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Towards inclusive semileptonic decays from Lattice QCD

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in collaboration with

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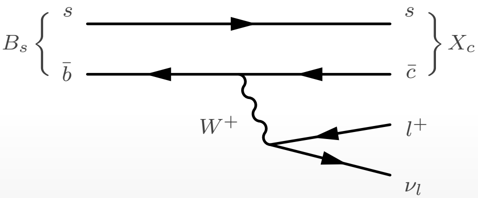
Semileptonic decays: dictionary

Focus on **weak semileptonic** $B_{(s)}$-meson decays
*All the techniques presented can be applied to similar semileptonic decay, e.g. $B_{(s)} \to X_{c/u}\, l\nu_l$ or $D_{(s)} \to X_{d/s}\, l\nu_l$

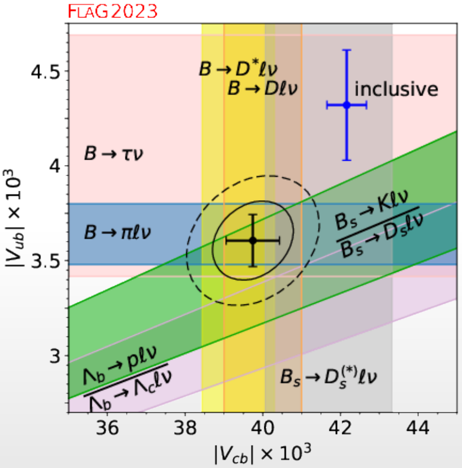


mediated by the weak hamiltonian

$$H_W = \frac{4G_F}{\sqrt{2}} V_{cb} \left[\bar{b}_L \gamma^\mu c_L\right] \left[\bar{\nu}_l \gamma_\mu l\right]$$

- **EXCLUSIVE**: $B_s \to \boldsymbol{D_s}\, l\nu_l$, with just one hadron in the final state
- **INCLUSIVE**: $B_s \to \boldsymbol{X_c}\, l\nu_l$, with multi-particle states

Introduction and motivations



[Aoki et al. (2021)[1]]

- $\sim 3\sigma$ discrepancy (in the plot) between inclusive/exclusive determination;
- lattice QFT represents a fully non-perturbative theoretical approach to QCD;
- no current predictions from lattice QCD for the inclusive decays.

This talk: Pilot study $B_s \to X_c l \nu_l$ [Barone et al. (2023)[2]]

- status and strategies for inclusive decays on the lattice;
- future directions.

Outline of the talk

Inclusive decays for the decay rate:

- discussion of the strategy ($\sim$ addressing the inverse problem)
- comparison of two methods for the analyisis $\begin{cases} \text{Chebyshev-polynomial approach} \\ \text{modified Backus-Gilbert (HLT)} \end{cases}$
- some technical details on the Chebyshev-polynomial approach

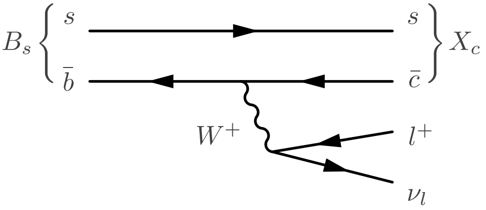
Steps towards studies to:

- understand the systematics (Chebyshey-polynomial approach)
- (address finite-volume effects)

Related directions:

- ground-state limit (and P-waves)
- kinematic moments and comparison with continuum approaches (e.g. OPE)

Differential decay rate



$$\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2\mathrm{d}q_0\mathrm{d}E_l} = \frac{G_F^2|V_{cb}|^2}{8\pi^3}\boxed{L_{\mu\nu}}\boxed{W^{\mu\nu}},$$

Leptonic tensor

Hadronic tensor

$$\boxed{W^{\mu\nu}} = \sum_{X_c}(2\pi)^3\delta^{(4)}(p-q-r)\frac{1}{2E_B}\langle B_s(\boldsymbol{p})|\, J^{\mu\dagger}\, |X_c(\boldsymbol{r})\rangle \, \langle X_c(\boldsymbol{r})|\, J^{\nu}\, |B_s(\boldsymbol{p})\rangle \, .$$

contains all the **non-perturbative** QCD

Hadronic tensor

The Hadronic tensor can be decomposed into 5 **Lorentz invariant structure functions**

$$W_i(q^2, v\cdot q) = W_i(q^2, \omega)\,, \qquad \omega = E_{X_c}\,,$$

$$W^{\mu\nu} = -g^{\mu\nu}\boxed{W_1} + v^\mu v^\nu \boxed{W_2} - i\epsilon^{\mu\nu\alpha\beta}v_\alpha q_\beta \boxed{W_3} + q^\mu q^\nu \boxed{W_4} + (v^\mu q^\nu + v^\nu q^\mu)\boxed{W_5}\,.$$

contribute to total decay rate

disappear after integration over E_l (massless limit)

relevant only for τ i.e. $m_l \neq 0$

Total decay rate

$$\boxed{\Gamma = \frac{G_F^2 |V_{cb}|^2}{24\pi^3} \int_0^{\boldsymbol{q}^2_{\max}} \mathrm{d}\boldsymbol{q}^2 \, \sqrt{\boldsymbol{q}^2} \, \bar{X}(\boldsymbol{q})^2}\,,$$

kinematics

$$\boxed{\bar{X}(\boldsymbol{q}^2)} = \int_{\omega_{\min}}^{\omega_{\max}} \mathrm{d}\omega \, \boxed{k_{\mu\nu}} \times \boxed{W^{\mu\nu}} = \sum_{l=0}^{2} \int_{\omega_{\min}}^{\omega_{\max}} \mathrm{d}\omega X^{(l)}(\boldsymbol{q}, \omega)\,, \quad \omega = E_{X_c}$$

portal to compute the $\Gamma/|V_{cb}|^2$

from lattice?

$$X^{(0)} = \boldsymbol{q}^2 W_{00} + \sum_i (q_i^2 - \boldsymbol{q}^2) W_{ii} + \sum_{i \neq j} q^i W_{ij} q^j\,,$$

$$X^{(1)} = -q_0 \sum_i q^i (W_{0i} + W_{i0})\,,$$

$$X^{(2)} = q_0^2 \sum_i W_{ii}\,.$$

Inclusive decays on the lattice

[Hansen et al. (2017)[3], Hashimoto (2017)[4], Gambino and Hashimoto (2020)[5]]

We need the non-perturbative calculation of the hadronic tensor

$$W^{\mu\nu}(\boldsymbol{q},\omega) \sim \sum_{X_c} \langle B_s| J^{\mu\dagger} |X_c\rangle \langle X_c| J^{\nu} |B_s\rangle \,.$$

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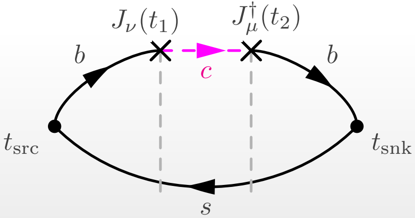
On the lattice, this is achieved with a **4pt correlation function**:



- $t_{\rm src}$, t_2, $t_{\rm snk}$ fixed
- $t_{\rm src} \le t_1 \le t_2$
- $t = t_2 - t_1$
- t small $\rightarrow$ signal-to-noise ratio deteriorate with t

$$\boxed{C^{\mu\nu}(t) \simeq \frac{C^{\mu\nu}_{\rm 4pt}(t_{\rm snk}, t_2, t_1, t_{\rm src})}{C_{\rm 2pt}(t_{\rm snk}, t_2) C_{\rm 2pt}(t_1, t_{\rm src})} \quad \leftrightarrow \quad \langle B_s|\tilde{J}^{\mu\dagger}(\boldsymbol{q},0)\, e^{-t\hat{H}}\, \tilde{J}^{\nu}(\boldsymbol{q},0)|B_s\rangle}\,.$$

Lattice data (Euclidean)

$$0 \quad 1 \quad 2 \quad \cdots \quad T \longrightarrow t/a$$

finite/discrete number of
time-slices $t = -i\tau$

lattice data
(correlation function)

$$\boxed{C(t)} = \int_0^\infty \mathrm{d}\omega \, \boxed{\rho(\omega)} \, e^{-\omega t}$$

spectral function:
$\sum_j \langle 0| \mathcal{O} |j\rangle \langle j| \mathcal{O}^\dagger |0\rangle \, \delta(\omega - E_j)$

$$\rho(\omega) \underset{\text{ill-posed problem}}{\overset{\text{trivial}}{\rightleftarrows}} C(t)$$

Lattice data (Euclidean)

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$$\rho(\omega) \quad \underset{\text{ill-posed problem}}{\overset{\text{trivial}}{\rightleftarrows}} \quad C(t)$$

Extracting the hadronic tensor is an ill-posed problem (**inverse problem**)

lattice data for inclusive $\leftarrow$

$$\boxed{C_{\mu\nu}(t)} = \int_0^\infty \mathrm{d}\omega \, \boxed{W_{\mu\nu}(\boldsymbol{q}, \omega)} \, e^{-\omega t}$$

$\sum_{X_c} \langle B_s | J_\mu^\dagger | X_c \rangle \langle X_c | J_\nu | B_s \rangle \, \delta(\omega - E_{X_c})$

Decay rate from lattice data

kinematics factors

$$\bar{X}(\boldsymbol{q}^2) = \int_{\omega_{\min}}^{\omega_{\max}} \mathrm{d}\omega\, W^{\mu\nu} \boxed{k_{\mu\nu}(\boldsymbol{q},\omega)}$$

$$= \int_{\boxed{\omega_0}}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} \boxed{k_{\mu\nu}(\boldsymbol{q},\omega)\theta(\omega_{\max}-\omega)} \rightarrow \text{kernel operator}$$

$0 \leq \omega_0 \leq \omega_{\min}$

$$= \int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} K_{\mu\nu}(\boldsymbol{q},\omega)$$

Here we are NOT extracting the hadronic tensor $W_{\mu\nu}$!
We are addressing directly the integral $\bar{X}$ using techniques common to a typical inverse problem.

