Interpolating to the B-scale

Simon Kuberski based on work with A. Conigli, J. Frison, P. Fritzsch, A. Gérardin, J. Heitger, G. Herdoíza, C. Pena, H. Simma, R. Sommer

Lattice@CERN 2024
July 16, 2024

ALPHA Collaboration

Funded by the European Union

CERN

WHY SHOULD WE DO B-PHYSICS ON THE LATTICE?

- Search for BSM physics at the high-precision frontier: Deviations between Standard Model predictions and experiment in flavor physics observables.

- Several *B-anomalies*, e.g.,
  - Ratios testing lepton flavor universality.
  - Branching fractions of rare decays.
  - Tension between inclusive and exclusive determinations of $|V_{\mathrm{ub}}|$ and $|V_{\mathrm{cb}}|$.

- Need precise determinations of hadronic matrix elements and quark masses.

$\rightarrow$ *Ab initio* Standard Model predictions from lattice QCD.

Multi-scale problems in lattice QCD

- By discretizing QCD in a finite volume, we introduce two cutoffs:
  - Infrared cutoff: $\Lambda_{\mathrm{IR}} \sim 1/L$
  - Ultraviolet cutoff: $\Lambda_{\mathrm{UV}} \sim 1/a$

- Finite-volume effects vanish exponentially $\propto \exp(-m_\pi L)$

  $\rightarrow$ require $m_\pi L \geq 4$.

- Cutoff effects vanish polynomially $\propto c_1 a + c_2 a^2 + c_3 a^3 + c_4 a^4 \ldots$, with logarithmic corrections [Husung et al., 1912.08498]!

  $\rightarrow$ For heavy quarks h: fulfill $am_h \ll 1$ for reliable continuum extrapolations.

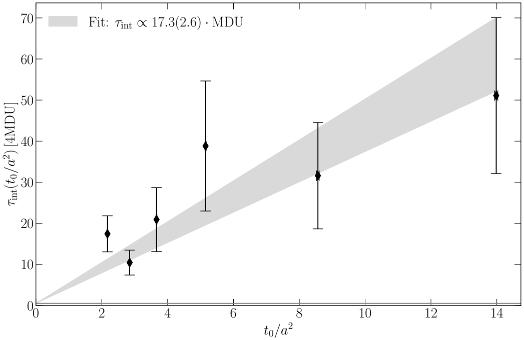
$$L^{-1} \ll m_\pi \approx 135\,\mathrm{MeV} \ll am_h \ll a^{-1}$$

$$L^{-1} \ll m_\pi \approx 135\,\mathrm{MeV} \ll am_h \ll a^{-1}$$

- The cost to generate ensembles scales at least with $(L/a)^5$...

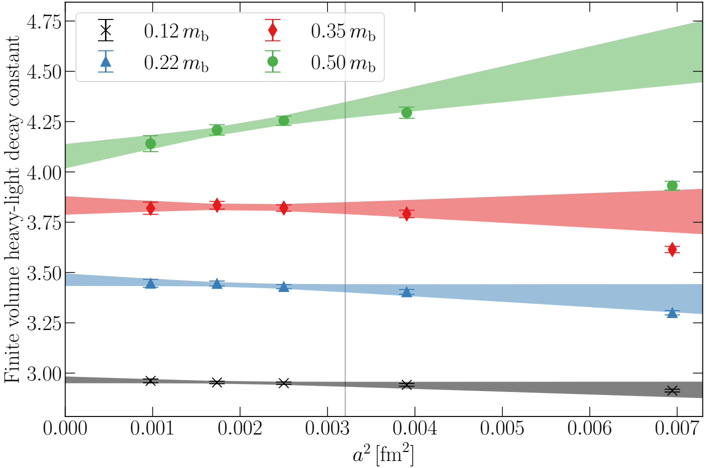
$$L^{-1} \ll m_\pi \approx 135\,\mathrm{MeV} \ll am_h \ll a^{-1}$$

- The cost to generate ensembles scales at least with $(L/a)^5$...
- ...and there is critical slowing down towards the continuum limit.
  Here: 2+1 flavor CLS ensembles, no topology freezing!



$$L^{-1} \ll m_\pi \approx 135\,\mathrm{MeV} \ll am_h \ll a^{-1}$$

- The cost to generate ensembles scales at least with $(L/a)^5$...
- ...and there is critical slowing down towards the continuum limit.
  Here: 2+1 flavor CLS ensembles, no topology freezing!
- What are the energy scales that can be reached at physical pion mass?
  - $m_\pi^{\mathrm{phys}} L \geq 4$ implies $L \geq 6\,\mathrm{fm}$ (assume $T \gg L$ here).
  - State of the art: $L/a = 96$ at $a = 0.06\,\mathrm{fm} \rightarrow a^{-1} \sim 3.3\,\mathrm{GeV}^{-1}$.
  - Largest on the market: $L/a = 144$ at $a = 0.04\,\mathrm{fm} \rightarrow a^{-1} \sim 4.9\,\mathrm{GeV}^{-1}$.
- Simulating a relativistic b on the finest large-volume ensembles: am_b is of $\mathrm{O}(1)$.



- Consider (finer than) conventional lattice spacings

$$0.031\,\mathrm{fm} \leq a \leq 0.083\,\mathrm{fm}$$

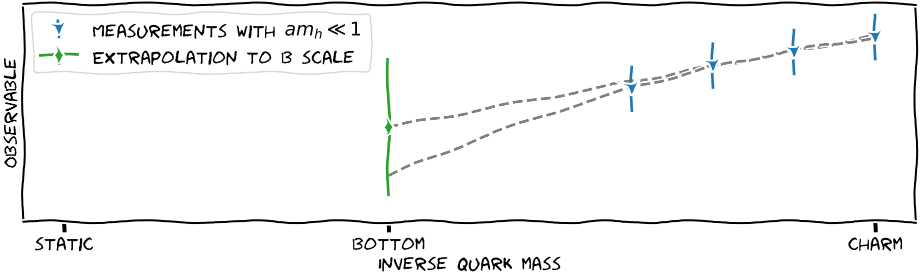
in finite-volume.

