Long distance physics and hadronic decays

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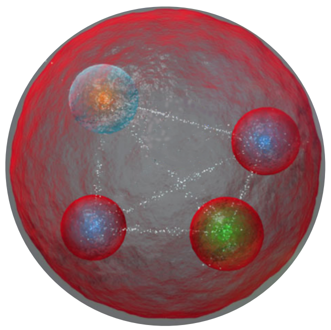
CERN

THE UNIVERSITY *of* EDINBURGH

Multi-hadron observables

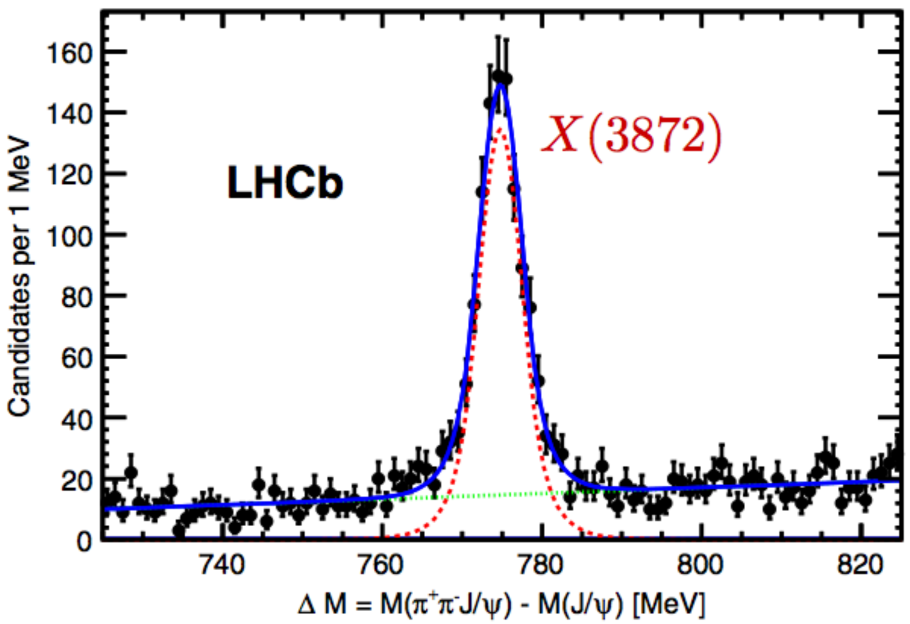
- Exotics, XYZs, tetra- and penta-quarks, *H* dibaryon



e.g. X(3872)

$$\sim |D^0\overline{D}^{*0} + \overline{D}^0 D^{*0}\rangle\,?$$

- LHCb (PRD92, 2015)



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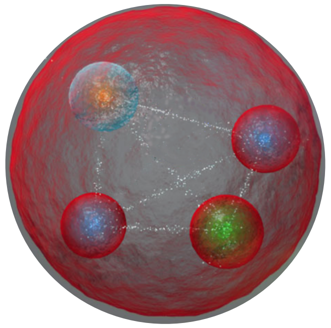

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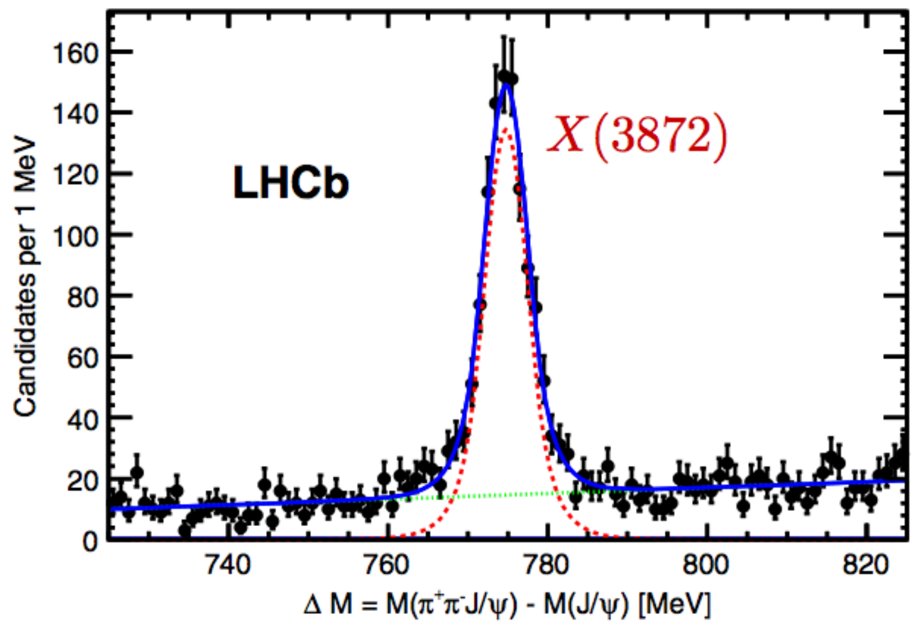

- Eletroweak, CP violation, resonant enhancement

CP violation in charm $\quad D \to \pi\pi, K\overline{K}$

$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4}$

LHCb (PRL, 2019)