To extract $W_{\mu\nu}(\boldsymbol{q},\bar{\omega})$ we would replace $K_{\mu\nu}(\boldsymbol{q},\omega) \rightarrow \delta(\omega-\bar{\omega})$

Decay rate from lattice data

kinematics factors

$$
\begin{aligned}
\bar{X}(\boldsymbol{q}^2) &= \int_{\omega_{\min}}^{\omega_{\max}} \mathrm{d}\omega \, W^{\mu\nu} \boxed{k_{\mu\nu}(\boldsymbol{q}, \omega)} \\
&= \int_{\boxed{\omega_0}}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \boxed{k_{\mu\nu}(\boldsymbol{q}, \omega)\theta(\omega_{\max} - \omega)} \longrightarrow \text{kernel operator} \\
&= \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} K_{\mu\nu}(\boldsymbol{q}, \omega)
\end{aligned}
$$

$0 \leq \omega_0 \leq \omega_{\min}$

Can we trade

$$
\int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} K_{\mu\nu}(\boldsymbol{q}, \omega) \quad \leftarrow \textbf{?} \rightarrow \quad \boxed{C^{\mu\nu}(t)} = \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} e^{-\omega t}
$$

lattice data

Decay rate from lattice data

kinematics factors

$$\bar{X}(\boldsymbol{q}^2) = \int_{\omega_{\min}}^{\omega_{\max}} \mathrm{d}\omega\, W^{\mu\nu} \boxed{k_{\mu\nu}(\boldsymbol{q},\omega)}$$

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$0 \leq \omega_0 \leq \omega_{\min}$

$$= \int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} K_{\mu\nu}(\boldsymbol{q},\omega)$$

We can approximate $K_{\mu\nu}$ with a polynomial in $e^{-\omega}$ (here lattice units $a = 1$)

$$K_{\mu\nu} \simeq c_{\mu\nu,0} + c_{\mu\nu,1} e^{-\omega} + \cdots + c_{\mu\nu,N} e^{-\omega N},$$

$$\Rightarrow \bar{X} \simeq c_{\mu\nu,0} \underbrace{\int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu}}_{C^{\mu\nu}(0)} + c_{\mu\nu,1} \underbrace{\int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} e^{-\omega}}_{C^{\mu\nu}(1)} + \cdots + c_{\mu\nu,N} \underbrace{\int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} e^{-\omega N}}_{C^{\mu\nu}(N)}.$$

Polynomial approximation strategies

$$K(\omega) : [\underbrace{\boxed{\omega_0}}_{\omega_0 \in [0, \omega_{\min})}, \infty) \to \mathbb{R}, \qquad K(\omega) \simeq \sum_j^N c_j \underbrace{\boxed{P_j(\omega)}}_{\text{family of polynomials in } e^{-\omega}} .$$

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$\omega_0 \in [0, \omega_{\min})$

family of polynomials in $e^{-\omega}$

Chebyshev approach (1)

Standard Chebyshev polynomials $T_k(\omega) : [-1, 1] \to [-1, 1]$

Shifted Chebyshev polynomials $\tilde{T}_k(\omega) : [\omega_0, \infty] \to [-1, 1]$

$$\Rightarrow \quad \boxed{\tilde{T}_k(\omega)} = T_k(\boxed{h(\omega)})$$

generic shifted Chebyshev

$\tilde{T}_k(\omega) \; = \; \sum_{j=0}^k \tilde{t}_j^{(k)} e^{-j\omega}$

$$h : [\omega_0, \infty] \to [-1, 1] \,, \quad \boxed{h(\omega) = Ae^{-\omega} + B} \,, \quad \begin{cases} h(\omega_0) & = -1 \\ h(\infty) & = 1 \end{cases}$$

map between domains

$$K(\omega) \simeq \frac{\tilde{c}_0}{2} + \sum_{k=1}^N \tilde{c}_k \tilde{T}_k(\omega) \,, \qquad \tilde{c}_k = \langle K, \tilde{T}_k \rangle = \int_{\omega_0}^\infty \mathrm{d}\omega \, K(\omega) \tilde{T}_j(\omega) \tilde{\Omega}(\omega) \,.$$

Polynomial approximation strategies

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$\omega_0 \in [0, \omega_{\min})$

family of polynomials in $e^{-\omega}$

HLT-like approach (2)

We minimize the functional (L_2-norm)

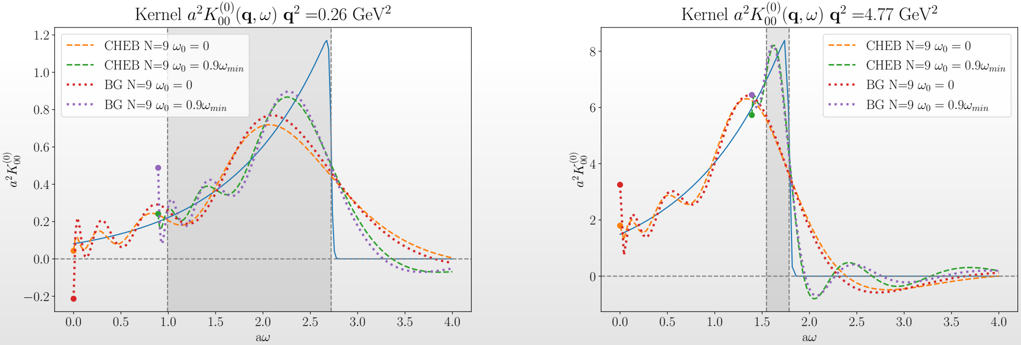
$$A[g] = \int_{\omega_0}^{\infty} \mathrm{d}\omega \left[K(\omega) - \sum_{j=1}^{N} g_j e^{-j\omega} \right]^2 ,$$

$$g_j \quad \leftrightarrow \quad \frac{\delta A}{\delta g_j} = 0\,.$$

Kernel: polynomial approximation

$$K_{\mu\nu}(\boldsymbol{q},\omega;t_0) = e^{2\omega t_0}\, k_{\mu\nu}(\boldsymbol{q},\omega) \boxed{\theta_\sigma\left(\omega_{\max}-\omega\right)} \longrightarrow$$

smooth step-function (sigmoid):
cut the unphysical states above $\omega_{\max}$,
here we fix $\sigma = 0.02$



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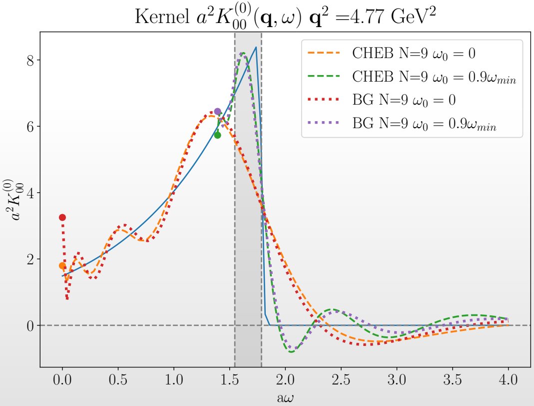

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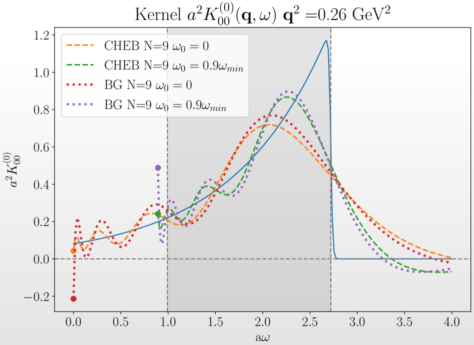
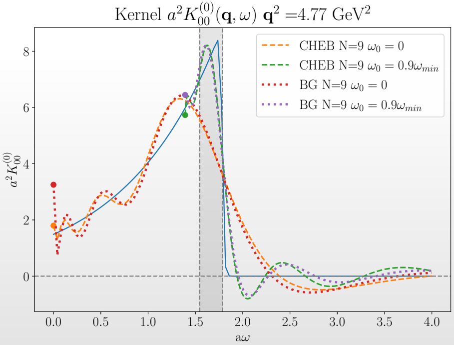


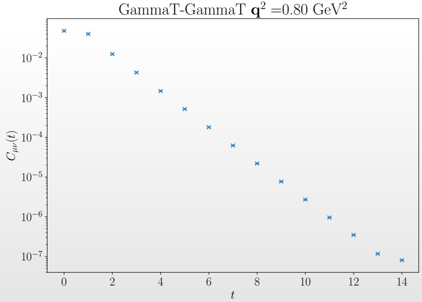


Analysis strategy

$$\bar{X}^{\text{naive}}(\boldsymbol{q}^2) = \int_{\omega_0}^{\infty} W^{\mu\nu} K_{\mu\nu}(\boldsymbol{q},\omega) = \sum_j^N c_{\mu\nu,j} \int_{\omega_0}^{\infty} W^{\mu\nu} \underbrace{\boxed{P_j(\omega)}}_{= \sum_k p_k^{(j)} e^{-k\omega}} \simeq \sum_j^N \bar{c}_{\mu\nu,j} C^{\mu\nu}(j)$$

Problems

- noise from the data adds up and error on $\bar{X}$ explodes;
- time-slice $t = 0$ must be avoided (receives contribution from both $b \to c$ and $b \to \bar{c}bb$).



Analysis strategy (2)

degree of polynomial approximation
limited by the available time-slices j

contains $e^{+2t_0\omega}$

$$\bar{X} = \int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} \boxed{K_{\mu\nu}(\omega, \boldsymbol{q}; t_0)}\, e^{-2t_0\omega} \quad \Rightarrow \quad \boxed{\bar{X} \simeq \sum_{j=0}^{\boxed{N}} \bar{c}_{\mu\nu,j} C^{\mu\nu}(j + 2t_0)}$$

- $j \leftrightarrow t$: degree corresponds to a certain time-slice, so N is limited by the available data (i.e. the choice of $t_{\mathrm{snk}} - t_{\mathrm{rsc}}$ and $t_2 - t_{\mathrm{rsc}}$) and the noise of the signal;
- we take $2t_0 = 1$, i.e. as small as possible.

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To **control the noise** we have 2 options:

Analysis strategy (2)

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degree of polynomial approximation limited by the available time-slices j

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- act on the **coefficients** $c_{\mu\nu,j}$ (HLT);

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To **control the noise** we have 2 options:

- act on the **coefficients** $c_{\mu\nu,j}$ (HLT);
- act on the **data** $C^{\mu\nu}$ (Chebyshev-polynomial approach).

Analysis strategy: Chebyshev

[Barata and Fredenhagen (1991)[6], Bailas et al. (2020)[7]]

We can expand the kernel $K_{\mu\nu}(\omega, \boldsymbol{q}; t_0)$ with shifted Chebyshev polynomials as

$$\bar{X}(\boldsymbol{q}^2) = \int_{\omega_0}^{\infty} W^{\mu\nu} K_{\mu\nu}(\boldsymbol{q}, \omega; t_0) e^{-2t_0\omega} = \sum_{j}^{N} \tilde{c}_{\mu\nu,j} \int_{\omega_0}^{\infty} W^{\mu\nu} \boxed{\tilde{T}_j(\omega)} e^{-2t_0\omega}, \qquad \tilde{T}_j(\omega) = \sum_k \tilde{t}_k^{(j)} e^{-k\omega}$$

where

$$\int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \tilde{T}_k(\omega) e^{-2\omega t_0} = \sum_{j=0}^{k} \tilde{t}_j^{(k)} C^{\mu\nu}(j + 2t_0) \, .$$

Chebyshev polynomials are **bounded**, so we normalize

$$\int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \tilde{T}_j(\omega) e^{-2\omega t_0} \qquad \rightarrow \qquad -1 \leq \frac{\int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \tilde{T}_j(\omega) e^{-2\omega t_0}}{\boxed{\int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \underbrace{\tilde{T}_0(\omega)}_{\equiv 1} e^{-2\omega t_0}}} \leq 1 \, .$$

$$\rightarrow C^{\mu\nu}(2t_0)$$

Analysis strategy: Chebyshev (2)

[Bailas et al. (2020)[7], Gambino and Hashimoto (2020)[5]]

The new relation is

$$\underbrace{\boxed{\langle \tilde{T}_k \rangle_{\mu\nu}}}_{\text{Chebyshev matrix elements}} \equiv \frac{\int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} \tilde{T}_j(\omega) e^{-2\omega t_0}}{\int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} \tilde{T}_0(\omega) e^{-2\omega t_0}} = \sum_{j=0}^{k} \tilde{t}_j^{(k)} \overbrace{\boxed{\frac{C_{\mu\nu}(j+2t_0)}{C_{\mu\nu}(2t_0)}}}^{\bar{C}_{\mu\nu}(j)}, \qquad |\langle \tilde{T}_j \rangle_{\mu\nu}| \leq 1\,,$$

and the value of $\bar{X}_{\mu\nu}(\boldsymbol{q}^2)$ in each channel can be obtained as (no sum over $\mu\nu$) here

$$\bar{X}_{\mu\nu}(\boldsymbol{q}^2) = C_{\mu\nu}(2t_0) \boxed{\sum_{j}^{N} \tilde{c}_{\mu\nu,j} \frac{\int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} \tilde{T}_j(\omega) e^{-2\omega t_0}}{\int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu} \tilde{T}_0(\omega) e^{-2\omega t_0}}} \rightarrow \equiv \bar{X}_{\bar{C}_{\mu\nu}}, \text{ depends on } \bar{C}_{\mu\nu}$$