- Continuum extrapolation of the pseudoscalar heavy-light decay constant at fixed (renormalized) quark masses.

- For illustration: Use three finest resolutions $\leq 0.05\,\mathrm{fm}$ to extrapolate with

$$f_\mathrm{hl}(a) = p_0 + p_1 \cdot a^2$$

- Simulating relativistic bottom quarks at several resolutions is not possible in large volumes in the near future!



- Extrapolation to the B scale is difficult, possibly mixing extra-/interpolations in a, $1/m_h$ and q^2 for semi-leptonic form factors.

- Simulating relativistic bottom quarks at several resolutions is not possible in large volumes in the near future!

Employ effective field theories for low-energy physics

$$a\,|\vec{p}| \ll 1\,, \qquad |\vec{p}| \ll m_{\mathrm{b}}$$

- here: Heavy Quark Effective Theory (HQET)

- Renormalizable effective theory $\leftrightarrow$ continuum limit.

Heavy Quark Effective Theory

- Integrate out heavy degrees of freedom of QCD Lagrangian for one heavy quark.
- Expand the Lagrangian in powers of $1/m_{\mathrm{h}}$.

→ Possible to describe bottom physics at next-to-leading order in HQET.

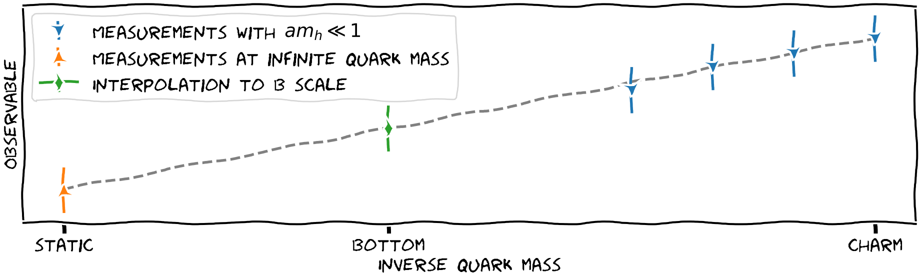
$$\mathcal{L}_{\mathrm{heavy}} = \bar{h}_v D_0 h_v - \omega_{\mathrm{kin}} \mathcal{O}_{\mathrm{kin}} - \omega_{\mathrm{spin}} \mathcal{O}_{\mathrm{spin}} \,, \qquad \mathcal{O}_{\mathrm{kin}} = \bar{h}_v \mathbf{D}^2 h_v \,, \quad \mathcal{O}_{\mathrm{spin}} = \bar{h}_v \sigma \cdot \mathbf{B} h_v$$

- Perturbative matching at order g_0^{2l} leads to power divergences in the coefficients [Nucl.Phys.B 368 (1992) 281-292, Maiani et al.]

$$\Delta c_k \sim g_0^{2(l+1)} a^{-p} \sim a^{-p} \left[\ln(a\Lambda)\right]^{-(l+1)} \overset{a\to 0}{\longrightarrow} \infty$$

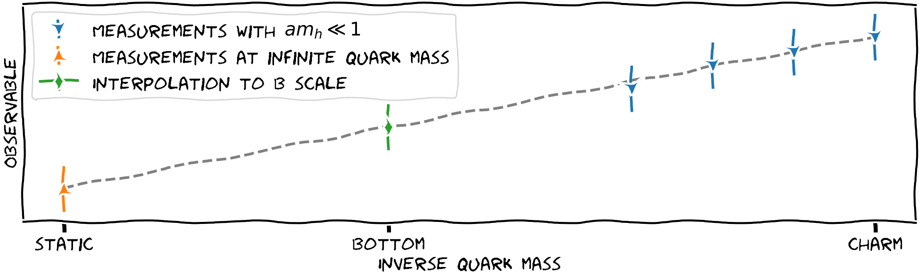
due to mixing of operators differing in dimensions by p.

(Non-perturbative) HQET



- Can we just perform an interpolation between results in static HQET and results in relativistic QCD below m_{b} where $am_h \ll 1$?

(Non-perturbative) HQET



- Can we just perform an interpolation between results in static HQET and results in relativistic QCD below m_b where $am_h \ll 1$?

$\rightarrow$ No! Even the static approximation requires non-trivial renormalisation and matching that would have to be **computed non-perturbatively**.

- Existing strategy to renormalize HQET non-perturbatively via step-scaling techniques [Heitger and Sommer, hep-lat/0310035].
  $\rightarrow$ Quite challenging since $1/m_\mathrm{b}$ effects are needed for precision.

- Given the static action $\mathcal{L}^{\rm stat} = \overline{\psi}_h D_0 \psi_h$, we have $E^{\rm stat} \sim \frac{1}{a} g_0^2$.
  - $\rightarrow$ $E^{\rm stat}$ divergent as $a \rightarrow 0$
- Renormalization $\rightarrow \delta m \sim \frac{1}{a} g_0^2$ and matching $\rightarrow m_b^{\rm finite}$.
  - $\rightarrow$ $E = E^{\rm stat} + \delta m + m_b^{\rm finite}$
- Heavy-light currents

$$
\begin{aligned}
V_k^{\rm stat} &= C_{V_k}(m_b) Z^{\rm stat}(g_0) \overline{\psi}_h \gamma_k \psi_l \\
V_0^{\rm stat} &= C_{V_0}(m_b) Z^{\rm stat}(g_0) \overline{\psi}_h \gamma_0 \psi_l
\end{aligned}
$$

  - $\rightarrow$ Matching coefficients $C_{V_{k(0)}}(m_b)$ log-divergent as $m_b \rightarrow \infty$ [Sommer, 1008.0710].

