Discussion session

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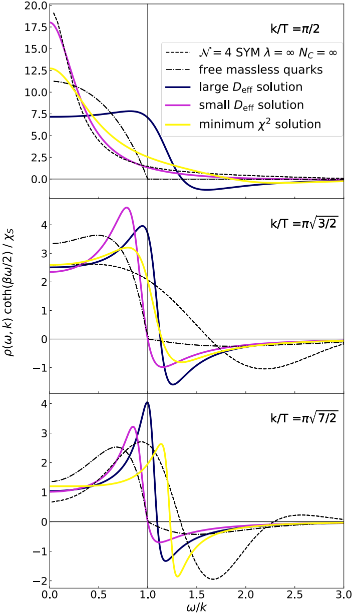
Essentially vector-vector correlator

thermal theory and zero temperature

different "basis" functions but practically similar inverse problem

don't smeare if you can: $e^{-\omega t}$ and imaginary-time correlators

localization in energy preferable depending on the context

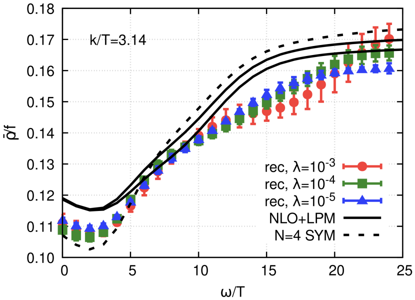


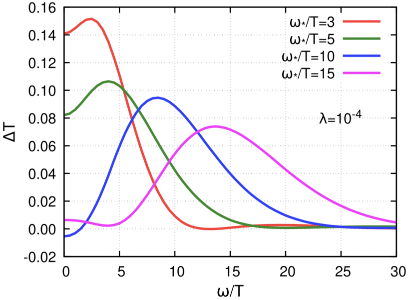
⟵ spec.funct. poorly constrained at small $k \simeq 0.4$ GeV

Representative spectral functions describing the lattice $2(T-L)$ correlator

⟵ spec.funct. better constrained at $k \approx 2\pi T \simeq 1.6$ GeV

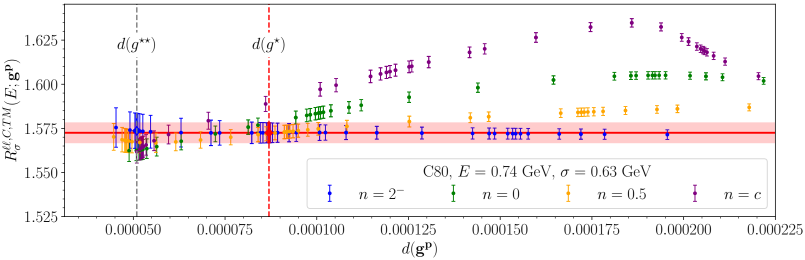
(I) Backus-Gilbert spectral function



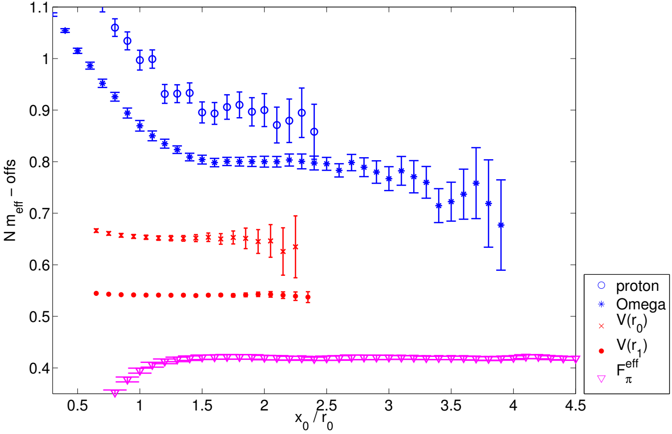


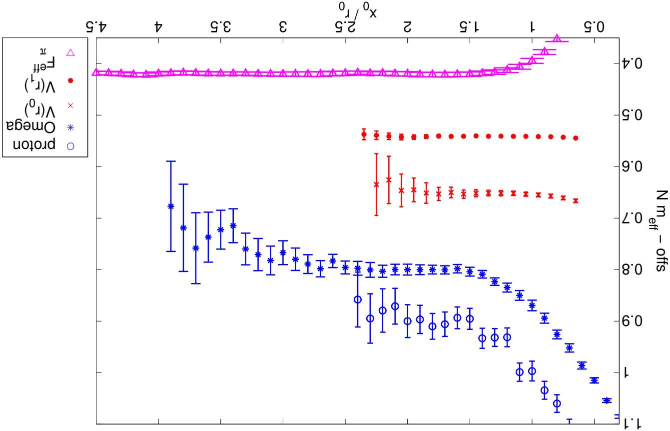
$$\frac{\bar{\rho}(\omega,k)}{f(\omega)} = \int_0^\infty d\omega' \; \Delta(\omega,\omega') \, \frac{\rho(\omega')}{f(\omega')} = \sum_i g_i(\omega) G_{\rm T}(\tau_i).$$