We can then calculate it through the Chebyshev matrix elements as

$$\bar{X}_{\bar{C}_{\mu\nu}}(\boldsymbol{q}^2) \simeq \frac{\tilde{c}_{\mu\nu,0}}{2} + \sum_{j=1}^{N} \tilde{c}_{\mu\nu,j} \boxed{\langle \tilde{T}_j \rangle_{\mu\nu}}\,.$$

We need to determine these from the data

Chebyshev fit

The relations between data and Chebyshev matrix elements are

$$\langle \tilde{T}_k \rangle_{\mu\nu} = \sum_{j=0}^{k} \tilde{t}_j^{(k)} \bar{C}_{\mu\nu}(j)\,, \qquad \bar{C}_{\mu\nu}(k) = \sum_{j=0}^{k} \tilde{a}_j^{(k)} \langle \tilde{T}_j \rangle_{\mu\nu}$$

So far this are related by a linear transformation

$$\begin{pmatrix} \bar{C}_{\mu\nu}(0) \\ \bar{C}_{\mu\nu}(1) \\ \vdots \\ \vdots \\ \bar{C}_{\mu\nu}(N) \end{pmatrix} = \begin{pmatrix} \tilde{a}_0^{(0)} & 0 & \cdots & \cdots & 0 \\ \tilde{a}_0^{(1)} & \tilde{a}_1^{(1)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \tilde{a}_0^{(N)} & \tilde{a}_1^{(N)} & \cdots & \cdots & \tilde{a}_N^{(N)} \end{pmatrix} \begin{pmatrix} \langle \tilde{T}_0 \rangle_{\mu\nu} \\ \langle \tilde{T}_1 \rangle_{\mu\nu} \\ \vdots \\ \vdots \\ \langle \tilde{T}_N \rangle_{\mu\nu} \end{pmatrix}$$

This is not taking into account the bounds on the Chebyshev matrix elements $\langle \tilde{T}_k \rangle_{\mu\nu}$.
$\Rightarrow$ We address it through a **Bayesian fit with constraints**.

Chebyshev fit (2)

We address the extraction of the Chebyshev through a fit with the following steps

1. start from a frequentist fit
2. enforce the bounds
3. stabilise the fit augmenting the χ^2 with some priors

The χ^2 (**Maximum Likelihood**) looks like (we drop the indices $\mu\nu$ for simplicity)

$$\boxed{\chi^2} = \sum_{i,j=1}^{N} \left(\bar{C}(i) - \sum_{\alpha=1}^{i} \tilde{a}_\alpha^{(i)} \langle \tilde{T}_\alpha \rangle \right) \mathsf{Cov}_{ij}^{-1} \left(\bar{C}(j) - \sum_{\alpha=1}^{j} \tilde{a}_\alpha^{(j)} \langle \tilde{T}_\alpha \rangle \right)$$

$\longrightarrow$ frequentist approach: $\frac{\partial \chi^2}{\partial \langle \tilde{T}_\alpha \rangle} = 0 \leftrightarrow$ solving the linear system

Chebyshev fit (2)

We address the extraction of the Chebyshev through a fit with the following steps

1. start from a frequentist fit
2. enforce the bounds
3. stabilise the fit augmenting the χ^2 with some priors

The χ^2 (**Maximum Likelihood**) looks like (we drop the indices $\mu\nu$ for simplicity)

$$\chi^2 = \sum_{i,j=1}^{N} \left(\bar{C}(i) - \sum_{\alpha=1}^{i} \tilde{a}_\alpha^{(i)} \boxed{\langle \tilde{T}_\alpha \rangle} \right) \mathrm{Cov}_{ij}^{-1} \left(\bar{C}(j) - \sum_{\alpha=1}^{j} \tilde{a}_\alpha^{(j)} \boxed{\langle \tilde{T}_\alpha \rangle} \right)$$

$\equiv f(\tau_\alpha)$, τ_α are now fit parameters, $\tau_\alpha \in (-\infty, \infty)$

We can enforce the bounds substituting $\langle \tilde{T}_\alpha \rangle = f(\tau_\alpha)$ with

$$f : (-\infty, +\infty) \to [-1, 1]$$

Chebyshev fit (2)

We address the extraction of the Chebyshev through a fit with the following steps

1. start from a frequentist fit
2. enforce the bounds
3. stabilise the fit augmenting the χ^2 with some priors

The χ^2_{aug} (**Maximum a Posteriori Probability**) looks like

$$\chi^2_{\text{aug}} = \sum_{i,j=1}^{N} \left(\bar{C}(i) - \sum_{\alpha=1}^{i} \tilde{a}_\alpha^{(i)} \boxed{f(\tau_\alpha)} \right) \mathsf{Cov}_{ij}^{-1} \left(\bar{C}(j) - \sum_{\alpha=1}^{j} \tilde{a}_\alpha^{(j)} \boxed{f(\tau_\alpha)} \right) + \boxed{\chi^2_{\text{prior}}}$$

$$-1 \leq f(\tau_\alpha) \leq 1$$

gaussian prior on $\tau_\alpha \sim \mathcal{N}(\bar{\tau}_\alpha, \bar{\sigma}_\alpha)$

$\tau_\alpha \in (-\infty, \infty)$

which stabilizes the fit

Chebyshev fit (3)

In practice we choose

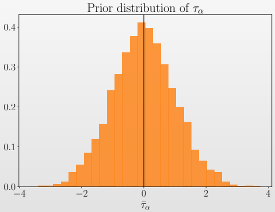
$$\chi^2_{\text{aug}} = \sum_{i,j=1}^{N} \left(\bar{C}(i) - \sum_{\alpha=1}^{i} \tilde{a}^{(i)}_{\alpha} \operatorname{erf}\left(\frac{\tau_\alpha}{\sqrt{2}}\right) \right) \mathsf{Cov}^{-1}_{ij} \left(\bar{C}(j) - \sum_{\alpha=1}^{j} \tilde{a}^{(j)}_{\alpha} \operatorname{erf}\left(\frac{\tau_\alpha}{\sqrt{2}}\right) \right) + \chi^2_{\text{prior}}$$

$$\chi^2_{\text{prior}} = \sum_{\alpha=1}^{N} \frac{(\tau_\alpha - \boxed{\bar{\tau}_\alpha})^2}{\boxed{\bar{\sigma}^2_\alpha}}, \quad \tau_\alpha \sim \mathcal{N}(0,1)$$

$\longrightarrow$ sampled from $\mathcal{N}(0,1)$ $\forall$ bootstrap bin

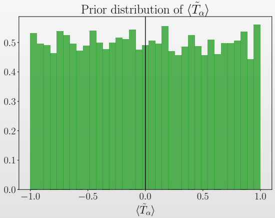
$\longrightarrow = 1$ (weak prior)

We are fitting under a bootstrap, so it's important to resample the prior for every bootstrap bin!



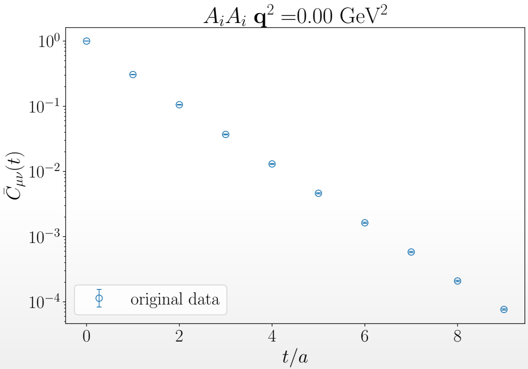
$$\longrightarrow \langle \tilde{T}_\alpha \rangle = \operatorname{erf}\left(\frac{\tau_\alpha}{\sqrt{2}}\right) \longrightarrow$$

$$\mathcal{N}(\mu,\sigma) \xrightarrow{\text{erf}} U(a,b)$$



Chebyshev fit: practical example

We consider the reconstruction of the correlator $\bar{C}_{\mu\nu}$ for the channel A_iA_i at $\boldsymbol{q}^2 = 0$



$$\bar{C}_{\mu\nu}(\boldsymbol{q},t) \rightarrow \bar{C}^{\text{fit}}_{\mu\nu}(\boldsymbol{q},t) = \sum_{j=0}^{k} \tilde{a}_j^{(k)} \langle \tilde{T}_j \rangle_{\mu\nu}$$

Chebyshev fit: practical example

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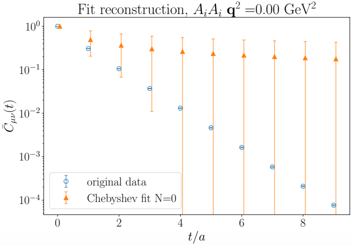


Fit N = 0 (i.e. just priors)

$$\bar{C}_{\mu\nu}(0) = \langle \tilde{T}_0 \rangle_{\mu\nu} \equiv 1$$

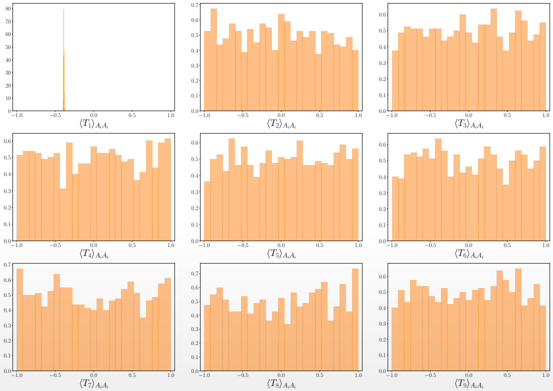
In practice we are trading our original data for a refitted version

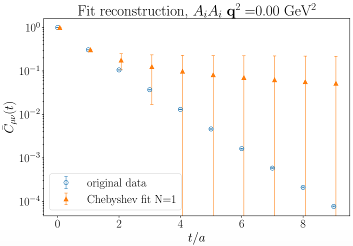
$$\bar{C}_{\mu\nu}(\boldsymbol{q},t) \rightarrow \bar{C}^{\rm fit}_{\mu\nu}(\boldsymbol{q},t) = \bar{C}_{\mu\nu}(\boldsymbol{q},t) + \boxed{\delta\bar{C}_{\mu\nu}(\boldsymbol{q},t)} \longrightarrow$$

correction that accounts for the Chebyshev bounds

Chebyshev fit: practical example

We consider the reconstruction of the correlator $\bar{C}_{\mu\nu}$ for the channel A_iA_i at $\boldsymbol{q}^2 = 0$





Fit $\mathbf{N = 1}$

$$\bar{C}_{\mu\nu}(k) = \sum_{j=0}^{k} \tilde{a}_j^{(k)} \langle \tilde{T}_j \rangle_{\mu\nu} \,, \quad 0 \leq k \leq 1$$

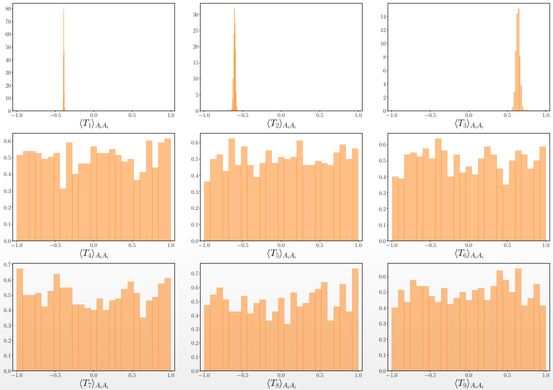
In practice we are trading our original data for a refitted version

