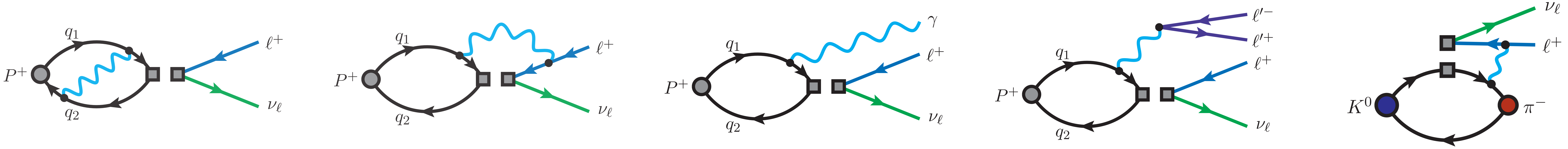


QED effects on the lattice

Matteo Di Carlo

18th July 2024

Funded by
the European Union

Lattice@CERN 2024
8-19 July 2024

Outline of the talk

1. **Why** are **isospin-breaking** and **QED** corrections **relevant**?

2. **How** are these effects **included in lattice calculations**?

3. **What** **observables** have been / can be computed?

4. **Where** do we stand and **where** do we go?

1. Why

1. Why

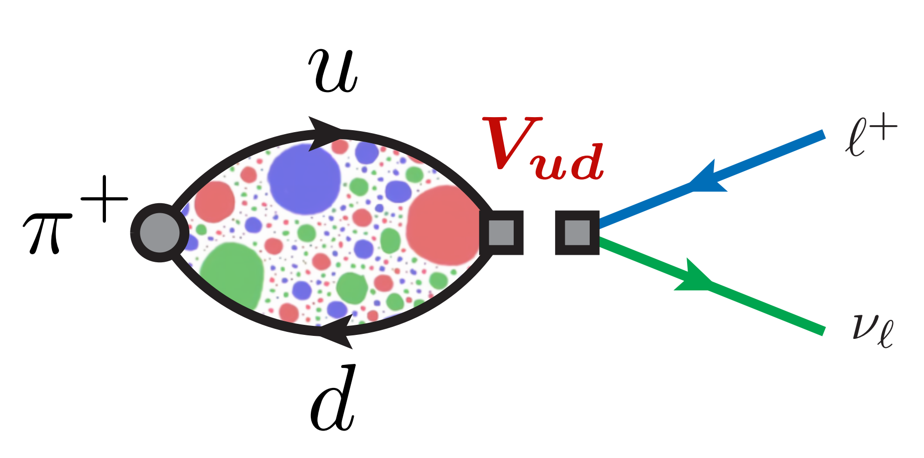
Testing the Standard Model with flavour physics

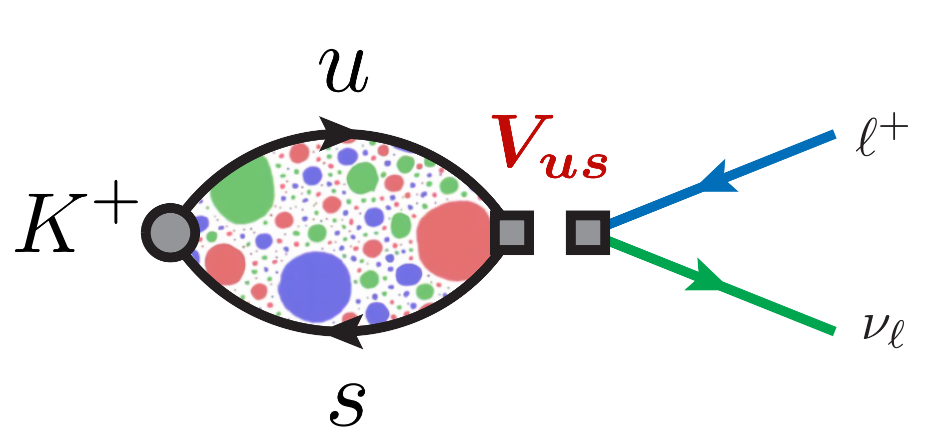
Unitarity of the CKM matrix $\iff$ test the validity of the Standard Model

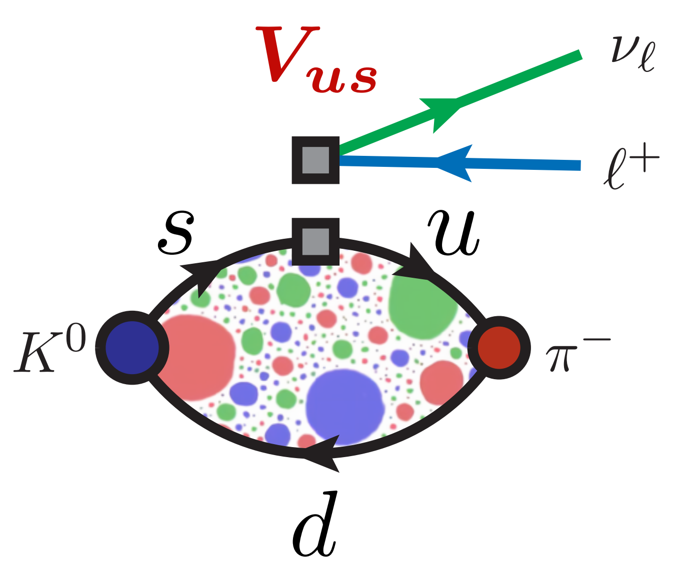
$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

in the Standard Model:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$







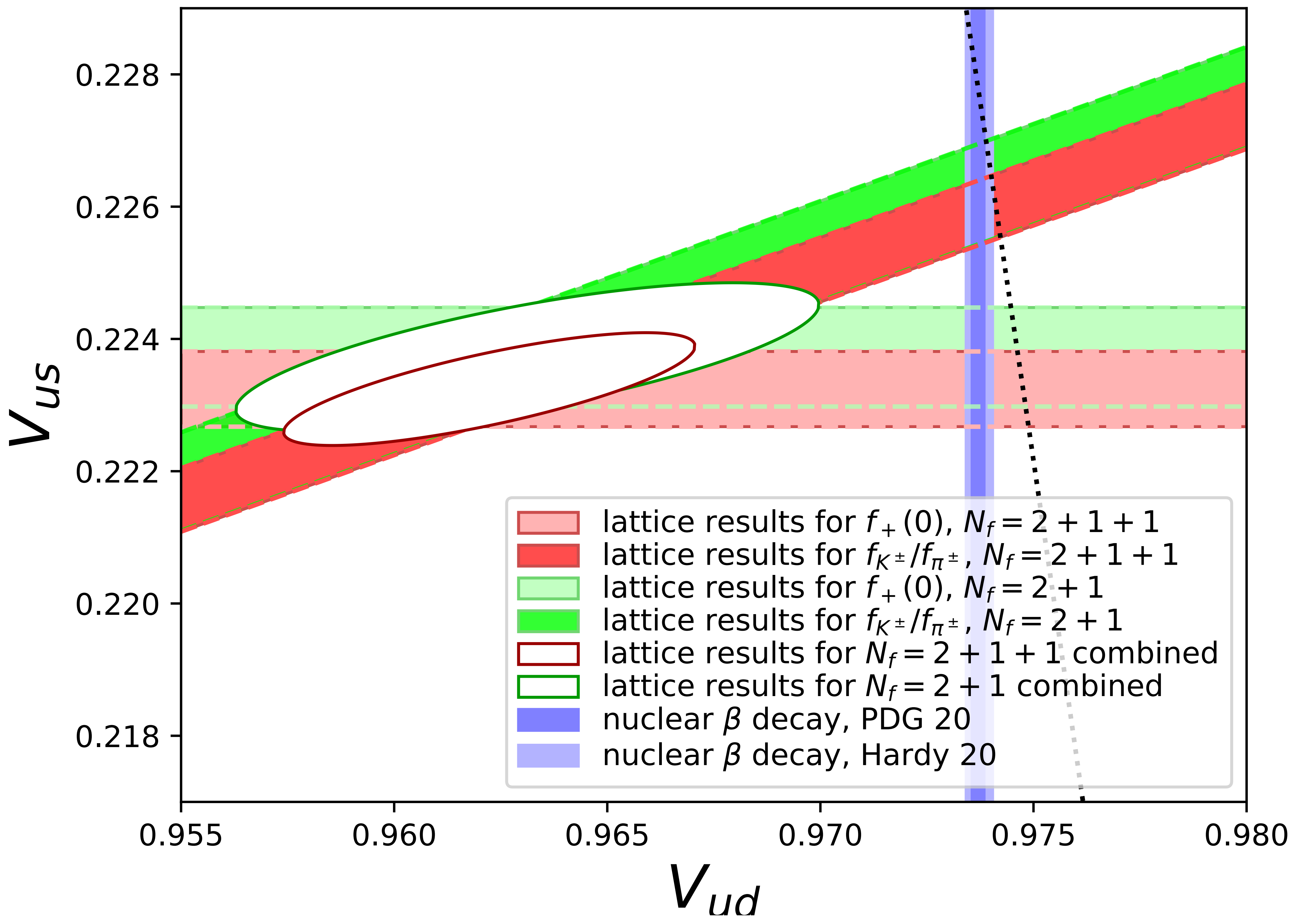
FLAG Review 2021.
EPJC **82**, 869 (2022)

$$\boldsymbol{f_{K^\pm}/f_{\pi^\pm} = 1.1934\,(19)}$$

$$\boldsymbol{f_+^{K\pi}(0) = 0.9698\,(17)}$$

First-row CKM unitarity tests

FLAG 2021 FLAG Review 2021. EPJC **82**, 869 (2022)



Different **tensions** in the V_{us}-V_{ud} plane:

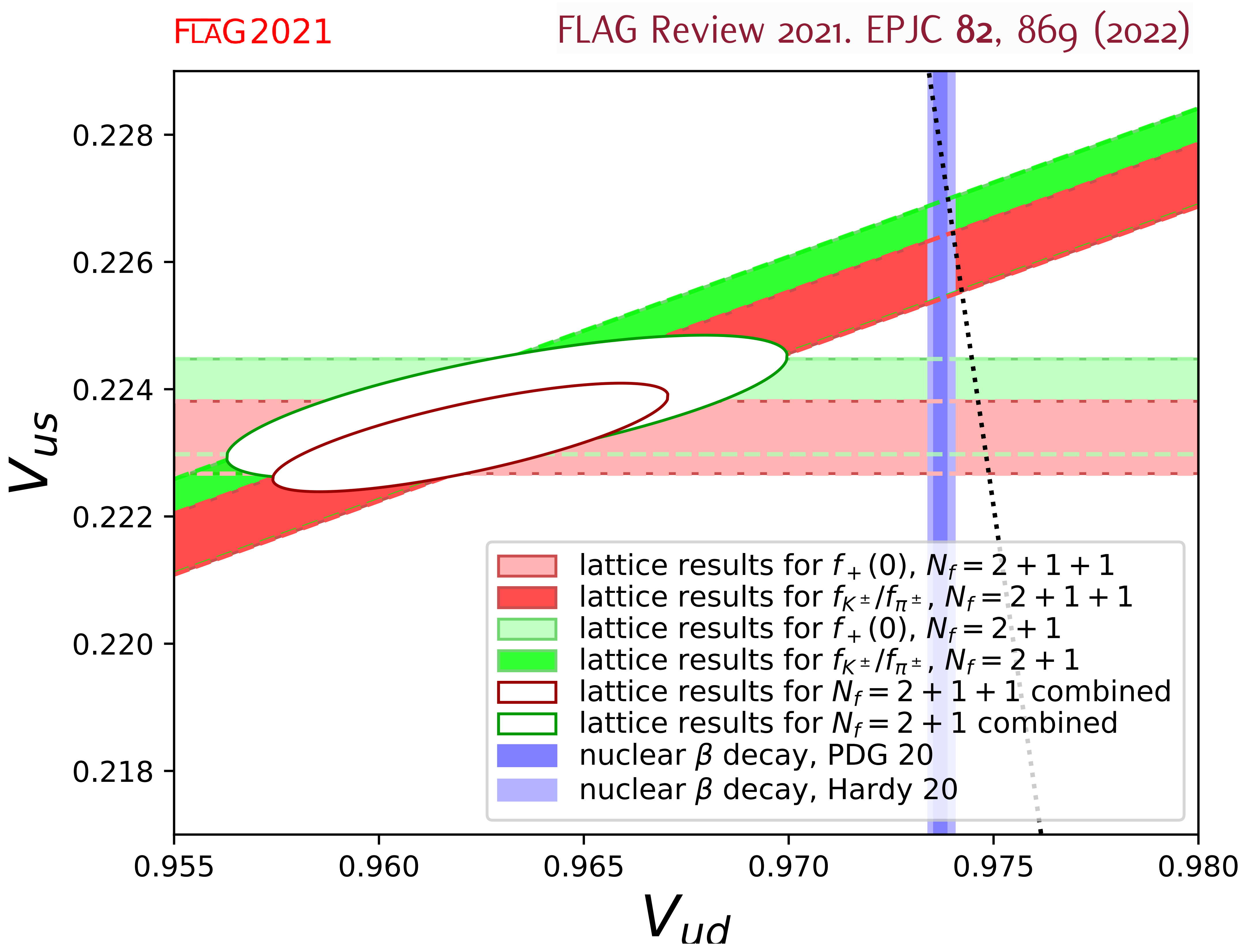
$$|V_u|^2 - 1 = 2.8\sigma$$

$$|V_u|^2 - 1 = 5.6\sigma \qquad |V_u|^2 - 1 = 3.3\sigma$$

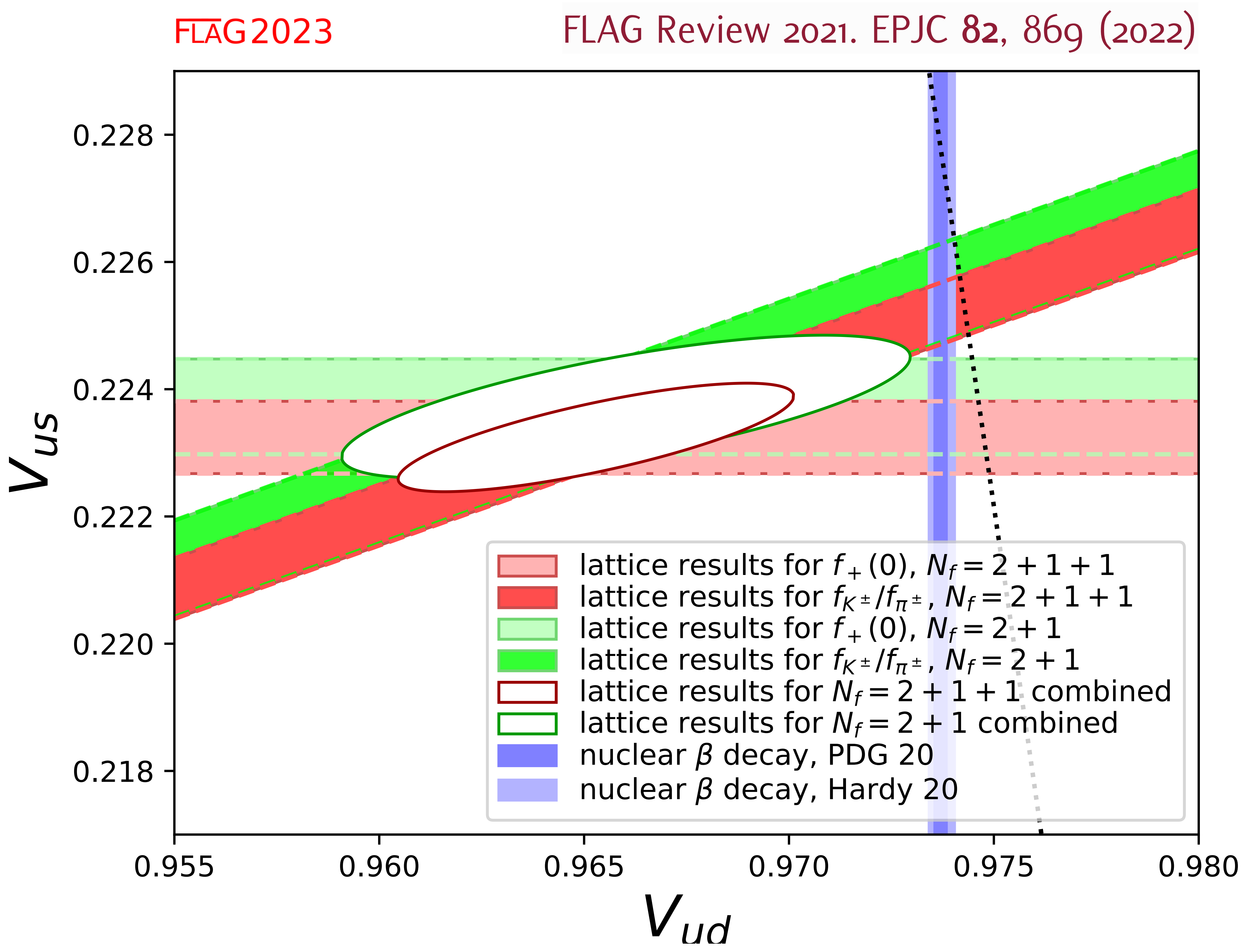
$$|V_u|^2 - 1 = 3.1\sigma \qquad |V_u|^2 - 1 = 1.7\sigma$$

Experimental and **theoretical** control of these quantities is of crucial importance to solve the issue

First-row CKM unitarity tests



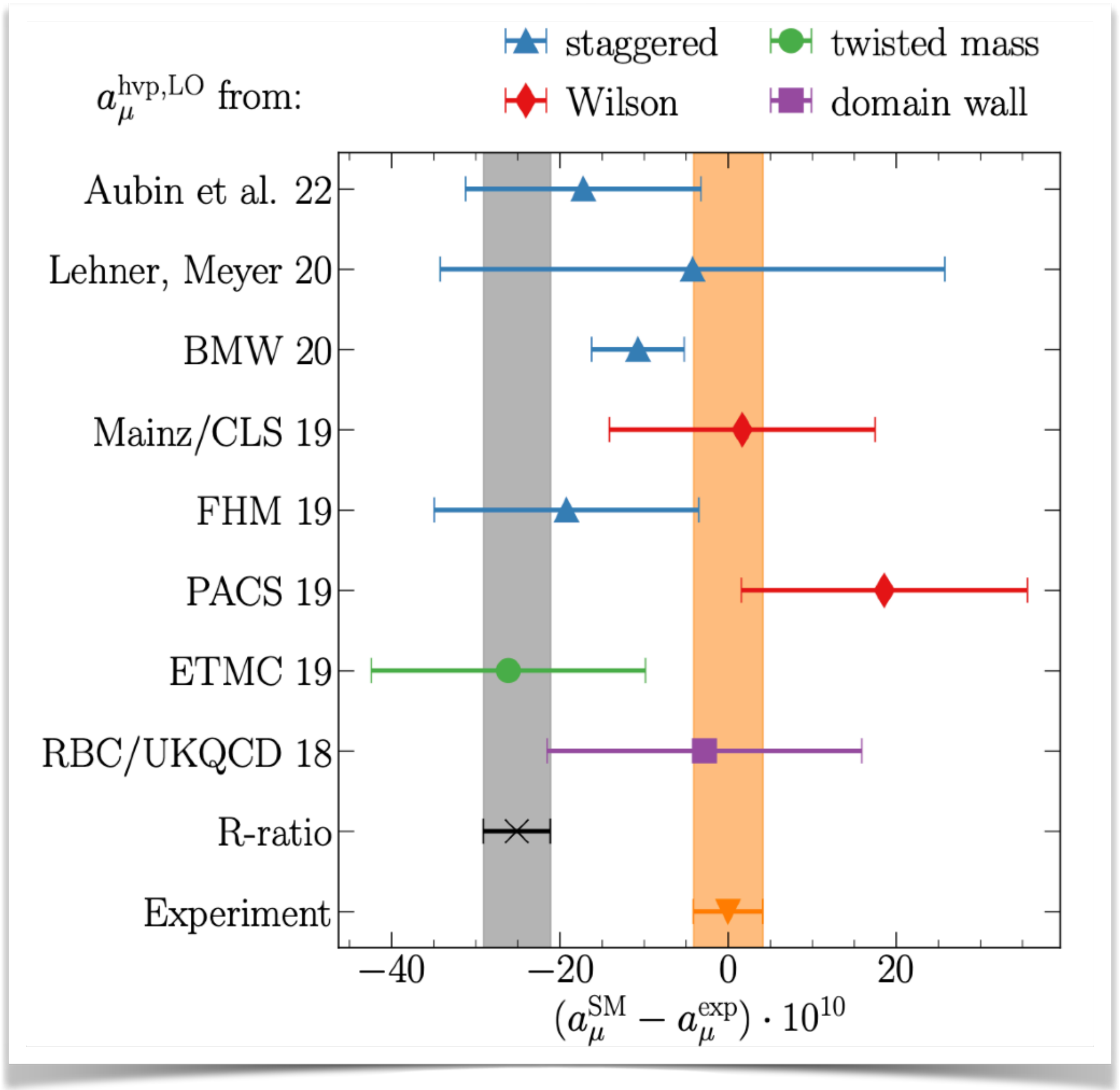
with QED corrections
from lattice calculation



without QED corrections
from lattice calculations

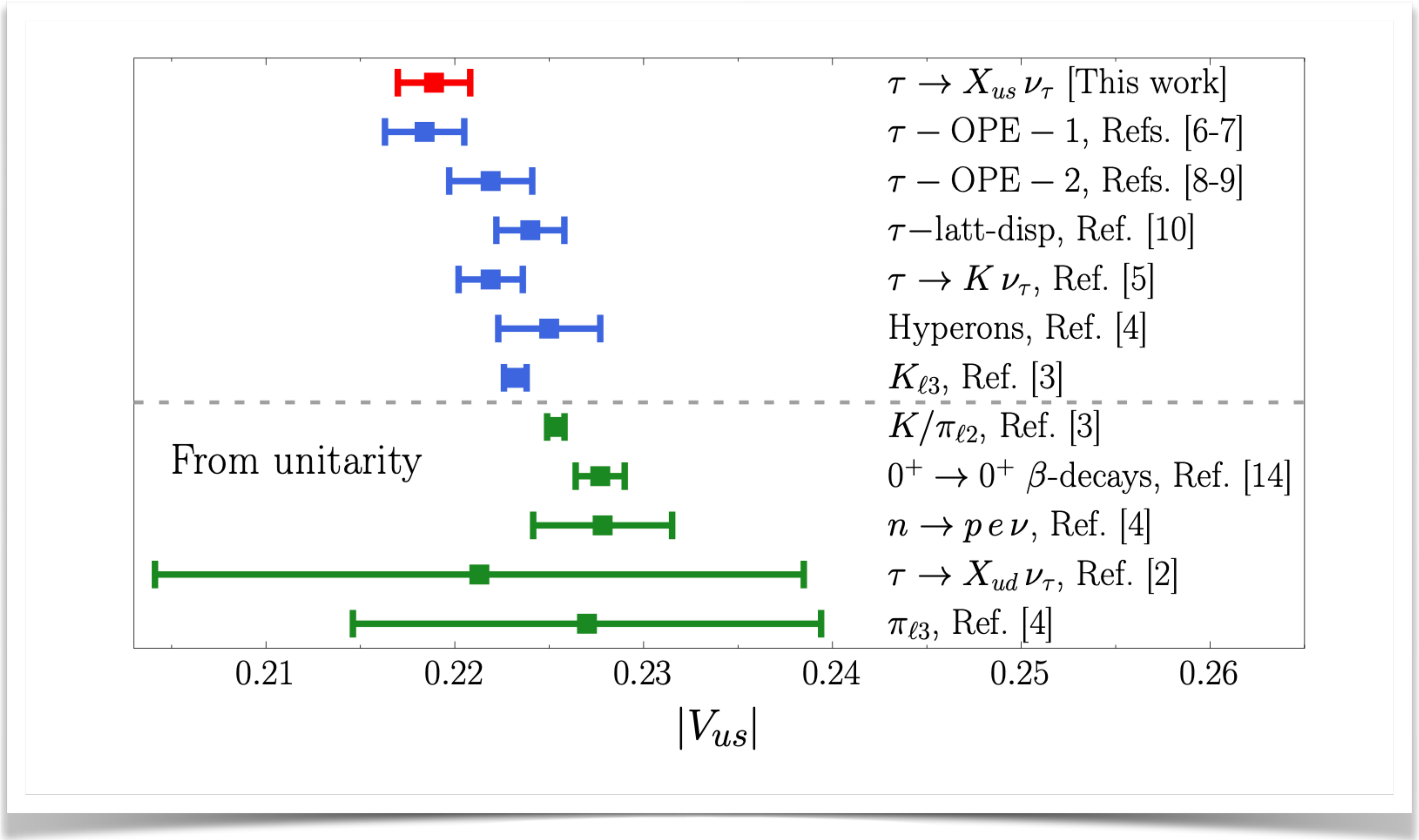
Other motivations...