$f_0(1710)$ could enhance ΔA_{CP}

Soni (2017)

Resonant B decays $\quad B \to K^* \ell\ell \to K\pi\,\ell\ell$

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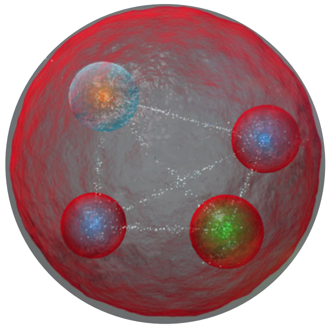

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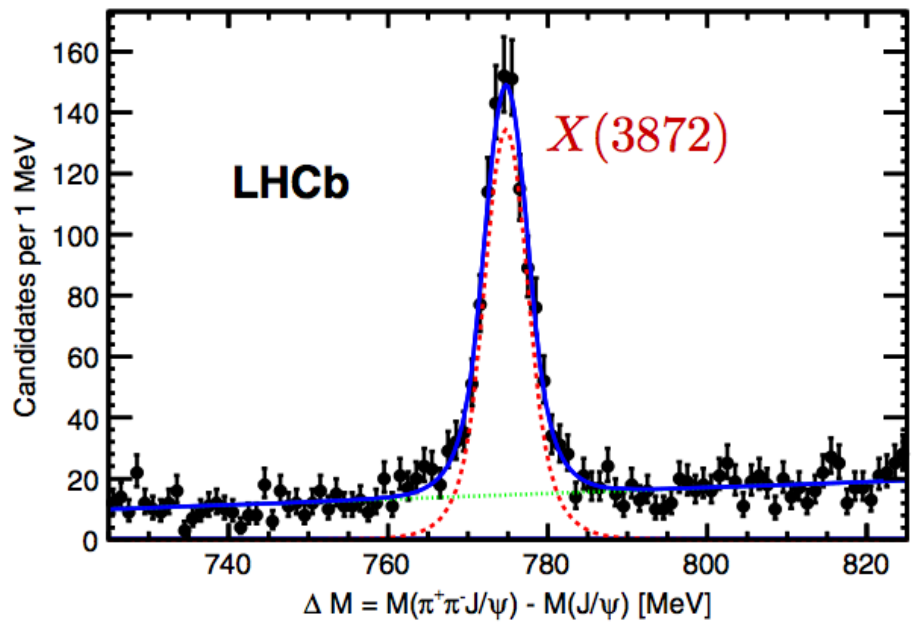

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Resonant B decays $B \to K^*\,\ell\ell \to K\pi\,\ell\ell$

$$|X\rangle, |\rho\rangle\,, |K^*\rangle\,, |f_0\rangle \quad \notin \quad \textbf{QCD Fock space}$$

S-matrix (and unitarity)

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$
- Overlaps of multi-hadron ***asymptotic states*** → S matrix

$$[\mathbb{S}\,\mathbb{S}^\dagger]_{\alpha\beta} = \sum_\gamma \langle \alpha, \mathrm{out}|\gamma, \mathrm{in}\rangle\langle\gamma, \mathrm{in}|\beta, \mathrm{out}\rangle = \mathbb{I}_{\alpha\beta}$$

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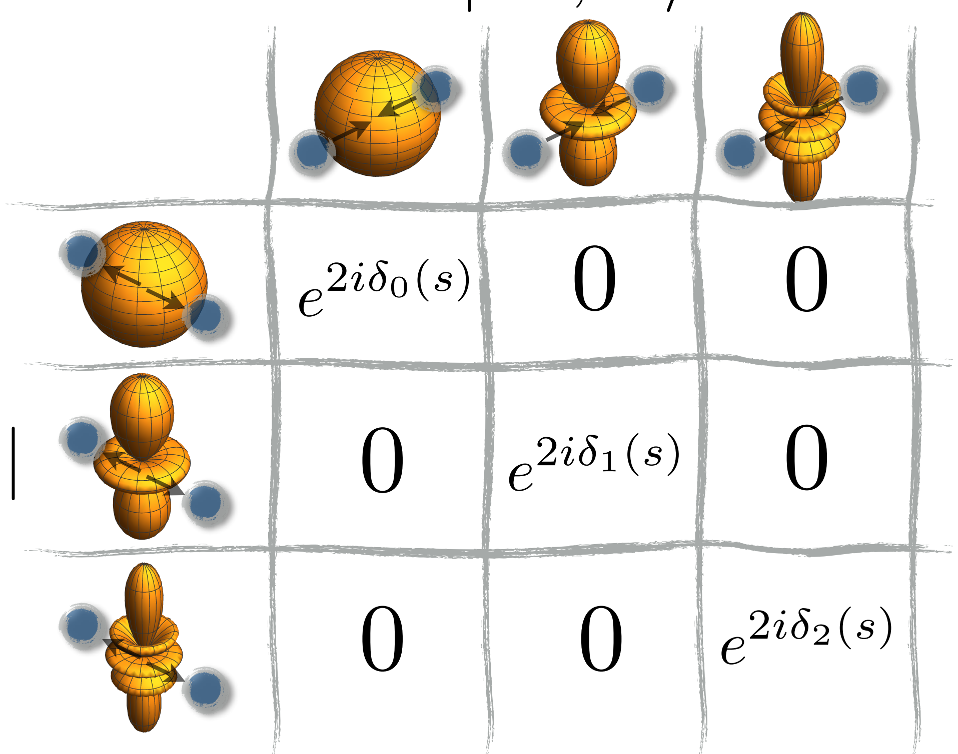
$|\pi\pi, \mathrm{in}\rangle$