Our strategy, based on [Guazzini et al., 0710.2229]:

Cancel renormalization and matching [Sommer et al., 2312.09811].

Semi-leptonic form factors

- Phenomenologically relevant: the q^2 dependence of semi-leptonic form factors.
- Form factor decomposition in the B-meson rest frame

$$(\sqrt{2}p_k^\pi)^{-1}\langle\pi(p^\pi)|V_k(0)|B(\vec{p}=0)\rangle = h_\perp(E_\pi) = h_\perp^{\rm stat}(E_\pi) + O(1/m_{\rm b})$$

- Cancel matching and renormalization for $h_\perp^{\rm stat}$,

$$\frac{h_\perp(E_\pi)}{h_\perp(E_\pi^{\rm ref})} = \frac{h_\perp^{\rm stat}(E_\pi)}{h_\perp^{\rm stat}(E_\pi^{\rm ref})} + O(1/m_{\rm b})\,.$$

- Connection with $f_{B^\star}$: Normalize to the vector decay constant

$$h_\perp(E_\pi^{\rm ref}) = \hat{f}_{\rm V}\frac{h_\perp(E_\pi^{\rm ref})}{\hat{f}_{\rm V}} = \hat{f}_{\rm V}\left[\frac{h_\perp^{\rm stat}(E_\pi^{\rm ref})}{\hat{f}_{\rm V}^{\rm stat}} + {\rm O}(1/m_{\rm b})\right]$$

$\rightarrow$ Problem solved for $h_\perp$. How to compute $\hat{f}_{\rm V}$?

Step-scaling: Use the finite-volume as a tool.

- Cancel matching and renormalization via ratios of observables $O(L_2)/O(L_1)$ or differences of logs computed in two volumes :

$$\sigma_{\rm V} = \left[\log[L_{\rm ref}^{3/2}\,\hat{f}_{\rm V}(L_2)] - \log[L_{\rm ref}^{3/2}\,\hat{f}_{\rm V}(L_1)]\right]$$

- Same ansatz to cancel the additive divergence in the static energy

$$\sigma_m = L_{\rm ref}\left[m_{\rm PS}(L_2) - m_{\rm PS}(L_1)\right]$$

- Connect large-volume (CLS) ensembles with small volumes in two steps:

$$L_\infty \to L_2 = 1\,{\rm fm} \qquad \text{and} \qquad L_2 = 1\,{\rm fm} \to L_1 = 0.5\,{\rm fm}$$

- Small volume $L_1 = 0.5\,{\rm fm}$: Simulate **relativistic b quarks** with $am_{\rm b} \ll 1$

- Vector meson decay constant $\hat{f}_V = f_V\sqrt{m_V}$,

$$\hat{f}_V = \sqrt{2}\langle 0|V_k(0)|V(\vec{p}=0,k)\rangle_{\rm NR} = \hat{f}_V^{\rm stat} + O(1/m_b)\,,$$

- For the step-scaling, we define

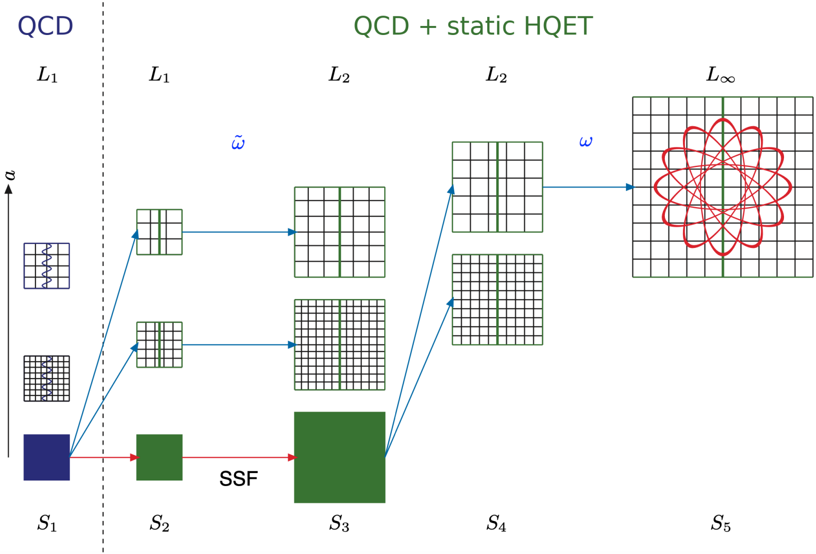
$$\Phi_{\vec{V}}(L) \equiv \ln\left(\frac{L_{\rm ref}^{3/2}\,\hat{f}_V(L)}{2}\right)$$

- Compute the large-volume (physical) quantity via

$$\Phi_{\vec{V}} = \Phi_{\vec{V}}(L_1) + [\Phi_{\vec{V}}(L_2) - \Phi_{\vec{V}}(L_1)] + [\Phi_{\vec{V}} - \Phi_{\vec{V}}(L_2)]$$

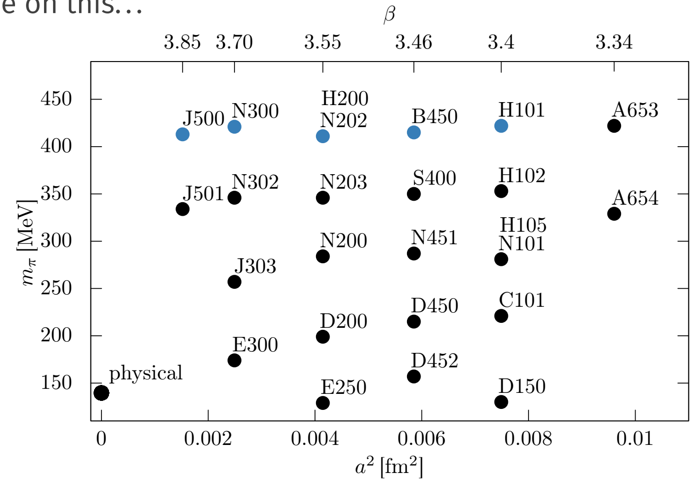
- Each observable is continuum extrapolated.