S.Kuberski @Lattice2023



HVP contribution to muon g-2

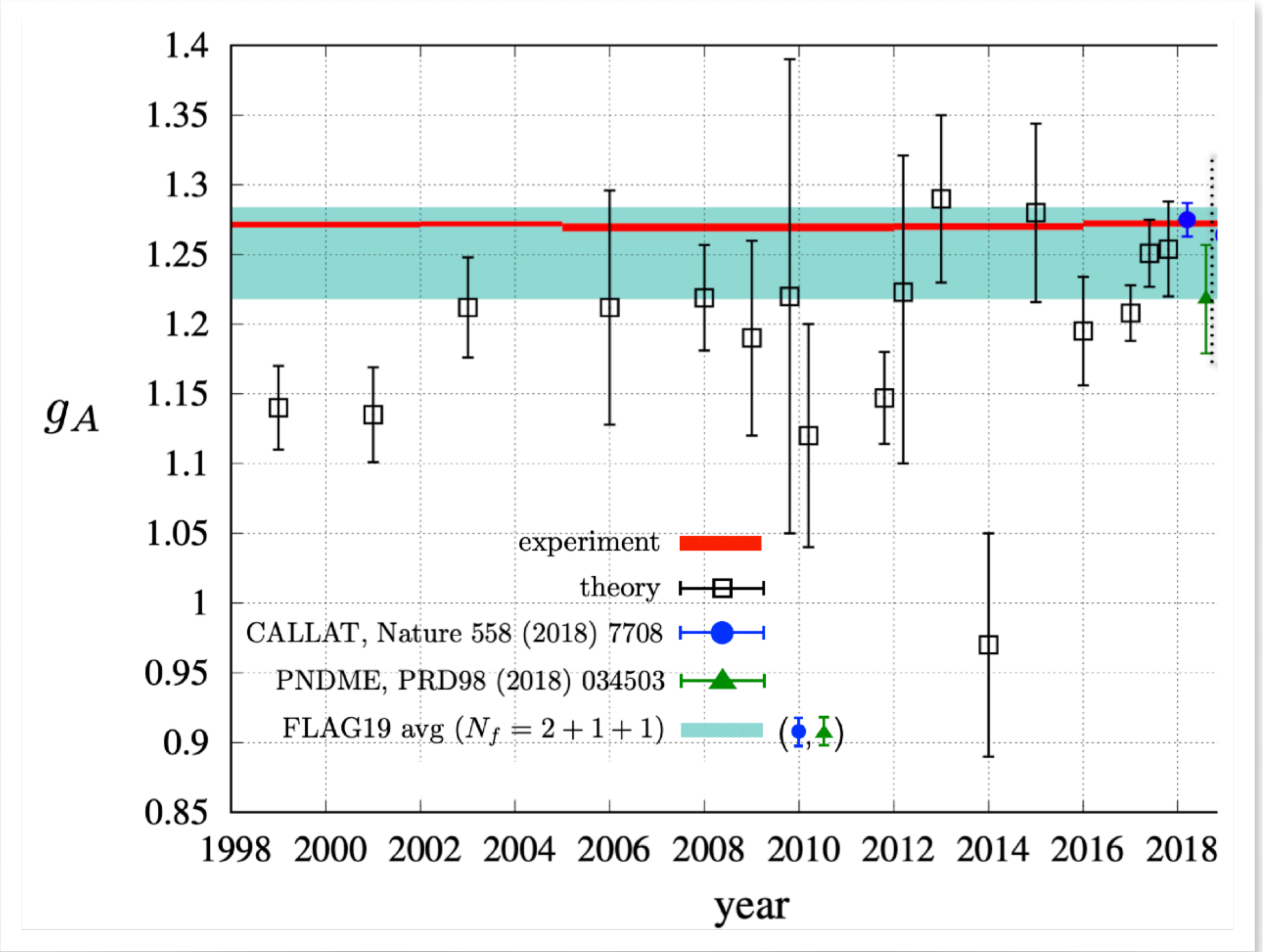
ETMC, PRL 132 (2024)



Inclusive hadronic decay of τ lepton

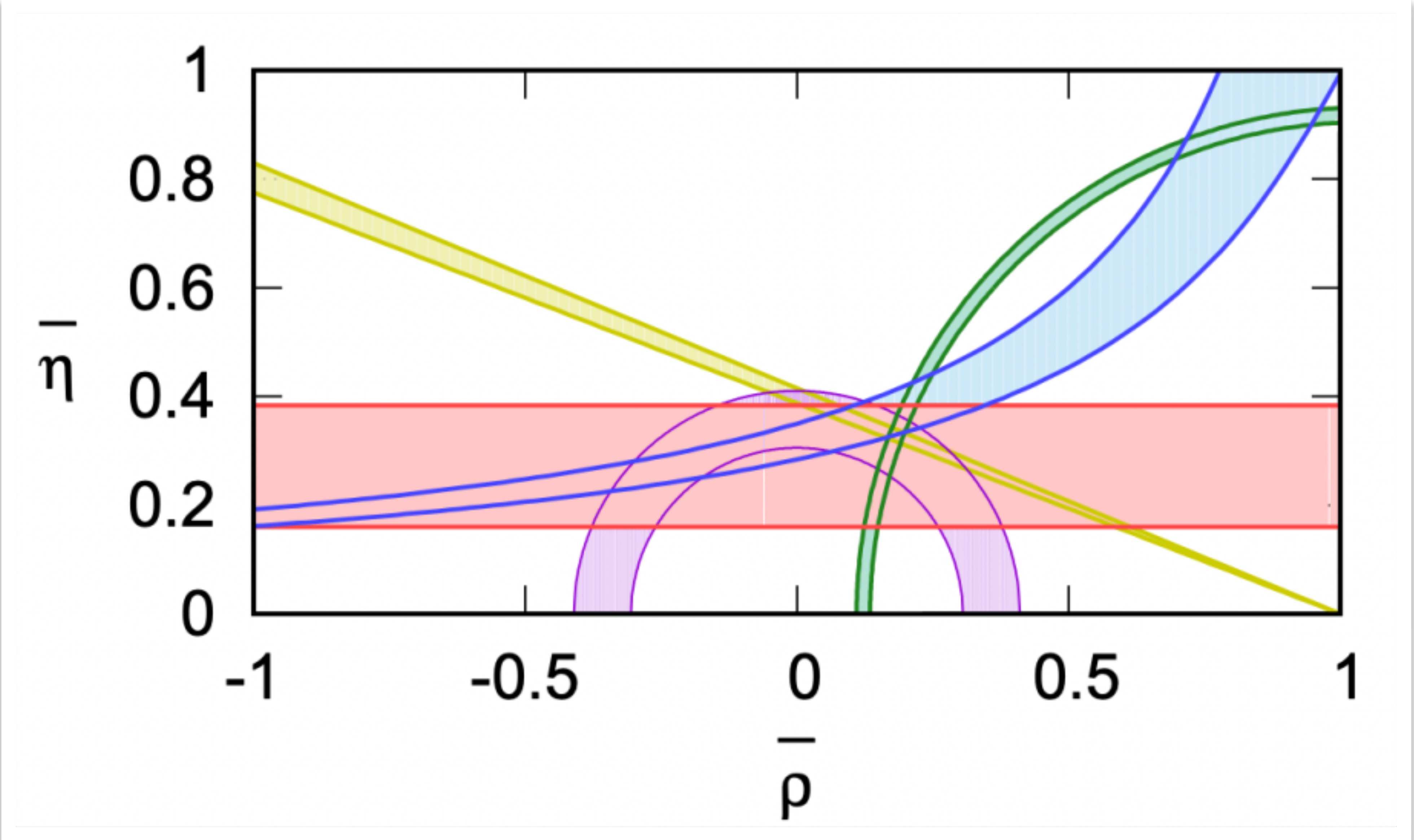
and more...

A.Nicholson, Lattice@CERN2024



Nucleon axial charge

R.Abbott et al., PRD **102** (2020)



Study of CP violation in the SM

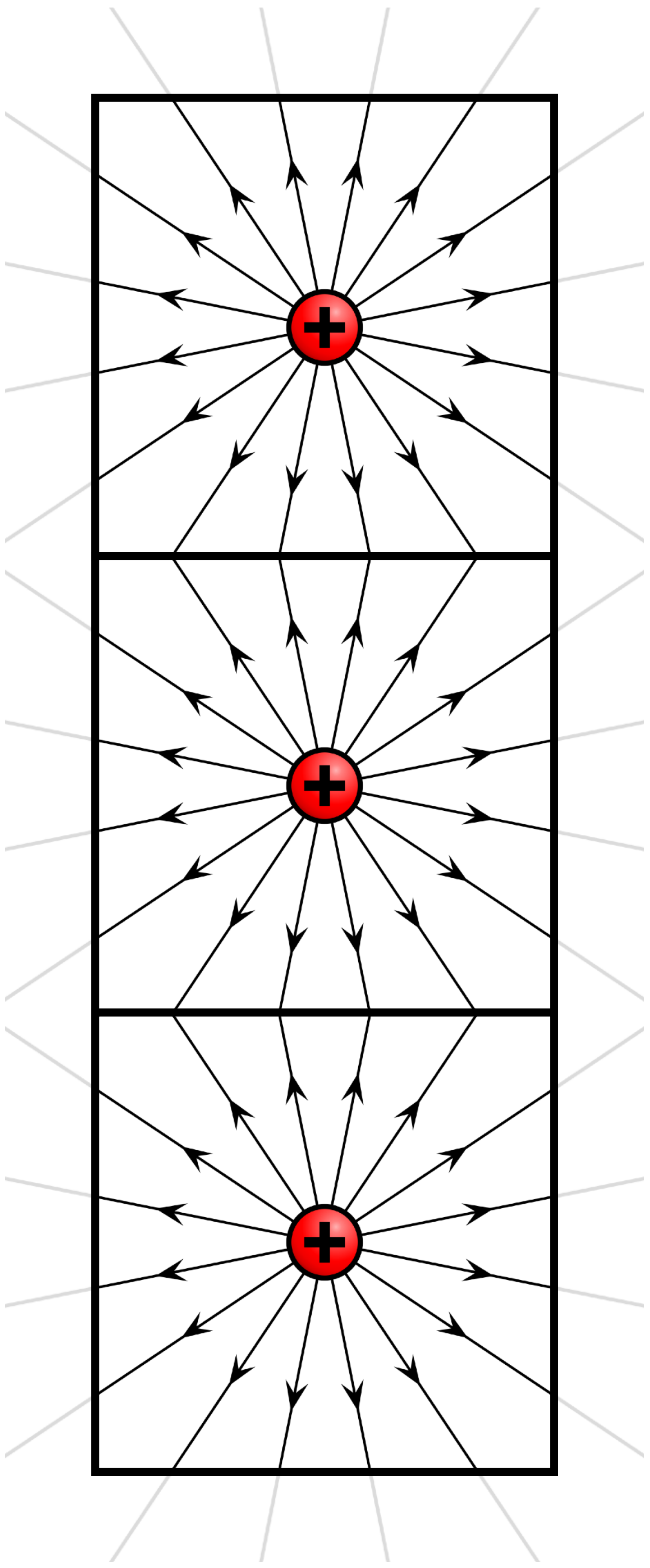
2. How

Computing **QED corrections on a <u>finite-sized</u> lattice** is challenging:

- **long-range interactions** don't like **finite volumes** with **periodic boundary conditions**
- **finite-volume effects** can be sizeable and power-like

  M.Hayakawa & S.Uno, PTP **120** (2008) / Z.Davoudi & M.Savage, PRD **90** (2014) / S.Borsanyi et al., Science **347** (2015)
- logarithmic **infrared divergences** arise in virtual/real decay rates

  V.Lubicz et al., PRD **95** (2017)



There are also recent proposals to compute radiative corrections as **convolutions** of hadronic correlators with **<u>infinite-volume</u>** QED kernels

N.Asmussen et al., [1609.08454] / T.Blum et al., PRD **96** (2017) / X.Feng & L.Jin, PRD **100** (2019) / N.Christ et al., [2304.08026]

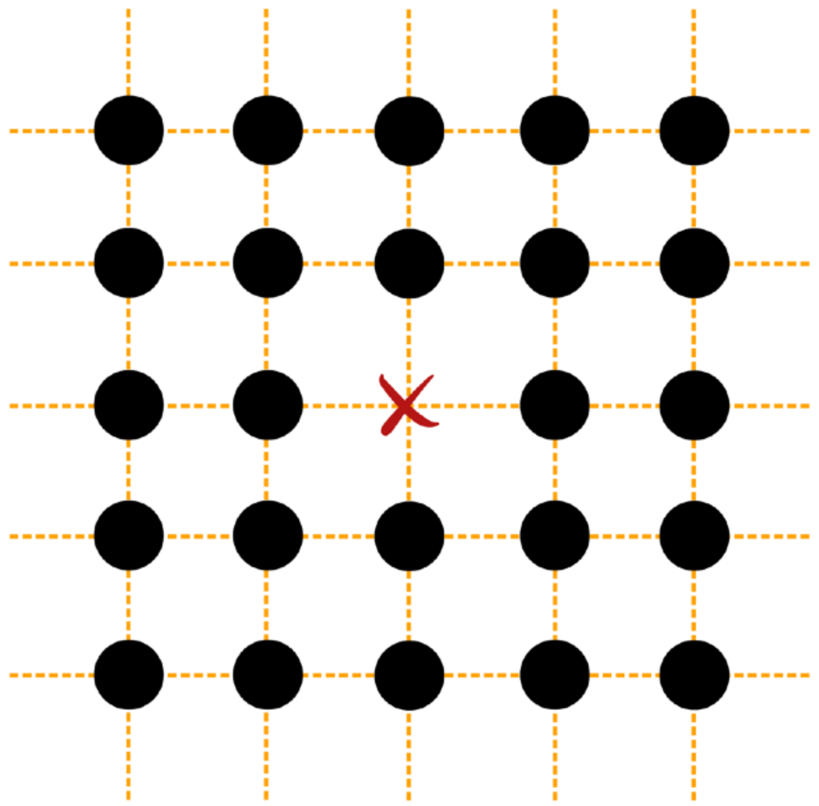
Charged states in a finite box

Gauss law: only zero net charge is allowed in a finite volume with periodic boundary conditions

$$Q = \int_{\text{p.b.c.}} \mathrm{d}^3\mathbf{x}\; j_0(t,\mathbf{x}) = \int_{\text{p.b.c.}} \mathrm{d}^3\mathbf{x}\; \boldsymbol{\nabla}\cdot\boldsymbol{E}(t,\mathbf{x}) = 0$$

Possible solutions:

$\mathrm{QED_L}$

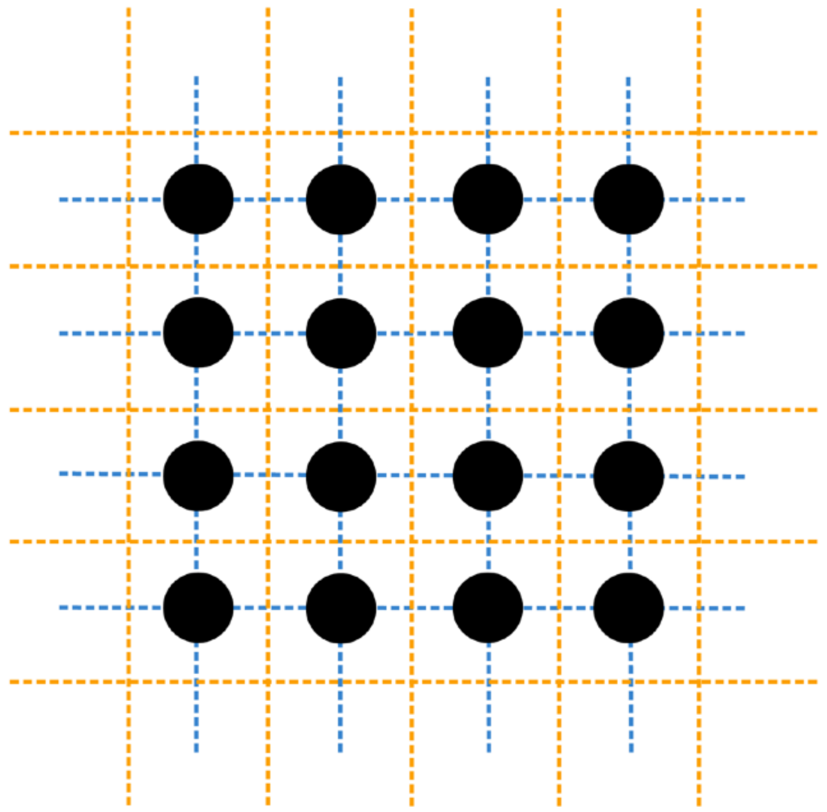


$\Omega_3 = 2\pi\mathbb{Z}^3/L$

remove spatial zero-mode of the photon field

M.Hayakawa & S.Uno, PTP **120** (2008)

$\mathrm{QED_{C^*}}$

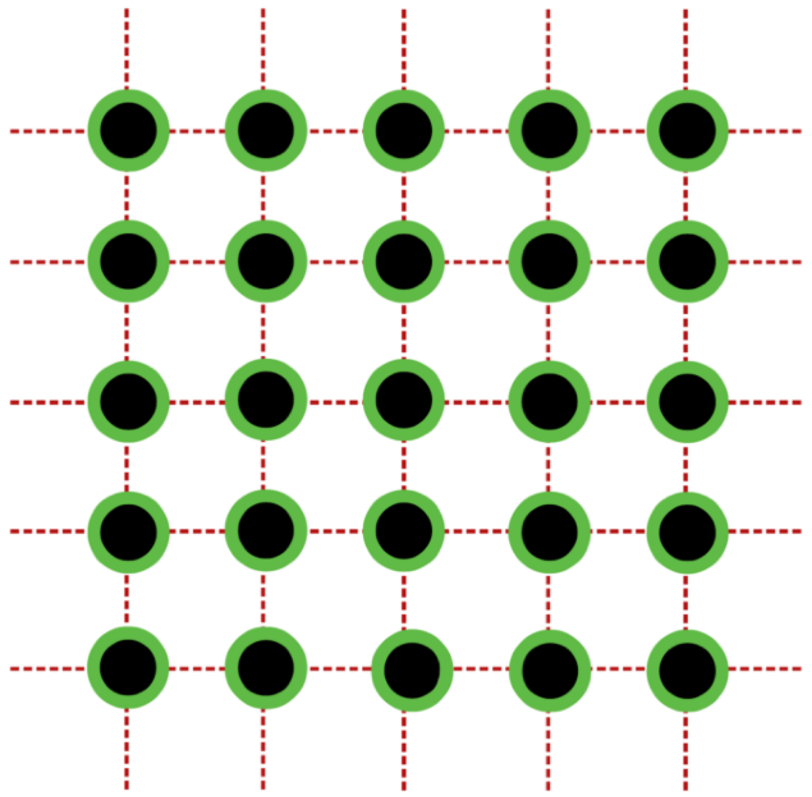


$\Omega_3 = 2\pi\mathbb{Z}^3/L$ $\quad \Omega_3' = (2\,\mathbb{Z}^3 + \bar{\mathbf{n}})\pi/L$

employ C* boundary conditions

A.S.Kronfeld & U.-J.Wiese, NPB **357** (1991)
B.Lucini et al., JHEP **02** (2016)

$\mathrm{QED_m}$

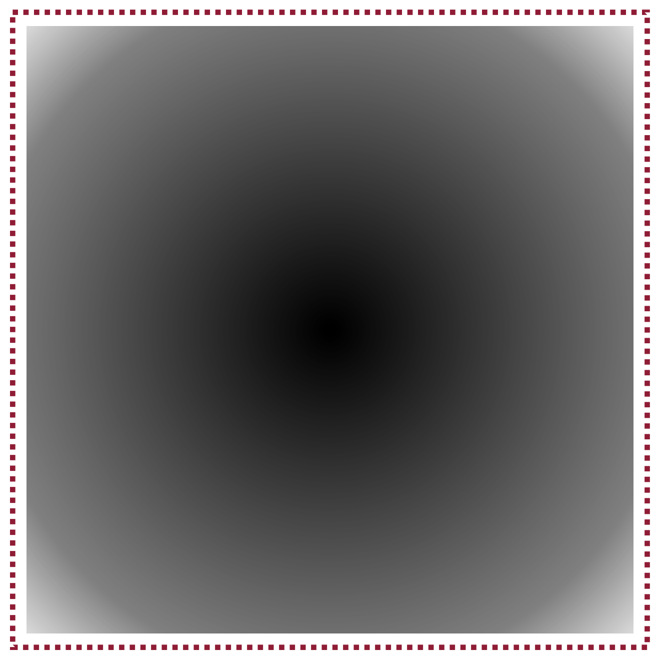


$\Omega_4 = 2\pi\{\mathbb{Z}^3/L, \mathbb{Z}/T\}$

use massive photon m_γ

M.G.Endres et al., [1507.08916]

$\mathrm{QED_\infty}$



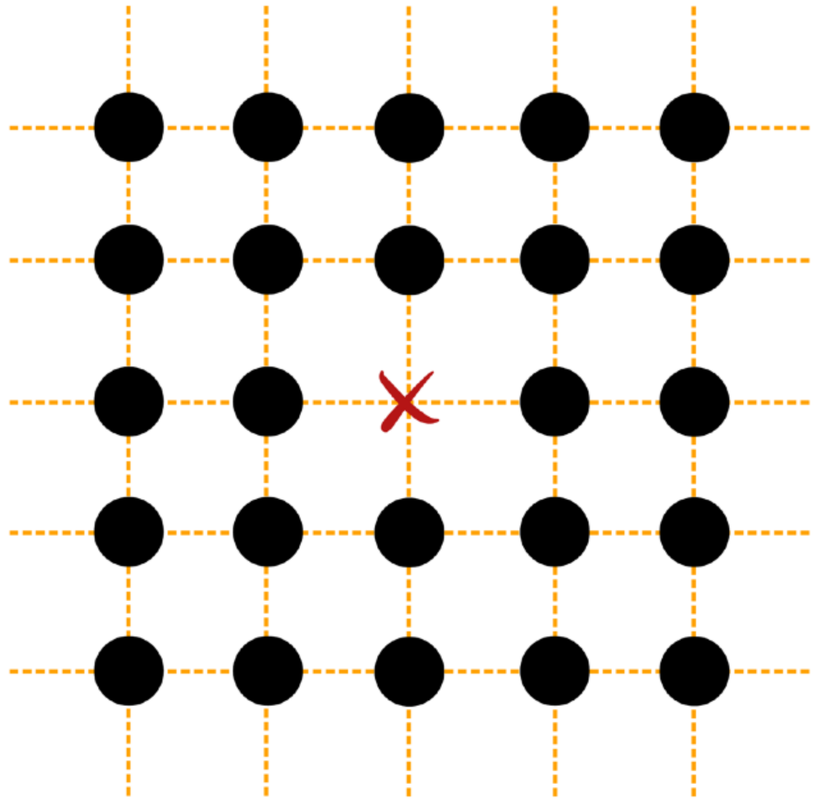
$\Omega_4 = \mathbb{R}^4$

infinite-volume reconstruction

X.Feng & L.Jin, PRD **100** (2019)
N.Christ et al., [2304.08026]