$S(s) \equiv \langle \pi\pi, \mathrm{out}|$



depends on $s = E_{\mathrm{cm}}^2$ and angular variables

diagonal in angular momentum

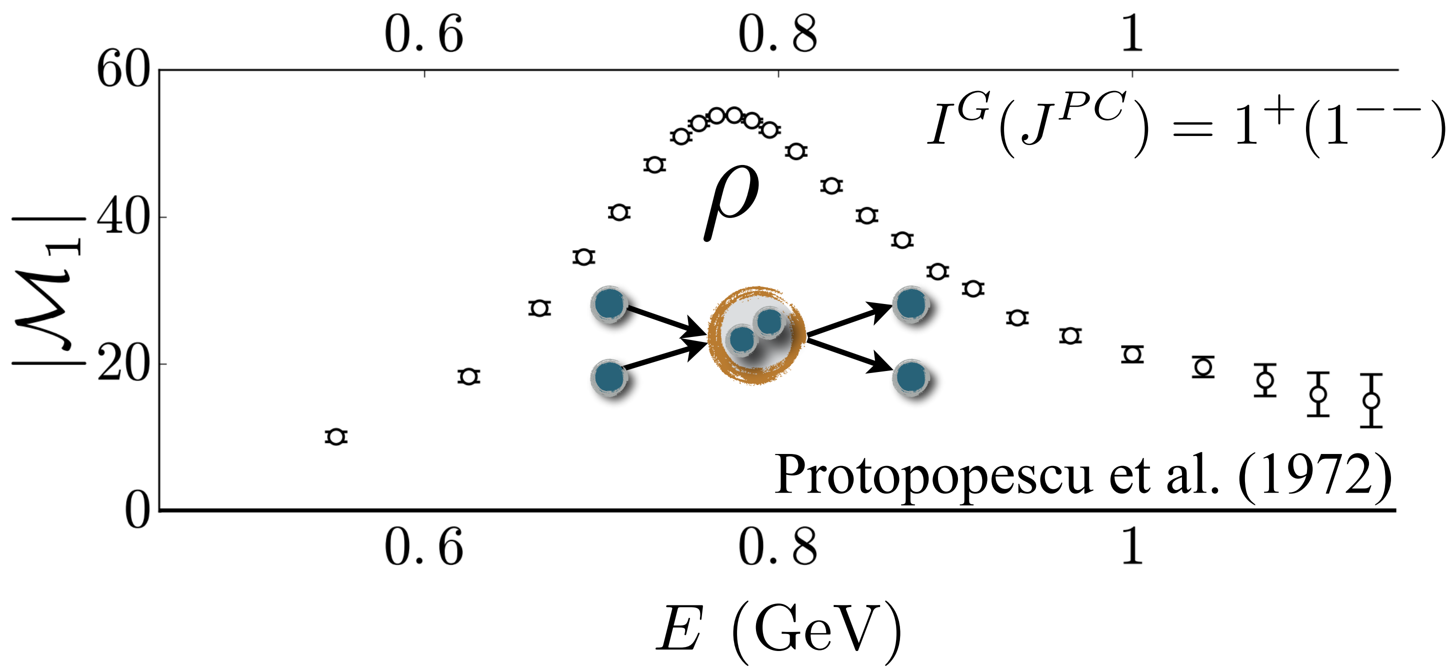
$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

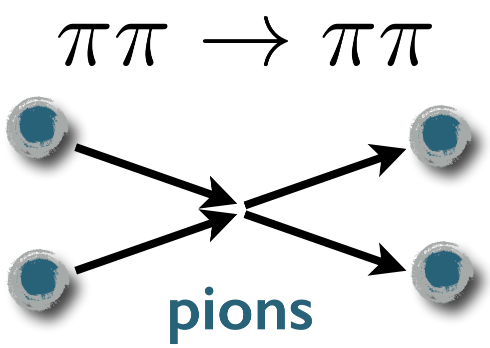
- ***An enormous space of information*** $|\pi\pi\pi\pi, \mathrm{in}\rangle \quad |K\overline{K}, \mathrm{in}\rangle \quad \cdots$

QCD resonances (and unitarity)