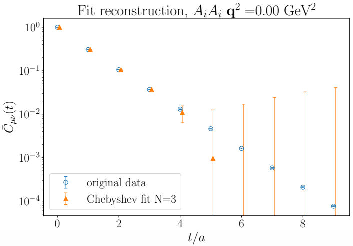
$$\bar{C}_{\mu\nu}(\boldsymbol{q},t) \to \bar{C}^{\mathrm{fit}}_{\mu\nu}(\boldsymbol{q},t) = \bar{C}_{\mu\nu}(\boldsymbol{q},t) + \boxed{\delta\bar{C}_{\mu\nu}(\boldsymbol{q},t)} \longrightarrow$$

correction that accounts for the Chebyshev bounds

Chebyshev fit: practical example

We consider the reconstruction of the correlator $\bar{C}_{\mu\nu}$ for the channel A_iA_i at $\boldsymbol{q}^2 = 0$





Fit N = 3

$$\bar{C}_{\mu\nu}(k) = \sum_{j=0}^{k} \tilde{a}_j^{(k)} \langle \tilde{T}_j \rangle_{\mu\nu}\,, \quad 0 \leq k \leq 3$$

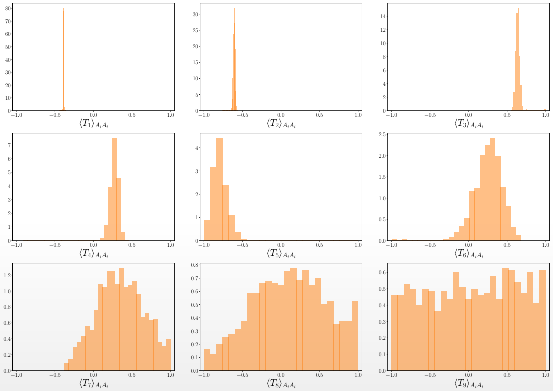
In practice we are trading our original data for a refitted version

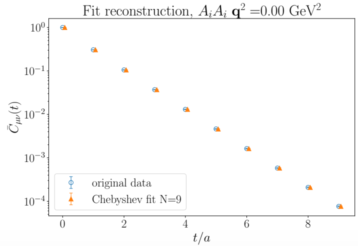
$$\bar{C}_{\mu\nu}(\boldsymbol{q}, t) \to \bar{C}^{\text{fit}}_{\mu\nu}(\boldsymbol{q}, t) = \bar{C}_{\mu\nu}(\boldsymbol{q}, t) + \boxed{\delta\bar{C}_{\mu\nu}(\boldsymbol{q}, t)} \longrightarrow$$

correction that accounts for the Chebyshev bounds

Chebyshev fit: practical example

We consider the reconstruction of the correlator $\bar{C}_{\mu\nu}$ for the channel A_iA_i at $\boldsymbol{q}^2 = 0$





Fit N = 9

$$\bar{C}_{\mu\nu}(k) = \sum_{j=0}^{k} \tilde{a}_j^{(k)} \langle \tilde{T}_j \rangle_{\mu\nu} \,, \quad 0 \le k \le 9$$

In practice we are trading our original data for a refitted version

$$\bar{C}_{\mu\nu}(\boldsymbol{q}, t) \rightarrow \bar{C}^{\text{fit}}_{\mu\nu}(\boldsymbol{q}, t) = \bar{C}_{\mu\nu}(\boldsymbol{q}, t) + \boxed{\delta\bar{C}_{\mu\nu}(\boldsymbol{q}, t)} \longrightarrow$$

correction that accounts for the Chebyshev bounds

Analysis strategy: modified Backus-Gilbert (HLT)

[Backus and Gilbert (1968)[8], Hansen et al. (2019)[9], Bulava et al. (2021)[10]]

Aside from the functional $A[g]$, which approximates the target function (kernel), we include some information on the data

$$A[g] = \int_{\omega_0}^{\infty} \mathrm{d}\omega \left[K_{\mu\nu}(\boldsymbol{q}, \omega; t_0) - \sum_{j=1}^{N} g_j \boxed{e^{-j\omega}} \right]^2 \longrightarrow \text{exponential basis}$$

$$B[g] = \sigma_{\bar{X}}^2 = \sum_{i,j=1}^{N} g_i \,\mathrm{Cov}\left[\bar{C}_{\mu\nu}(i), \bar{C}_{\mu\nu}(j)\right] g_j \,, \quad \bar{C}_{\mu\nu}(i) = \frac{C_{\mu\nu}(i + 2t_0)}{C_{\mu\nu}(2t_0)}$$

We minimise

$$\boxed{W_\lambda[g] = (1 - \lambda) \frac{A[g]}{A[0]} + \lambda B[g]} \,.$$

The parameter λ control the interplay between the 2 functionals, i.e. the balance between **statistical** and **systematic** errors.

Analysis strategy: modified Backus-Gilbert more in general

[Alexandrou et al. (2022)[11]]

We can generalise to allow the use of an arbitrary basis of polynomials $\tilde{P}_k(\omega)$ in $e^{-\omega}$

$$\tilde{P}_k(\omega) = \sum_{j=0}^{k} \tilde{p}_j^{(k)} e^{-j\omega}\,, \quad \omega \in [\omega_0, \infty)$$

such that the functionals read

$$A[g] = \int_{\omega_0}^{\infty} \mathrm{d}\omega\, \underbrace{\boxed{\tilde{\Omega}(\omega)}}_{\text{smooth weight function}} \left[K_{\mu\nu}(\boldsymbol{q}, \omega; t_0) - \sum_{j=0}^{N} g_j \underbrace{\boxed{\tilde{P}_j(\omega)}}_{\text{arbitrary basis}} \right]^2$$

$$B[g] = \sum_{k,l=1}^{N} g_k \mathrm{Cov}\left[\boxed{\bar{C}^P_{\mu\nu}(k)}, \bar{C}^P_{\mu\nu}(l) \right] g_l$$

$$\bar{C}^P_{\mu\nu}(k) = \sum_{j=0}^{k} \tilde{p}_j^{(k)} \bar{C}_{\mu\nu}(j)$$

Analysis strategy: Backus-Gilbert generalised (2)

Why generalising? The solution of the Backus-Gilbert problem with $\lambda = 0$ is given by

$$\boldsymbol{A} \cdot \boldsymbol{g} = \boldsymbol{K} \quad \leftrightarrow \quad \boldsymbol{g} = \boldsymbol{A}^{-1} \cdot \boldsymbol{K}$$

with

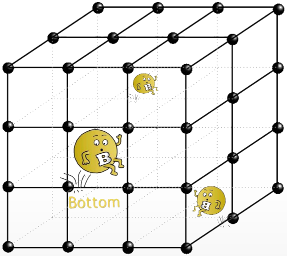
$$A_{ij} = \int_{\omega_0}^{\infty} \mathrm{d}\omega \, \tilde{\Omega}(\omega) \tilde{P}_i(\omega) \tilde{P}_j(\omega) \, ,$$
$$K_i = \int_{\omega_0}^{\infty} \mathrm{d}\omega \, \tilde{\Omega}(\omega) \tilde{P}_i(\omega) K(\omega)$$

In general the the matrix $\boldsymbol{A}$ is ill-conditioned and its inverse requires arbitrary precision. If we choose the $\tilde{\Omega}$ and $\tilde{P}_j = \tilde{T}_j$ from the Chebyshev we can take advantage of their orthogonality property such that $\boldsymbol{A}$ is diagonal! The solution for $\lambda \neq 0$ is

$$\boldsymbol{g}_\lambda = \boldsymbol{W}_\lambda^{-1} \cdot \boldsymbol{K} \, , \quad \boldsymbol{W}_\lambda = (1-\lambda)\boldsymbol{A} + \lambda A[0] \boldsymbol{B}$$

Inclusive decays on the lattice: setup

Simulations carried out on the DiRAC Extreme Scaling service at the University of Edinburgh using the **Grid** [Boyle et al.[12]] and **Hadrons** [Portelli et al.[13]] software packages



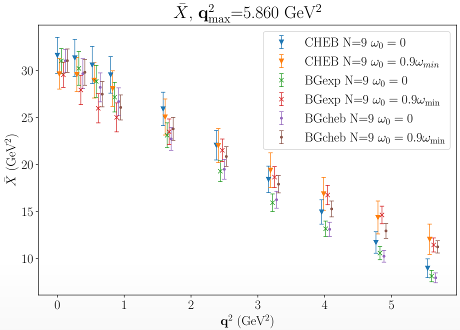
Pilot study with RBC/UKQCD ensembles [Allton et al. (2008)[14]]:

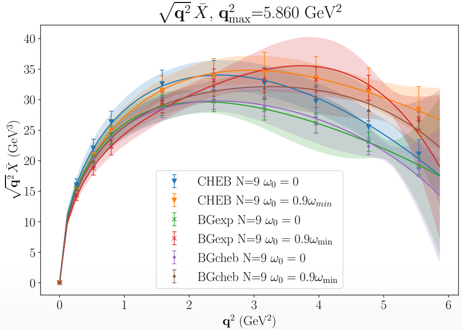
- lattice $24^3 \times 64$;
- lattice spacing $a^{-1} = 1.79\,$GeV;
- $M_\pi \simeq 330$ MeV;
- 120 gauge configurations, 8 sources;
- 8+2 momenta (Twisted BC).

Simulation:

- RHQ action for b quark [El-Khadra et al. (1997)[15], Christ et al. (2007)[16], Lin and Christ (2007)[17]]:
  - based on clover action with anisotropic terms;
  - 3 parameters non-perturbatively tuned to remove higher order discretization errors;
  - b quark simulated at its **physical** mass;
- DWF action for s, c quarks with **near-to-physical** mass.

Results and comparison





Key points:

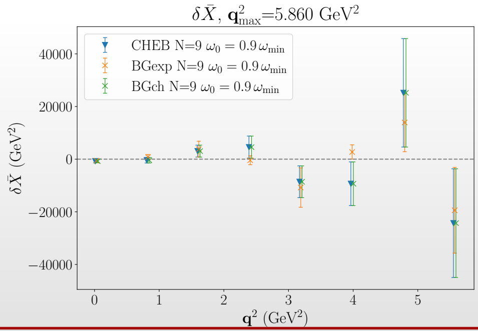
- Chebyshev and Backus-Gilbert approaches are fully compatible;
- pilot study:
  - values are in the right ballpark (compared to B decay rate, based on $SU(3)$ flavour symmetry);
  - low statistics, roughly $5-10\%$ error.

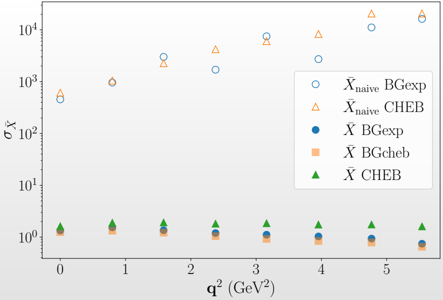
First interpretation of the two methods

Recalling $\bar{X}(\boldsymbol{q^2}) = C^{\mu\nu}(2t_0)\bar{X}_{\bar{C}_{\mu\nu}}$ we can interpret the two methods as

$$\bar{X}_{\bar{C}_{\mu\nu}} = \bar{X}^{\text{naive}}_{\bar{C}_{\mu\nu}} + \delta\bar{X}_{\bar{C}_{\mu\nu}}\,, \quad \begin{cases} \delta\bar{X}^{CHEB}_{\bar{C}_{\mu\nu}} & = \sum_{k=0}^{N} \tilde{c}_{\mu\nu,k}\,\delta\bar{C}_{\mu\nu}(k)\,, \\ \delta\bar{X}^{BG}_{\bar{C}_{\mu\nu}} & = \sum_{k=0}^{N} \delta g_{\mu\nu,k}\,\bar{C}_{\mu\nu}(k) \end{cases}$$

i.e. a "naive" piece, where we just blindely apply the polynomial approximation, and a correction term, which is essentially a noisy zero that takes care of the variance reduction.