B-physics from step-scaling



- QCD observables with relativistic b quarks in finite volume at $L_1 = 0.5$ fm where $a^{-1} \in [9.5,\ 25]\,\mathrm{GeV}^{-1}$.
- Step-scaling for observables with:
  - static quarks
  - relativistic quarks with $m_h < m_\mathrm{b}$
- Contact with large-volume simulations.

- Simulate $N_{\mathrm{f}} = 3$ massless QCD in the sea in small volumes.
- Current status in large volume: SU(3) symmetric point, $m_\pi = m_K \approx 420$ MeV. Later more on this...



First results

for the vector decay constant

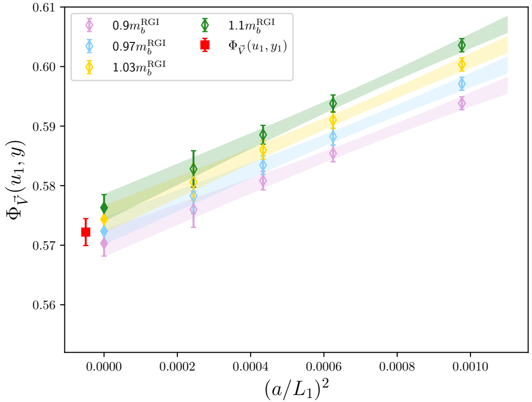
[2312.09811] [2312.10017]

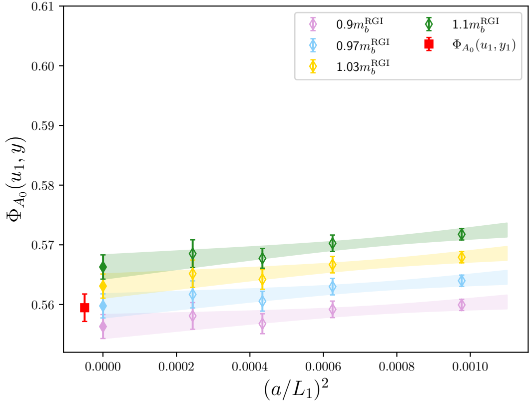
Interpolate observables to the B-scale:

- Interpolate relativistic measurements around m_b:
  - In small volumes: Observables such as $\hat{f}_\mathrm{V}$, m_B/m_b.
- Interpolate between the static limit and $m_h \ll m_\mathrm{b}$:
  - Step-scaling functions such as σ_V, σ_m.
  - In large volume: Ratios of observables like $h_\perp(E_\pi)/h_\perp(E_\pi^\mathrm{ref})$ or $h_\perp(E_\pi^\mathrm{ref})/\hat{f}_\mathrm{V}$.
- Interpolations in $1/m_h$ are performed in the continuum limit:
  - Continuum extrapolations at the B-scale only for $a^{-1} \in [9.5,\ 25]\,\mathrm{GeV}^{-1}$
  - Cutoff effects partially cancel in differences.

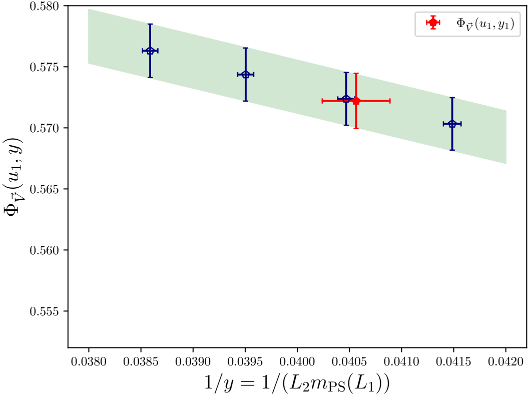
$$\Phi_{\vec{V}} = \Phi_{\vec{V}}(L_1) + [\Phi_{\vec{V}}(L_2) - \Phi_{\vec{V}}(L_1)] + [\Phi_{\vec{V}} - \Phi_{\vec{V}}(L_2)]$$

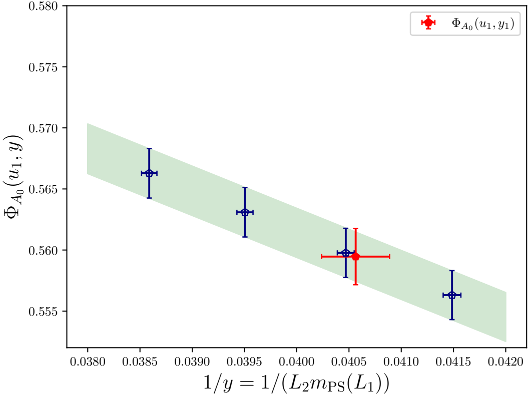
Continuum extrapolation at the bottom scale





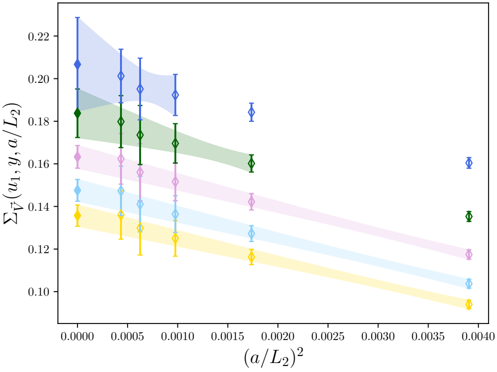
- B-physics on fine lattices in small volumes.
- Continuum extrapolations for vector (left) and axial (right) decay constants.
- Four heavy valence quark masses encompass the bottom quark mass.