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

scattering rate





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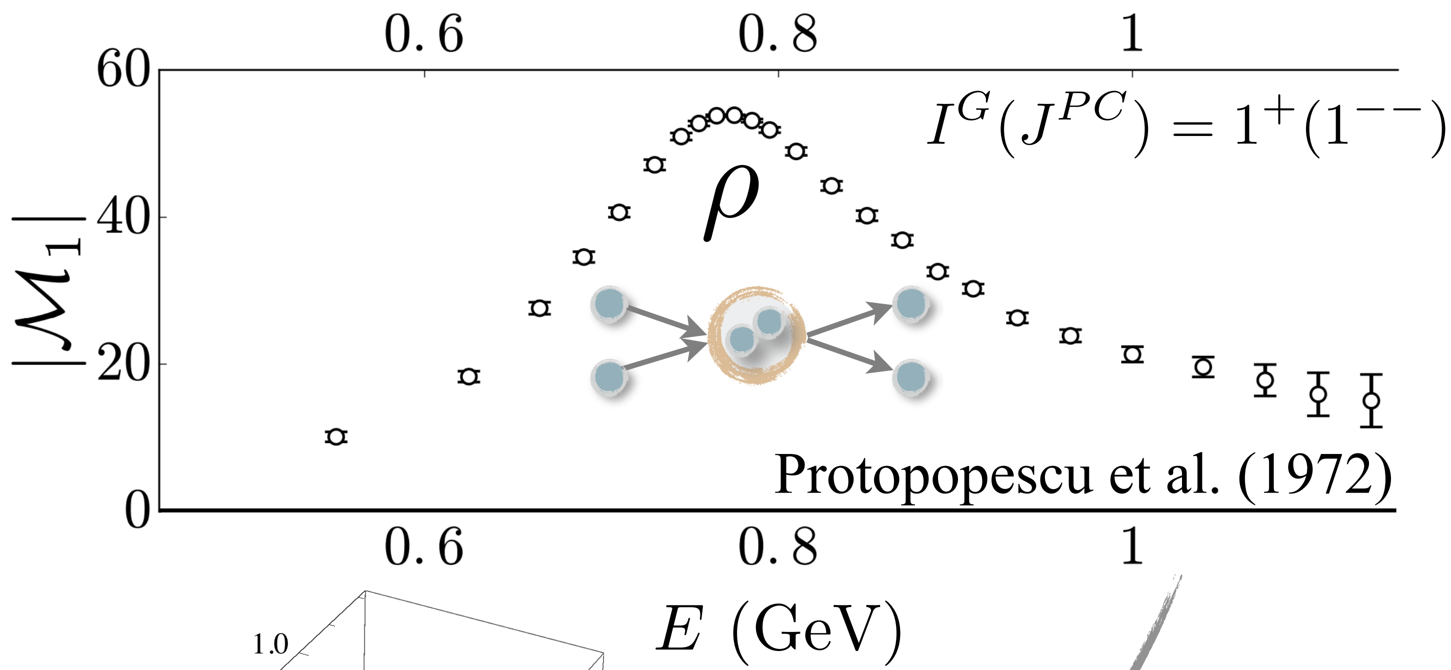