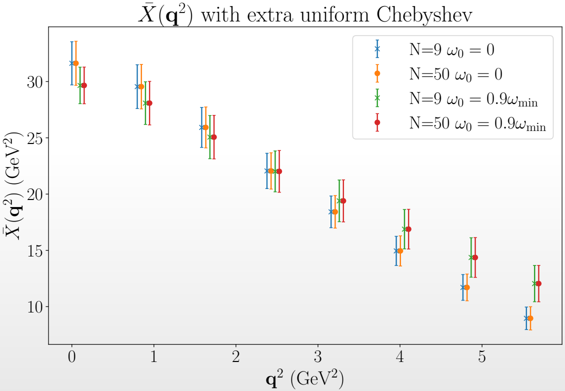
Systematic errors: polynomial approximation (1)

1) Truncation error of the polynomial approximation

Limited by the available time-slices of $C_{\mu\nu}$, here $N = 9$

$$\bar{X}(\boldsymbol{q}^2) = C_{\mu\nu}(2t_0)\left[\sum_{j=0}^{\boxed{N}} \tilde{c}_{\mu\nu,j}\langle\tilde{T}_j\rangle_{\mu\nu} + \boxed{\sum_{j=N+1}^{\infty} \tilde{c}_{\mu\nu,j}\langle\tilde{T}_j\rangle_{\mu\nu}}\right]$$

We add extra term assuming $\langle\tilde{T}_j\rangle_{\mu\nu} \sim U(-1,1)$



$\Rightarrow$ plot suggests that the truncation error is mild (with the current statistical precision).

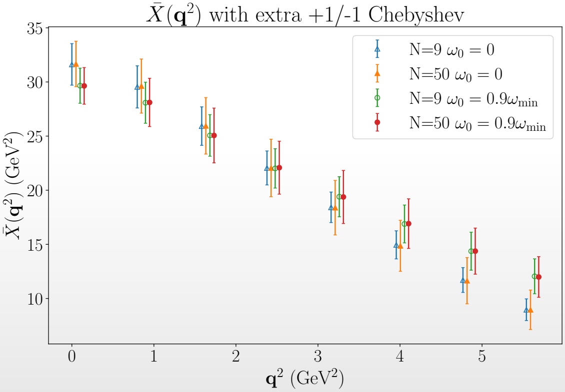
Systematic errors: polynomial approximation (1)

1) Truncation error of the polynomial approximation (more aggressive)

Limited by the available time-slices of $C_{\mu\nu}$, here $N = 9$

$$\bar{X}(\boldsymbol{q}^2) = C_{\mu\nu}(2t_0)\left[\sum_{j=0}^{\boxed{N}} \tilde{c}_{\mu\nu,j}\langle \tilde{T}_j\rangle_{\mu\nu} + \boxed{\sum_{j=N+1}^{\infty} \tilde{c}_{\mu\nu,j}\langle \tilde{T}_j\rangle_{\mu\nu}}\right]$$

We add extra term assuming $\langle \tilde{T}_j\rangle_{\mu\nu} \in \{-1, 1\}$

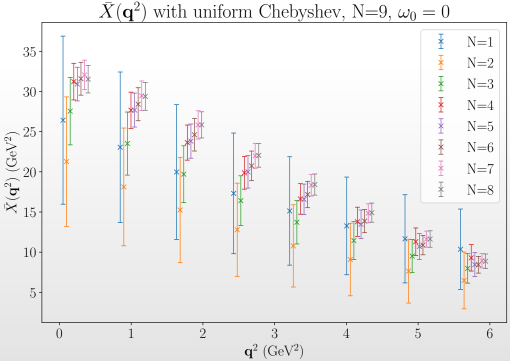


$\Rightarrow$ plot suggests that the truncation error is mild (with the current statistical precision)

Systematic errors: polynomial approximation (1)

To understand the saturation:

- we take a priori all the (N=9) Chebyshev to be uniform (result is expected to be correct, but noisy);
- we introduce the actual Chebyshev determined from the fit step by step (starting from the lowest degrees) to see how the situation changes.

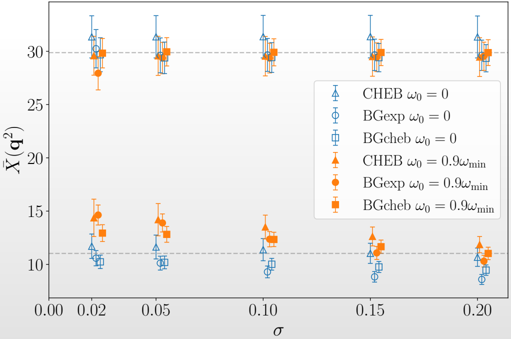


Systematic errors: polynomial approximation (2)

2) Limit $\sigma \to 0$ **of the sigmoid** $K_{\sigma,\mu\nu}(\boldsymbol{q},\omega;t_0) \propto \theta_\sigma(\omega_{\max}-\omega)$

$$\bar{X}(\boldsymbol{q}^2) = \lim_{\sigma\to 0}\left(\lim_{V\to\infty}\right)\bar{X}_\sigma(\boldsymbol{q}^2)$$

The order of the limit does not commute. Here only one volume, so the second limit is neglected.

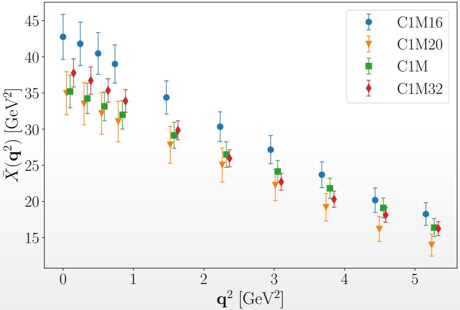


upper: $\boldsymbol{q}^2 = 0.26\text{GeV}^2$
lower: $\boldsymbol{q}^2 = 4.77\text{GeV}^2$

$\Rightarrow$ here very mild σ dependence for larger $\boldsymbol{q}^2$ ($N = 9$ is small, hence the quality of the reconstruction for different σ is not strongly affected)
NB: limit must be taken with care together with $N \to \infty$

A glimpse at possible finite-volume effects

To address the volume limit we need to take into account finite volume effects. Here, we address those directly repeating the computation on a set ensemble $L^3 \times 64$ differring only by the volume $L = 16, 20, 24, 32$



The data suggests mild dependence, BUT we cannot really resolve it with the current statistical precision. We can also try to model them analytically (case $D_s \to X\, l\nu \Rightarrow$ Ryan Kellermann's talk @Lattice2024)

Ground-state limit

We can consider the limit where only the ground state dominates, i.e.

$$W_{\mu\nu} \quad \to \quad \delta(\omega - E_{D_s^{(*)}}) \frac{1}{4M_{B_s} E_{D_s^{(*)}}} \langle B_s | J_\mu^\dagger | D_s^{(*)} \rangle \langle D_s^{(*)} | J_\nu | B_s \rangle$$

If we decompose

$$\bar{X} = \bar{X}^{\parallel} + \bar{X}^{\perp}$$

and restrict to the **vector currents (VV)** the matrix element can be decomposed as

$$\langle D_s | V_\mu | B_s \rangle = f_+(q^2)(p_{B_s} + p_{D_s})_\mu + f_-(q^2)(p_{B_s} - p_{D_s})_\mu$$

and we can show that

$$\bar{X}_{VV}^{\parallel} \quad \to \quad \frac{M_{B_s}}{E_{D_s}} \boldsymbol{q}^2 |f_+(q^2)|^2$$

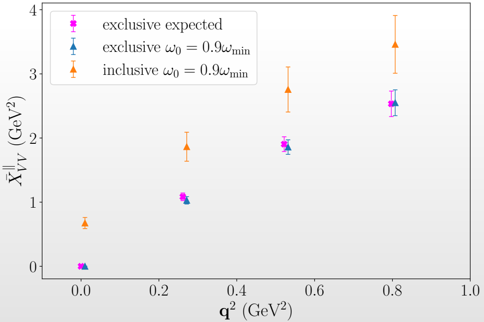
$\Rightarrow$ cross-check for the analysis

Ground-state limit: exclusive decay

The matrix element and form factors can be extracted from **3pt-correlation functions**. From that we can generate mock data for the 4pt functions (where only the ground state contributes)

$$C^G_{\mu\nu} = \frac{1}{4M_{B_s}E_{D_s}} \langle B_s| V^\dagger_\mu |D_s\rangle\langle D_s| V_\nu |B_s\rangle \, e^{-E_{D_s}t}$$

and run the analysis!



A good control over the ground states (S-waves, D_s, D_s^*) will allow us to subtract it from the 4pt correlator and then address the study of the P-wave contributions ($\Rightarrow$ Zhi Hu's talk @Lattice2024)

Moments: continuum

[with @Matteo Fael]

So far we focused on the decay rate, but we can consider other observables for inclusive decays. We consider q^2 kinematical moments

$$Q_n(q^2_{\text{cut}}) = \int_{q^2_{\text{cut}}}^{q^2_{\text{max}}} (q^2)^n \left[\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2\,\mathrm{d}q_0\,\mathrm{d}E_l}\right] \mathrm{d}q^2\,\mathrm{d}q_0\,\mathrm{d}E_l$$

It is usual to compare **centralized moments** $q_n(q^2_{\text{cut}})$ of the differential distributions, since they are more sensitive to the power corrections and independent from the value of $|V_{cb}|$:

$$\begin{aligned} q_1(q^2_{\text{cut}}) &= \left\langle q^2 \right\rangle_{q^2 \geq q^2_{\text{cut}}}, && \text{for } n = 1, \\ q_n(q^2_{\text{cut}}) &= \left\langle (q^2 - \langle q^2 \rangle)^n \right\rangle_{q^2 \geq q^2_{\text{cut}}}, && \text{for } n \geq 2. \end{aligned}$$

with

$$\langle (q^2)^n \rangle_{q^2 \geq q^2_{\text{cut}}} = \frac{Q_n}{Q_0},$$

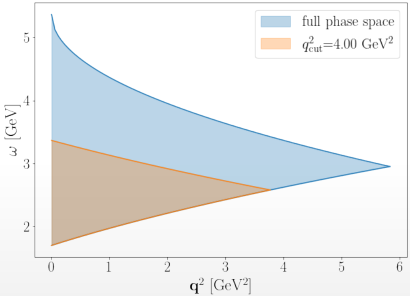
Moments: lattice

[Gambino et al. (2022)[28], Barone et al. (2024)[29]]

On the lattice we need to change the kinematic variables $(q_0, q^2) \to (\omega, \boldsymbol{q}^2)$

$$Q_n(q^2_{\rm cut}) = \int_{\boldsymbol{q}^2_{\rm cut}}^{\boldsymbol{q}^2_{\rm max}} (q^2)^n \left[\frac{\mathrm{d}\Gamma}{\mathrm{d}\boldsymbol{q}^2\,\mathrm{d}\omega\,\mathrm{d}E_l}\right] \mathrm{d}\boldsymbol{q}^2 \mathrm{d}\omega \mathrm{d}E_l\,,$$

$$\begin{cases} \boldsymbol{q}^2 & = (M_{B_s} - \omega)^2 - q^2 \\ \omega & = M_{B_s} - q_0 \end{cases}$$