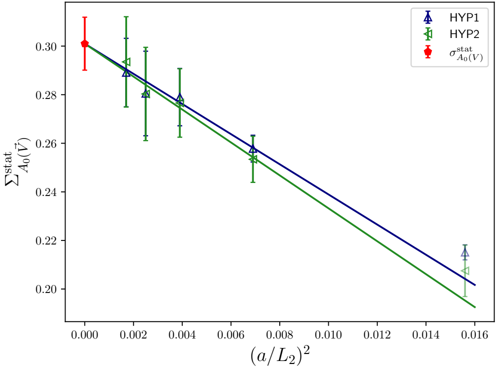


- B-physics on fine lattices in small volumes.
- Interpolate in inverse heavy-light meson mass $1/y = 1/(L_{\mathrm{ref}} m_{\mathrm{PS}}(L_1)) \propto 1/m_h$.
- Straight-forward interpolation to $m_h = m_{\mathrm{b}}$.

- Continuum extrapolation of relativistic and static step-scaling functions for $\Phi_{\vec{V}}$.
- $L = 0.5\,\mathrm{fm}$ to $L = 1\,\mathrm{fm}$. Only include $am_h^{\mathrm{RGI}} < 0.8$.

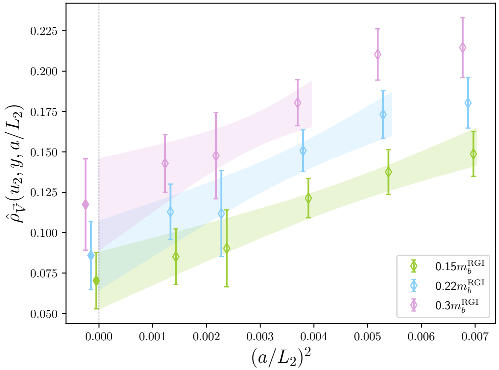


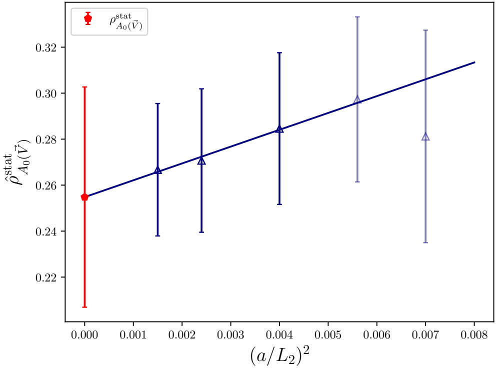
relativistic: valence masses $m_h^{\mathrm{RGI}} < 0.5\, m_b^{\mathrm{RGI}}$



static: two heavy quark actions

- Continuum extrapolation of relativistic and static step-scaling functions for $\Phi_{\vec{V}}$.
- $L = 1\,\mathrm{fm}$ to L_{CLS}. Only include $am_h^{\mathrm{RGI}} < 0.8$.

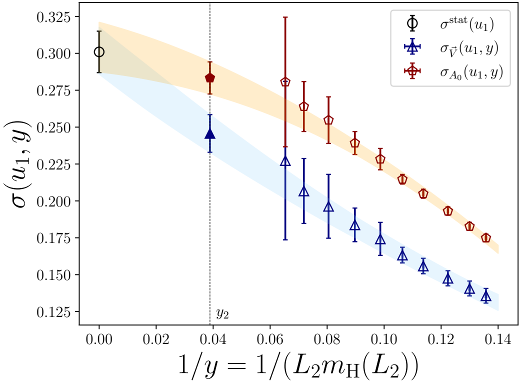




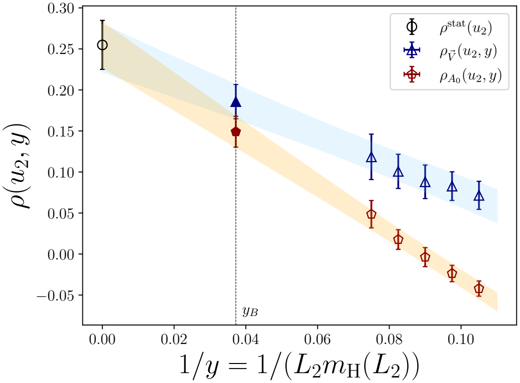
relativistic: valence masses $m_h^{\mathrm{RGI}} < 0.3\,m_b^{\mathrm{RGI}}$

static

- Interpolation to $1/m_B$: highly constrained by the static result.
- Step-scaling functions of pseudoscalar Φ_{A_0} and vector $\Phi_{\vec{V}}$ decay constant have the same static limit (heavy quark symmetry).