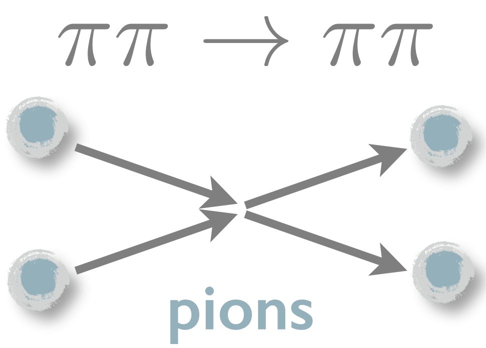

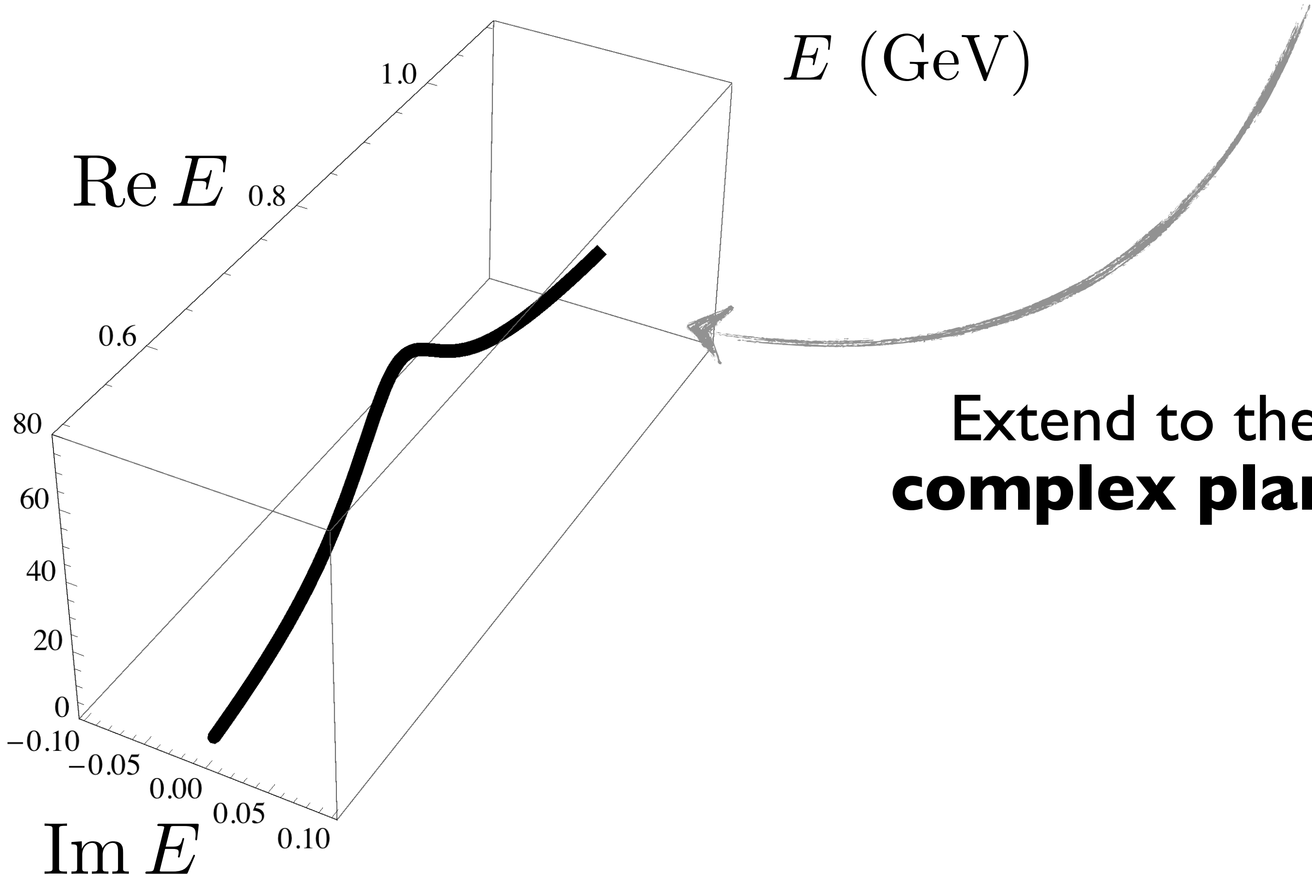

Extend to the **complex plane**

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