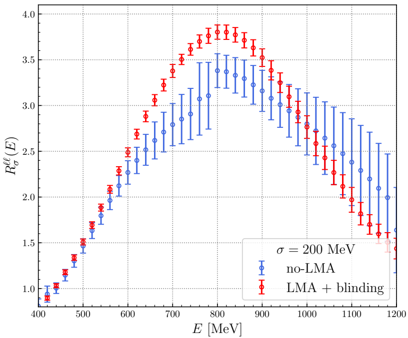
Choice made here: $f(\omega) = \frac{\omega^2}{\tanh(\beta\omega/2)}$ in order to make $\frac{\rho(\omega')}{f(\omega')}$ a 'slowly varying function', since the BG method is exact if $\frac{\rho(\omega')}{f(\omega')}$ is constant.



$d(g^{\star\star})$
$d(g^{\star})$
$R_{\sigma}^{\ell\ell,C,TM}(E;\mathbf{g}^{\mathbf{P}})$
1.625
1.600
1.575
1.550
1.525
0.000050
0.000075
0.000100
0.000125
0.000150
0.000175
0.000200
0.000225
$d(\mathbf{g}^{\mathbf{P}})$
C80, $E = 0.74$ GeV, $\sigma = 0.63$ GeV
$n = 2^-$
$n = 0$
$n = 0.5$
$n = c$


proton
Omega
V(r0)
V(r1)
Fπeff
N meff − offs
x0 / r0
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.5
1
1.5
2
2.5
3
3.5
4
4.5


x_0 / r_0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
$N\, m_{eff}$ – offs
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
proton
Omega
$V(r_0)$
$V(r_1)$
F_π^{eff}



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[MB, Giusti, Saccardi ’24]

The problem

Correlator $C(t) \leftrightarrow$ spectral densities $\rho(\omega) \leftrightarrow$ experiment

$$C(t) = \int_\omega \rho(\omega)\, e^{-\omega t}\,, \quad \text{with} \int_\omega \equiv \int_0^\infty d\omega\,.$$

Goal: calculation of $\rho(\omega)$

HVP contribution to $(g-2)_\mu$: $a_\mu^{\mathrm{HVP}} = \int dt\, K(t, m_\mu)\, C(t)$

kernel $K(t, m_\mu)$ defined in the continuum infinite-volume theory

$\rightarrow$ finite-volume effects (Lüscher analysis) [Hansen-Patella]

$\rightarrow$ (enhanced) cutoff effects (Symanzik's EFT) [Mainz][Sommer]

Note: $t \to Mt$ and $\omega \to \omega/M$

1. Fredholm integral equation
$$\int_t e^{-\omega' t}\, C(t) = \int_\omega \mathcal{H}(\omega', \omega)\, \rho(\omega) \quad \mathcal{H}(\omega, \omega') = \frac{1}{\omega + \omega'}$$

2. diagonalize $\mathcal{H}$ w/ "Mellin" basis
$$\mathcal{H}(\omega, \omega') = \int_s u_s^*(\omega)\, |\lambda_s|^2\, u_s(\omega') \quad u_s(t) = \frac{e^{is\log(t)}}{\sqrt{2\pi t}}$$

3. $\mathcal{H}$ ill-conditioned
$$|\lambda_s|^2 = \frac{\pi}{\cosh(\pi s)} \overset{s\to\pm\infty}{\to} 0$$

4. Tikhonov's regulator (other choices possible)
$$\mathcal{H}_\alpha^{-1}(\omega, \omega') = \int_s u_s^*(\omega) \frac{1}{|\lambda_s|^2 + \alpha} u_s(\omega')$$