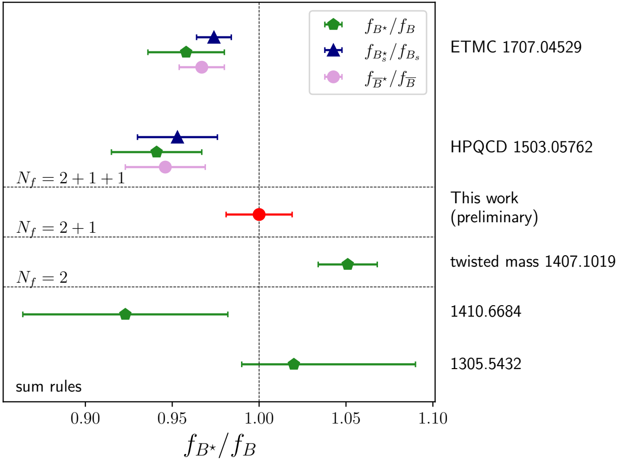
$L_1 \to L_2$



$L_2 \to L_{\rm CLS}$

Results for $f_{B^\star}/f_B$

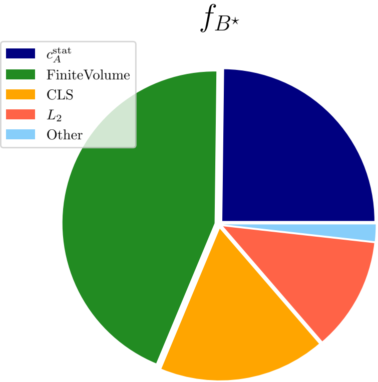
- Combine all pieces to arrive at the final result.
- N.b.: We (currently) work at the SU(3) symmetric point.
  Expect light quark dependence in the **ratio** $f_{B^\star}/f_B$ to be small.



- Puzzling situation for the ratios $f_{B_{(s)}}/f_{B^\star_{(s)}}$.
- Systematically improvable result with competitive uncertainties.

Results for $f_{B^\star}/f_B$

- Combine all pieces to arrive at the final result.
- N.b.: We (currently) work at the SU(3) symmetric point.
  Expect light quark dependence in the **ratio** $f_{B^\star}/f_B$ to be small.



- Puzzling situation for the ratios $f_{B_{(s)}}/f_{B^\star_{(s)}}$.
- Systematically improvable result with competitive uncertainties.
- Decay constants currently at about 2.5% precision, dominated by finite-volume statistical uncertainties.

First results

FOR THE MASS OF THE BOTTOM QUARK

[2312.09811] [2312.10017]

- In small volume, compute

$$m_h^{\mathrm{RGI}} = \frac{M}{m_{\mathrm{R}}(1/L_0)} \frac{Z_{\mathrm{A}}}{Z_{\mathrm{P}}(L_0)} [1 + (b_{\mathrm{A}} - b_{\mathrm{P}}) a m_h] \, m_h^{\mathrm{PCAC}}(L_1) \quad \text{and} \quad \pi_m = \frac{m_{\mathrm{PS}}(L_1)}{m_h^{\mathrm{RGI}}}$$

  with the running factor from [ALPHA, 1802.05243] and the renormalization and improvement from [Fritzsch, Heitger, SK].

- Compute the bottom quark mass via

$$L_{\mathrm{ref}} m_h^{\mathrm{RGI}} = \Big(L_{\mathrm{ref}} m_{\mathrm{PS}} - L_{\mathrm{ref}} [m_{\mathrm{PS}} - m_{\mathrm{PS}}(L_2)] - L_{\mathrm{ref}} [m_{\mathrm{PS}}(L_2) - m_{\mathrm{PS}}(L_1)] \Big) \frac{m_h^{\mathrm{RGI}}}{m_{\mathrm{PS}}(L_1)}$$

$$\equiv \frac{L_{\mathrm{ref}} m_{\mathrm{PS}} - \rho_m(L_2) - \sigma_m(L_1)}{\pi_m(L_1)}$$

  with the physical input for m_{PS}. We choose $m_{\mathrm{PS}} = m_{\overline{B}} \equiv \frac{2}{3} m_B + \frac{1}{3} m_{B_s}$ for $h = \mathrm{b}$.

- We have omitted the light quark dependence. Let's expand

$$\begin{aligned}\rho_m(L_2) &= L_{\mathrm{ref}}[m_{\mathrm{PS}} - m_{\mathrm{PS}}(L_2)] \\ &= L_{\mathrm{ref}}\left[m_{\mathrm{PS}} - m_{\mathrm{PS}}^{\mathrm{SU(3)}}\right] + L_{\mathrm{ref}}\left[m_{\mathrm{PS}}^{\mathrm{SU(3)}} - m_{\mathrm{PS}}(L_2)\right]\end{aligned}$$

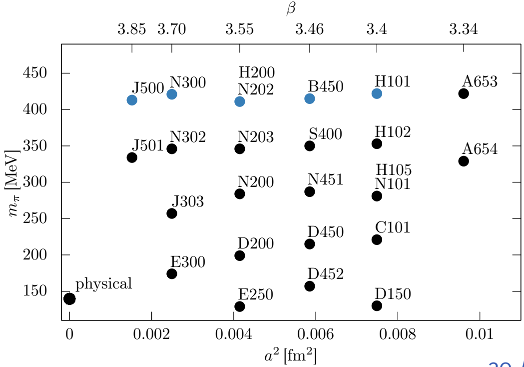
  where $m_{\mathrm{PS}}^{\mathrm{SU(3)}} \equiv m_{\mathrm{PS}}(m_\pi = m_K \approx 420\,\mathrm{MeV})$ is the heavy-light meson mass at the SU(3) symmetric point.

- Normalize step-scaling to the SU(3) symmetric point ($2+1$ flavor CLS).