And repeat the same steps as for the decay rate

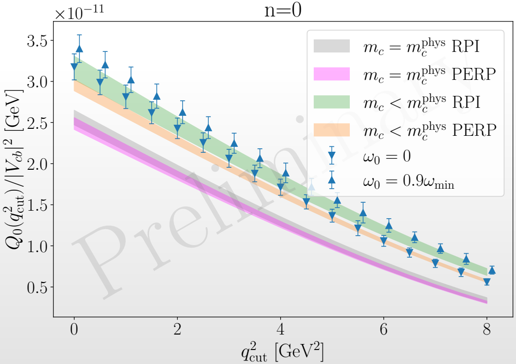
kinematics ($\to$ kernel $K^{(n)}_{Q,\mu\nu}$)

$$Q_n(q^2_{\rm cut}) = \int_{\boldsymbol{q}^2_{\rm cut}}^{\boldsymbol{q}^2_{\rm max}} \mathrm{d}\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\,\bar{X}_{Q_n}(\boldsymbol{q}^2)\,, \quad \bar{X}_{Q_n}(\boldsymbol{q}^2) = \int_{\omega_{\rm min}}^{\omega_{\rm max}} \mathrm{d}\omega \boxed{k_{Q_n,\mu\nu}} \times W^{\mu\nu}$$

Moments: lattice "VS" continuum

Continuum predictions for $B \to D\, l\nu_l$, we assume $SU(3)$-flavour symmetry. Lattice data come from only one ensemble with only close-to-physical hadrons masses:

$$M_{B_s}^{\rm PDG} = 5.367\,\text{GeV}\,, \quad M_{B_s}^{\rm lat} = 5.3670(20)\,\text{GeV}$$
$$M_{D_s}^{\rm PDG} = 1.968\,\text{GeV}\,, \quad M_{D_s}^{\rm lat} = 1.6994(11)\,\text{GeV}$$



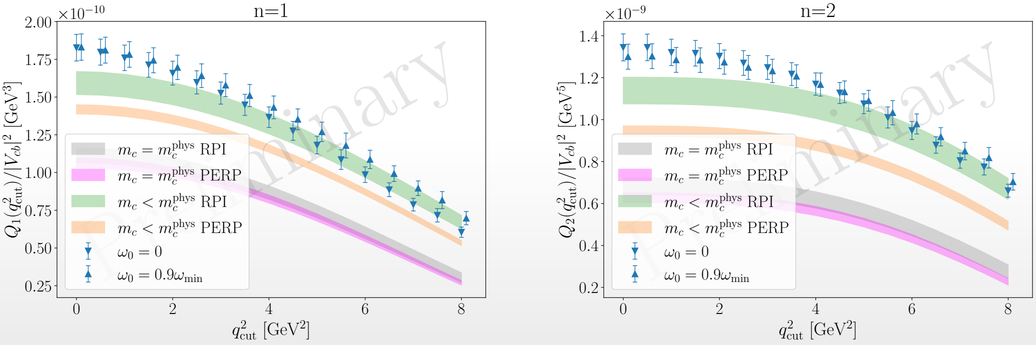
We rescale the charm mass using relations between heavy meson and heavy quarks in HQET:

$$M_{D_s} = m_c + \overline{\Lambda} + \frac{\mu_\pi^2 - d_H/2\mu_G^2}{2m_c} + O\left(\frac{1}{m_c^2}\right)$$

We consider two basis for the HQET parameters, RPI [Bernlochner et al. (2022)[27]] and PERP [Finauri and Gambino (2024)[30]]

Moments: lattice “VS” continuum

The agreement gets worse as we increase n for both RPI/PERP and lattice, but incertainty on the determination of the rescaled m_c not included.

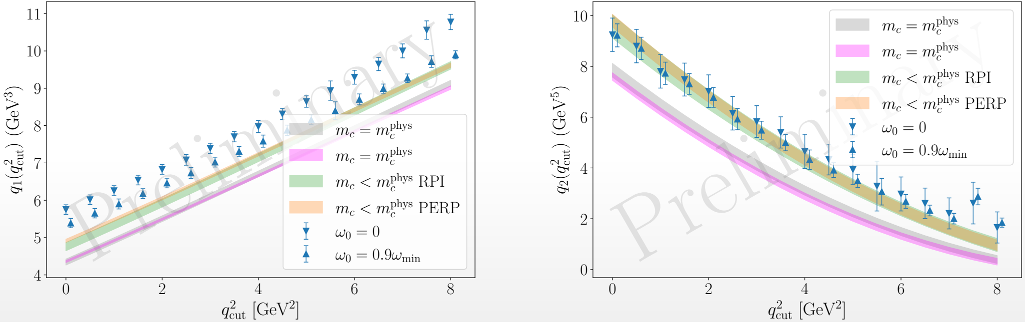


NB: on the lattice side, small q^2_{cut} contains large $\boldsymbol{q}^2$, hence larger cut-off effects .

$\Rightarrow$ better agreement expected on the tails

Centralized moments: lattice "VS" continuum

Preliminary results - feasibility study.



Lattice data (after extrapolation to the physical world) can be used to extract HQET parameters used in the OPE expansion

Summary and outlook

Summary:

- promising prospects for inclusive decays on the lattice;
- solid approach for the analysis: Chebyshev and Backus-Gilbert approaches compatible within error.

Coming next:

- continue to work towards understanding the systematics involved in solving the inverse problem;
- dedicated simulations to address the systematics for polynomial approximation, finite volume effects, continuum limit,...;
- understand better the ground state limit (compare with form factors) and address the excited states (P-waves);
- compute more observables (kinematic moments) to compare with experiments (LHCb, Belle II) and continuum approaches.

$\Rightarrow$ prepare for a full study B_s/B (and in parallel also D_s/D).

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THANK YOU!

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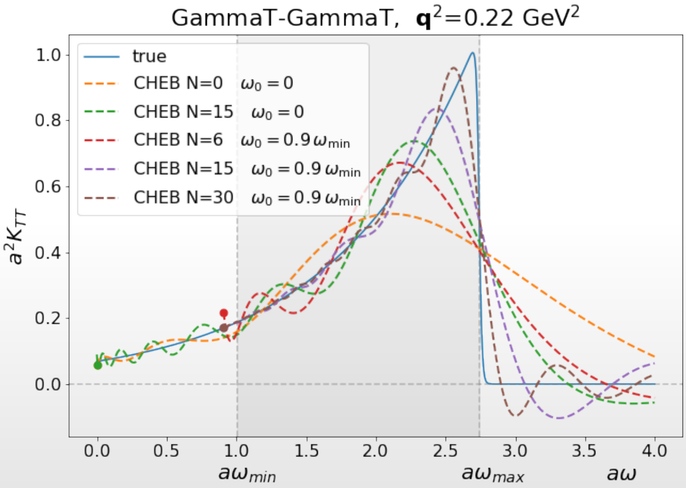
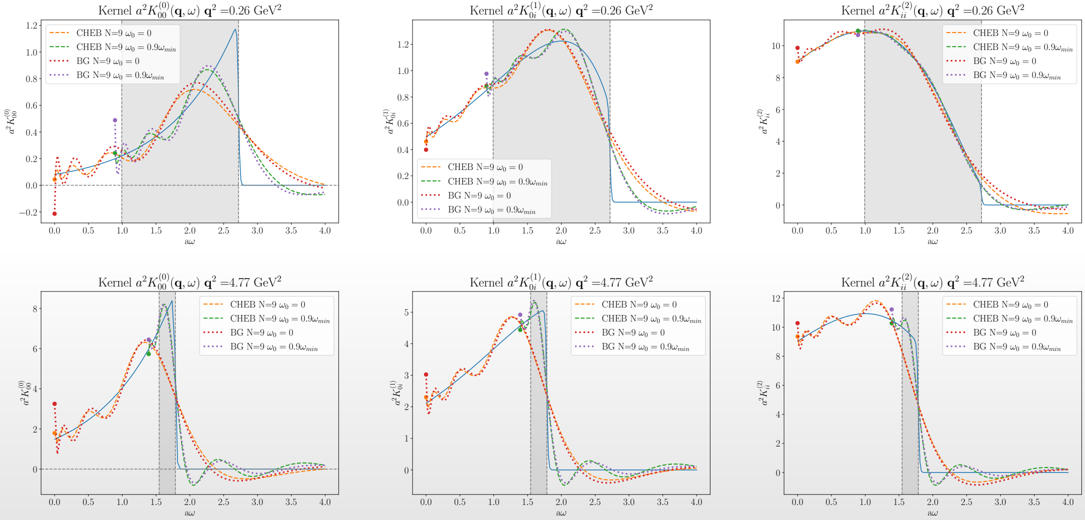
Backup

BACKUP

Chebyshev polynomial approximation: more

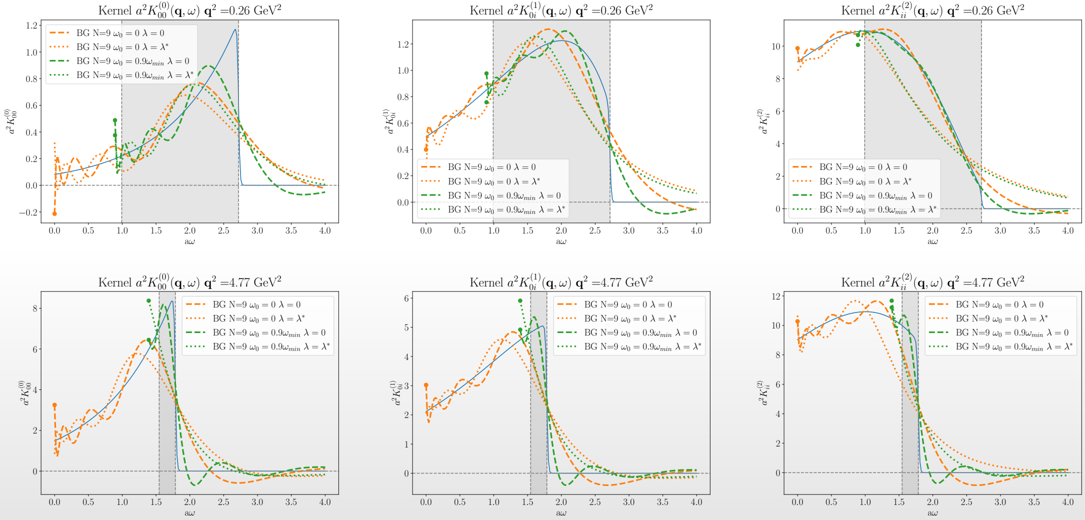


Kernels



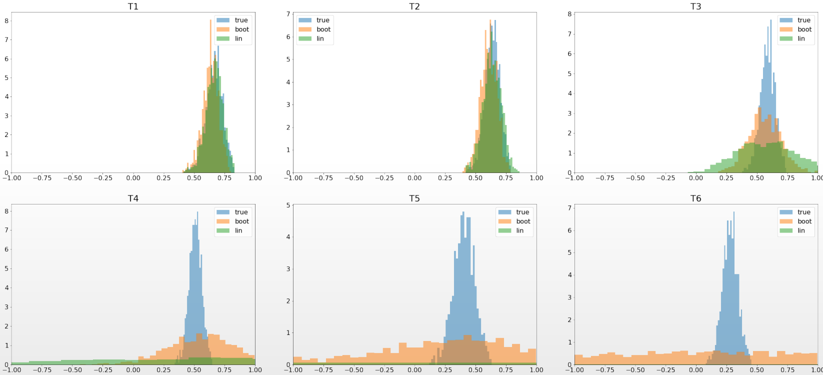
Kernel $a^2K_{00}^{(0)}(\mathbf{q},\omega)$ $\mathbf{q}^2$ =0.26 GeV^2
Kernel $a^2K_{0i}^{(1)}(\mathbf{q},\omega)$ $\mathbf{q}^2$ =0.26 GeV^2
Kernel $a^2K_{ii}^{(2)}(\mathbf{q},\omega)$ $\mathbf{q}^2$ =0.26 GeV^2
Kernel $a^2K_{00}^{(0)}(\mathbf{q},\omega)$ $\mathbf{q}^2$ =4.77 GeV^2
Kernel $a^2K_{0i}^{(1)}(\mathbf{q},\omega)$ $\mathbf{q}^2$ =4.77 GeV^2
Kernel $a^2K_{ii}^{(2)}(\mathbf{q},\omega)$ $\mathbf{q}^2$ =4.77 GeV^2
CHEB N=9 $\omega_0 = 0$
CHEB N=9 $\omega_0 = 0.9\omega_{min}$
BG N=9 $\omega_0 = 0$
BG N=9 $\omega_0 = 0.9\omega_{min}$
$a^2K_{00}^{(0)}$
$a^2K_{0i}^{(1)}$
$a^2K_{ii}^{(2)}$
$a\omega$