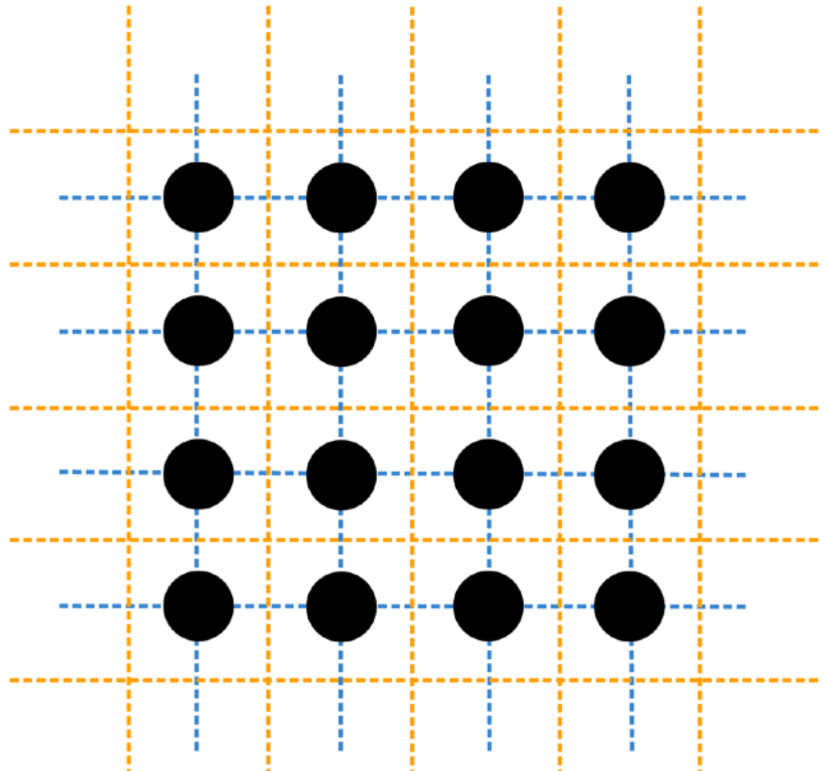
Charged states in a finite box

$\mathrm{QED_L}$



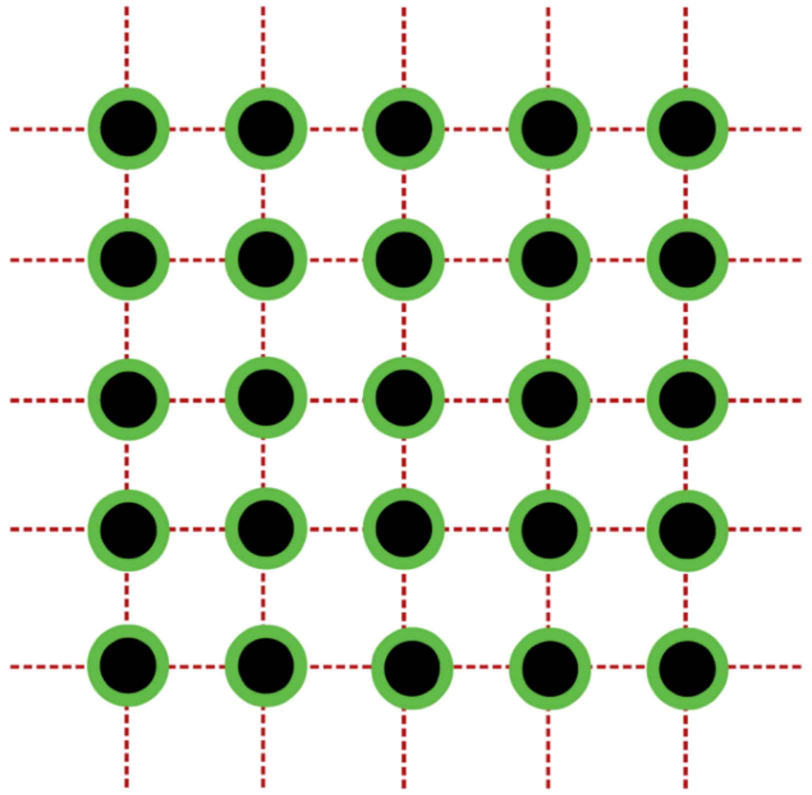
$\Omega_3 = 2\pi\mathbb{Z}^3/L$

$\mathrm{QED_{C^*}}$



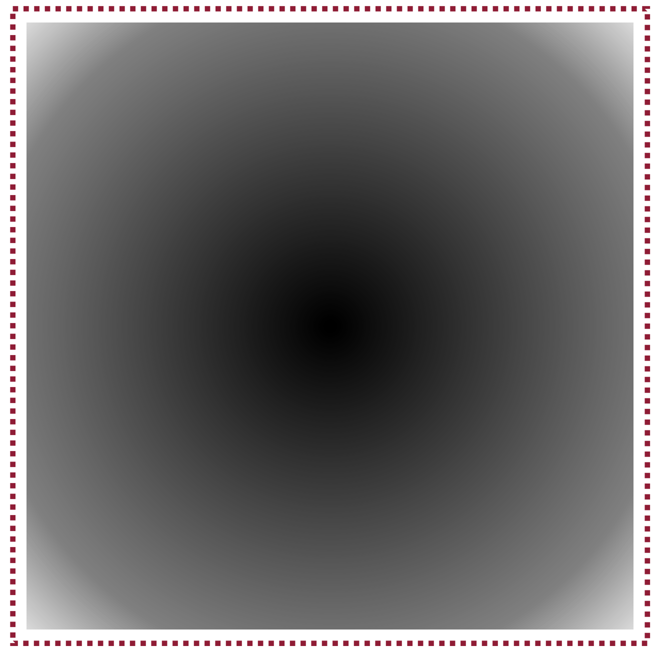
$\Omega_3 = 2\pi\mathbb{Z}^3/L$ $\quad \Omega_3' = (2\,\mathbb{Z}^3 + \bar{\mathbf{n}})\pi/L$

$\mathrm{QED_m}$



$\Omega_4 = 2\pi\{\mathbb{Z}^3/L, \mathbb{Z}/T\}$

$\mathrm{QED_\infty}$



$\Omega_4 = \mathbb{R}^4$

finite-volume photon

non-local | local

∞-volume photon

power-like finite-volume effects

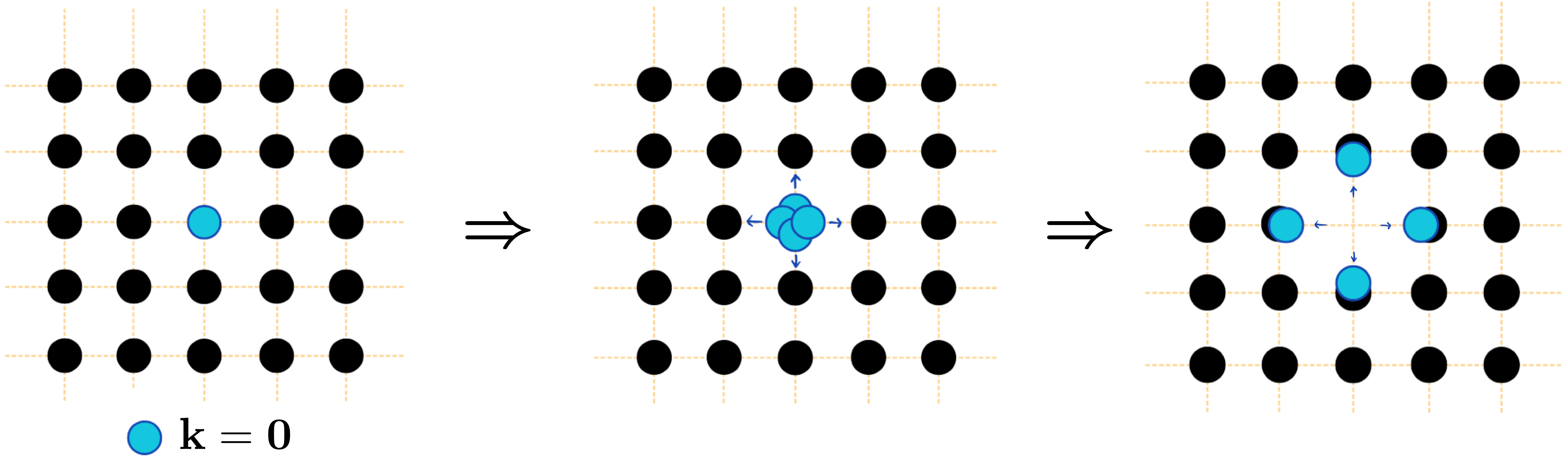
exponential finite-volume effects

UV / IR mixing | dedicated ensembles | two IR regulators | observable-dependent

QEDr regularization

Special case of "IR-improvement"

Z.Davoudi et al., PRD 99 (2019)
MDC, PoS LATTICE2023 (2024) [2401.07666]



The spatial zero mode is not **removed** but **redistributed over the neighbouring modes** on a shell of radius $|\mathbf{p}| = \frac{2\pi}{L}|\mathbf{r}| \quad (\mathbf{r} \in \mathbb{Z}^3)$

QED$_\text{L}$: $D_L^{\mu\nu}(k_0,\mathbf{k}) = \delta^{\mu\nu}\frac{1-\delta_{\mathbf{k},\mathbf{0}}}{k_0^2+\mathbf{k}^2}$ $\Rightarrow$ **QED$_\text{r}$:** $D_{\mathbf{p}}^{\mu\nu}(k_0,\mathbf{k}) = \delta^{\mu\nu}\frac{1-\delta_{\mathbf{k},\mathbf{0}}}{k_0^2+\mathbf{k}^2} + \frac{\delta_{\mathbf{k}^2,\mathbf{p}^2}}{n(\mathbf{p}^2)}\frac{\delta^{\mu\nu}}{k_0^2+\mathbf{p}^2}$

3. What

Lattice QCD+QED calculations can provide IB corrections for several hadronic observables:

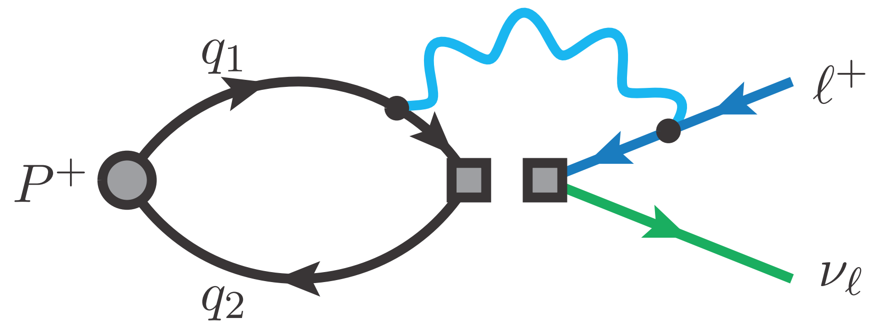
- hadron masses & quark masses
- HVP contribution to muon g-2
- leptonic & semileptonic weak decay rates
- CP violation parameters
- ...

As hadronic uncertainties **decrease**, such corrections become more and more **relevant**!

This is a growing research field: improvements expected in the near future

3. What

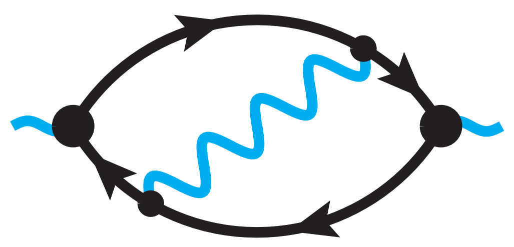
leptonic decays



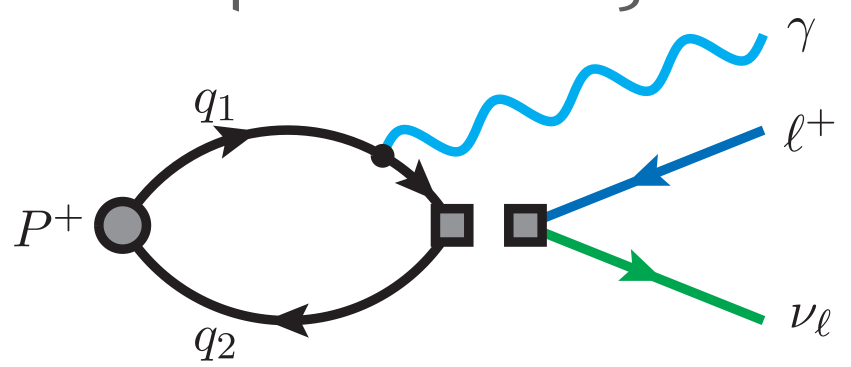
N. Carrasco et al., PRD 91 (2015)
V. Lubicz et al., PRD 95 (2017)
N.Tantalo et al., [1612.00199v2]
D. Giusti et al., PRL 120 (2018)
MDC et al., PRD 100 (2019)
MDC et al., PRD 105 (2022)
P.Boyle, MDC et al., JHEP 02 (2023)
N.Christ et al., [2304.08026]

White Paper: Phys. Rept. 887 (2020)

HVP contribution
to muon g-2



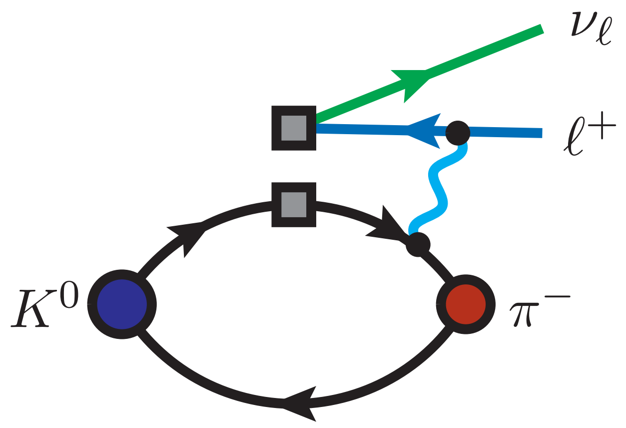
real photon emission
leptonic decays



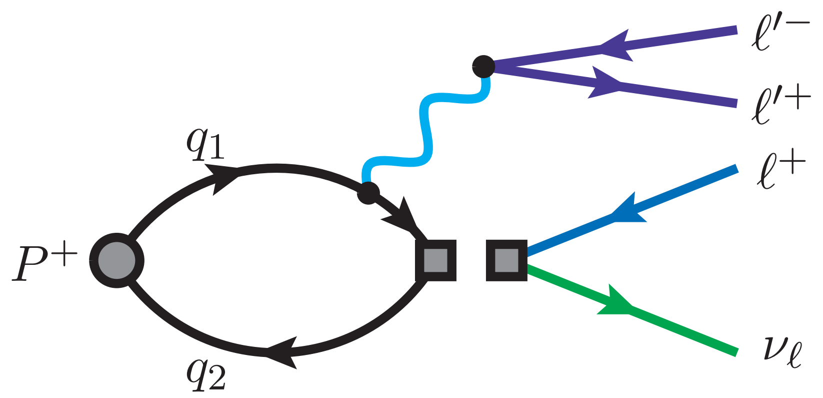
G.M. de Divitiis et al., [1908.10160]
C. Kane et al., [1907.00279 & 2110.13196]
R. Frezzotti et al., PRD 103 (2021)
A.Desiderio et al., PRD 102 (2021)
D. Giusti et al., [2302.01298]
R.Frezzotti et al., [2306.05904]

C.Sachrajda et al., [1910.07342]
N.Christ et al., [2304.08026]

semileptonic decays



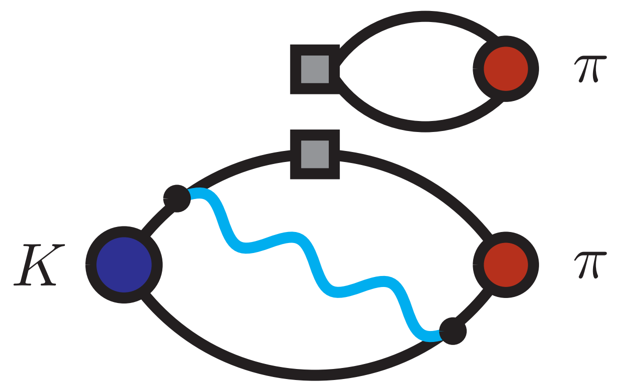
virtual photon emission
leptonic decays



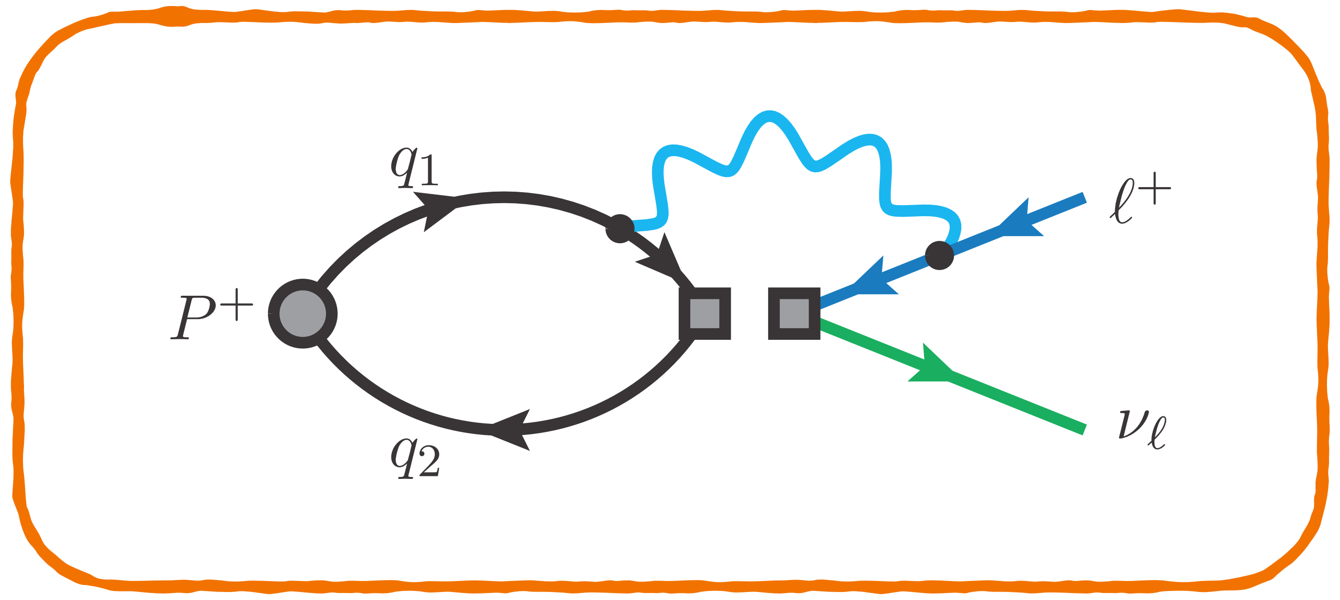
G.Gagliardi et al., Phys. Rev. D 105 (2022)
R.Frezzotti et al., [2306.07228]

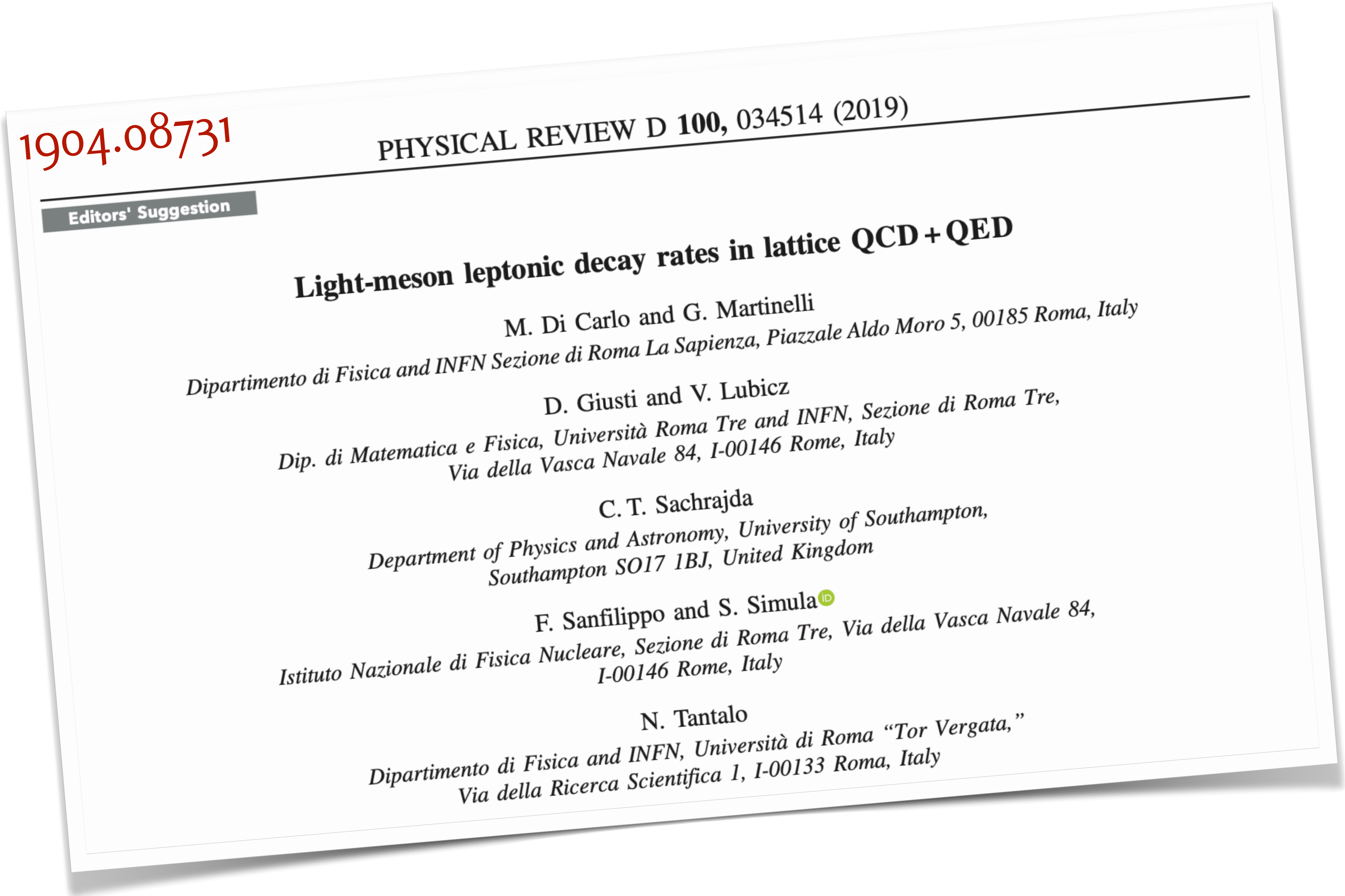
R.Abbott et al., PRD 102 (2020)
Z.Bai et al., PRL 115 (2015)
N.Christ et al., PRD 106 (2022)
N.Christ & X.Feng, EPJ Web Conf. 175 (2018)
Y.Cai & Z.Davoudi, [1812.11015]

hadronic decays



leptonic decays of light pseudoscalar mesons




1904.08731

PHYSICAL REVIEW D **100,** 034514 (2019)