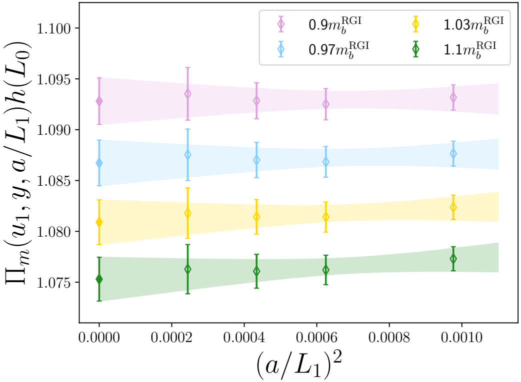
- Reach physical sea quarks via

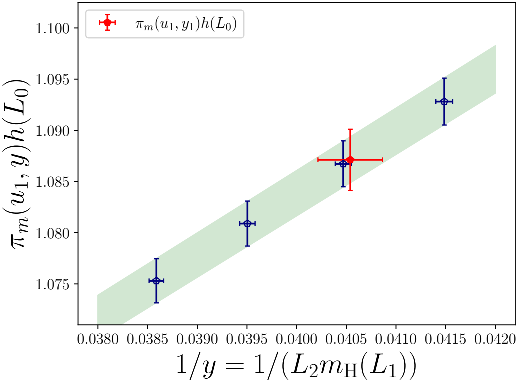
$$\lim_{m_\pi \to m_\pi^{\mathrm{phys}}} \lim_{a \to 0} L_{\mathrm{ref}}[m_{\mathrm{PS}} - m_{\mathrm{PS}}^{\mathrm{SU(3)}}]$$

  where $m_K^2 + \frac{1}{2}m_\pi^2 \approx \mathrm{const.}$.



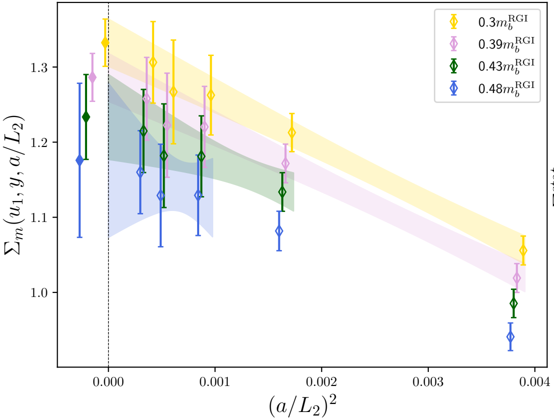
Continuum extrapolation at the bottom scale



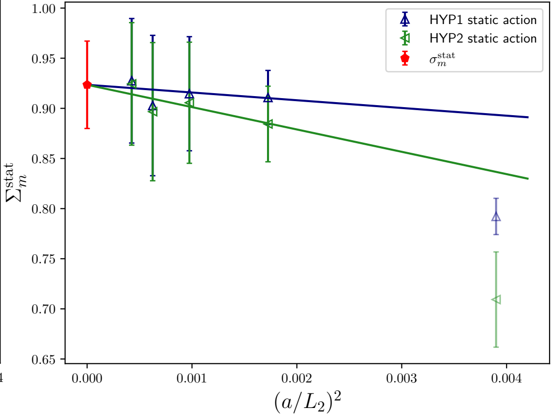


- Ratio of heavy-light meson mass and heavy quark mass m_H/m_h^{RGI} around the b.
- Left: Continuum extrapolation for four valence quark masses.
- Right: Interpolation to $1/m_{\overline{\mathrm{B}}}$

- Continuum extrapolation of relativistic and static step-scaling functions for the quark mass $\Sigma_m = L_2\,[m_H(L_2) - m_H(L_1)]$ and $\Sigma_m^{\rm stat}$ from $L = 0.5\,\rm fm$ to $L = 1\,\rm fm$ with $m_h^{\rm RGI} < 0.5\,m_b^{\rm RGI}$.



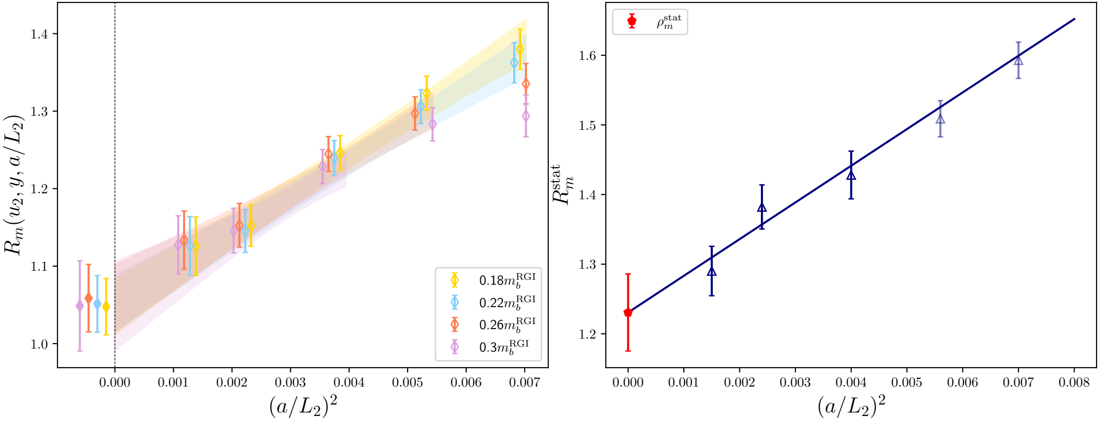
relativistic: several valence masses



static: two actions

L_2 TO CLS: CONTINUUM LIMIT

- Continuum extrapolation of relativistic and static step-scaling functions for the quark mass $R_m = L_2\,[m_H - m_H(L_2)]$ with $m_h^{\mathrm{RGI}} < 0.3\, m_b^{\mathrm{RGI}}$ and R_m^{stat} from $L = 1\,\mathrm{fm}$ to CLS .

