Discussion about the lattice EM quark current-current correlator and the R-ratio in the context of the muon $g - 2$

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Lattice ↔ R-ratio

\[ C(t) = \frac{1}{24\pi^2} \int_0^\infty ds \sqrt{s} R(s) e^{-|t|\sqrt{s}} \]

[Bernecker et al '11]

- R-ratio → lattice: “straightforward”
  - integrate R-ratio

- Lattice → R-ratio: inverse Laplace transform on limited data
  - ill-posed problem
Why solve inverse problem?

Don’t need inverse methods to show disagreement between lattice and data-driven approaches

Situation before CMD-3 $e^+e^- \rightarrow \pi^+\pi^-$ measurement

- BMW’20 computation of $a_{\mu,\text{win}}^{\text{LO-HVP}}$
  confirmed by 3 other groups (8 other calculations for $a_{\mu,\text{win},ud}^{\text{LO-HVP}}$)

- unacceptable discrepancy

- $\sim 60\%$ of $a_{\mu,\text{win}}^{\text{LO-HVP}}$ ($\sim 70\%$ of $a_{\mu}^{\text{LO-HVP}}$) comes from $\rho$ peak

- already suggests that $\rho$ peak could be the culprit in R-ratio measurements

Aside (results in $\times 10^{-10}$ units from [BMW’20, Colangelo et al ‘22, BMW/DMZ’23, Mainz’24, ETM’22, . . . ])

$$[a_{\mu,10-\infty}^{\text{LO-HVP}}]_{\text{lat-R}} \equiv [a_{\mu}^{\text{LO-HVP}} - a_{\mu,\text{win}}^{\text{LO-HVP}} - a_{\mu,00-04}^{\text{LO-HVP}}]_{\text{lat-R}} \approx 13.5(6.2) - 9.4(1.9) - 0.45(67) = 6.2(7.0)$$

(naive combination of errors)
Why solve inverse problem?

\( a_{\mu}^{LO-HVP} \) and \( a_{\mu,\text{win}}^{LO-HVP} \) can already help “eliminate” measurements

**CMD-3 R-ratio** is \( \sim 5\% \) larger than previous WA around \( \rho \)-peak!

Tension of KLOE w/ lattice is \( 2.0\sigma \)

(Problems w/ radiative corrections in KLOE (& BES III) not covered by systematic uncertainties? [BaBar ’23])

**Aside (now CMD-3 alone, . . .)**

\[
[a_{\mu,10-\infty}^{LO-HVP}]_{\text{lat-R}} \approx -5.5(7.5) - 2.5(2.1) - 0.45(67) = -8.5(7.8)
\]

(naive combination of errors)

**CMD-3** agrees w/ lattice for \( a_{\mu}^{LO-HVP} \) and \( a_{\mu,\text{win}}^{LO-HVP} \)

To agree w/ R-ratio prediction, lattice LD window > or < than R-ratio one (see previous page)?
Why solve inverse problem?

Want to see more specifically what problem may be w/ measured \( R(s) \)

**HLT’19 by ETMC’22 (cf. Nazario)**

- Can’t get R-ratio point-by-point
  - ⇒ get smeared R-ratio in model-independent way from \( C(t) \)

- Work w/ fixed smearing fn for all simulations: Gaussian in \( \sqrt{s} \) w/ fixed \( \sigma \) centered around same set of \( \sqrt{s} \)
  - → get same *physical* quantities for all simulations
  - → can take \( m_q \rightarrow m_q^{\text{phys}}, a \rightarrow 0 \) (and \( L \rightarrow \infty \)) of those quantities instead of \( C(t) \) which is more complicated
Limitations:

- QED effects not included → challenging at large $t$

- Limited # of $t$ available in lattice computation of $C(t)$ and significant correlations in $\delta C(t)$

  \[ \Rightarrow \text{large correlations and limited information in reconstructed smeared } R(s) \]

- e.g. there is a $\lesssim 3\sigma$ tension in $\pm 600 \text{ MeV around } \sqrt{s} = 800 \text{ MeV}$ ($M_\rho \simeq 775 \text{ MeV w/ } \Gamma_\rho \simeq 150 \text{ MeV}$)

- very challenging to reduce $\sigma$

Mattia: how must $C(t)$ be improved to move forward?
Should one be less ambitious?