Kernels Backus-Gilbert

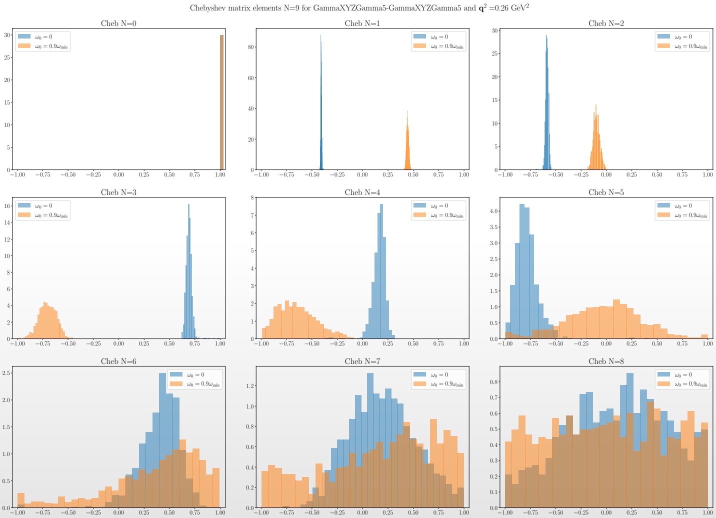


Chebyshev fit: example

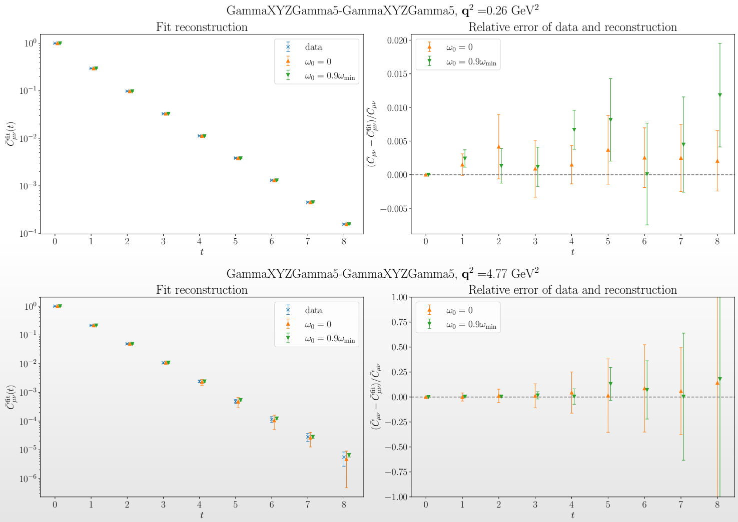
True chebyshev distribution $\rightarrow$ data $\rightarrow$ data + noise $\rightarrow$ analysis



Chebyshev data reconstruction - distribution



Chebyshev data reconstruction - data



Analysis strategy: Backus-Gilbert - a different perspective

With the previous idea we can put things in a equivalent but different perspective. We can write the coefficients as

$$g_i = \boxed{c_i} + \epsilon_i \quad \text{case } \lambda = 0$$

and require that the correction ϵ_i approximate the null function through the minimisation of $W_\lambda[\epsilon]$

$$W_\lambda[\epsilon] = (1-\lambda)A[\epsilon] + \lambda B[\epsilon]$$

$$A[\epsilon] = \int_{\omega_0}^{\infty} \mathrm{d}\omega\, \tilde{\Omega}(\omega) \left[\sum_{j=0}^{N} \epsilon_j \tilde{P}_j(\omega)\right]^2 ,$$

$$B[\epsilon] = \sum_{i,j=1}^{N} \left[2\epsilon_i \sigma_{ij}^P c_j + \epsilon_i \sigma_{ij}^P \epsilon_j\right] , \quad \sigma_{ij}^P = \text{Cov}\left[\bar{C}_{\mu\nu}^P(i), \bar{C}_{\mu\nu}^P(j)\right]$$

NB: this is equivalent to the previous case! It's just a different perspective which may give more insight in particular with the comparison with the Chebyshev case.

Analysis strategy: Backus-Gilbert constraints

[Bulava et al. (2021)[10]]

On top of that, we also include a constraint on the area:

$$\int_{\omega_0}^{\infty} \mathrm{d}\omega\, \tilde{\Omega}(\omega) \sum_{j=0}^{N} g_j \tilde{P}_j(\omega) = \int_{\omega_0}^{\infty} \mathrm{d}\omega\, \tilde{\Omega}(\omega) K_{\mu\nu}(\boldsymbol{q}, \omega)\,.$$

The value of λ can be in principle tuned arbitrarily. In practice, we choose the value of optimal balance λ^* between statistical and systematic errors with

$$W(\lambda) = W_\lambda[g^\lambda]\,, \qquad \left.\frac{\mathrm{d}W(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda^*} = 0$$

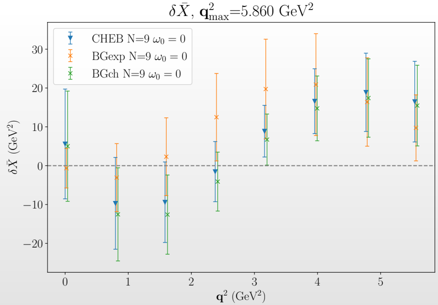
$$\Rightarrow \frac{A[g^{\lambda^*}]}{A[0]} = B[g^{\lambda^*}]$$

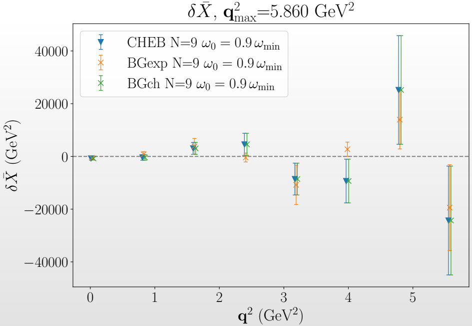
First interpretation of the two methods

Recalling $\bar{X}(\boldsymbol{q^2}) = C^{\mu\nu}(2t_0)\bar{X}_{\bar{C}_{\mu\nu}}$ we can interpret the two methods as

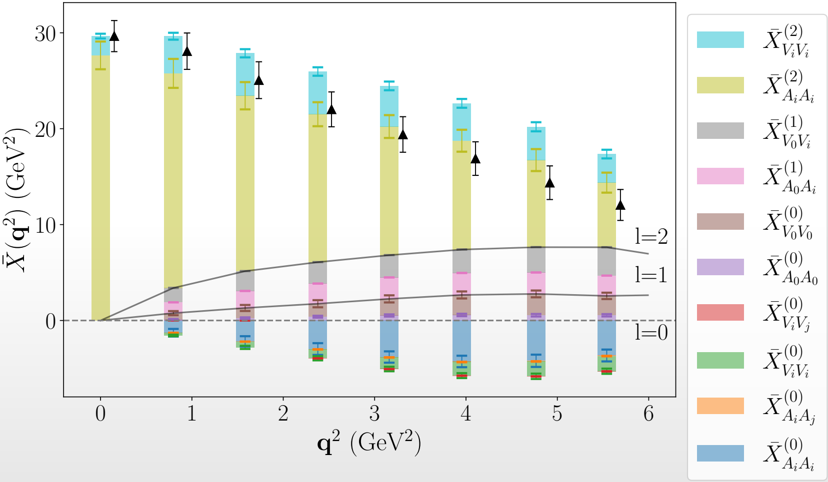
$$\bar{X}_{\bar{C}_{\mu\nu}} = \bar{X}^{\text{naive}}_{\bar{C}_{\mu\nu}} + \delta\bar{X}_{\bar{C}_{\mu\nu}}\,, \quad \begin{cases} \delta\bar{X}^{CHEB}_{\bar{C}_{\mu\nu}} & = \sum_{k=0}^{N} c_{\mu\nu,k}\delta C_{\mu\nu}(k)\,, \\ \delta\bar{X}^{BG}_{\bar{C}_{\mu\nu}} & = \sum_{k=0}^{N} \delta g_{\mu\nu,k} C_{\mu\nu}(k) \end{cases}$$

i.e. a "naive" piece, where we just blindely apply the polynomial approximation, and a correction term, which is essentially a noisy zero that takes care of the variance reduction.

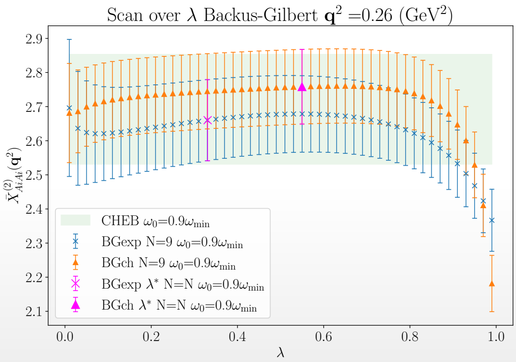


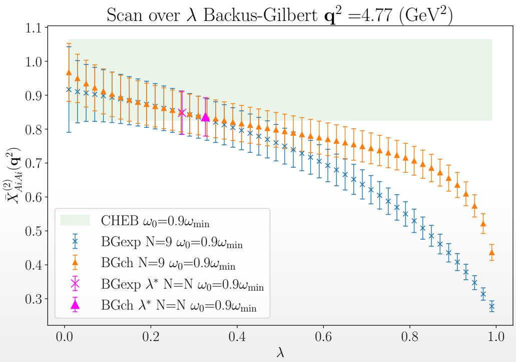


$\bar{X}$ contributions



Scan over λ (Backus-Gilbert)





Systematics errors: finite-volume effects in $D_s \to X_s\, l\nu_l$ (1)

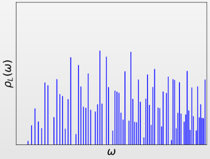
[Kellermann et al. (2024)[18]]

⚠ Digression on the study of a parallel project for $D_s \to X_s\, l\nu_l$

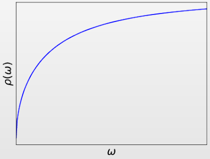
We address finite-volume effects combining lattice data and analytical model by:

- consider 2-body final states ($K\bar{K} \to$ dominant contribution)
- vary the limit of the energy integral ω, with a treshold $\omega_{\rm th} > \omega_{\max}$ to include higher energy states

We model the spectral function (here for the case $J = 0$)



$$\rho_L^{J=0}(\omega) = \frac{\pi}{V}\sum_{\boldsymbol{q}} \frac{1}{4(\boldsymbol{q}^2+m^2)}\,\delta\left(\omega - 2\sqrt{\boldsymbol{q}^2+m^2}\right) \xrightarrow{V\to\infty}$$

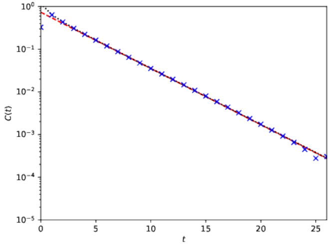


Systematics errors: finite-volume effects in $D_s \to X_s \, l\nu_l$ (2)

Given the assumption, we model the correlator to be

$$\bar{C}(t) = A_0 e^{-E_0 t} + s(L) \sum_i A_i e^{-E_i t} \frac{1}{E_i^2 - m_J^2} \,,$$

$\Rightarrow$ good agreement with data ✓ (here $\boldsymbol{q}^2 = 0$)!



