Discussion about the lattice EM quark current-current correlator and the R-ratio in the context of the muon $g - 2$

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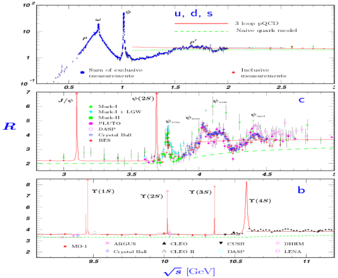
Lattice ↔ R-ratio

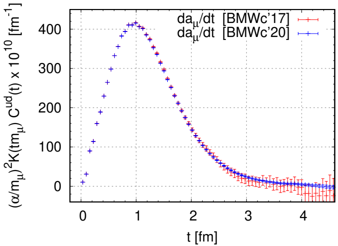
$$C(t) = \frac{1}{24\pi^2}\int_0^\infty ds\sqrt{s}\,R(s)\,e^{-|t|\sqrt{s}}$$

[Bernecker et al ’11]



(PDG compilation)

$\longleftrightarrow$



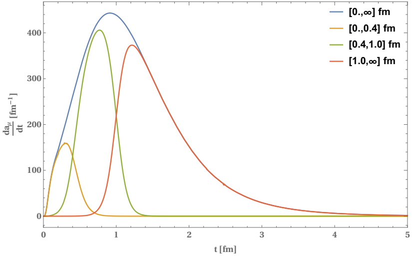
- R-ratio $\longrightarrow$ lattice: “straightforward”

  $\rightarrow$ integrate R-ratio

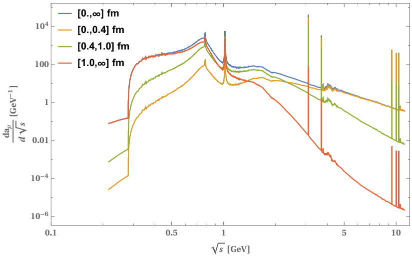
- Lattice $\longrightarrow$ R-ratio: inverse Laplace transform on limited data

  $\rightarrow$ ill-posed problem

Windows [RBC-UKQCD'18]



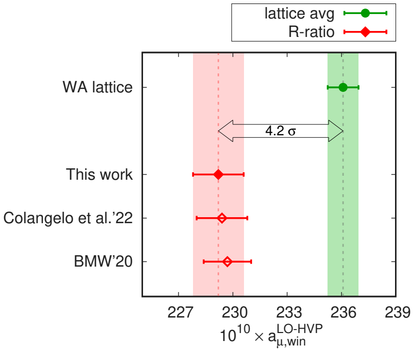
$\updownarrow$



Why solve inverse problem?

Don't need inverse methods to show disagreement between lattice and data-driven approaches

Situation before CMD-3 $e^+e^- \to \pi^+\pi^-$ measurement



- BMW'20 computation of $a_{\mu,\text{win}}^{\text{LO-HVP}}$ confirmed by 3 other groups (8 other calculations for $a_{\mu,\text{win},ud}^{\text{LO-HVP}}$)
- unacceptable discrepancy
- $\sim 60\%$ of $a_{\mu,\text{win}}^{\text{LO-HVP}}$ ($\sim 70\%$ of $a_{\mu}^{\text{LO-HVP}}$) comes from ρ peak
- already suggests that ρ peak could be the culprit in R-ratio measurements

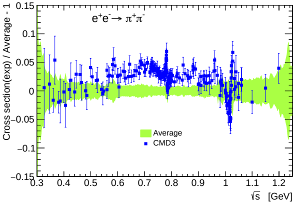
Aside (results in $\times 10^{-10}$ units from [BMW'20, Colangelo et al '22, BMW/DMZ'23, Mainz'24, ETM'22, ...])

$$[a_{\mu,10\text{-}\infty}^{\text{LO-HVP}}]_{\text{lat-R}} \equiv [a_{\mu}^{\text{LO-HVP}} - a_{\mu,\text{win}}^{\text{LO-HVP}} - a_{\mu,00\text{-}04}^{\text{LO-HVP}}]_{\text{lat-R}} \simeq 13.5(6.2) - 9.4(1.9) - 0.45(67) = 6.2(7.0)$$

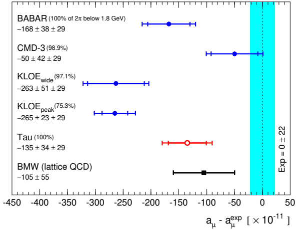
(naive combination of errors)

Why solve inverse problem?