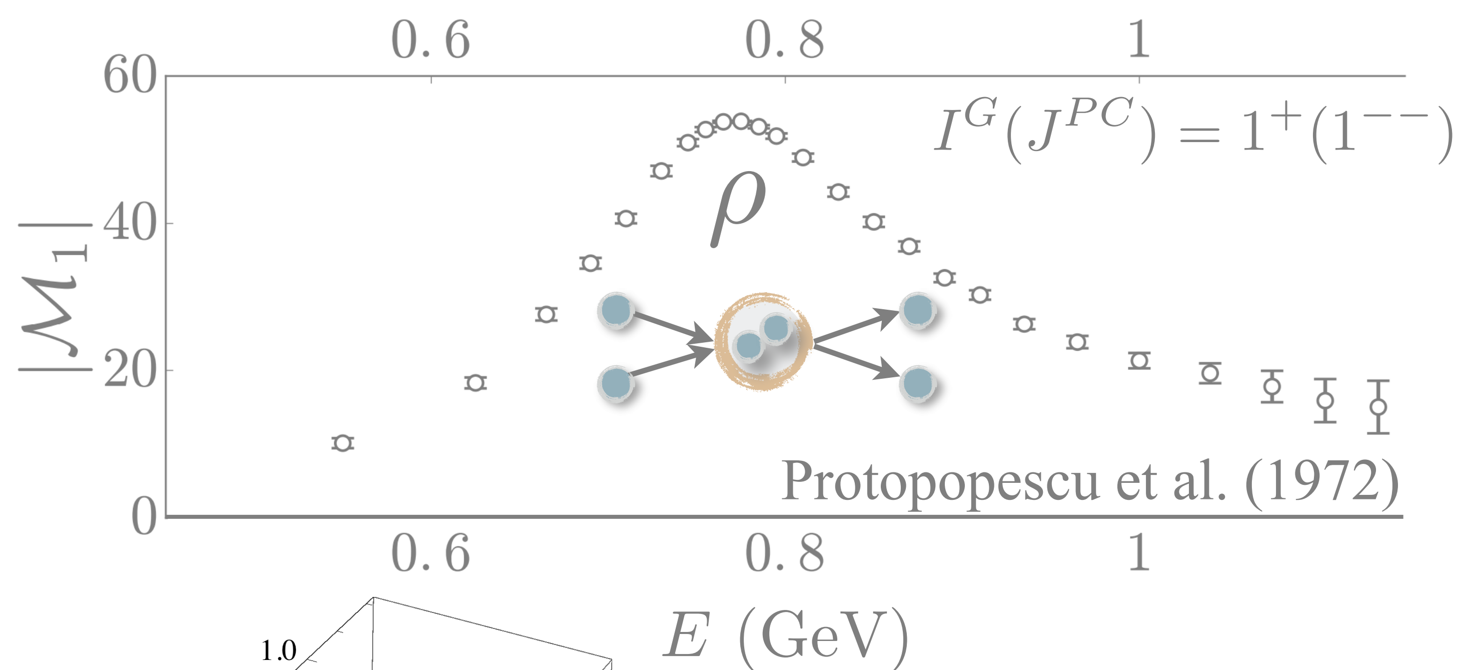
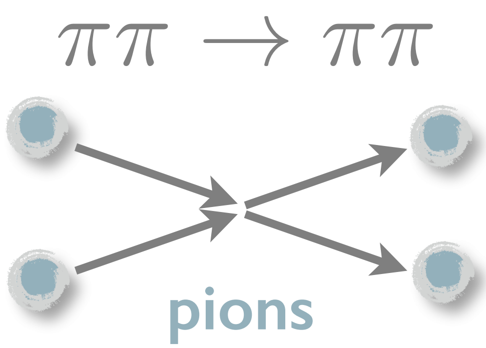
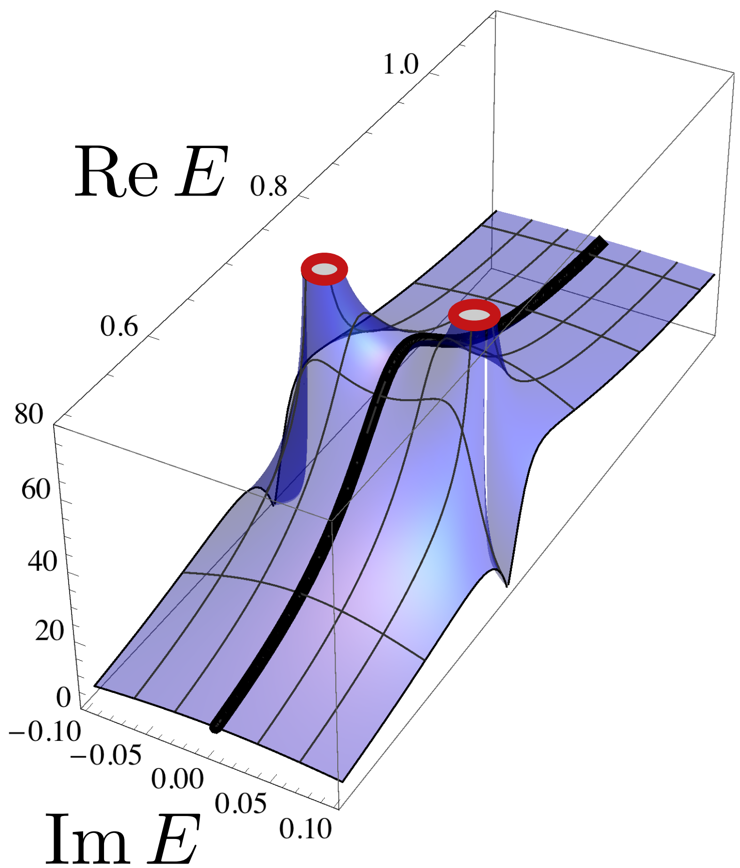
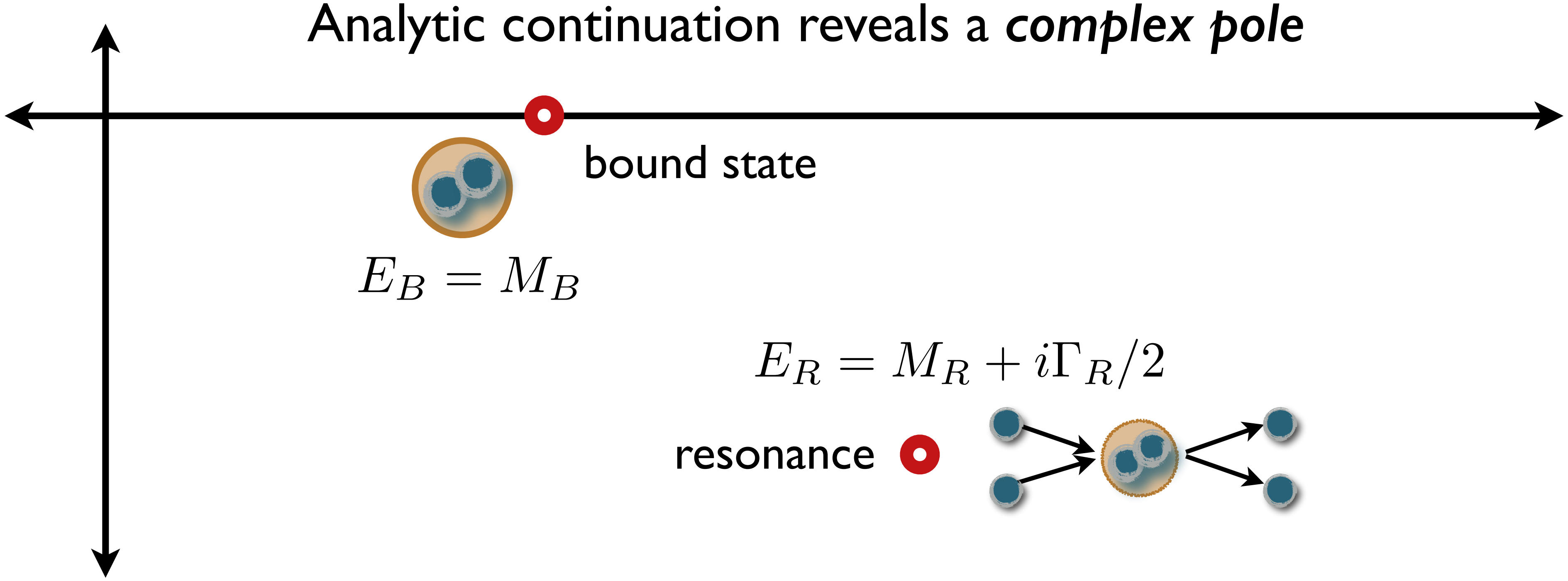




