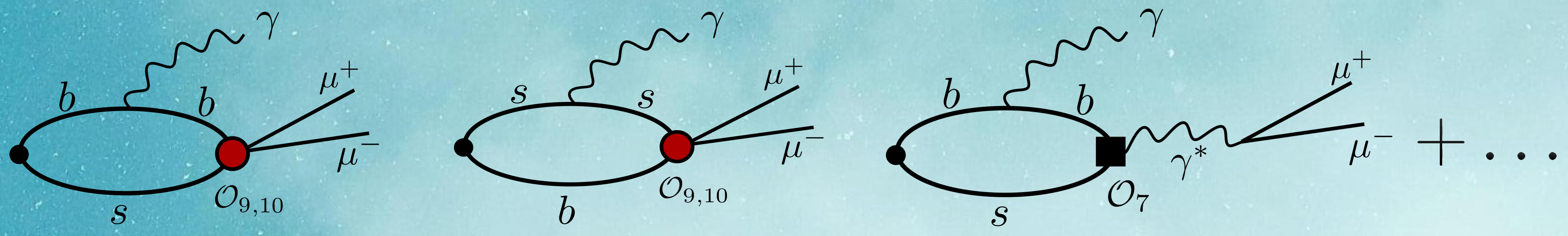

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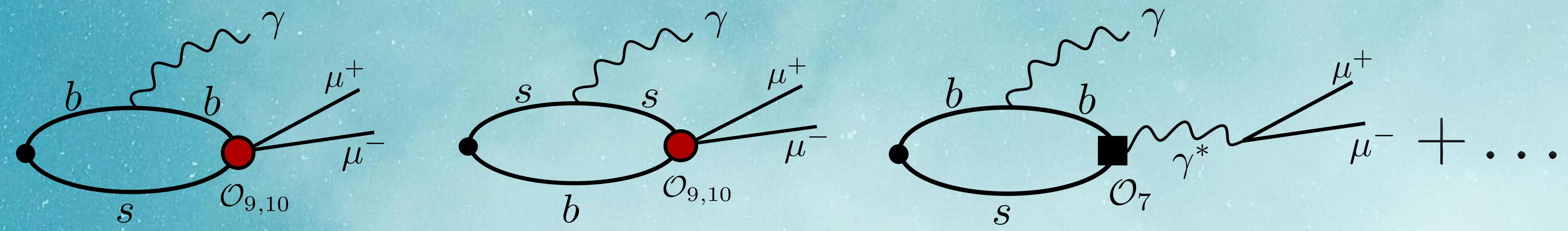

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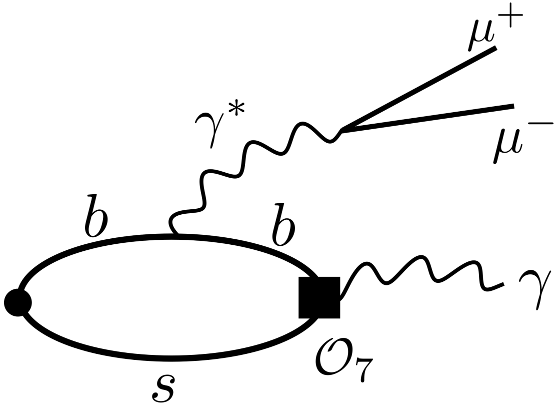

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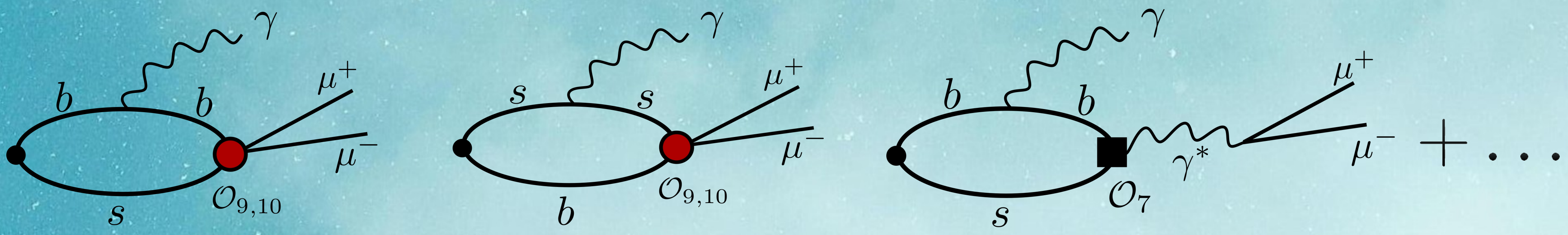