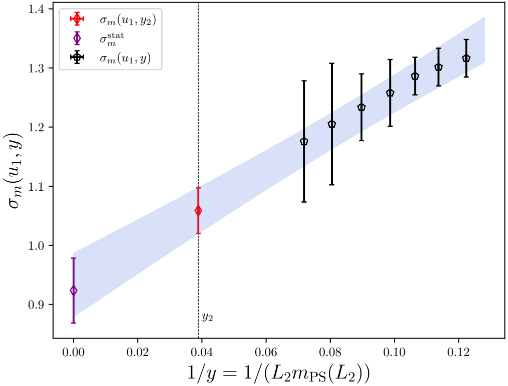


relativistic

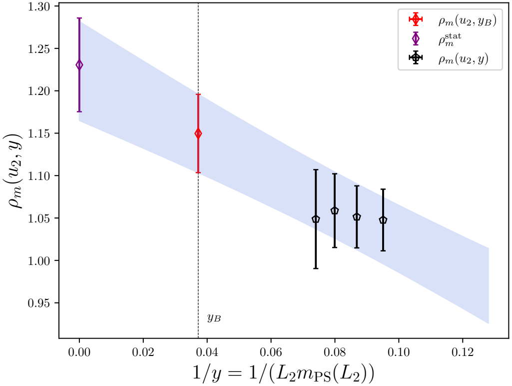
static

- Interpolate SSFs to the bottom scale in the continuum, where

$$\sigma_m = \lim_{a\to 0} \Sigma_m \text{ and } \rho_m = \lim_{a\to 0} R_m \,.$$

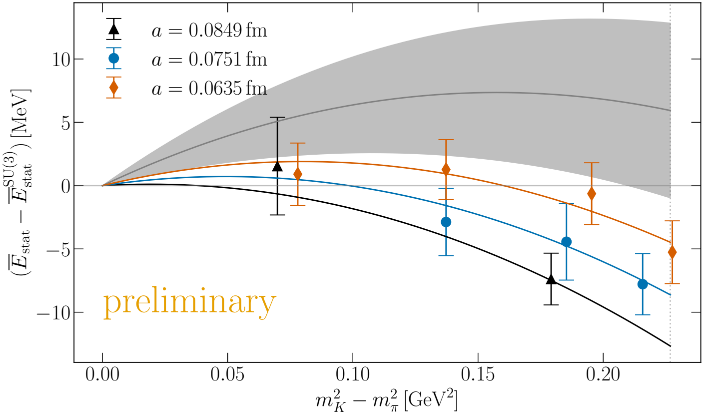


$0.5\,\text{fm} \to 1\text{fm}$



$1\,\text{fm} \to \infty$

- Need to compute $\lim\limits_{m_{\mathrm{PS}} \to m_{\overline{B}}} \lim\limits_{m_\pi \to m_\pi^{\mathrm{phys}}} \lim\limits_{a \to 0} L_{\mathrm{ref}}[m_{\mathrm{PS}} - m_{\mathrm{PS}}^{\mathrm{SU}(3)}]$.
- Small effects expected: $2m_{\mathrm{l}} + m_{\mathrm{s}} = \mathrm{const.}$ and cutoff effects largely cancel.



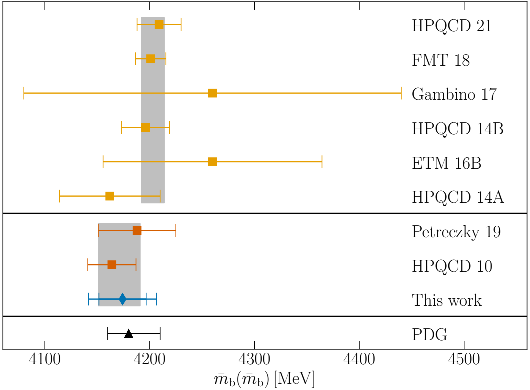
- Look at quark mass and cutoff effects in the static energy

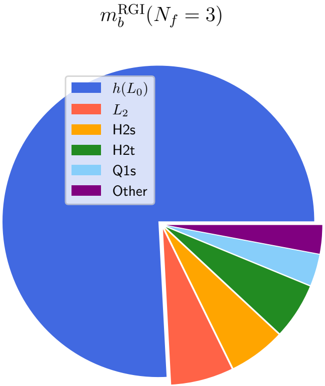
$$\overline{E}_{\mathrm{stat}} = \frac{2}{3} E_{\mathrm{stat}}^{\mathrm{hl}} + \frac{1}{3} E_{\mathrm{stat}}^{\mathrm{hs}}$$

- Effect at the physical point is at most $13\,\mathrm{MeV}$ for a static heavy quark.
- Very small w.r.t. our uncertainties (next slide)!

$$m_b^{\mathrm{RGI}}(N_f = 3) = \frac{L_{\mathrm{ref}} m_{\mathrm{PS}} - \rho_m(L_2) - \sigma_m(L_1)}{L_{\mathrm{ref}} \pi_m(L_1)} = 6.608(49)\ \mathrm{GeV}\ [0.7\%]$$

- Result at the SU(3) symmetric point, neglected shift of $\lesssim 0.017\,\mathrm{GeV}$.
- Uncertainty dominated by running to RGI $\rightarrow$ improvable external quantity.





What is already done

- Simulations of relativistic bottom quarks in small volumes.
- Step-scaling functions of static HQET and relativistic QCD with $am_h \ll 1$.
- Continuum step-scaling chain to the SU(3) symmetric point in large-volume.

What remains to be done (by **anyone** who likes to make use of this work)

- Simulations of static HQET and QCD in large volumes for $m_\pi \to m_\pi^{\text{phys}}$.
- (Log-)difference with zero-momentum quantities at the SU(3) symmetric point.

Conclusions

- Precision B-physics from well controlled continuum extrapolations, keeping $am_h \ll 1$, is possible by including static HQET, canceling matching and renormalization.
- We need to further explore the parameter space (finite-volume definitions, $1/m_{\mathrm{h}}^2$ effects, cuts in the continuum extrapolations).
- Next step: Extension to semi-leptonics (correlation functions have been computed) and inclusion of physical light quark masses.

Take home message

- Step-scaling functions can be used by any collaboration with any action.
- For form factors, only the large-volume computation and $\hat{f}_B^\star$ are needed.