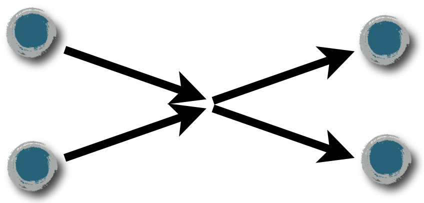


Analytic continuation reveals a ***complex pole***



Analyticity

- Instead of $|\mathcal{M}(s)|^2$ → analytically continue the ***amplitude*** itself



For two-particle energies $(2m)^2 < s < (4m)^2$, what is the analytic structure?

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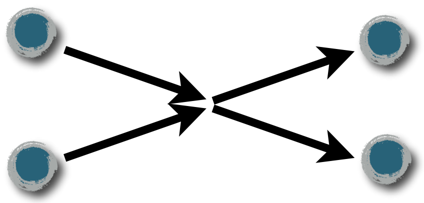

- The optical theorem tells us…

$$\rho(s)|\mathcal{M}_\ell(s)|^2 = \mathrm{Im}\,\mathcal{M}_\ell(s)$$

where $\rho(s) = \frac{\sqrt{1-4m^2/s}}{32\pi}$ is the two-particle phase space

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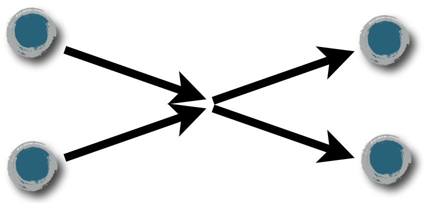

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- Unique solution is…

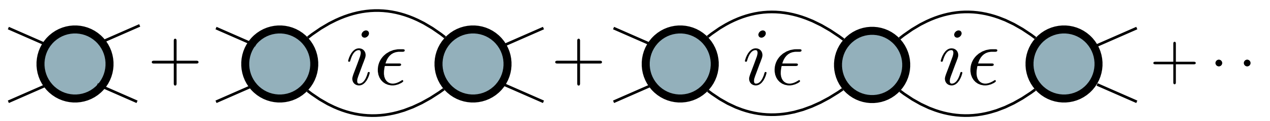
$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)}$$

K matrix (short distance)

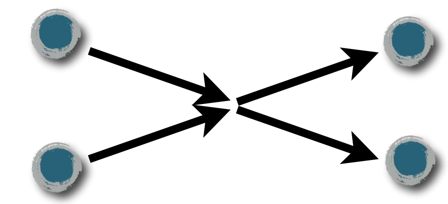
phase-space cut (long distance)

Key message: *The scattering amplitude has a square-root branch cut*

Analyticity (diagrammatic)

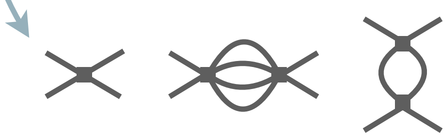
$\mathcal{M}(s) \equiv$ 

on-shell particles = singularities:
non-analytic for $(2m)^2 < s < (4m)^2$



propagating pion

Bethe-Salpeter kernel

 $= \int [\text{real, analytic}]$

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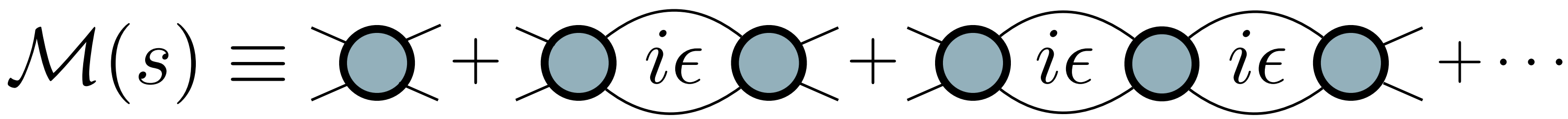

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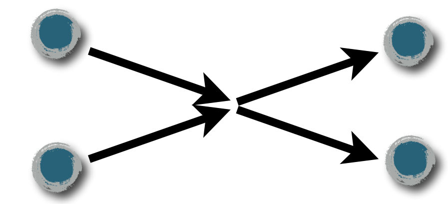