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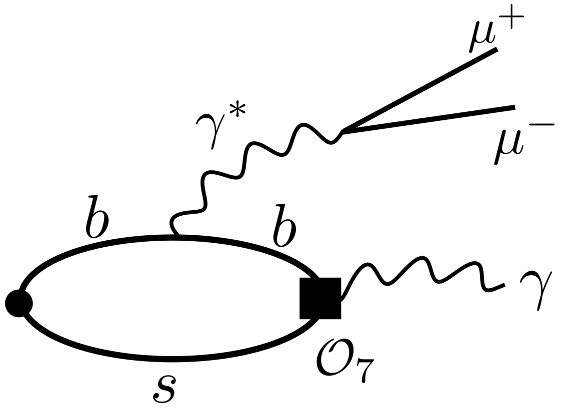
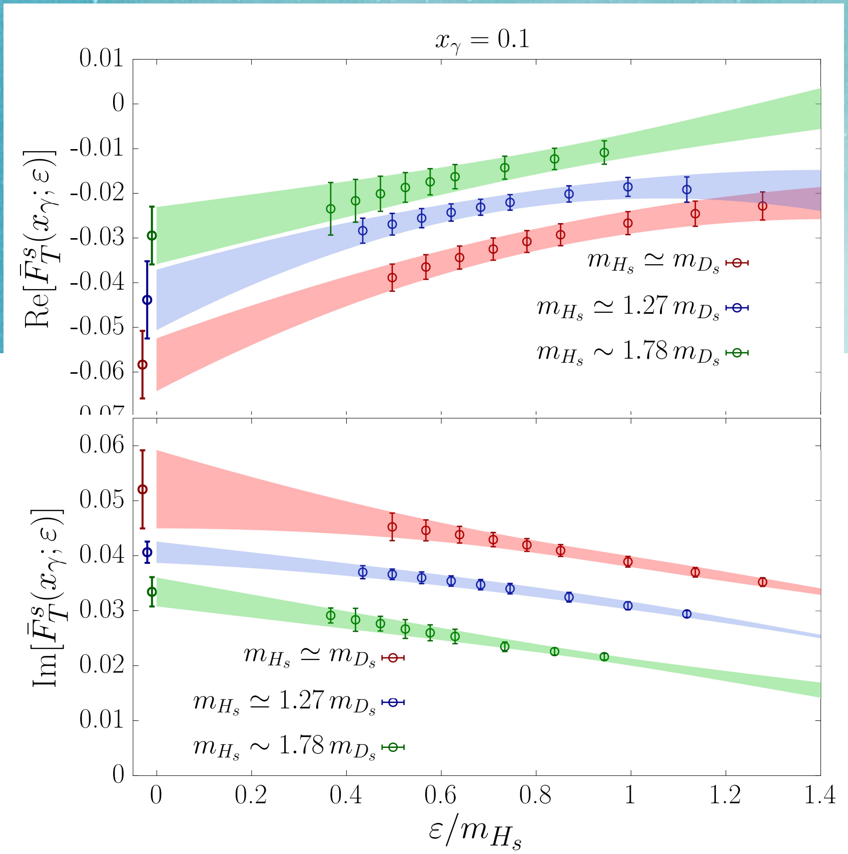
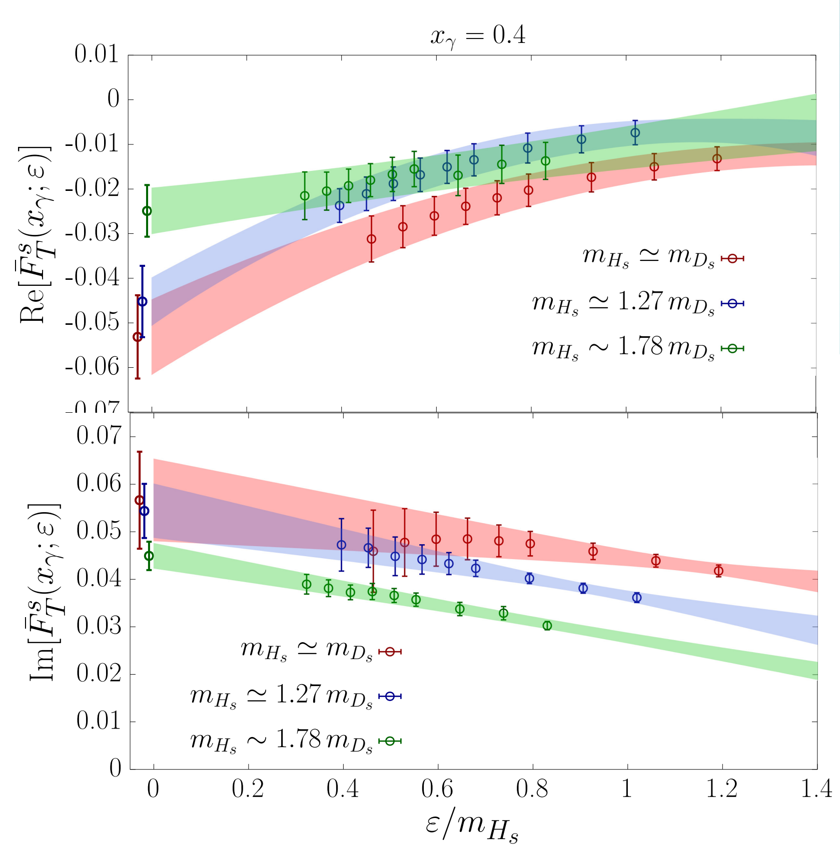