Answer a more simple and targeted question \[ \text{[BMW/DMZ '23]}: \]

What part of the experimentally measured spectrum may have to be modified to resolve disagreement with the lattice and how?

Present situation:

- Very few HVP quantities computed on lattice with:
  - all contributions to $C(t)$: flavors, quark Wick contractions, QED and SIB corrections
  - all limits taken: $a \to 0$, $L \to \infty$, $m_q \to m_{q^\phi}$, ...
  - typically $a_{\mu}^{\text{LO-HVP}}$ windows, the hadronic running of $\alpha$ and other quantities of phenomenological importance
Should one be less ambitious?

Want approach that:

- makes use of available results (generically called $a_j$ here)
- provides useful information w/ limited lattice input
- can be systematically improved w/ more lattice input
- can incorporate theoretical constraints [e.g. Colangelo et al '20]
- includes measure of agreement of lattice & data-driven results w/ comparison hypothesis
- accounts for all correlations in lattice and data-driven observables . . . including uncertainties on these
  $\rightarrow$ needed for quantitative comparison
Testing R-ratio: methodology

- Chop $a_j^R$ into contributions $a_{jb}^R$ from same $\sqrt{s}$-intervals $l_b$ for all $j$

  $$a_j^R = \sum_b a_{jb}^R$$

- To accommodate lattice results $a_j^{\text{lat}}$, allow common rescaling of $a_{jb}^R$, for all $j$, in certain $l_b$

  $$a_j^{\text{lat}} = \sum_b \gamma_b a_{jb}^R = \sum_b (1 + \delta_b) a_{jb}^R$$

  → can take some $\gamma_b = 1$

  → simplest interpretation: R-ratio rescaled by $\gamma_b$ in $l_b$

  → however, constrains shape of R-ratio modification in limited way

  → $\Phi$ deformation may be allowed

- Minimize w.r.t. parameters $\gamma_b$ & $a_{jb}$

  $$\chi^2(a_{jb}, \gamma_b) = \sum_{j,k} \left[ a_j^{\text{lat}} - \sum_b \gamma_b a_{jb} \right] \left[ C_{\text{lat}}^{-1} \right]_{jk} \left[ a_k^{\text{lat}} - \sum_c \gamma_c a_{kc} \right] + \sum_{(jb)(kc)} \left[ a_{jb}^R - a_{jb} \right] \left[ C_R^{-1} \right]_{(jb)(kc)} \left[ a_{kc}^R - a_{kc} \right]$$
Consider $a_1 = a_{\mu}^{\text{LO-HVP}}$, $a_2 = a_{\mu,\text{win}}^{\text{LO-HVP}}$ (2 obs.) w/(out) $a_3 = \delta(\Delta_{\text{had}}^{(5)} \alpha)$ (3 obs.)

- Stat and syst uncertainties on lattice covariance matrices do not change overall picture.
Conclusions / Questions

- No inverse methods needed to show disagreement between lattice and data-driven approaches
- Nor for discriminating measurements
- However needed to more clearly identify possible problems with measurements
- Many solutions to inverse problem: Backus-Gilbert/HLT, NN, Bayesian approaches, MEM, . . .
- Certainly less good methods . . .
- . . . but no “best” or “one-size-fits-all” solution
- “Model-independent” solutions are not the holy grail: we know a lot about the R-ratio
  → additional knowledge should be used
  → will help mitigate the fact of limited independent lattice information
Conclusions / Questions

- Only go for “full” $R(s)$ vs $s$ if needed (e.g. LHC needs $x$ dependence of PDFs)

- Often better to focus on more specific quantities

- Important to carefully formulate question(s) to be answered . . .

- . . . and develop methods to best answer it

- If lattice and data-driven methods end up agreeing, important to combine to gain in precision (will need $0.2\%$ on $a^{\text{LO-HVP}}_\mu$ in 2025!)

  $\Rightarrow$ methods described in [BMW/DMZ '23] can be used to do so effectively
BACKUP