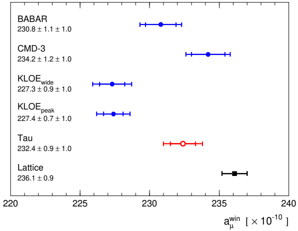
$a_\mu^{\text{LO-HVP}}$ and $a_{\mu,\text{win}}^{\text{LO-HVP}}$ can already help “eliminate” measurements



CMD-3 R-ratio is $\sim$ 5% larger than previous WA around ρ-peak !



Tension of KLOE w/ lattice is 2.0σ ...



Tension of KLOE w/ lattice is 5.4σ !

(Problems w/ radiative corrections in KLOE (& BES III) not covered by systematic uncertainties ? [BaBar ’23])

Aside (now CMD-3 alone, . . .)

$$[a_{\mu,10\text{-}\infty}^{\text{LO-HVP}}]_{\text{lat-R}} \simeq -5.5(7.5) - 2.5(2.1) - 0.45(67) = -8.5(7.8)$$

(naive combination of errors)

CMD-3 agrees w/ lattice for $a_\mu^{\text{LO-HVP}}$ and $a_{\mu,\text{win}}^{\text{LO-HVP}}$

To agree w/ R-ratio prediction, lattice LD window $>$ or $<$ than R-ratio one (see previous page)?

Why solve inverse problem?

Want to see more specifically what problem may be w/ measured $R(s)$

HLT'19 by ETMC'22 (cf. Nazario)

- Can't get R-ratio point-by-point

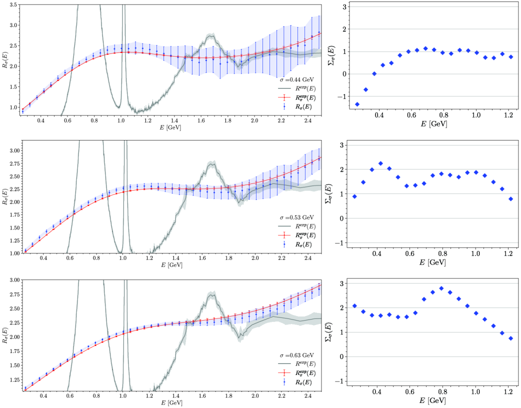
  $\Rightarrow$ get smeared R-ratio in model-independent way from $C(t)$

- Work w/ fixed smearing fn for all simulations: Gaussian in $\sqrt{s}$ w/ fixed σ centered around same set of $\sqrt{s}$

  $\rightarrow$ get same *physical* quantities for all simulations

  $\rightarrow$ can take $m_q \rightarrow m_q^{\text{phys}}$, $a \rightarrow 0$ (and $L \rightarrow \infty$) of those quantities instead of $C(t)$ which is more complicated

HLT by ETMC



Beautiful methods and results

Limitations:

- QED effects not included $\rightarrow$ challenging at large t
- Limited # of t available in lattice computation of $C(t)$ and significant correlations in $\delta C(t)$

  $\Rightarrow$ large correlations and limited information in reconstructed smeared $R(s)$
- e.g. there is a $\lesssim 3\sigma$ tension in $\pm 600\,\mathrm{MeV}$ around $\sqrt{s} = 800\mathrm{MeV}$ ($M_\rho \simeq 775\,\mathrm{MeV}$ w/ $\Gamma_\rho \simeq 150\,\mathrm{MeV}$)
- very challenging to reduce σ

  Mattia: how must $C(t)$ be improved to move forward?

Should one be less ambitious?

Answer a more simple and targeted question [BMW/DMZ '23]:

What part of the experimentally measured spectrum may have to be modified to resolve disagreement with the lattice and how?

Present situation:

- Very few HVP quantities computed on lattice w/:
  - all contributions to $C(t)$: flavors, quark Wick contractions, QED and SIB corrections
  - all limits taken: $a \to 0$, $L \to \infty$, $m_q \to m_q^\phi$, ...
  - typically $a_\mu^{\text{LO-HVP}}$ windows, the hadronic running of α and other quantities of phenomenological importance

Should one be less ambitious?