Solution

$$\int_t e^{-\omega' t}\, C(t) = \int_\omega \mathcal{H}(\omega',\omega)\,\rho(\omega)$$

$$\begin{aligned}
\rho_\alpha(\omega) &\equiv \mathcal{H}_\alpha^{-1} \int_t e^{-\omega' t} C(t) \\
&= \int_s u_s^*(\omega) \frac{\lambda_s}{|\lambda_s|^2+\alpha} \int_t u_s^*(t)\, C(t) \\
&= \int_t \left[\int_s u_s^*(\omega) \frac{\lambda_s}{|\lambda_s|^2+\alpha}\, u_s^*(t) \right] C(t) = \int_t g_\alpha(t|\omega)\, C(t) \\
&= \int_{\omega',\omega''} \mathcal{H}_\alpha^{-1}(\omega,\omega')\, \mathcal{H}(\omega',\omega'')\, \rho(\omega) = \int_{\omega''} \delta_\alpha(\omega,\omega'')\, \rho(\omega'')
\end{aligned}$$

Inverse Laplace Transform (ILT)

$\rho(\omega) = \lim_{\alpha\to 0} \rho_\alpha(\omega)$ and $\delta(\omega-\omega'') = \lim_{\alpha\to 0} \delta_\alpha(\omega,\omega'')$

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Since $\delta_\alpha(\omega,\omega') = \int_t g_\alpha(t|\omega)e^{-\omega' t}$

coefficients $g_\alpha(t|\omega)$ minimize the functional

$$\int_\omega \left[\delta(\omega-\omega') - \delta_\alpha(\omega,\omega')\right]^2 + \alpha \int_t g_\alpha(t|\omega)^2$$

Equivalently $\rho_\alpha(\omega) = \int_t r_\alpha(t)e^{-\omega t}$

coefficients $r_\alpha(t)$ minimize the functional

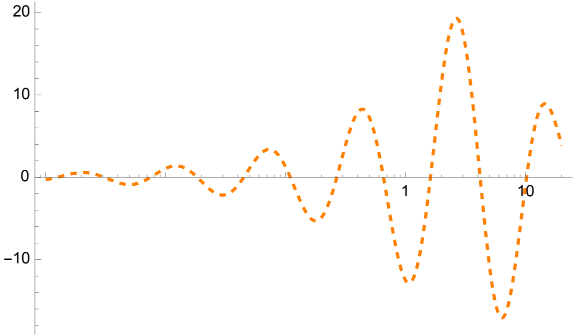
$$\int_\omega \left[\rho(\omega) - \rho_\alpha(\omega)\right]^2 + \alpha \int_t r_\alpha(t)^2$$

Smearing

$$\rho_\kappa \equiv \int_\omega \rho(\omega)\,\kappa(\omega)$$

$$\rho_{\kappa,\alpha} = \int_\omega \rho_\alpha(\omega)\,\kappa(\omega) = \int_t g_\alpha(t|\kappa)\,C(t)$$

Example



y-axis $= g_\alpha(t|\omega = 0.5)$, x-axis $= t$

Lattice

0. $\int_t \to a\sum_{t=a}$ and $\overline{C}_a(t) = C(t) + O(a^2)$

1. Fredholm integral equation becomes

$$a\sum_{t=a} e^{-\omega' t}\overline{C}_a(t) = \int_\omega \overline{\mathcal{H}}_a(\omega',\omega)\overline{\rho}_a(\omega) \quad \overline{\mathcal{H}}_a = \frac{ae^{-(\omega+\omega')a}}{1-e^{-a(\omega+\omega')}}$$

2. diagonalization $\overline{\mathcal{H}}_a$ possible!! (hypergeometric functions)

$\int_{\omega'} \overline{\mathcal{H}}_a(\omega,\omega')v_s(\omega',a) = |\lambda_s|^2 v_s(\omega,a) \quad s > 0$

3. same spectrum as continuum and

$\lim_{a\to 0} v_s(\omega,a) \propto \mathrm{Re} u_s(a\omega) + O(a^2)$

4. $L^2(0,\infty,d\omega) \neq \ell^2(\mathbb{Z}^+)$ leads to $v_s(\omega,a) \neq \overline{v}_s(t,a)$

5. new coefficients $\overline{g}_\alpha(t|\omega) = \int_{s>0} v_s(\omega,a)\frac{|\lambda_s|}{|\lambda_s|^2+\alpha}\overline{v}_s(t,a)$

Discrete ILT

Inverse Laplace Transfrom on infinite regular lattice (explicitly) solved

- no discretization errors from ILT ($\alpha \sim 0$), only from $\overline{C}_a(t)$
- $\lim_{\alpha\to 0}$ and $\lim_{a\to 0}$ commute