Editors' Suggestion

Light-meson leptonic decay rates in lattice QCD + QED

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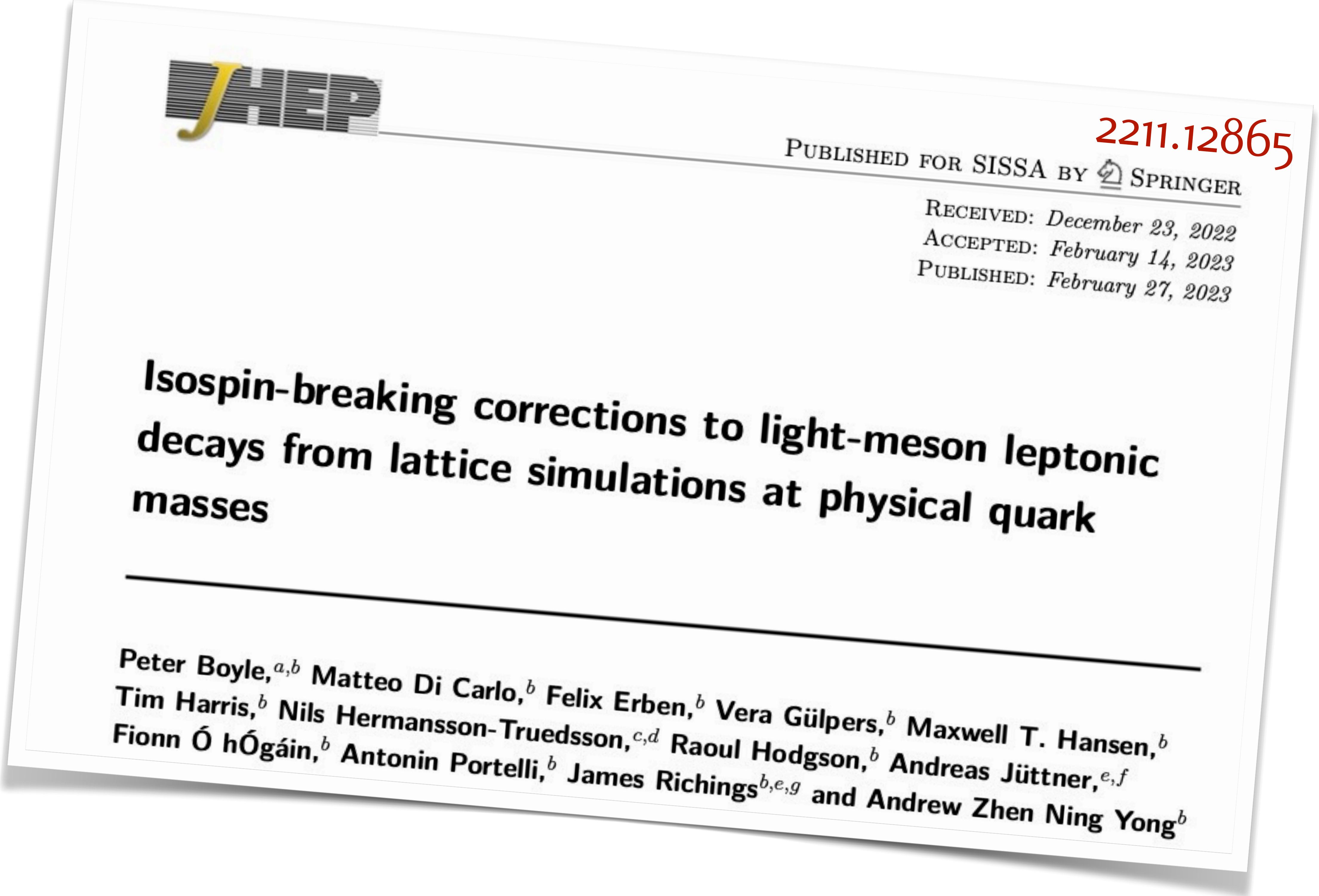
2211.12865

JHEP

PUBLISHED FOR SISSA BY SPRINGER

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PUBLISHED: *February 27, 2023*

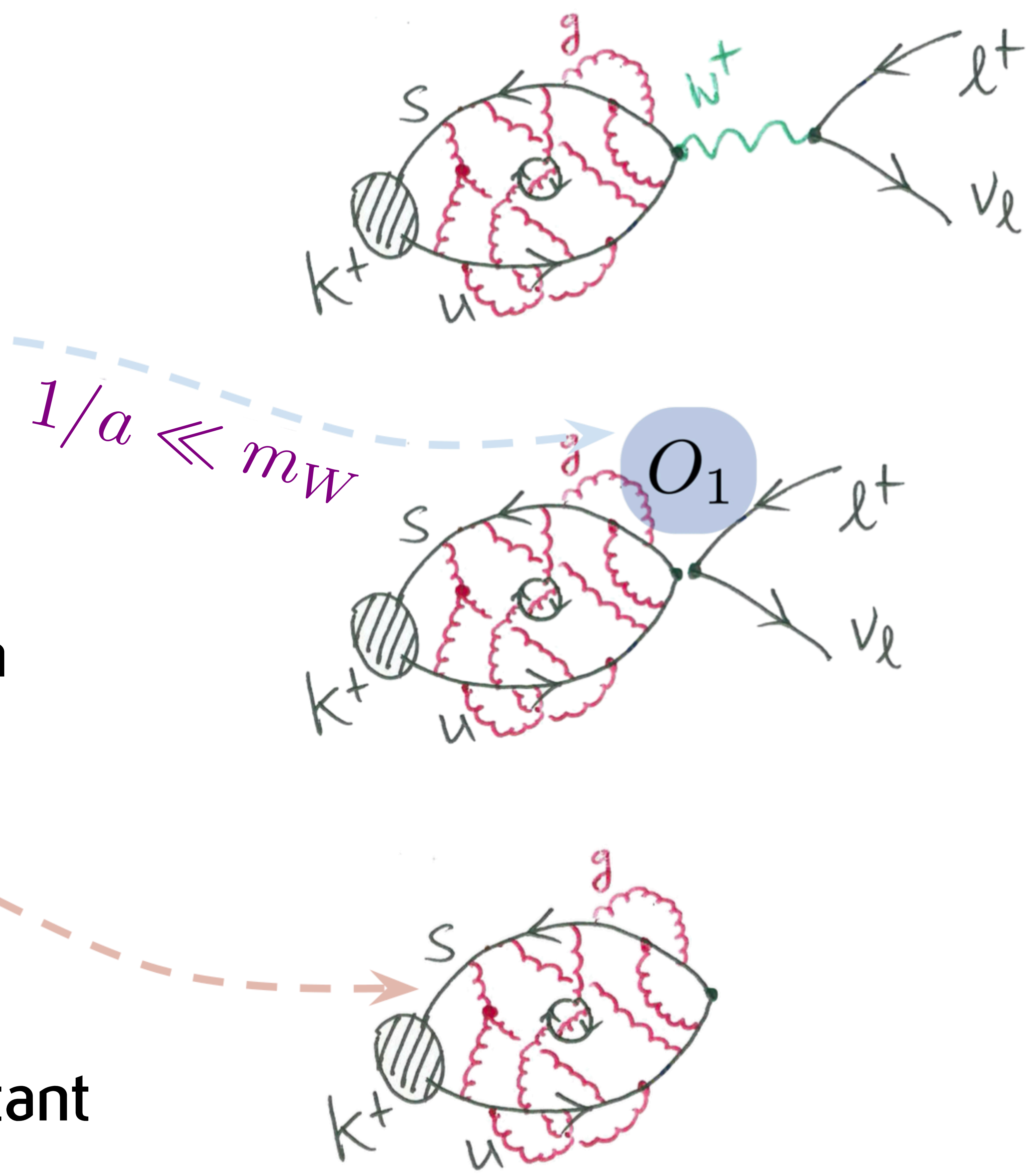
Isospin-breaking corrections to light-meson leptonic decays from lattice simulations at physical quark masses

Peter Boyle,[a,b] Matteo Di Carlo,[b] Felix Erben,[b] Vera Gülpers,[b] Maxwell T. Hansen,[b] Tim Harris,[b] Nils Hermansson-Truedsson,[c,d] Raoul Hodgson,[b] Andreas Jüttner,[e,f] Fionn Ó hÓgáin,[b] Antonin Portelli,[b] James Richings[b,e,g] and Andrew Zhen Ning Yong[b]

Leptonic decays of pseudoscalar mesons

Can be studied in an **effective Fermi theory** with the W-boson integrated out and the local interaction described by

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left[\bar{q}_2\, \gamma_\mu (1-\gamma_5)\, q_1\right] \left[\bar{\nu}_\ell\, \gamma^\mu (1-\gamma_5)\, \ell\right]$$



In the **PDG convention**, the tree-level decay rate takes the form

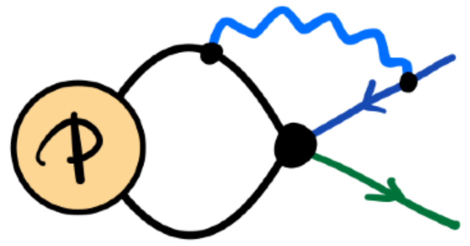
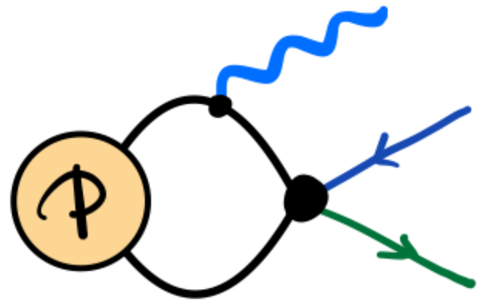
$$\Gamma_P^{\text{tree}} = \frac{G_F^2}{8\pi} m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2 m_P \left[f_{P,0}\right]^2$$

with the non-perturbative dynamic encoded in the **decay constant**

$$\mathcal{Z}_0 \langle 0 | \bar{q}_2\, \gamma_0 \gamma_5\, q_1 | P, \mathbf{0} \rangle^{(0)} = i\, m_{P,0}\, f_{P,0}$$

Leptonic decay rate at $\mathcal{O}(\alpha)$

- **IR divergences** appear in intermediate steps of the calculation — F. Bloch & A. Nordsieck, PR 52 (1937) 54

$$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\mathrm{IR}}\to 0} \Big\{$$

$$+$$

$$\Big\}$$

IR finite — IR divergent — IR divergent

- **UV divergences**: need to include QED corrections to the renormalization of the weak Hamiltonian

$$\mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left(1 + \frac{\alpha_{\rm em}}{\pi} \ln\left(\frac{M_Z}{M_W}\right)\right) \underbrace{[\,\bar{q}_2\, \gamma_\mu (1-\gamma_5)\, q_1\,]\,[\,\bar{\nu}_\ell\, \gamma^\mu (1-\gamma_5)\, \ell\,]}_{O_1^{\rm W\text{-}reg}(M_W)}$$

A.Sirlin, NPB **196** (1982)
E.Braaten & C.S.Li, PRD **42** (1990)

$$O_1^{\rm W\text{-}reg}(M_W) = Z^{\rm W\text{-}S}\left(\frac{M_W}{\mu}, \alpha_s(\mu), \alpha_{\rm em}\right) O_1^{\rm S}(\mu)$$

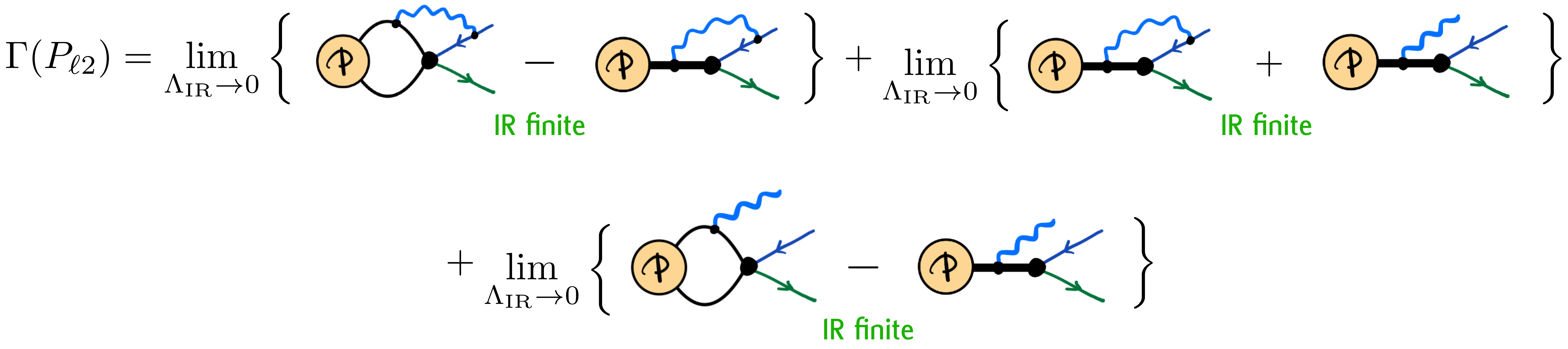
- perturbative @ 2 loops in QCD+QED
- non-perturbative in lattice QCD+QED

MDC et al., PRD **100** (2019)

Leptonic decay rate at $\mathcal{O}(\alpha)$

The RM123+Soton recipe

F. Bloch & A. Nordsieck, PR **52** (1937)
N. Carrasco et al., PRD **91** (2015)
V. Lubicz et al., PRD **95** (2017)
D. Giusti et al., PRL **120** (2018)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

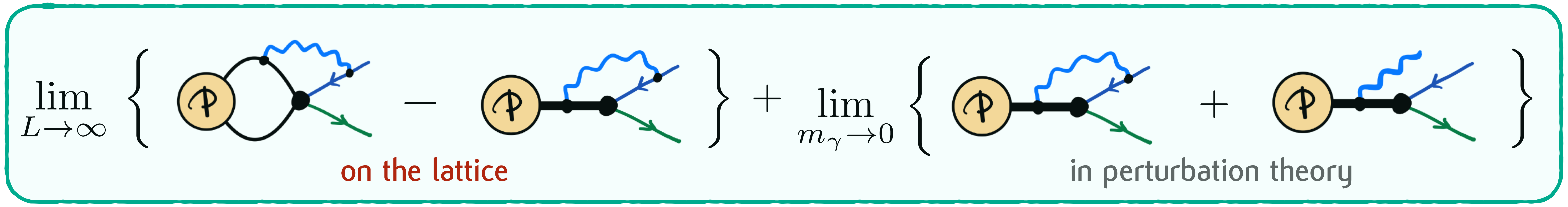


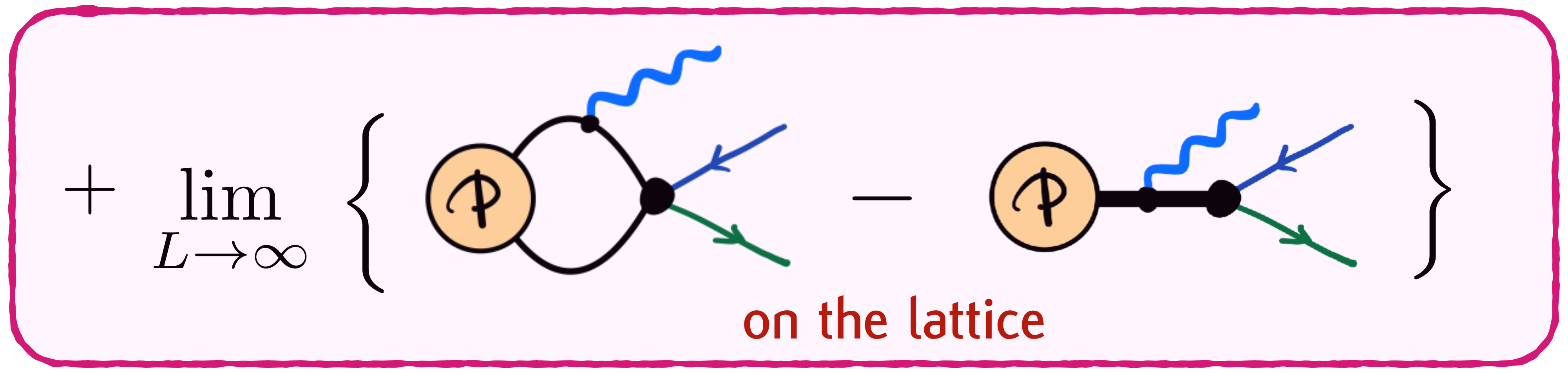
Leptonic decay rate at $\mathcal{O}(\alpha)$

The RM123+Soton recipe

F. Bloch & A. Nordsieck, PR **52** (1937)
N. Carrasco et al., PRD **91** (2015)
V. Lubicz et al., PRD **95** (2017)
D. Giusti et al., PRL **120** (2018)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

$\Gamma(P_{\ell 2}) =$





enough for $K_{\mu 2}$ and $\pi_{\mu 2}$

finite-volume scaling well studied

V.Lubicz et al., PRD **95** (2017) N.Tantalo et al., [1612.00199v2]
MDC et al., PRD **105** (2022)

relevant for K_{e2} and π_{e2}
& decays of heavier mesons

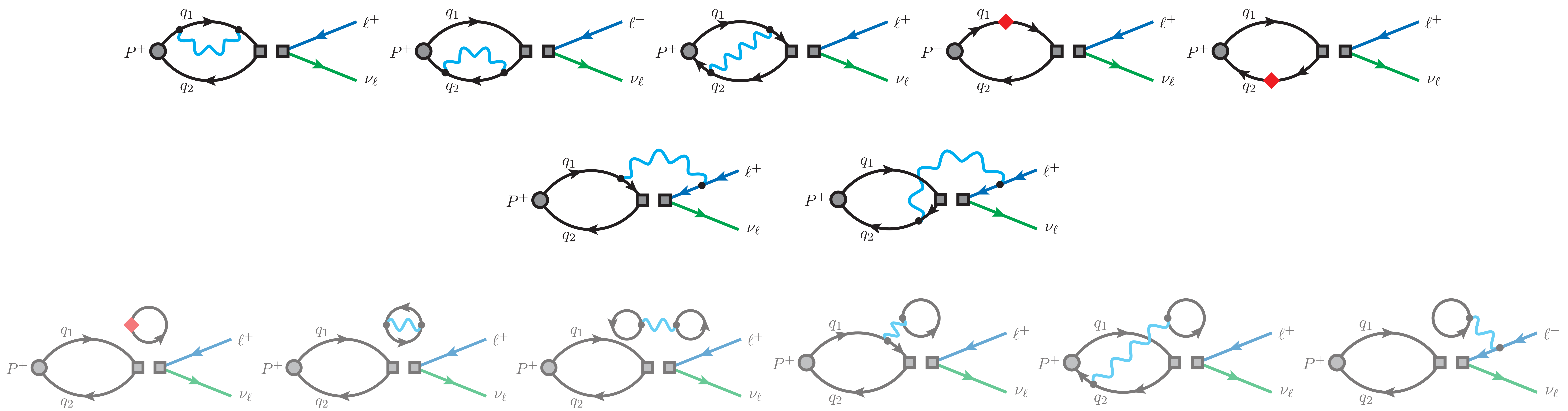
G.M. de Divitiis et al., [1908.10160]
R. Frezzotti et al., PRD **103** (2021)
A. Desiderio et al., PRD **102** (2021)
C. Kane et al., [1907.00279 & 2110.13196]
D. Giusti et al., [2302.01298]
R.Frezzotti et al., [2306.05904]

see C.Sachrajda's talk at 11.30

IB corrections to the decay amplitude

G.M.de Divitiis et al. [RM123], PRD **87** (2013)

RM123 perturbative method: expand lattice path-integral around isosymmetric point $\alpha = m_{\mathrm{u}} - m_{\mathrm{d}} = 0$



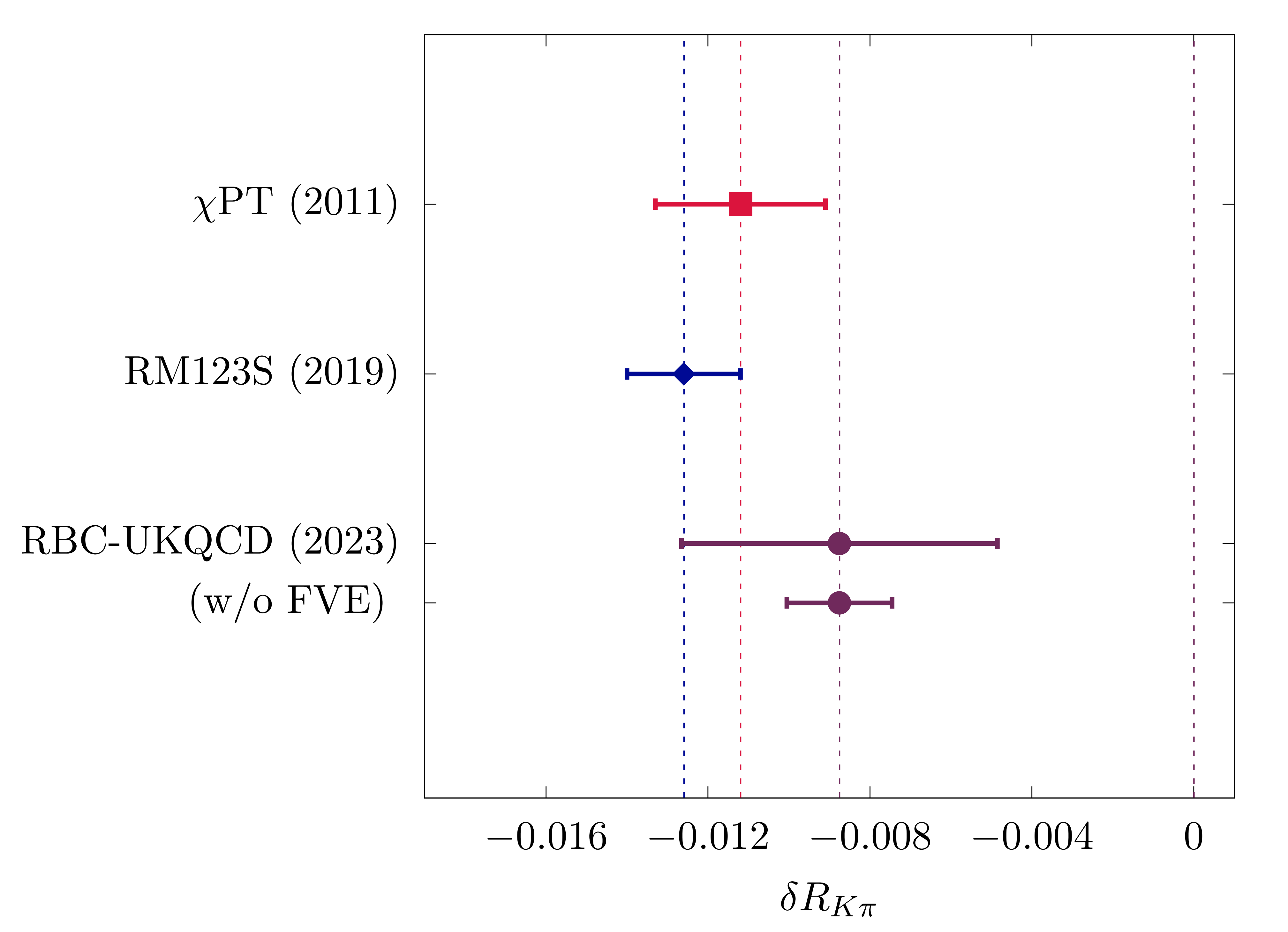
Both RM123S and RBC-UKQCD calculations are performed in the **electro-quenched approximation: sea quarks electrically neutral**

Results for $\delta R_{K\pi}$

V. Cirigliano et al., PLB **700** (2011)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

- ■ $\delta R_{K\pi} = -0.0112\,(21)$
- ◆ $\delta R_{K\pi} = -0.0126\,(14)$
- ● $\delta R_{K\pi} = -0.0086\,(13)(39)_{\mathrm{vol.}}$

$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi})$$



- Strong evidence that $\delta R_{K\pi}$ **can be computed from first principles non-perturbatively on the lattice!**
- Results highlight crucial role of **finite-volume effects**: ongoing effort to tame such systematic uncertainty
- **Errors on** $|V_{us}|/|V_{ud}|$ from **theoretical** inputs can become comparable with those from **experiments**

QED finite-volume effects

In **finite-volume** (massless) QED the photon zero modes require a regularisation

$$\Delta g(L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{\mathrm{d}k_0}{2\pi} \frac{\mathcal{G}(k_0, \mathbf{k})}{k_0^2 + |\mathbf{k}|^2}$$

$$D^{\mu\nu}(k_0, \mathbf{k}) = \delta^{\mu\nu} \frac{1}{k_0^2 + |\mathbf{k}|^2}$$

M. Hayakawa & S. Uno, PTP 120 (2008) ↓ $\mathrm{QED_L}$

$$\Delta' g(L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{\mathrm{d}k_0}{2\pi} \frac{\mathcal{G}(k_0, \mathbf{k})}{k_0^2 + |\mathbf{k}|^2}$$

$$D_L^{\mu\nu}(k_0, \mathbf{k}) = \delta^{\mu\nu} \frac{1 - \delta_{\mathbf{k},\mathbf{0}}}{k_0^2 + |\mathbf{k}|^2}$$

QED finite-volume effects

using the notation of
B.Lucini et al., JHEP 1602 (2016)