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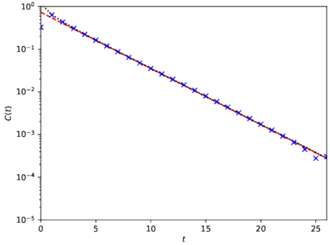
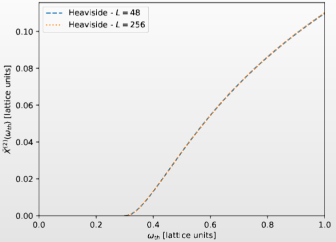
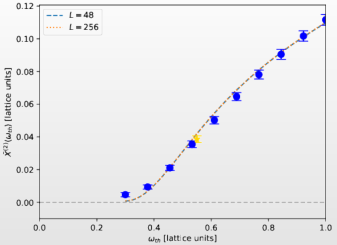


We test on $\bar{X}^{(2)}_{AA}(\boldsymbol{q}^2 = 0)$



Heaviside $\to$ smeared sigmoid

$$\bar{X}^{(2)}_{\sigma} = \int_{\omega_0}^{\omega_{\rm th}} \mathrm{d}\omega \left(\begin{matrix} \overbrace{\boxed{W_{\mu\nu}}}^{\text{lattice data}} \\ \underbrace{\boxed{\rho_L}}_{\text{model}} \end{matrix} \right) K_{\sigma,\mu\nu} \Rightarrow$$



P-wave and "1/2 versus 3/2 puzzle"

[Bigi et al. (2007)[22]]

From the study of four-point correlation function we can address the decay

$$B_s \to D_s^{**}\, l\nu_l\,, \quad D_s^{**} = \{D_{s,1}, D^*_{s,2}, D^*_{s,0}, D^*_{s,1}\}$$

$$D_s^{**} \equiv \text{P-wave} \to \quad \begin{cases} j_c^P = (1/2)^+ + j_b^P \Rightarrow J^P = (0^+, 1^+) \\ j_c^P = (3/2)^+ + j_b^P \Rightarrow J^P = (1^+, 2^+) \end{cases}$$

Motivation: the composition of B semileptonic decay is [Aubert et al. (2008)[19], Liventsev et al. (2008)[20]]

$$B \to X_c\, l\nu_l \Rightarrow X_c = \begin{cases} D^{(*)} \sim 70\% & \text{(S-wave)} \\ D_1, D_2^* \sim 15\% & (j_c^P = (3/2)^+) \\ \boxed{? \sim 15\%} \longrightarrow \text{natural candidate is } j_c^P = (1/2)^+ \end{cases}$$

BUT the proposal is in contrast with predictions from sum rules [Uraltsev (2001)[21]] $\to$ puzzle!

P-wave from inclusive data

[Becirevic et al. (2005)[24], Atoui et al. (2015)[25], Bailas et al. (2019)[26]]

How do we extract the P-wave contribution from the lattice data?

$$C_{\mu\nu}(\boldsymbol{q}, t) = \sum_X \frac{1}{4M_{B_s}E_X} \langle B_s| J_\mu^\dagger |X\rangle \langle X| J_\nu |B_s\rangle \, e^{-E_X t}$$

- control the ground-state D_s, D_s^* (S-wave) from three-point functions
- subtract this ground state from the four-point functions

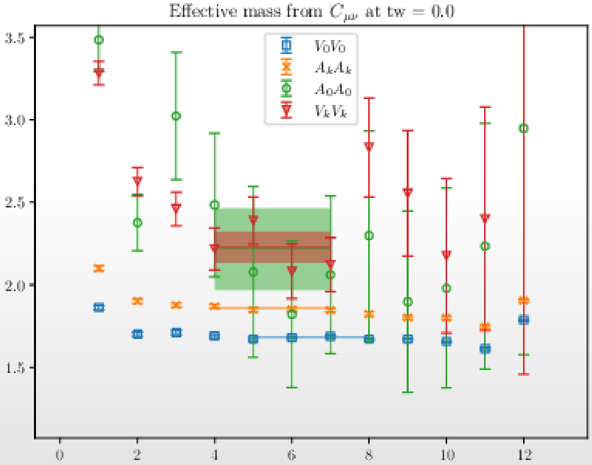
$$C_{\mu\nu}^{\mathrm{P-wave}}(\boldsymbol{q}, t) = \sum_{X=D_{s,i}^{(*)}} \frac{1}{4M_{B_s}E_X} \langle B_s| J_\mu^\dagger |X\rangle \langle X| J_\nu |B_s\rangle \, e^{-E_X t}$$

- combine different channel to disantangle as much as possible the underlying form factors (using HQET formalism) [Bernlochner and Ligeti (2017)[23]]

$$\langle D_{s,i}^{(*)}(v', \varepsilon)| J^\mu |B_s(v)\rangle = \frac{\langle D_{s,i}^{(*)}(p_{D_s}, \varepsilon)| J^\mu |B_s(p_{B_s})\rangle}{\sqrt{M_{D_s} M_{B_s}}}$$

P-wave: zero recoil example

"Easy" case: at $\boldsymbol{q}^2 = \mathbf{0}$ some channels contains only contributions from specific states.



1) EFFECTIVE MASS

S-wave

$$C_{V_0V_0}(\mathbf{0},t) = |h_+|^2 e^{-M_{D_s} t}$$

$$C_{A_kA_k}(\mathbf{0},t) = \frac{1}{4}(1+w)^2 |h_{A_1}|^2 e^{-M_{D_s^*} t}$$

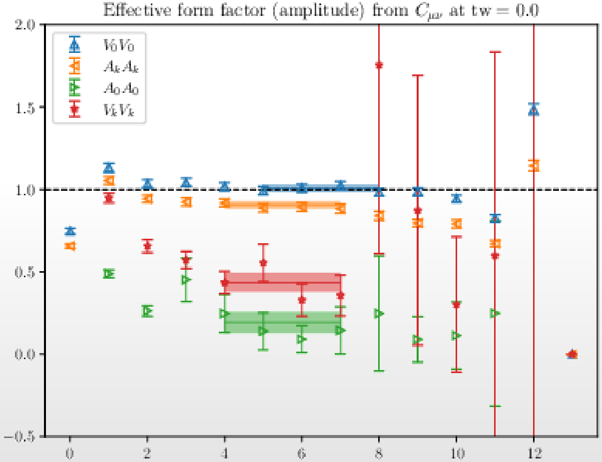
P-wave

$$C_{A_0A_0}(\mathbf{0},t) = |g_+|^2 e^{-M_{D^*_{s,0}} t}$$

$$C_{V_kV_k}(\mathbf{0},t) \simeq \frac{1}{4}|g_{V_1}|^2 e^{-M_{D^*_{s,1}} t}$$

P-wave: zero recoil example

"Easy" case: at $q^2 = 0$ some channels contains only contributions from specific states.



2) EFFECTIVE FF

S-wave

$$C_{V_0V_0}(\mathbf{0},t) = |h_+|^2 e^{-M_{D_s}t}$$

$$C_{A_kA_k}(\mathbf{0},t) = \frac{1}{4}(1+w)^2 |h_{A_1}|^2 e^{-M_{D_s^*}t}$$

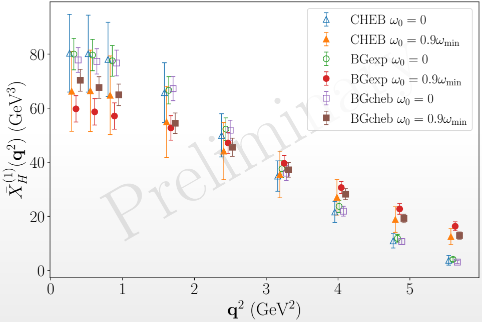
P-wave

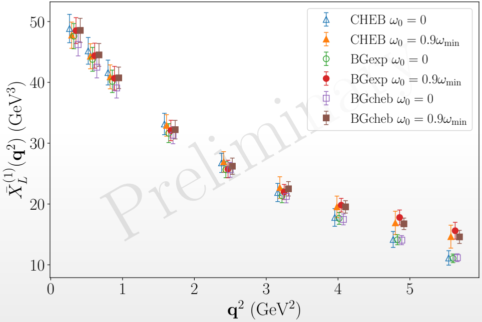
$$C_{A_0A_0}(\mathbf{0},t) = |g_+|^2 e^{-M_{D_{s,0}^*}t}$$

$$C_{V_kV_k}(\mathbf{0},t) \simeq \frac{1}{4}|g_{V_1}|^2 e^{-M_{D_{s,1}^*}t}$$

Moments

Hadronic mass and lepton energy moments.





Reference values for numerical evaluation of the HQE (1)

RPI basis from [Bernlochner et al. (2022)[27]]

For the evaluation using the RPI basis utilized in [Bernlochner et al. (2022)[27]] we have the following setup:

- We include power corrections up to $1/m_b^4$ with $r_E^4 \neq 0$ and $r_G^4 \neq 0$. We have $s_B = s_E = s_{qB} = 0$ as in the default fit of [Bernlochner et al. (2022)[27]].
  We use central values and uncertainties from Tab. 4 and the correlations from Fig. 9.
- We include the NNLO corrections in the free quark approximation and the NLO corrections to μ_G and ρ_D. This is at variance with Ref. [Bernlochner et al. (2022)[27]] where only the NLO correction in the free quark approximation were included for the moments.
- We adopt as reference values for the quark masses

$$m_b^{\mathrm{kin}}(1\,\mathrm{GeV}) = 4.562\,\mathrm{GeV}, \qquad \overline{m}_c(2\,\mathrm{GeV}) = 1.094\,\mathrm{GeV}.$$

Reference values for numerical evaluation of the HQE (1)

Historical basis (PERP basis) from [Finauri and Gambino (2024)[30]]

For the evaluation using the PERP basis utilized in [Finauri and Gambino (2024)[30]] we have the following setup:

- We include power corrections up to $1/m_b^3$ as in the default fit of [Finauri and Gambino (2024)[30]].
  We use central values, uncertainties and the correlations from Tab. 4.
- We include the NNLO corrections in the free quark approximation and the NLO corrections to μ_G and ρ_D. In [Finauri and Gambino (2024)[30]] it was included only the NNLO corrections proportional to $\alpha_s^2\beta_0$. It included the NLO corrections to μ_G and ρ_D.
- We adopt as reference values for the quark masses

$$m_b^{\rm kin}(1\,\text{GeV}) = 4.562\,\text{GeV}, \qquad \overline{m}_c(2\,\text{GeV}) = 1.094\,\text{GeV}.$$