Want approach that:

- makes use of available results (generically called a_j here)
- provides useful information w/ limited lattice input
- can be systematically improved w/ more lattice input
- can incorporate theoretical constraints [e.g. Colangelo et al '20]
- includes measure of agreement of lattice & data-driven results w/ comparison hypothesis
- accounts for all correlations in lattice and data-driven observables . . . including uncertainties on these

  $\rightarrow$ needed for quantitative comparison

Testing R-ratio: methodology

- Chop $a_j^{\rm R}$ into contributions $a_{jb}^{\rm R}$ from same $\sqrt{s}$-intervals I_b for all j

$$a_j^{\rm R} = \sum_b a_{jb}^{\rm R}$$

- To accommodate lattice results $a_j^{\rm lat}$, allow common rescaling of $a_{jb}^{\rm R}$, for all j, in certain I_b

$$a_j^{\rm lat} = \sum_b \gamma_b a_{jb}^{\rm R} = \sum_b (1+\delta_b) a_{jb}^{\rm R}$$

$\rightarrow$ can take some $\gamma_b = 1$

$\rightarrow$ simplest interpretation: R-ratio rescaled by γ_b in I_b

$\rightarrow$ however, constrains shape of R-ratio modification in limited way

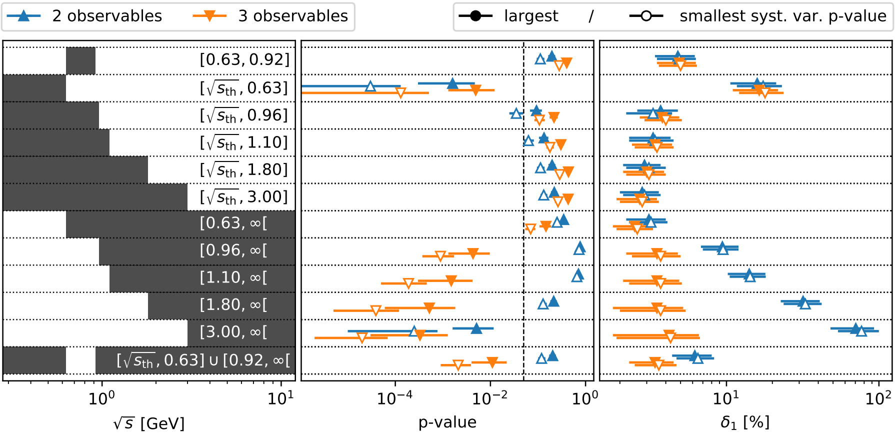
$\rightarrow$ Φ deformation may be allowed

- Minimize w.r.t. parameters γ_b & a_{jb}

$$\chi^2(a_{jb},\gamma_b) = \sum_{j,k}\left[a_j^{\rm lat} - \sum_b \gamma_b a_{jb}\right]\left[C_{\rm lat}^{-1}\right]_{jk}\left[a_k^{\rm lat} - \sum_c \gamma_c a_{kc}\right] + \sum_{(jb)(kc)}\left[a_{jb}^{\rm R} - a_{jb}\right]\left[C_{\rm R}^{-1}\right]_{(jb)(kc)}\left[a_{kc}^{\rm R} - a_{kc}\right]$$

Testing R-ratio: results

Consider $a_1 = a_\mu^{\text{LO-HVP}}$, $a_2 = a_{\mu,\text{win}}^{\text{LO-HVP}}$ (2 obs.) w(/out) $a_3 = \delta(\Delta_{\text{had}}^{(5)}\alpha)$ (3 obs.)



- Stat and syst uncertainties on lattice covariance matrices do not change overall picture

Conclusions / Questions

- No inverse methods needed to show disagreement between lattice and data-driven approaches
- Nor for discriminating measurements
- However needed to more clearly identify possible problems with measurements
- Many solutions to inverse problem: Backus-Gilbert/HLT, NN, Bayesian approaches, MEM, ...
- Certainly less good methods ...
- ... but no "best" or "one-size-fits-all" solution
- "Model-independent" solutions are not the holy grail: we know a lot about the R-ratio

  $\rightarrow$ additional knowledge should be used

  $\rightarrow$ will help mitigate the fact of limited independent lattice information

Conclusions / Questions

- Only go for “full” $R(s)$ vs s if needed (e.g. LHC needs x dependence of PDFs)
- Often better to focus on more specific quantities
- Important to carefully formulate question(s) to be answered …
- … and develop methods to best answer it
- If lattice and data-driven methods end up agreeing, important to combine to gain in precision (will need 0.2% on $a_\mu^{\text{LO-HVP}}$ in 2025!)

  $\Rightarrow$ methods described in [BMW/DMZ ’23] can be used to do so effectively

BACKUP