Hadron masses

Mass corrections can be obtained from **Compton amplitude** using **Cottingham formula**

$$\Delta m_P(L) = m_P(L) - m_P(\infty) = \frac{e^2}{4m_P}\Delta_{\mathbf{k}}\frac{M^{\mu\mu}(-\mathrm{i}|\mathbf{k}|,\mathbf{k})}{|\mathbf{k}|} \qquad M^{\mu\mu}(-\mathrm{i}|\mathbf{k}|,\mathbf{k}) = \frac{Z_{1\mathrm{P}}(0)}{|\mathbf{k}|} + \mathcal{M}(|\mathbf{k}|)$$

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using the notation of
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$$\Delta m_P(L) = \frac{e^2}{4m_P} \left[c_2(\boldsymbol{\theta}) \frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_1(\boldsymbol{\theta}) \frac{\mathcal{M}(0)}{2\pi L^2} + c_0(\boldsymbol{\theta}) \frac{\mathcal{M}'(0)}{L^3} + \sum_{\ell=0}^{\infty} \frac{(2\pi)^{\ell+1}}{L^{4+\ell}} \frac{c_{-1-\ell}(\boldsymbol{\theta})}{(\ell+2)!} \mathcal{M}^{(\ell+2)}(0) \right]$$

$$c_s(\boldsymbol{\theta}) = \left(\sum_{\mathbf{n} \in \Omega_{\boldsymbol{\theta}}} - \int \mathrm{d}^3\mathbf{n} \right) \frac{1}{|\mathbf{n}|^s}$$

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universal terms fixed by Ward identities

QED finite-volume effects

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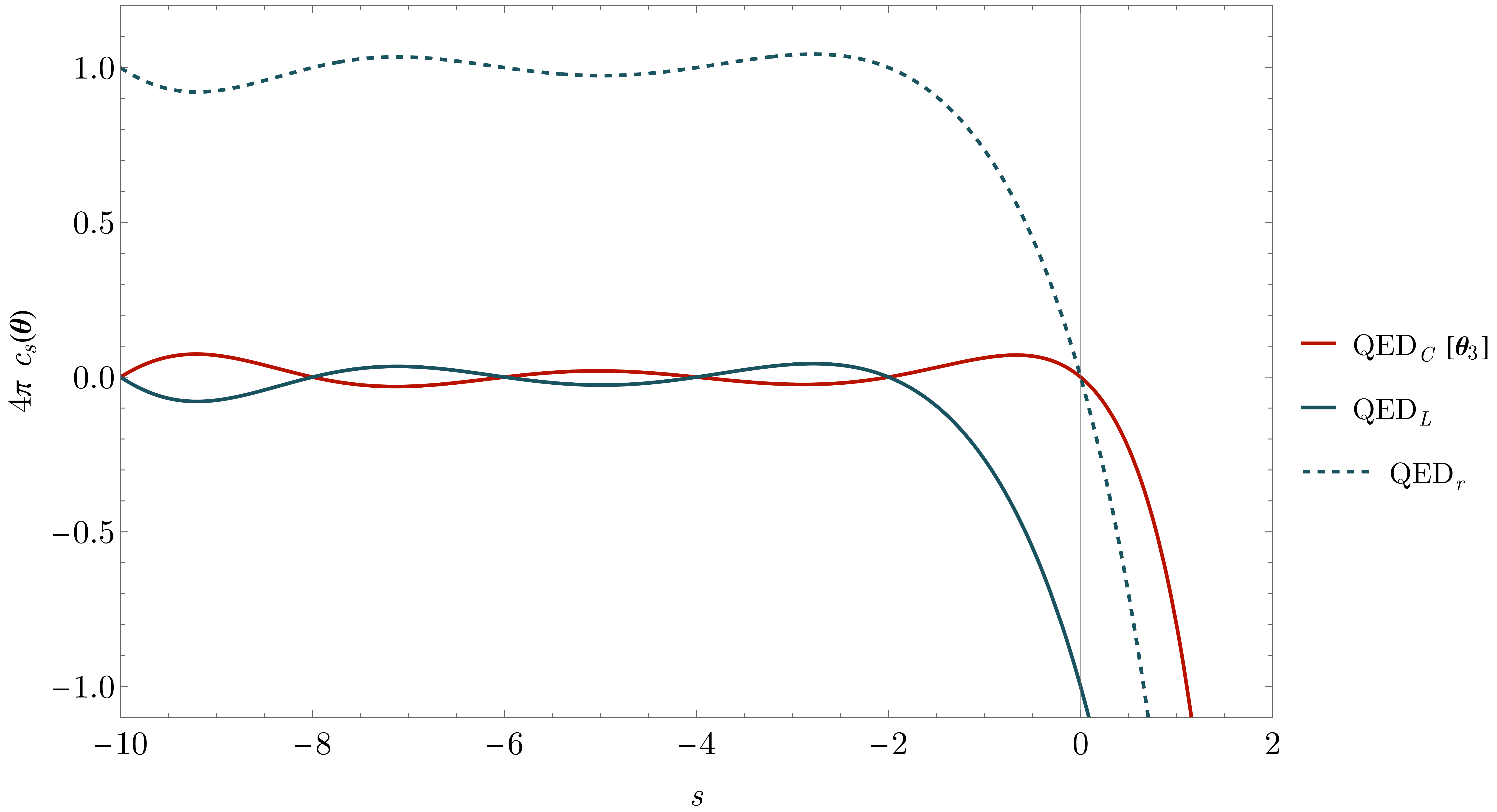
universal terms fixed by Ward identities — structure + multi-particle dependence

QED finite-volume effects

using the notation of
B.Lucini et al., JHEP 1602 (2016)

Hadron masses

$$\Delta m_P(L) = \frac{e^2}{4m_P}\left[c_2(\boldsymbol{\theta})\frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_1(\boldsymbol{\theta})\frac{\mathcal{M}(0)}{2\pi L^2} + c_0(\boldsymbol{\theta})\frac{\mathcal{M}'(0)}{L^3} + \sum_{\ell=0}^{\infty}\frac{(2\pi)^{\ell+1}}{L^{4+\ell}}\frac{c_{-1-\ell}(\boldsymbol{\theta})}{(\ell+2)!}\mathcal{M}^{(\ell+2)}(0)\right]$$



QED finite-volume effects

Leptonic decay amplitude

V. Lubicz et al., PRD 95 (2017)
N. Tantalo et al., [1612.00199v2]
MDC et al., PRD 105 (2022)
MDC et al., [2310.13358]

$$\Delta Y_P(L) = \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + 2\log\left(\frac{m_W L}{4\pi}\right) - 2A_1(\mathbf{v}_\ell)\left[\log\frac{m_P L}{2\pi} + \log\frac{m_\ell L}{4\pi} - 1\right] + \frac{c_3 - 2(c_3(\mathbf{v}_\ell) - B_1(\mathbf{v}_\ell))}{2\pi}$$

$$- \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\, c_2 - 4r_\ell^2 c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right]$$

universal

$$+ \frac{1}{(m_P L)^2}\left[-\frac{F_A(0)}{f_P}\,\frac{4\pi m_P[(1+r_\ell)^2\, c_1 - 4r_\ell^2\, c_1(\mathbf{v}_\ell)]}{1-r_\ell^4} + \frac{8\pi[(1+r_\ell^2)c_1 - 2c_1(\mathbf{v}_\ell)]}{(1-r_\ell^4)}\right]$$

$$+ \frac{1}{(m_P L)^3}\left[\frac{32\pi^2 c_0\,(2+r_\ell^2)}{(1+r_\ell^2)^3} + c_0\, C_\ell^{(1)} + c_0(\mathbf{v}_\ell)\, C_\ell^{(2)}\right]$$

$$+ \cdots$$

$$c_s(\mathbf{v}_\ell) = \left(\sum_{\mathbf{n}\neq\mathbf{0}} - \int d^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s\,(1-\mathbf{v}_\ell\cdot\hat{\mathbf{n}})}$$

- **Collinear** divergent terms as $|\mathbf{v}| \to 1$ and $\mathbf{v} \parallel \mathbf{k}$
- Dependence on the direction $\hat{\mathbf{v}}$ due to **rotational symmetry breaking**

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$$+ \cdots$$

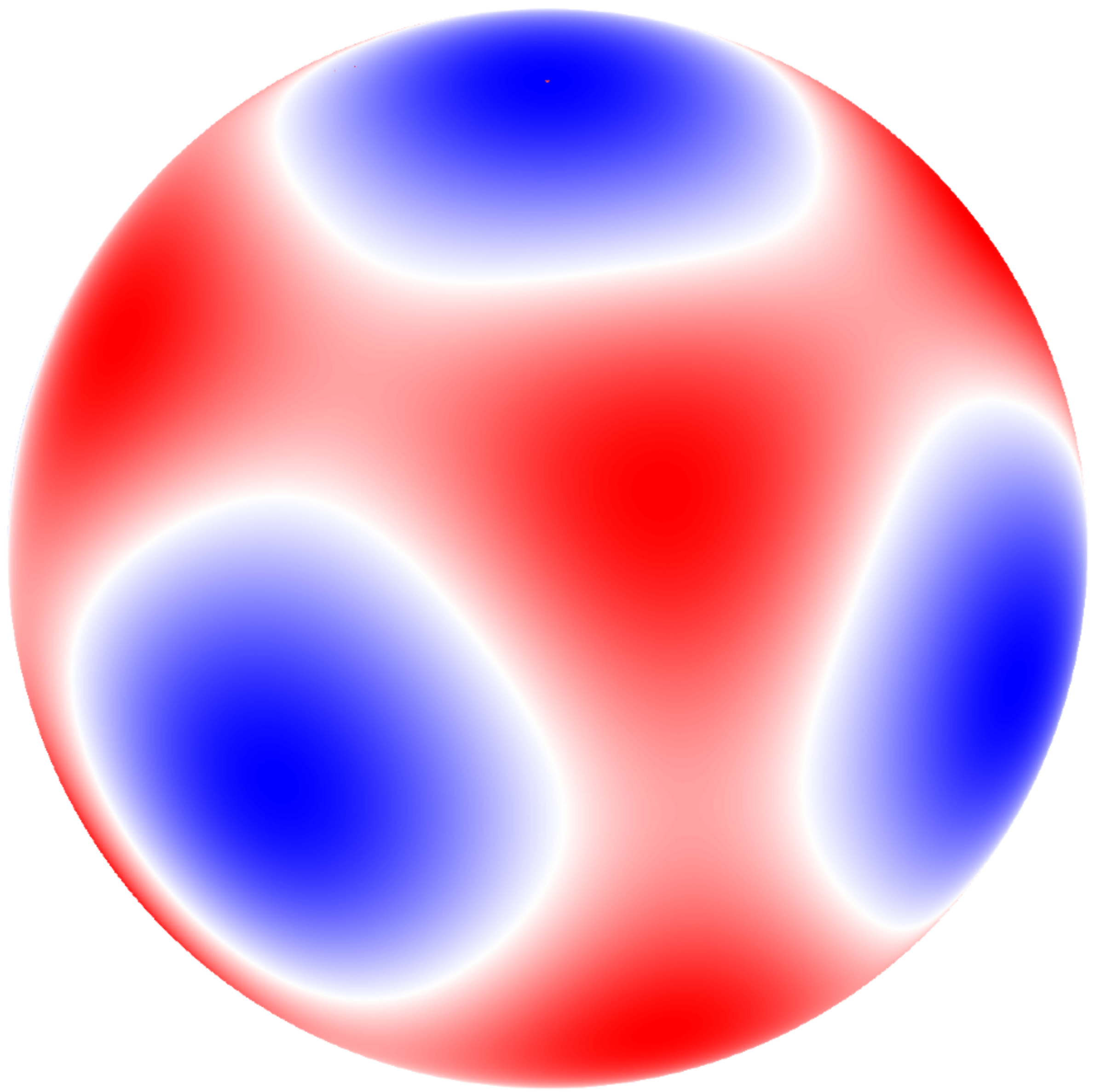
can $\mathrm{QED_r}$ help removing this unknown term?

$$c_s(\mathbf{v}_\ell) = \left(\sum_{\mathbf{n}\neq\mathbf{0}} - \int d^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s\,(1-\mathbf{v}_\ell\cdot\hat{\mathbf{n}})}$$

- **Collinear** divergent terms as $|\mathbf{v}| \to 1$ and $\mathbf{v} \parallel \mathbf{k}$
- Dependence on the direction $\hat{\mathbf{v}}$ due to **rotational symmetry breaking**