Gedanken experiment

- $C(t)$ w/o cutoff effects sampled at $t = an$, $n \in \mathbb{Z}^+$
  - $\alpha \to 0$ recover exact $\rho(\omega)$

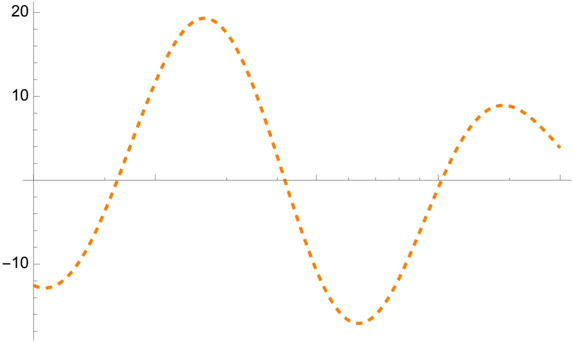
Why is that so?

- coefficients minimize distance $\int_\omega \left[\overline{\rho}_a(\omega) - \overline{\rho}_{a,\alpha}(\omega)\right]^2 + \cdots$
- ansatz $a \sum_{t=a} g_\alpha(t|\omega) C(t)$ does not minimize norm
  - presence of ordered $\lim_{\alpha\to 0} \lim_{a\to 0}$

Backus-Gilbert/HLT

HLT method but covariance $\leftrightarrow$ identity (Tikhonov)

coincide with $\overline{g}_\alpha(t|\omega)$ limit of infinite time slices



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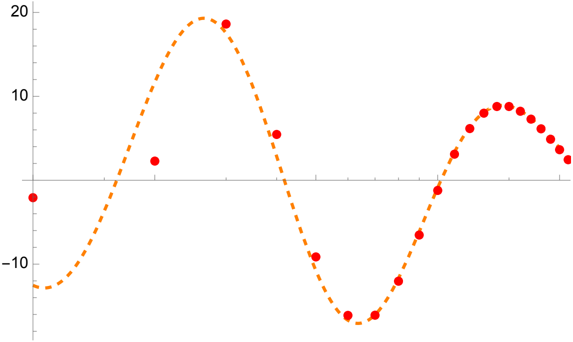

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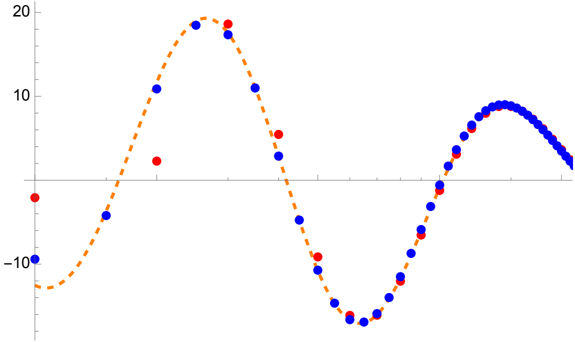

Denkpause

Continuum formulation of ILT

finite-size effects of ρ from $C(t)$

same for cutoff effects

truncation to finite temporal extent

scaling of stat errors and truncation to finite α

Discrete formulation of ILT

no additional cutoff effects beyond $\overline{C}_a(t) = C(t) + O(a^2)$

Hyphothesis: $\rho(\omega) \in L^2(0, \infty, d\omega)$
mass gap $\to$ support over $[\omega_0 > 0, \infty)$
short-distance divergences $\to \rho \simeq \omega^k \to$ cannot use $g_\alpha(t|\omega)$

Extension to QFT: change coefficients

$$g_\alpha(t|\omega, p) = \int_s u_s^*(\omega) \frac{\lambda_s}{\lambda_s \lambda_{s,p} + \alpha} u_s^*(t)\, t^p$$

p such that $t^{p-1/2} C(t)$ finite as $t \to 0$

define ρ_α from $g_\alpha(t|\omega, p)$, take $\alpha \to 0$

Example: vector correlator

$C(t) = \int_\omega e^{-\omega t} \omega^2 \bar{\rho}(\omega)$, $C(t) \overset{t \simeq 0}{\propto} 1/t^3$ and $\bar{\rho} \overset{\omega \gg 0}{\simeq}$ const

$p = \frac{5}{2} \quad \to \quad \bar{\rho}/\sqrt{\omega}$
$p = 3 \quad \to \quad \bar{\rho}/\omega$