...so we decided to apply the method to compute the diagram, let me give you just a flash on it

Extrapolating in $\epsilon \to 0$ at several heavy mass

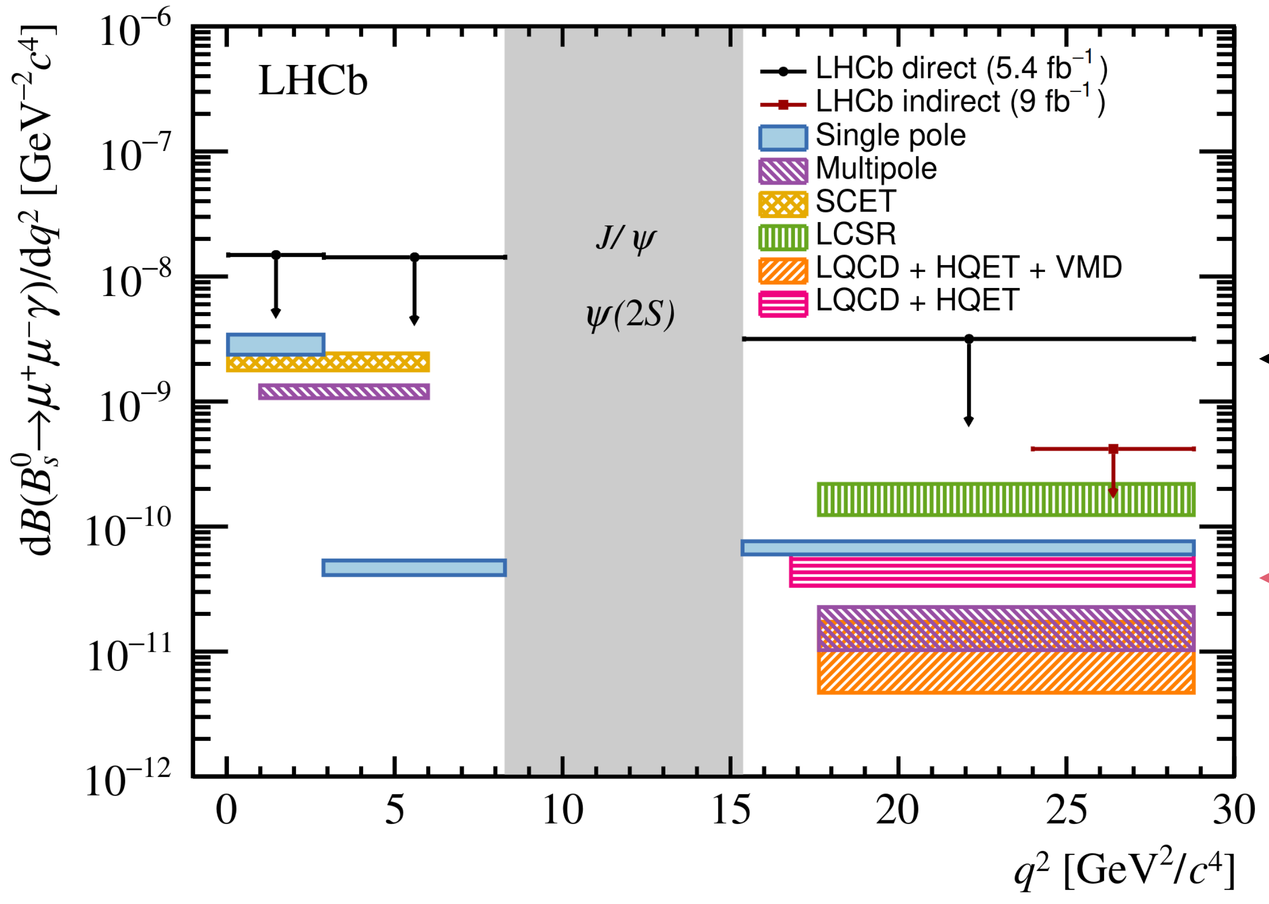




Much more controlled behaviour: no resonances present. A low degree polynomial or a simple model works well, allowing to take easily the vanishing smearing limit

A bit descouraging results from LHCb

LHCb published this year only upper limit [preprint arXiv:2404.03375]



LHCb UPPER BOUND

OUR PREDICTION

Conclusions _of this set of slides_...

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- Here applied to the case of two EW-currents, an hadronic state and the vacuum
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UPCOMING

- Continuum limit extrapolation for $K \to \bar{l}' l' l \nu_\ell$ where no narrow resonance is present
- To be presented at Lattice 2024 by Roberto Di Palma