Velocity-dependent coefficients in QED_r

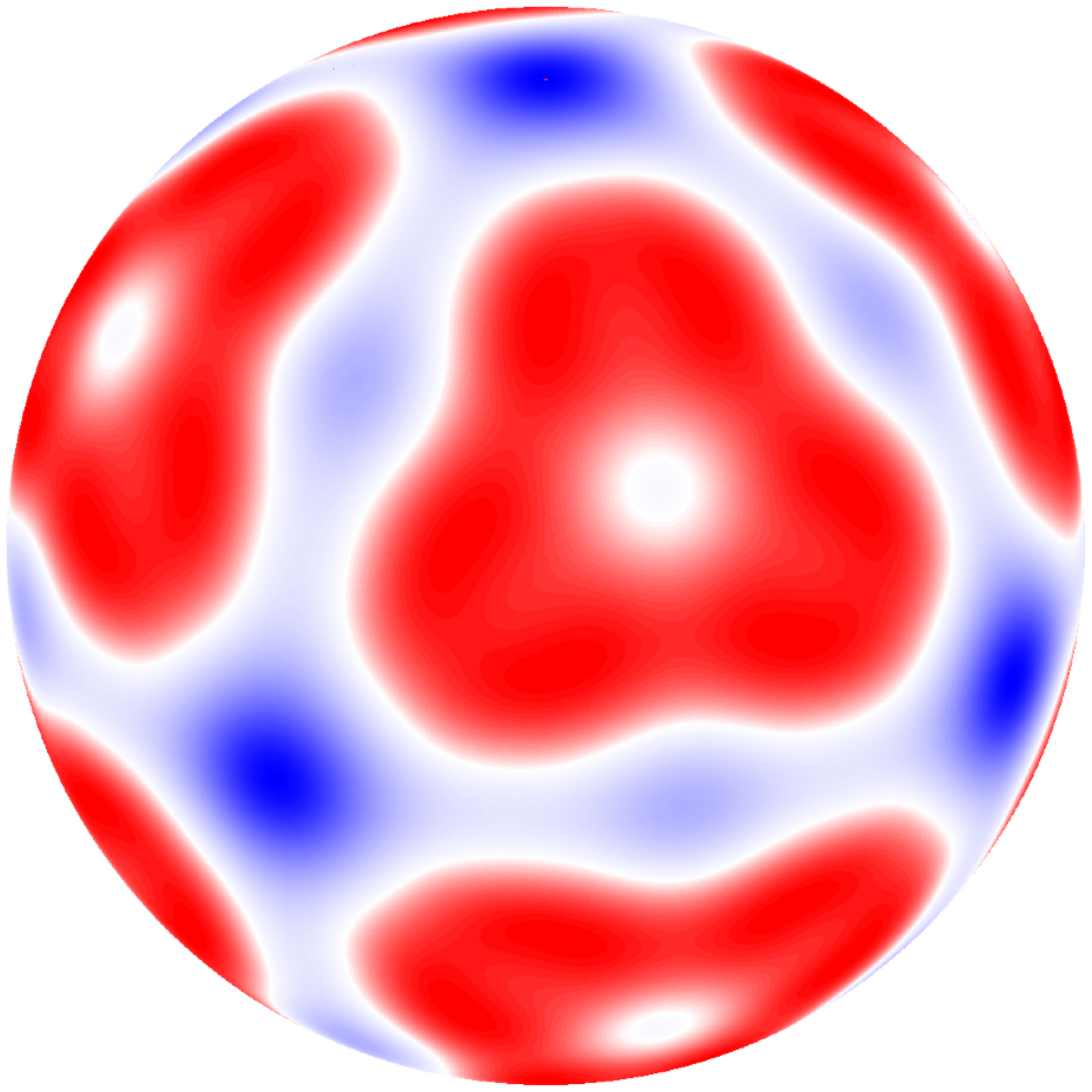
$|v| = 0.40$



$\max \bar{c}_0(\mathbf{v}) = 0.0171$

$\min \bar{c}_0(\mathbf{v}) = \text{-}0.0114$

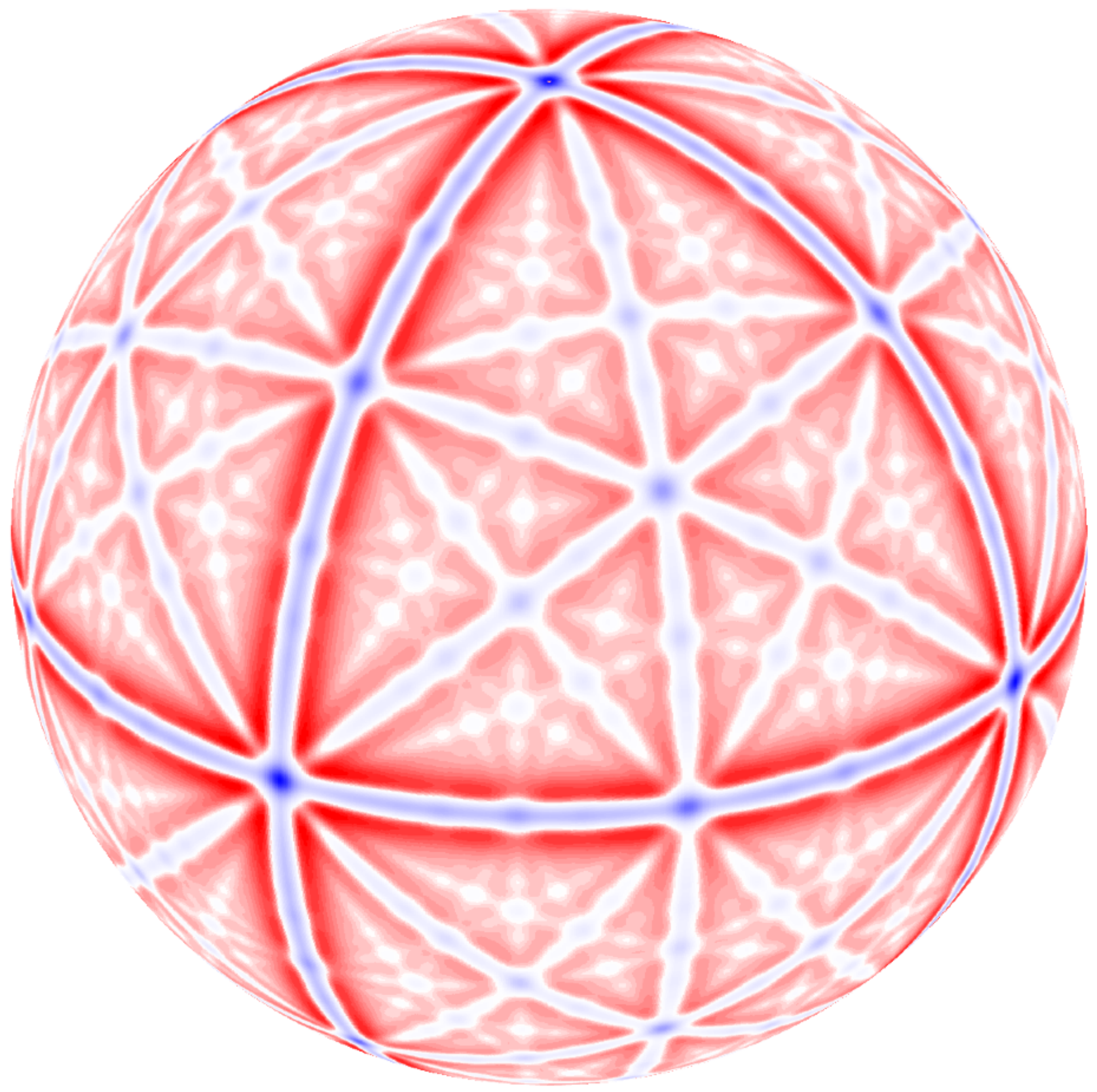
$|v| = 0.95$



$\max \bar{c}_0(\mathbf{v}) = 15.2832$

$\min \bar{c}_0(\mathbf{v}) = \text{-}2.8258$

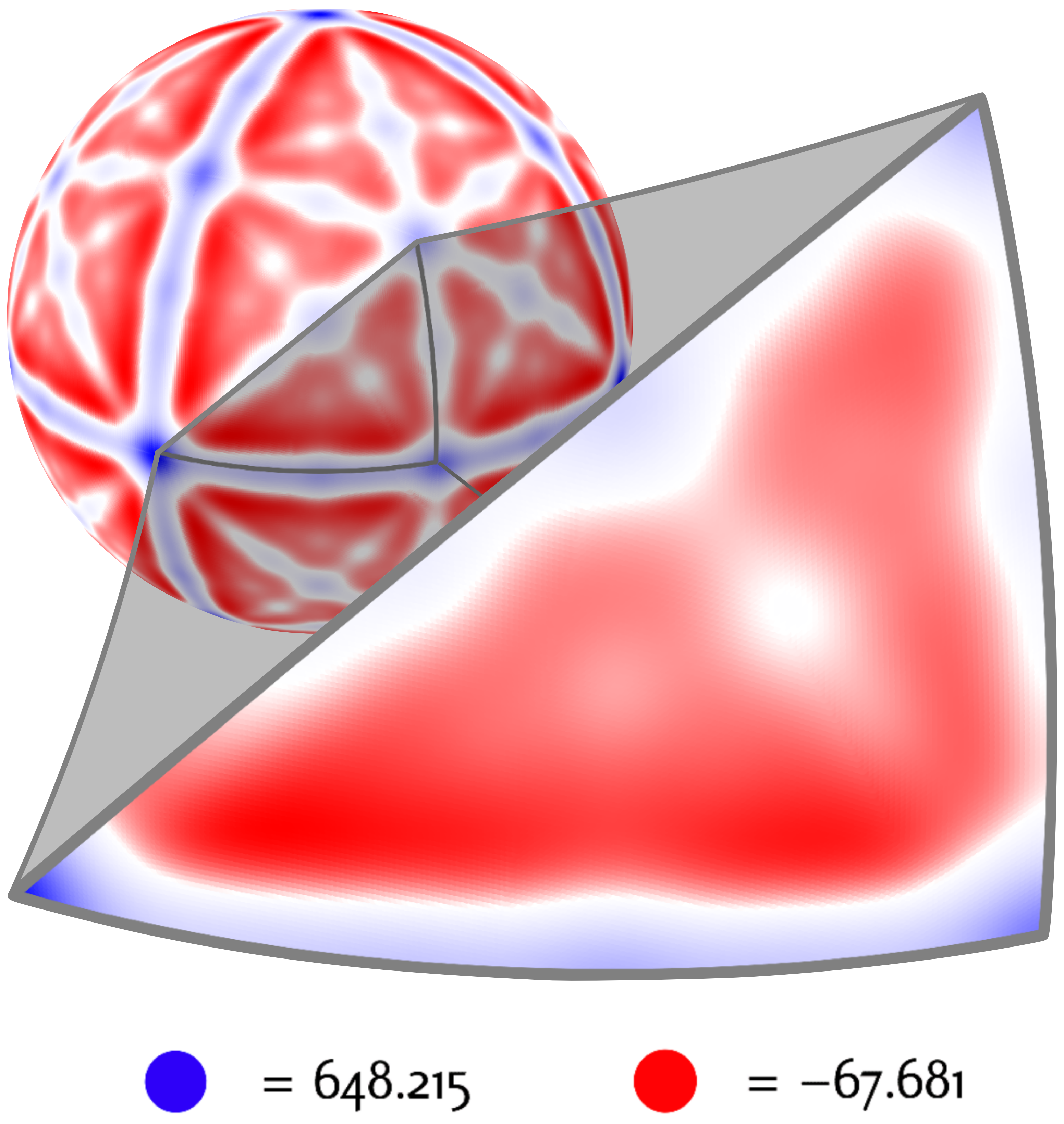
$|v| = 0.999$

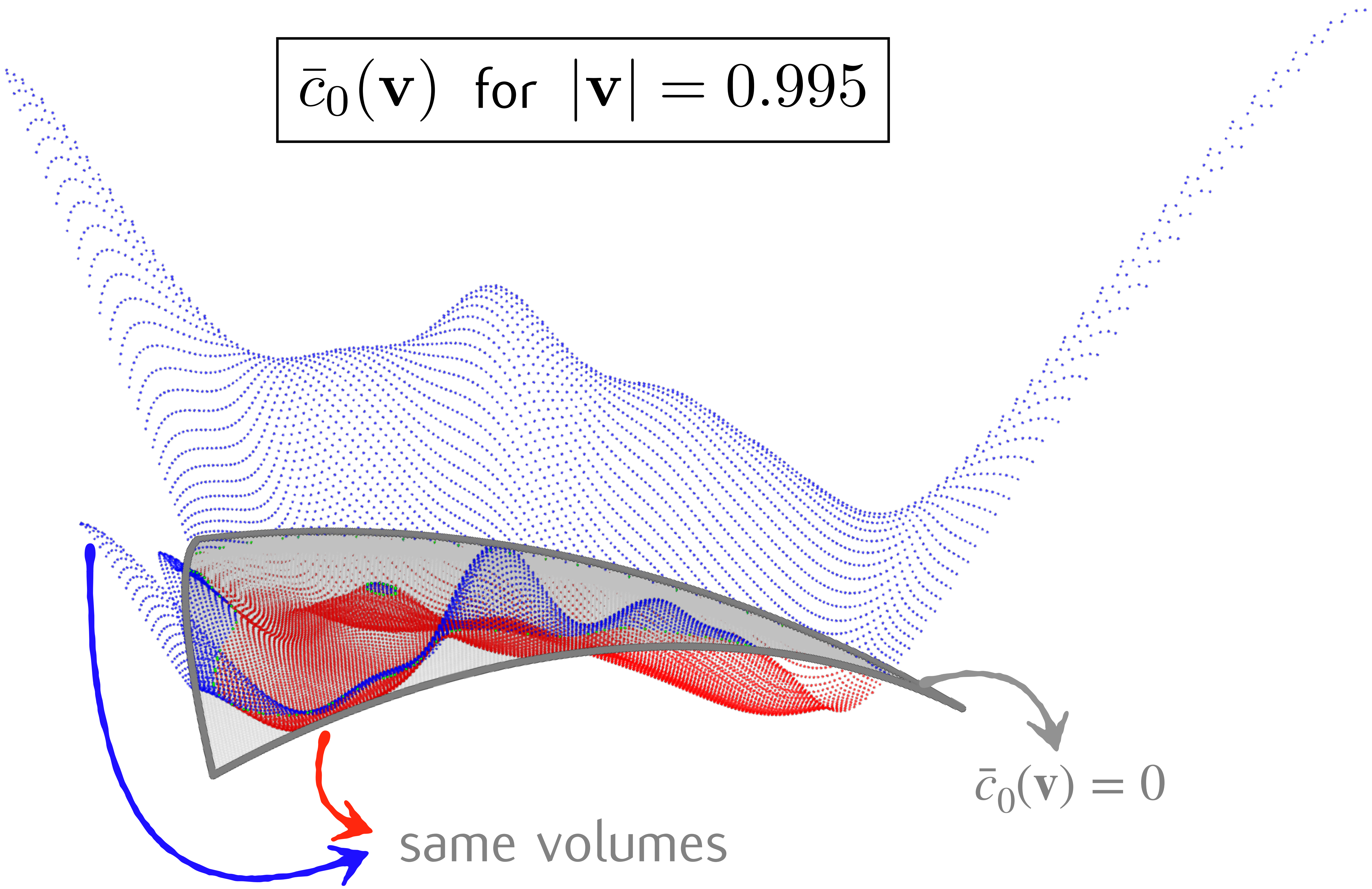


$\max \bar{c}_0(\mathbf{v}) = 9002.2317$

$\min \bar{c}_0(\mathbf{v}) = \text{-}807.4018$

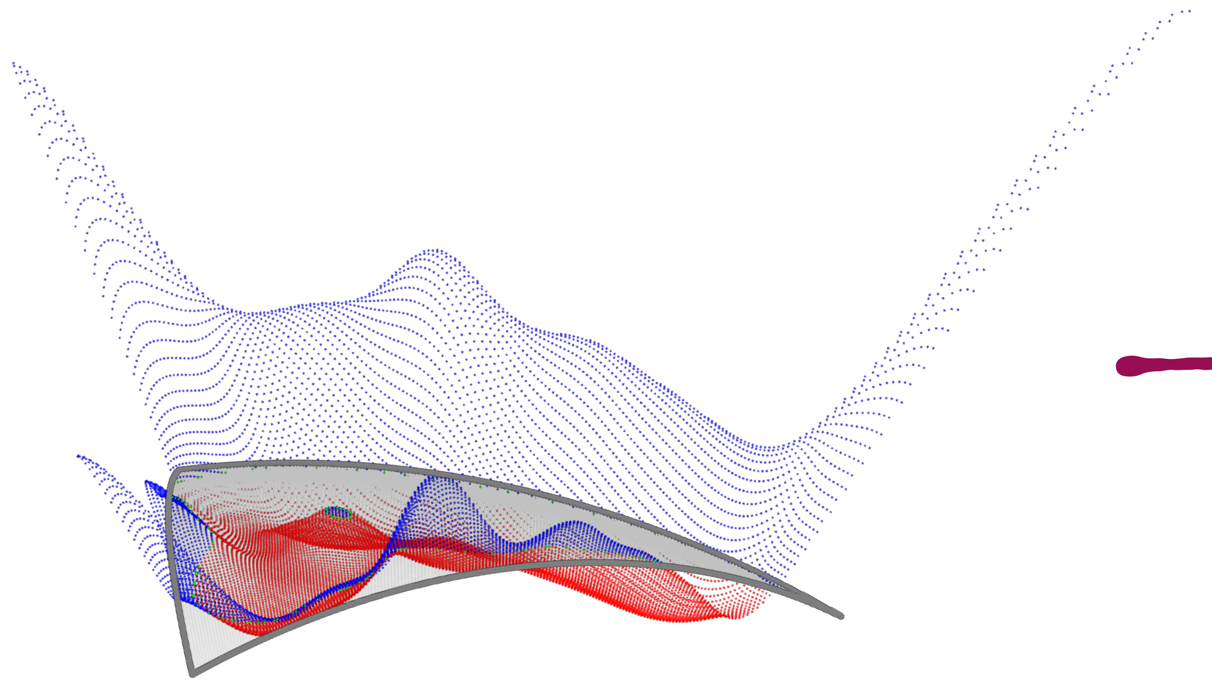
Velocity-dependent coefficients in QED$_r$





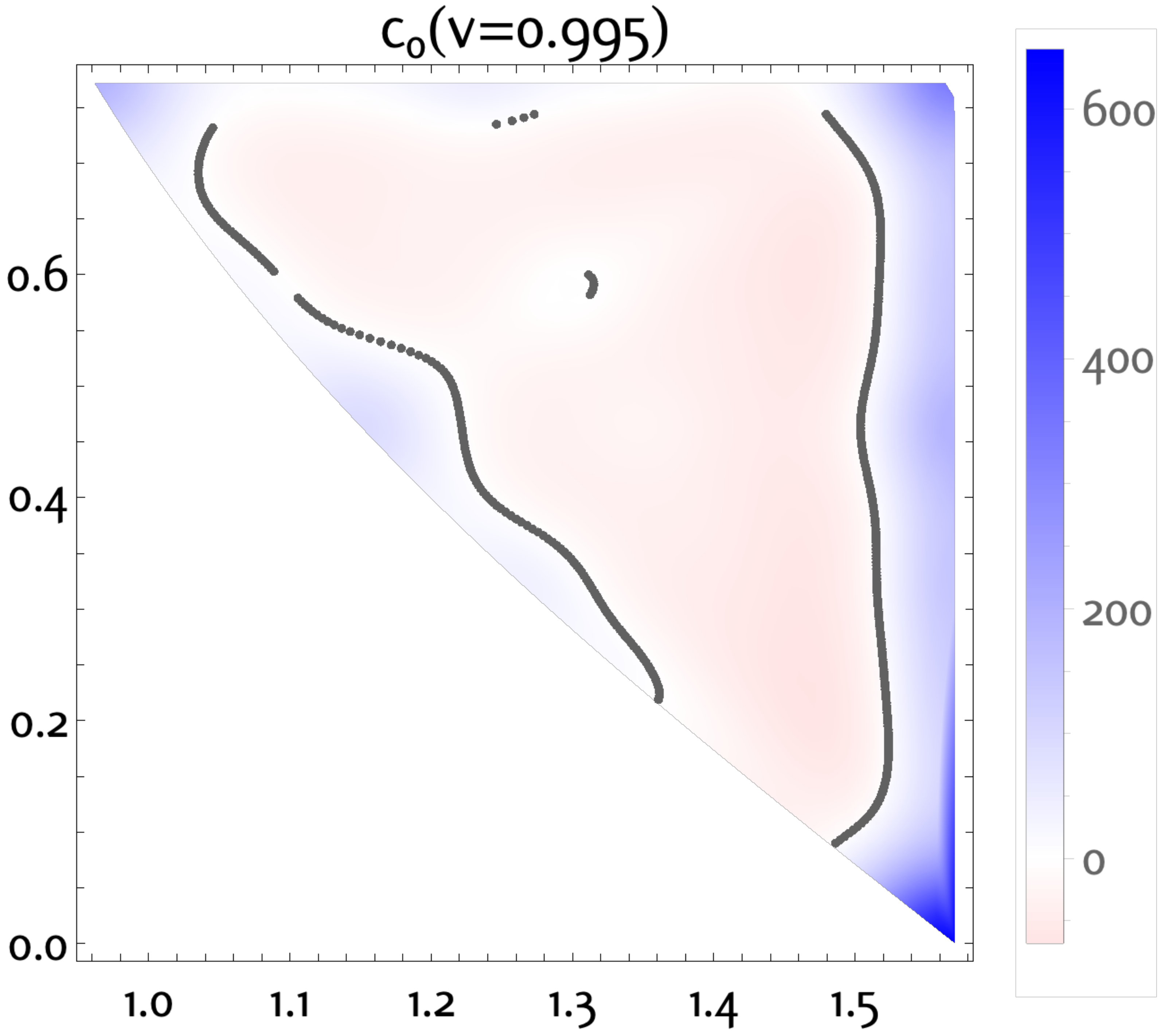
Velocity-dependent coefficients in QED_r

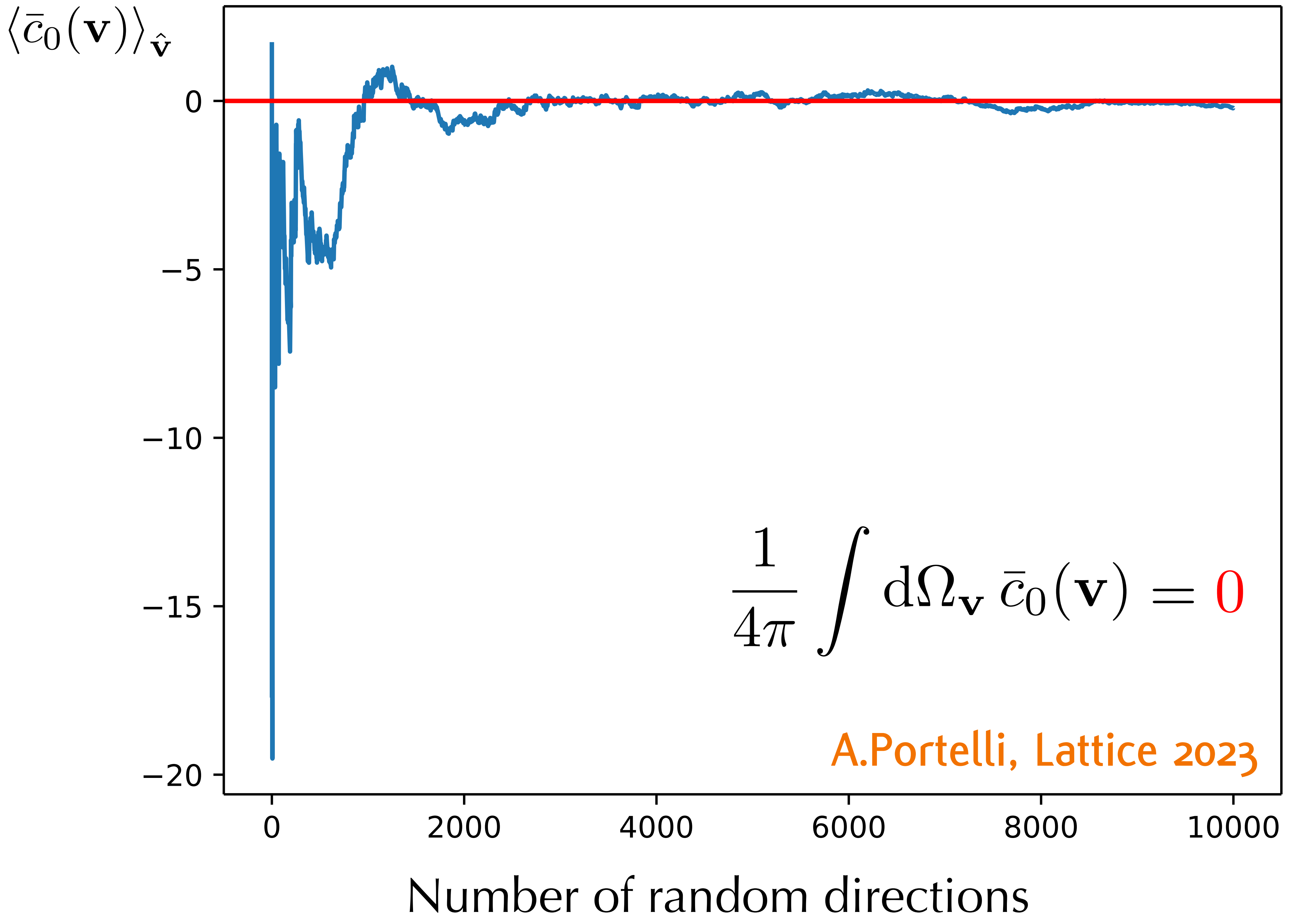
work in progress
in both directions



"magic angles"

Stochastic direction average





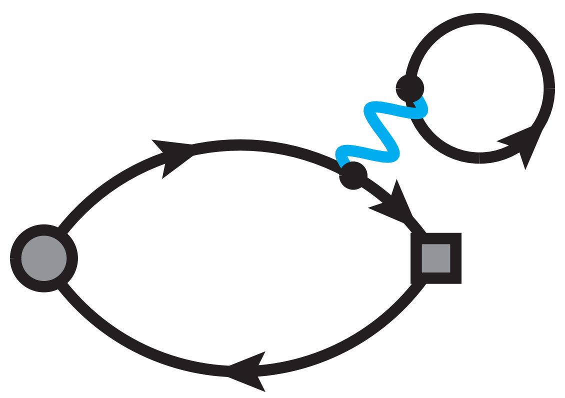
Take home messages on finite-volume effects?

- **Finite-volume expansions** studied for **masses** and **leptonic decays**
- **Unknown structure-dependent** contributions start at $O(1/L^3)$
- **QED$_r$ regularisation** could help pushing unknown effects to $O(1/L^4)$ [but requires more study!]
- **Velocity-dependent** effects potentially problematic for heavy meson decays (also in QED$_C$)
- Very important to compare with approaches with only **exponentially suppressed effects**

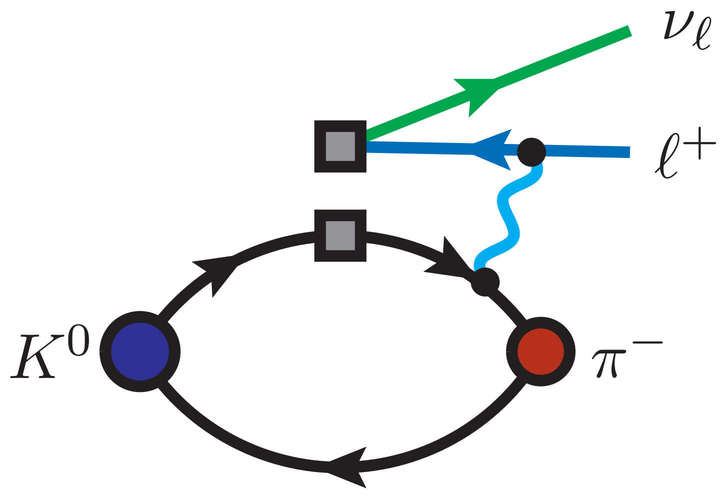
4. Where do we stand ...

- Current **tensions** in CKM unitarity require a combined effort of **theory** and **experiments**
- **Two lattice calculations** of IB and QED corrections to **light-meson leptonic decay rates**
- **Finite volume QED effects** have to be carefully investigated

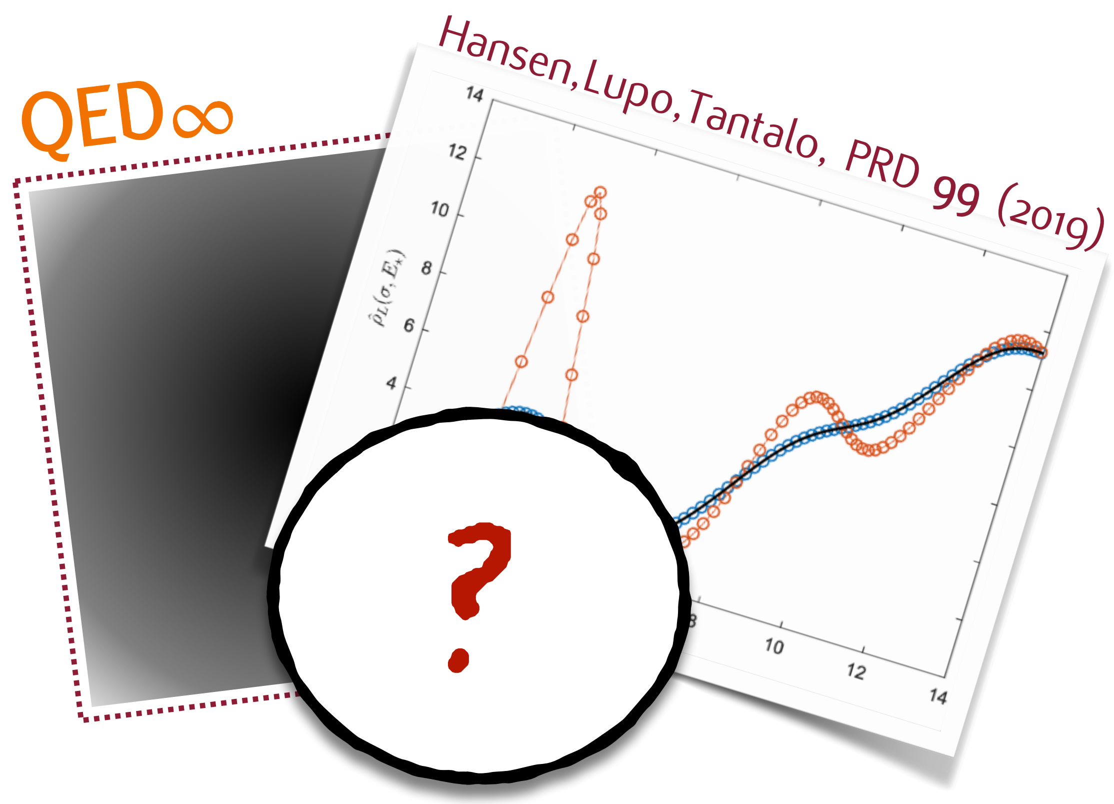
... and where to go?



move to unquenched calculations



study different weak processes



develop and apply new techniques

Thank you

This work has received funding from the European Union's Horizon Europe research and innovation programme under the Marie Sklodowska-Curie grant agreement No 101108006

Backup slides

Prospects for $|V_{us}/V_{ud}|$

A speculative exercise on the error budget

$$\left|\frac{V_{us}}{V_{ud}}\right|^2 = \left[\frac{\Gamma(K_{\ell 2})}{\Gamma(\pi_{\ell 2})}\frac{M_{K^+}^3}{M_{\pi^+}^3}\frac{(M_{K^+}^2 - M_{\mu^+}^2)^2}{(M_{\pi^+}^2 - M_{\mu^+}^2)^2}\right]_{\text{exp}} \cdot \left[\frac{f_{K,0}}{f_{\pi,0}}\right]^2 (1+\delta R_{K\pi})$$

- Let us use $\delta R_{K\pi} = -0.0086\,(13)(39)_{\text{vol.}}$

$[f_{K,0}/f_{\pi,0}]$		$\|V_{us}/V_{ud}\|$
FLAG21 2+1 average	1.1930 (33)	$0.23154\ (28)_{\text{exp}}\ (15)_{\delta R}\ (45)_{\delta R,\text{vol.}}\ (65)_{f_P}$

- From RM123+Soton calculation $\delta R_{K\pi} = -0.0126\,(14)$

$[f_{K,0}/f_{\pi,0}]$		$\|V_{us}/V_{ud}\|$
FLAG19 2+1+1 average	1.1966 (18)	$0.23131\ (28)_{\text{exp}}\ (17)_{\delta R}\ (35)_{f_P}$

- the uncertainty on $[f_{K,0}/f_{\pi,0}]$ dominates in the error budget
- if improved, precision from **lattice** starts being competitive with the **experimental** one

Infinite volume reconstruction

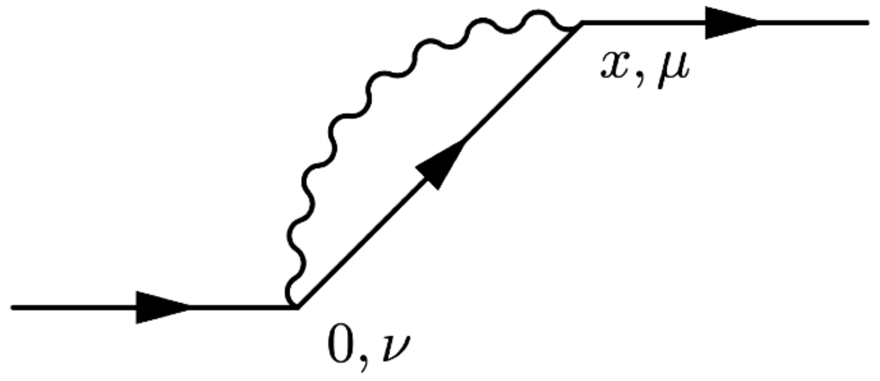
X.Feng & L.Jin, PRD 100 (2019)

QED$_\infty$

$$\Delta\mathcal{O} = \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\ \mathcal{H}(t,\mathbf{x})\ f_{\mathrm{QED}}(t,\mathbf{x})$$

Separate correlator into **short** and **long** distance part:

$$\Delta\mathcal{O} = \Delta\mathcal{O}^{(s)} + \Delta\mathcal{O}^{(l)}$$



$$\Delta\mathcal{O}^{(s)} = \frac{1}{2}\int_{-t_s}^{t_s} \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\ \mathcal{H}(t,\mathbf{x}) f_{\mathrm{QED}}(t,\mathbf{x}) \approx \frac{1}{2}\int_{-t_s}^{t_s} \mathrm{d}t \int_{L^3} \mathrm{d}^3\mathbf{x}\ \mathcal{H}^L(t,\mathbf{x}) f_{\mathrm{QED}}(t,\mathbf{x})$$

$$\Delta\mathcal{O}^{(l)} = \int_{t_s}^{\infty} \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\ \mathcal{H}(t,\mathbf{x}) f_{\mathrm{QED}}(t,\mathbf{x}) \approx \int_{L^3} \mathrm{d}^3\mathbf{x}\ \mathcal{H}^L(t_s,\mathbf{x})\ \mathcal{F}_{\mathrm{QED}}(t_s,\mathbf{x})$$

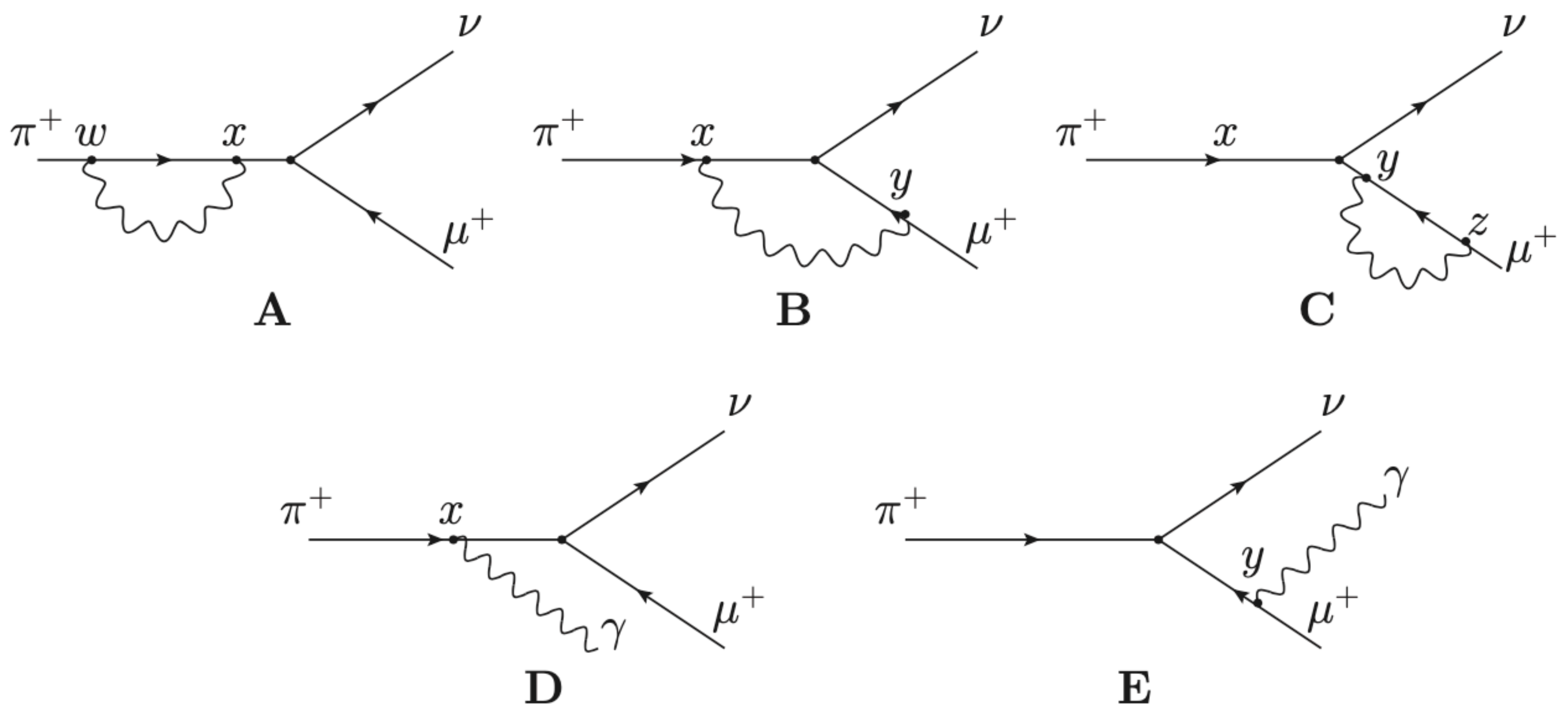
Exponentially suppressed

- finite-volume effects
- contributions of states with higher energy

Infinite volume reconstruction

QED$_\infty$

N.Christ et al., [2304.08026]



- Diagram A:

$$H^{(2)}_{\mu,\rho,\sigma}(t_1,t_2,\vec{x}) = \int d^3\vec{w}\, \langle 0|T\{J^W_\mu(0)J^{EM}_\rho(t_1,\vec{w}+\vec{x})J^{EM}_\sigma(t_2,\vec{w})\}|\pi(\vec{0})\rangle$$

- Diagram B and D:

$$H^{(1)}_{\mu,\rho}(x) = H^{(1)}_{\mu,\rho}(x_t,\vec{x}) = \langle 0|T\{J^W_\mu(0)J^{EM}_\rho(x)\}|\pi(\vec{0})\rangle$$

- Diagram C and E ($f_\pi \approx 130$ MeV):

$$H^{(0)}_\mu = H^{(0)}_t\delta_{\mu,t} = \langle 0|J^W_\mu(0)|\pi(\vec{0})\rangle = -im_\pi f_\pi \delta_{\mu,t}$$

from Luchang Jin's talk @Edinburgh May 30, 2023

Method applied to **leptonic decay rates:**

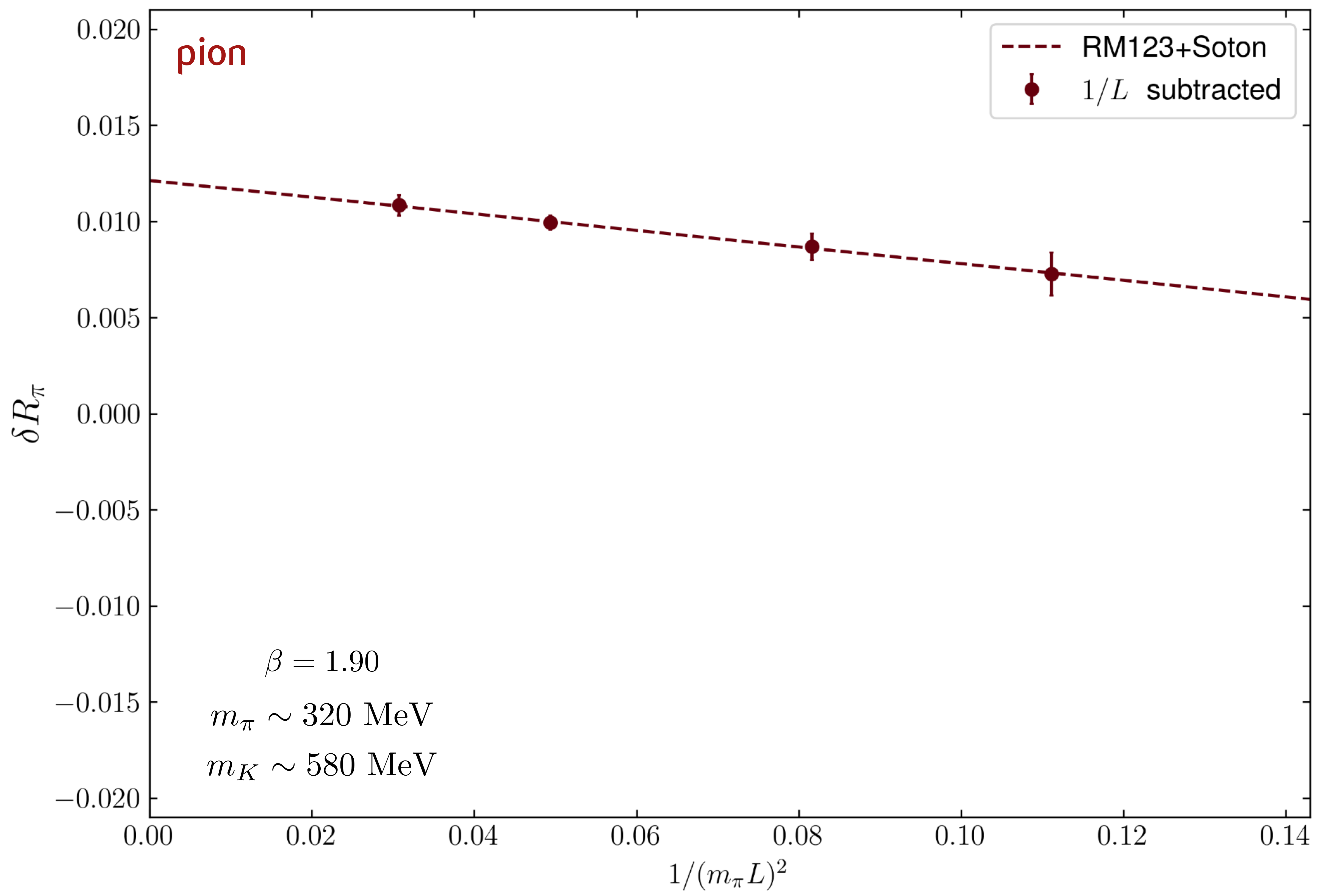
- **Logarithmic IR divergences** appear
- **BUT they cancel analytically** between diagrams
- numerical calculation is **ongoing...**

The method is appealing given the large finite-volume effects in QED$_L$ at $O(1/L^3)$

... systematics under control?

Comparing with RM123+Soton result

crucial role of finite volume effects?



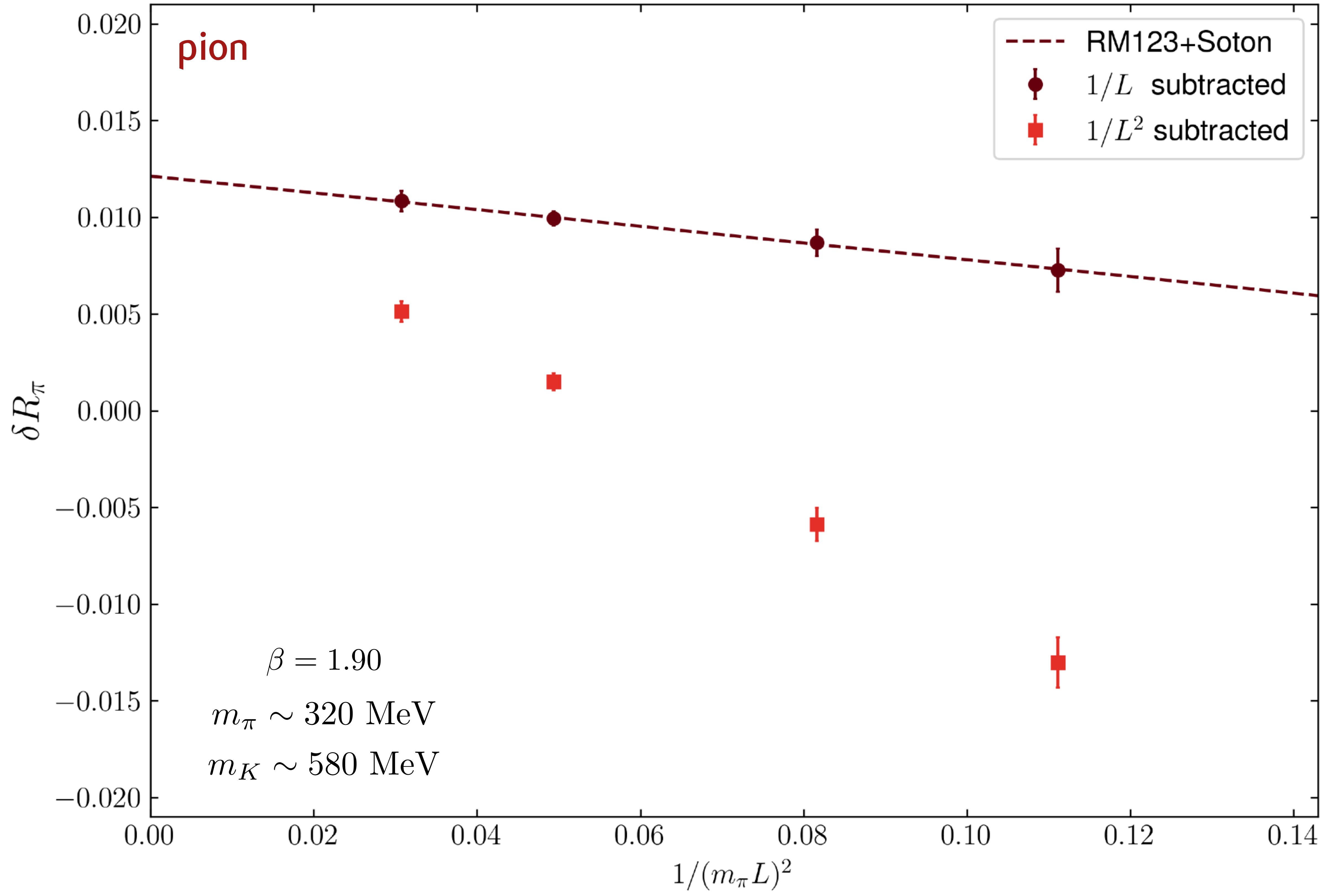
from MDC et al., PRD 100 (2019) [RM123+Soton]

Subtracting:

1. universal FVEs up to $1/L$

Comparing with RM123+Soton result

crucial role of finite volume effects?



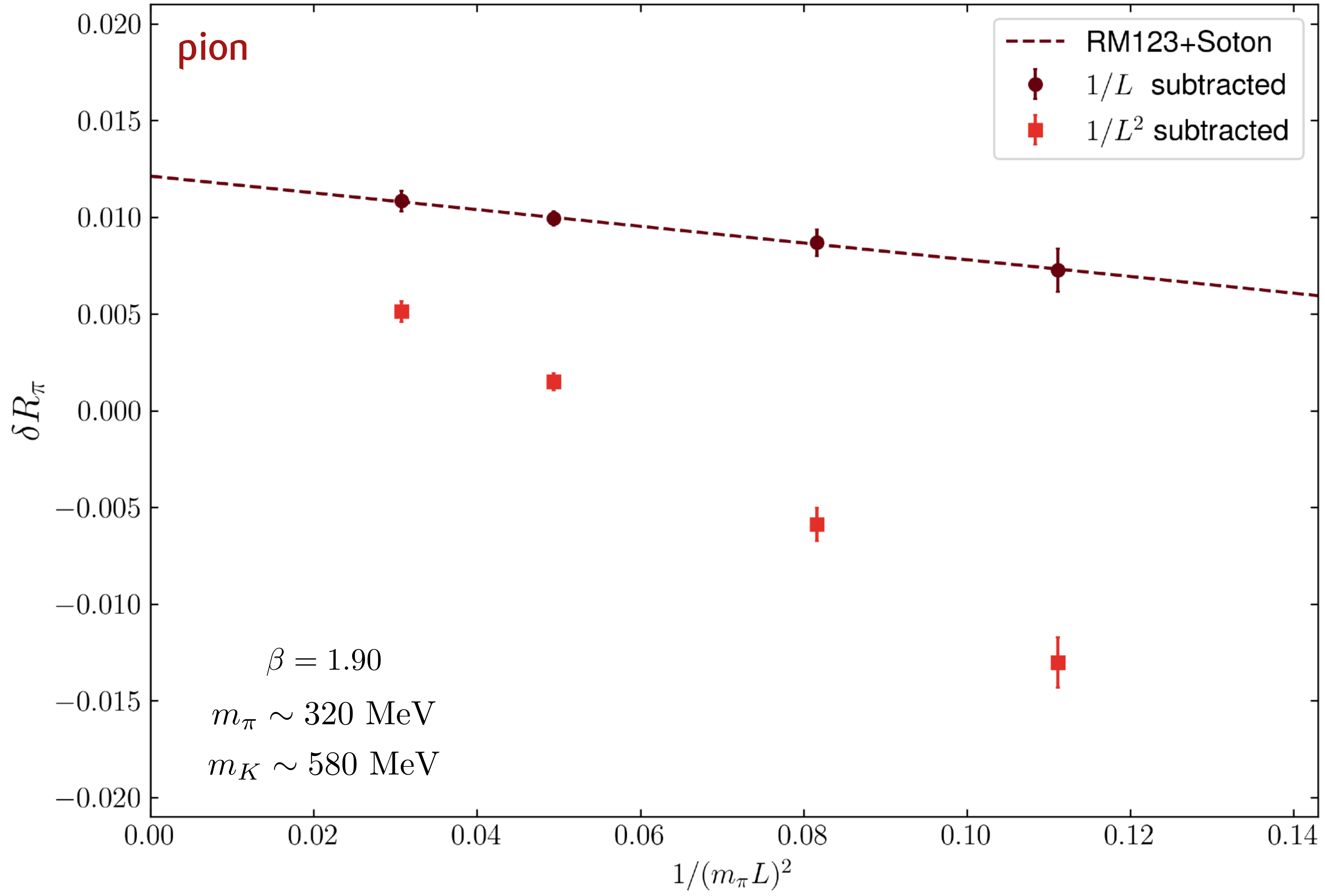
from MDC et al., PRD 100 (2019) [RM123+Soton]

Subtracting:

1. universal FVEs up to $1/L$
2. pointlike $1/L^2$

Comparing with RM123+Soton result

crucial role of finite volume effects?



from MDC et al., PRD 100 (2019) [RM123+Soton]

Subtracting:

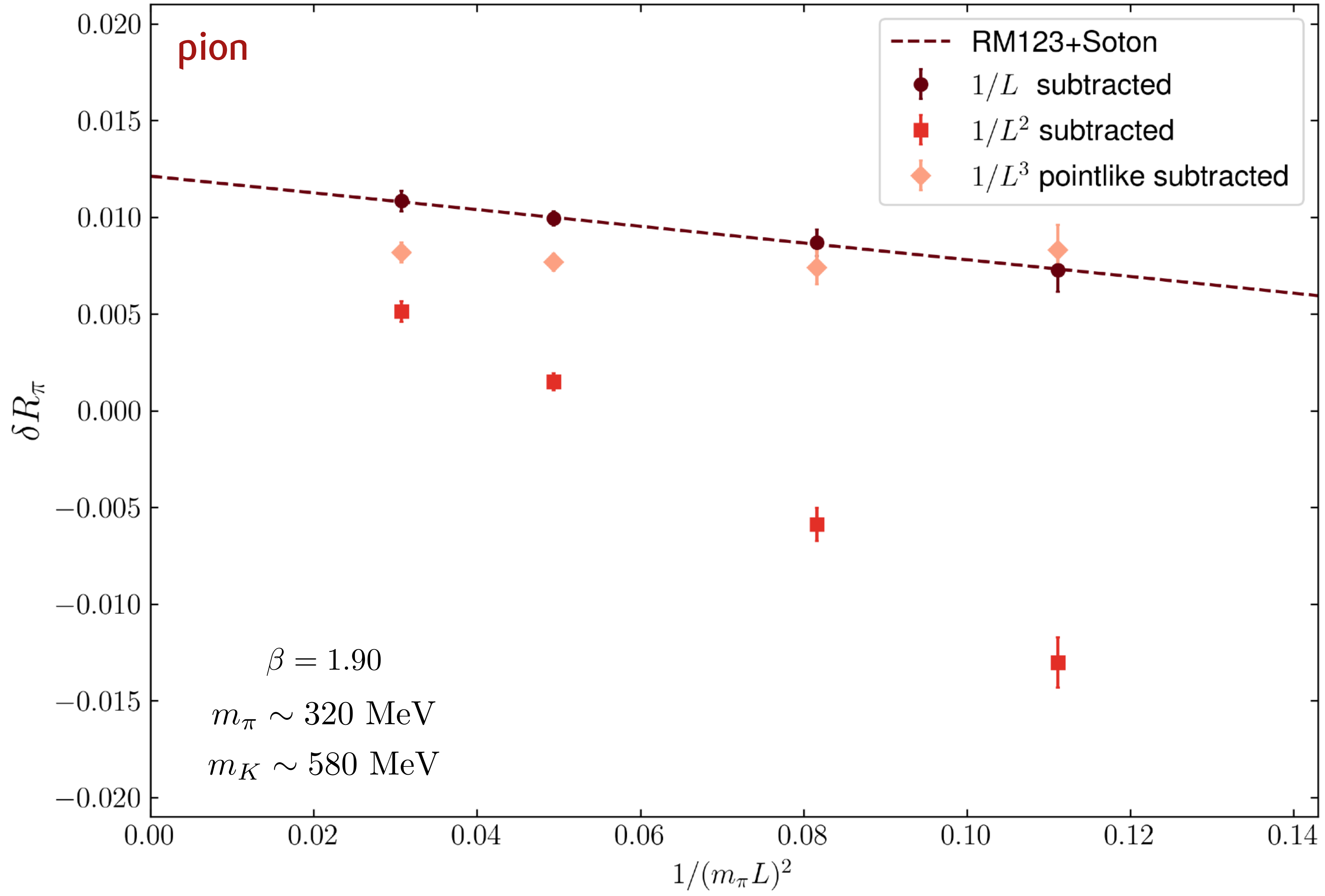
1. universal FVEs up to $1/L$
2. pointlike $1/L^2$
3. structure-dependent $1/L^2$

include the pointlike limit $Y^{(2)}_{\rm pt}(L)$ setting $F_A^\pi = 0$, and notice that the structure-dependent contribution at $\mathcal{O}(1/L^2)$ is negligible with respect to the pointlike one. In total, there

MDC et al., PRD 105 (2022)

Comparing with RM123+Soton result

crucial role of finite volume effects?



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MDC et al., PRD 105 (2022)

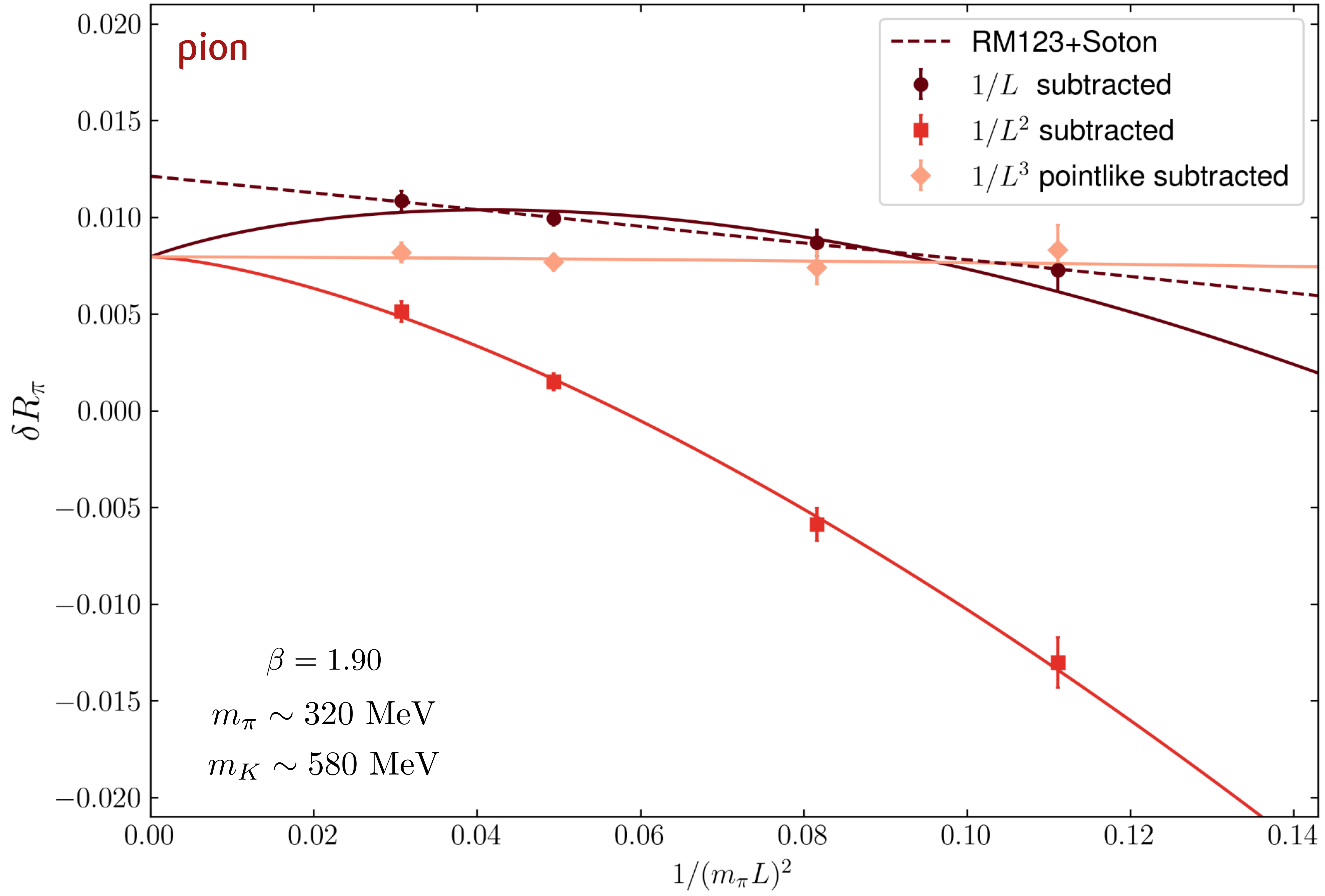
4. pointlike $1/L^3$

$$\Delta^{(3,\mathrm{pt})}(\delta R_P) = \left(\frac{2\alpha}{4\pi}\right)\frac{32\pi^2 c_0 (2 + r_\ell^2)}{(m_P L)^3 (1 + r_\ell^2)^3}$$

MDC et al., PRD 105 (2022)
N. Tantalo et al., [1612.00199v2]

Comparing with RM123+Soton result

crucial role of finite volume effects?



from MDC et al., PRD 100 (2019) [RM123+Soton]

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MDC et al., PRD 105 (2022)

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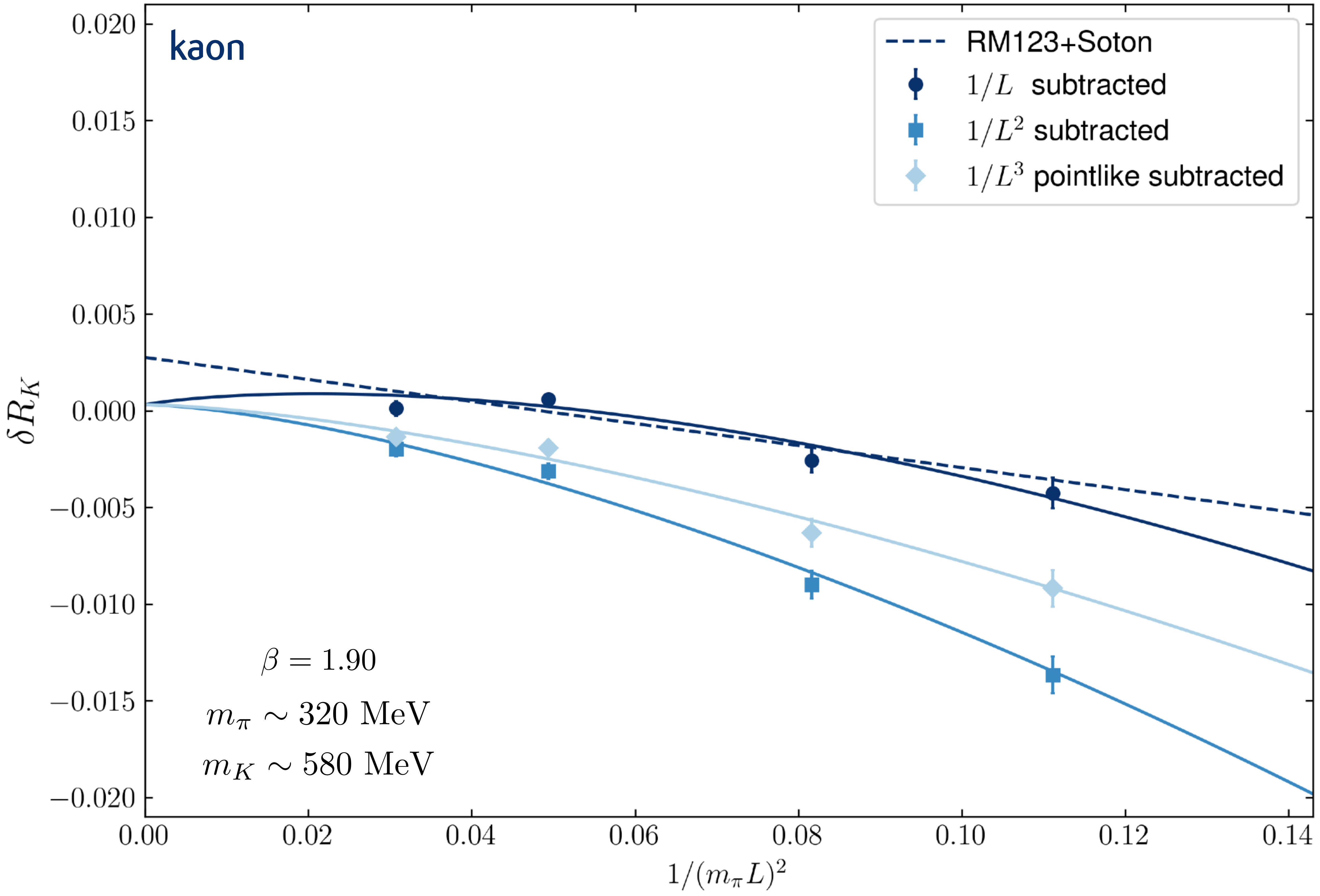
MDC et al., PRD 105 (2022)
N. Tantalo et al., [1612.00199v2]

Comparing with RM123+Soton result

crucial role of finite volume effects?



from MDC et al., PRD 100 (2019) [RM123+Soton]

Subtracting:

1. universal FVEs up to $1/L$
2. pointlike $1/L^2$
3. structure-dependent $1/L^2$

> include the pointlike limit $Y^{(2)}_{\rm pt}(L)$ setting $F^\pi_A = 0$, and notice that the structure-dependent contribution at $\mathcal{O}(1/L^2)$ is negligible with respect to the pointlike one. In total, there

MDC et al., PRD 105 (2022)

4. pointlike $1/L^3$

$$\Delta^{(3,\mathrm{pt})}(\delta R_P) = \left(\frac{2\alpha}{4\pi}\right) \frac{32\pi^2 c_0 (2 + r_\ell^2)}{(m_P L)^3 (1 + r_\ell^2)^3}$$

MDC et al., PRD 105 (2022)
N. Tantalo et al., [1612.00199v2]

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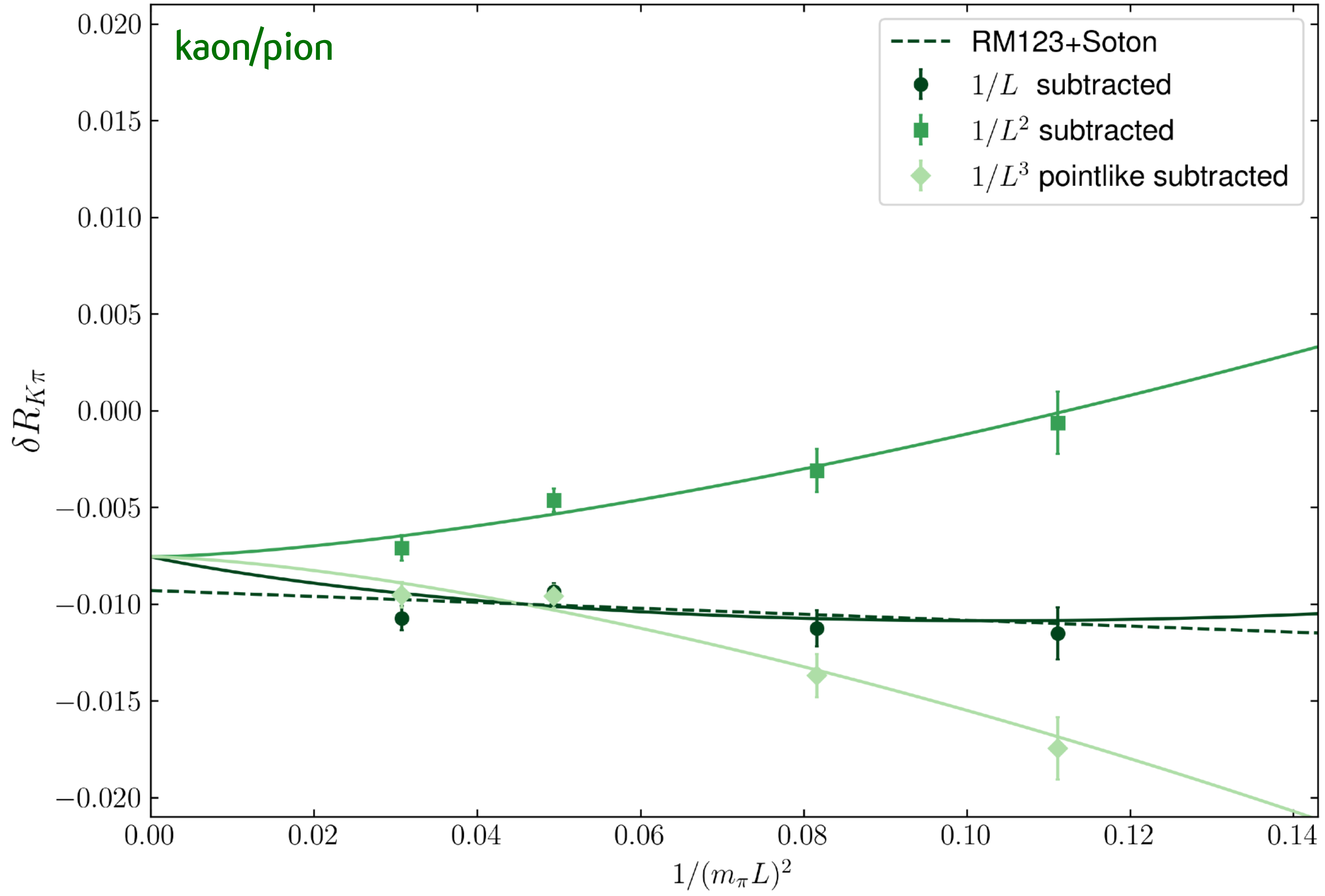

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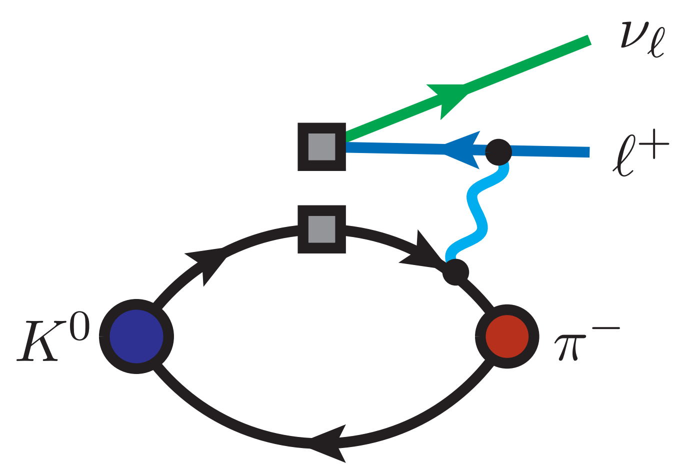
Semileptonic kaon decays

Goal: precision determination of $|V_{us}|$ & test of first-row unitarity

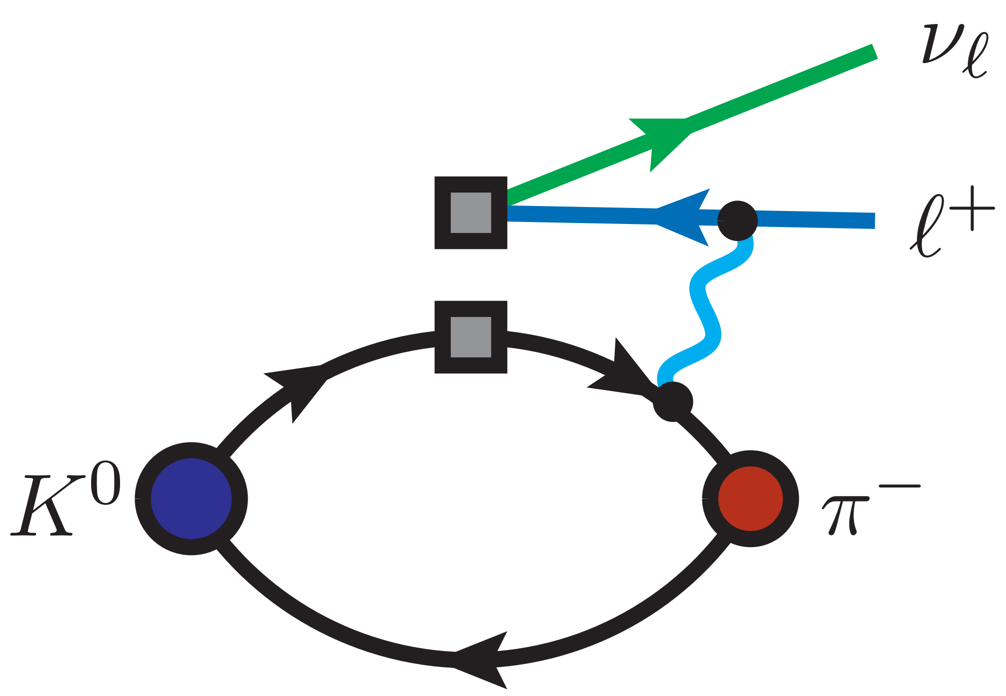


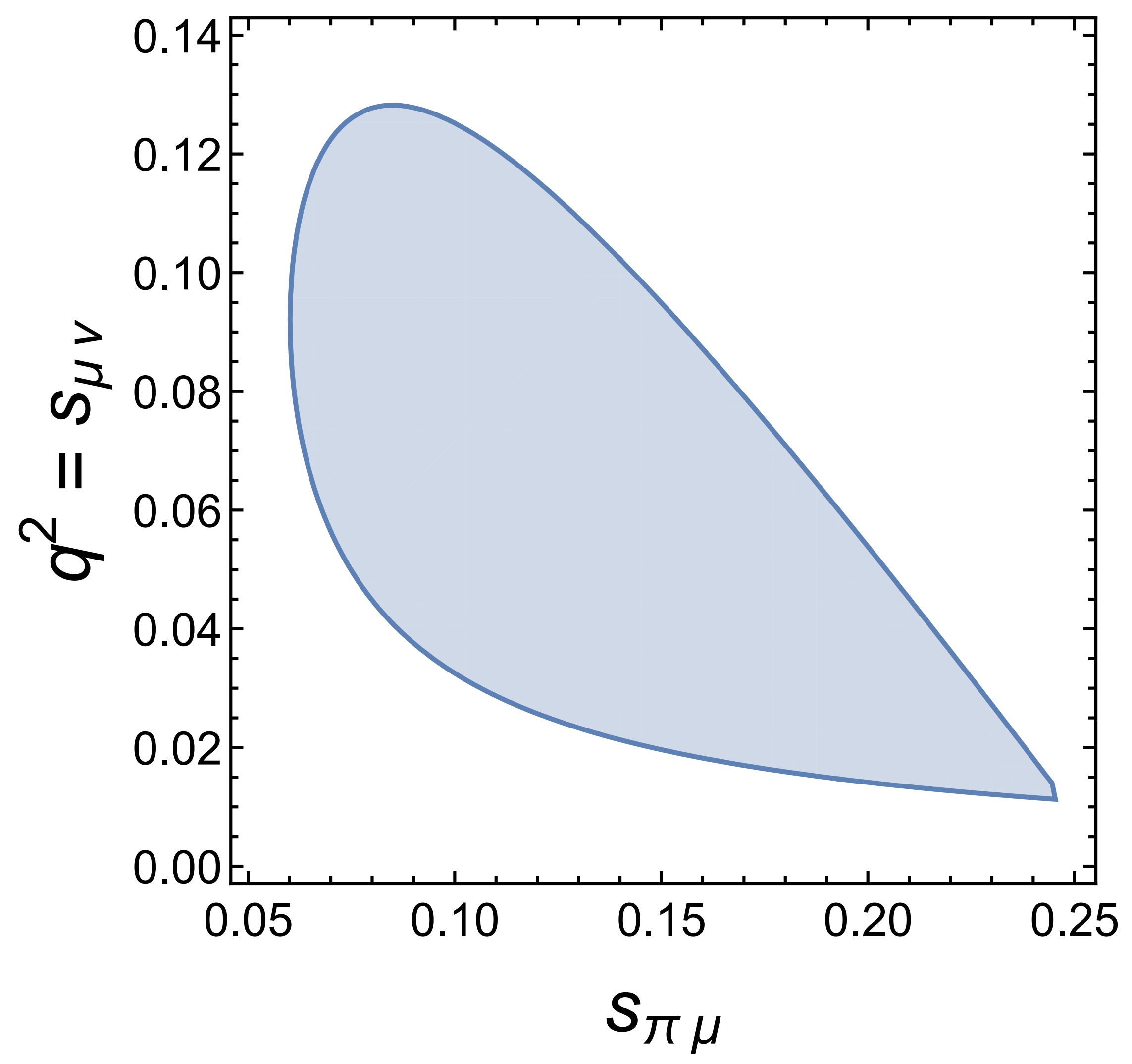
Relevance: sub-percent precision on $f^+(0)$ requires inclusion of IB effects

Status:

- no complete lattice QCD+QED calculations
- difficulties of finite-volume QED calculations identified — C.Sachrajda et al., [1910.07342]
- recent proposal using QED$_\infty$ method — N.Christ et al., [2304.08026] / N.Christ @Lattice2023

QED corrections to semileptonic decays





Although the RM123+Soton method could in principle be applied, additional **difficulties** arise compared to **leptonic decays**:

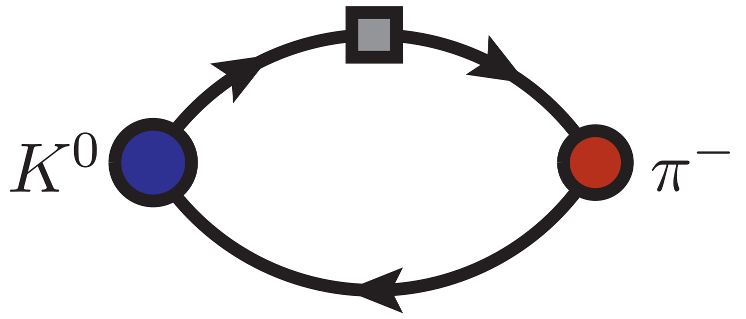
- integration over **three-body phase-space**
- problems of **analytical continuation** when intermediate on shell states are lighter than external ones
- evaluating **finite-volume corrections** potentially more complicated

Solutions to these issues are under study by different groups.

Hopefully we'll see progress in the next few years...

Extension of RM123S approach

- Without QED corrections:



$$\langle \pi(p_\pi)|\bar{s}\gamma^\mu u|K(p_K)\rangle = \boldsymbol{f_+(q^2)}\left[(p_\pi + p_K)^\mu - \frac{m_K^2 - m_\pi^2}{q^2}\, q^\mu\right] + \boldsymbol{f_0(q^2)}\,\frac{m_K^2 - m_\pi^2}{q^2}\, q^\mu$$

An appropriate observable to study is the **differential decay rate:** $s_{\pi\ell} = (p_\pi + p_\ell)^2,\ q^2 = (p_K - p_\pi)^2$

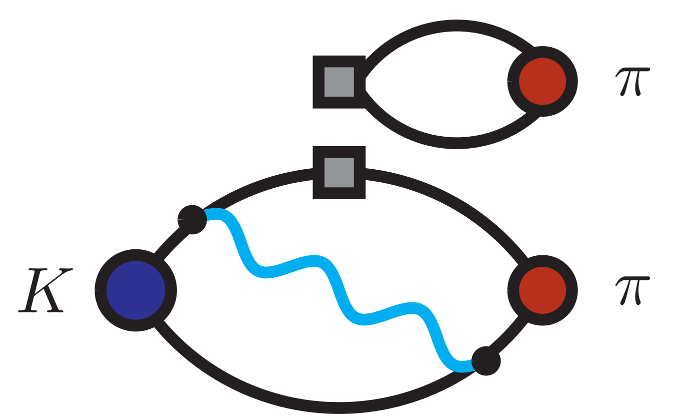
$$\frac{\mathrm{d}^2\Gamma^{(0)}}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} = G_F^2|V_{us}|^2\left[a_1(q^2, s_{\pi\ell})\,|\boldsymbol{f_+(q^2)}|^2 + a_2(q^2, s_{\pi\ell})\,\boldsymbol{f_+(q^2)f_0(q^2)} + a_3(q^2, s_{\pi\ell})\,|\boldsymbol{f_0(q^2)}|^2\right]$$

- Including QED, we can treat IR divergences using the **RM123S method**: C.Sachrajda et al., [1910.07342]

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} = \lim_{\Lambda_{\mathrm{IR}}\to 0}\left[\frac{\mathrm{d}^2\Gamma_0}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} - \frac{\mathrm{d}^2\Gamma_0^{\mathrm{pt}}}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}}\right] + \lim_{\Lambda_{\mathrm{IR}}\to 0}\left[\frac{\mathrm{d}^2\Gamma_0^{\mathrm{pt}}}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} + \frac{\mathrm{d}^2\Gamma_1}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}}\right]$$

Hadronic kaon decays

Goal: precision determination of $\mathrm{Re}(\epsilon'/\epsilon)$ & study of CP violation



Relevance: IB effects will be dominant source of systematic error, once continuum limit will be performed (work in progress)

Status:

- no complete lattice QCD+QED calculation
- lattice QCD calculations by RBC-UKQCD collaboration
- strategy for calculation of IB effects proposed
- first step: Coulomb corrections to $\pi^+\pi^+$ scattering

R.Abbott et al., PRD **102** (2020)
Z.Bai et al., PRL **115** (2015)

N.Christ et al., PRD **106** (2022)
N.Christ & X.Feng, EPJ Web Conf. **175** (2018)
Y.Cai & Z.Davoudi, [1812.11015]

Current status of ϵ'/ϵ

If **isospin-symmetry** is conserved, then the CP violation parameters can be expressed as

$$\frac{\epsilon'}{\epsilon} = \frac{i e^{i(\delta_2-\delta_0)}}{\sqrt{2}\epsilon}\frac{\mathrm{Re}(A_2)}{\mathrm{Re}(A_0)}\left[\frac{\mathrm{Im}(A_2)}{\mathrm{Re}(A_2)} - \frac{\mathrm{Im}(A_0)}{\mathrm{Re}(A_0)}\right]$$

$$A_I = \langle(\pi\pi)_I|H_W^{\Delta S=1}|K\rangle$$

$\delta_I =$ $\pi\pi$ scattering phase shifts

($I =$ isospin)

1. **RBC-UKQCD** performed **first calculation** of ϵ' in 2015 — Z.Bai et al., PRL **115** (2015)
2. **Improved result** in 2020: 3.5x more statistics + improved systematics — R.Abbott et al., PRD **102** (2020)

lattice: $\mathrm{Re}(\epsilon'/\epsilon) = 21.7\,(2.6)_{\mathrm{stat.}}(8.0)_{\mathrm{sys.}}\times 10^{-4}$

experiments: $\mathrm{Re}(\epsilon'/\epsilon) = 16.6\,(2.3)\times 10^{-4}$

Systematic error budget (from C.Kelly @Lattice2023)

- (~12%) Perturbation theory in Wilson coeffs to match 3f – 4f weak EFT at m_c
  - Improve with 4f calculation (active charm) : computationally infeasible?
  - Non-perturbative calculation of matching matrix : investigation underway

    [M.Tomii, *PoS* LATTICE2018 (2019) 216]

- (~23%) Lack of EM+isospin-breaking contributions in lattice calculation
  - Lattice measurement of these effects extremely challenging but approach is being formulated. [Phys.Rev.D 106 (2022) 1, 014508] [Christ, *PoS* LATTICE2021 (2022) 312]

    → estimated using χPT results
    V.Cirigliano et al., JHEP **02** (2020)

- (~12%) Use of single lattice spacing to compute I=0 amplitude
  - Repeat calculation with multiple, finer lattice spacings: **my current focus**

→ Intense work by RBC-UKQCD to reduce ~12% error due to use of single lattice spacing

—› IB correction will soon become relevant!

Isospin-breaking corrections

IB corrections are usually O(1%), but the "$\Delta I = 1/2$ rule" can give a ~20x enhancement in ϵ'/ϵ

A calculation of these effects is **very challenging!**

- **Lüscher & Lellouch-Lüscher formalisms** that relate finite-volume quantities (energy levels & correlation functions) to infinite-volume observables (scattering phase shifts & decay amplitudes) **need to be corrected for long range QED interactions**
- $\pi\pi$ final states with $I = 0$ and $I = 2$ are not independent anymore and **can mix**: it's a **coupled two-channel problem**

First step done: include QED corrections from Coulomb interaction to $\pi^+\pi^+$ scattering phase shift

Y.Cai & Z.Davoudi, [1812.11015] / N.Christ & X.Feng, EPJ Web Conf. **175** (2018) / N.Christ et al., PRD **106** (2022)