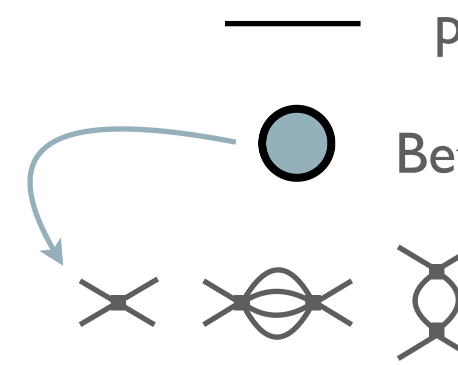

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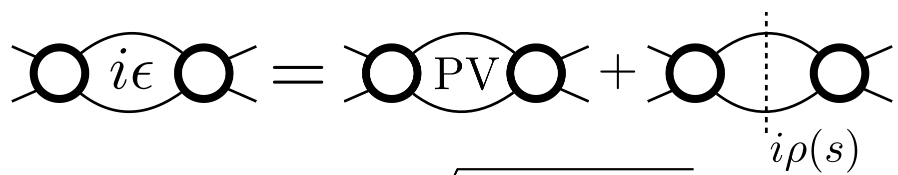
cutting rule



$\rho(s) \propto \sqrt{s-(2m)^2}$

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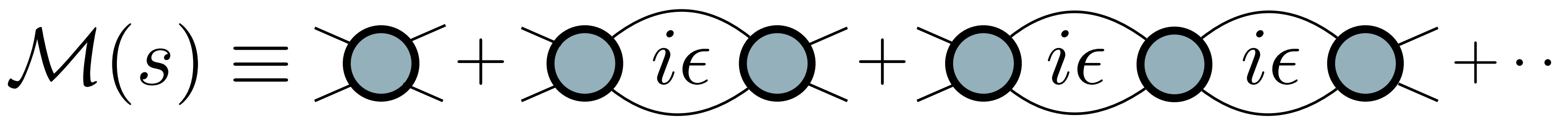

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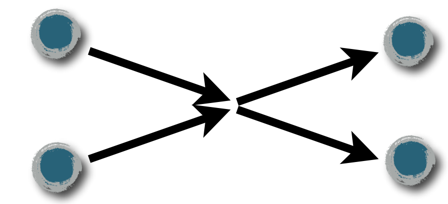

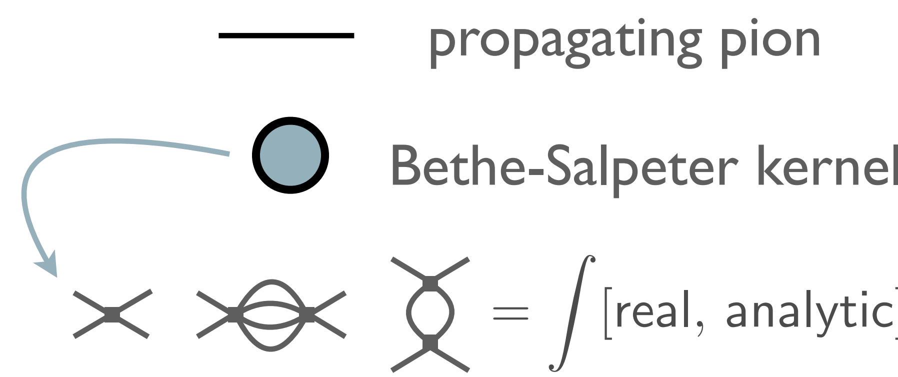

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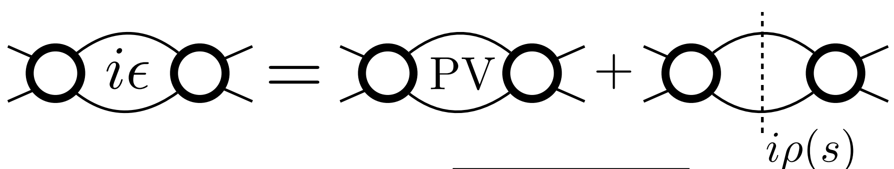

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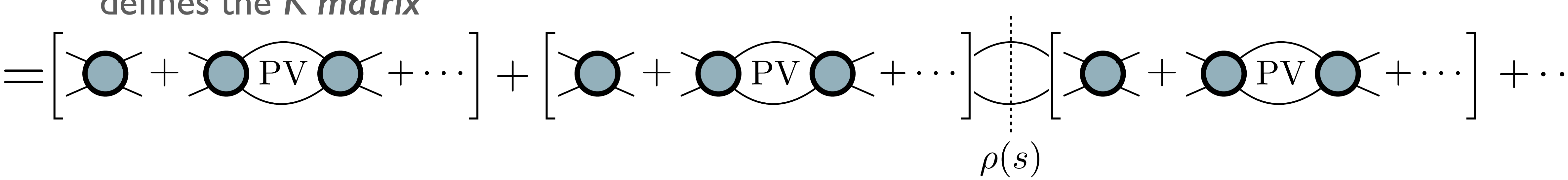
defines the K *matrix*



$$= \mathcal{K}(s) + \mathcal{K}(s) i\rho(s) \mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} - i\rho(s)}$$

K matrix (short distance)

phase-space cut (long distance)

Representations, cuts and sheets

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)} = \frac{1}{\rho(s)\cot\delta_\ell(s) - i\rho(s)} \qquad \rho(s) = \frac{\sqrt{s/4 - m^2}}{16\pi\sqrt{s}}$$

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- Simple relation between K-matrix, cot-delta, and S-matrix forms

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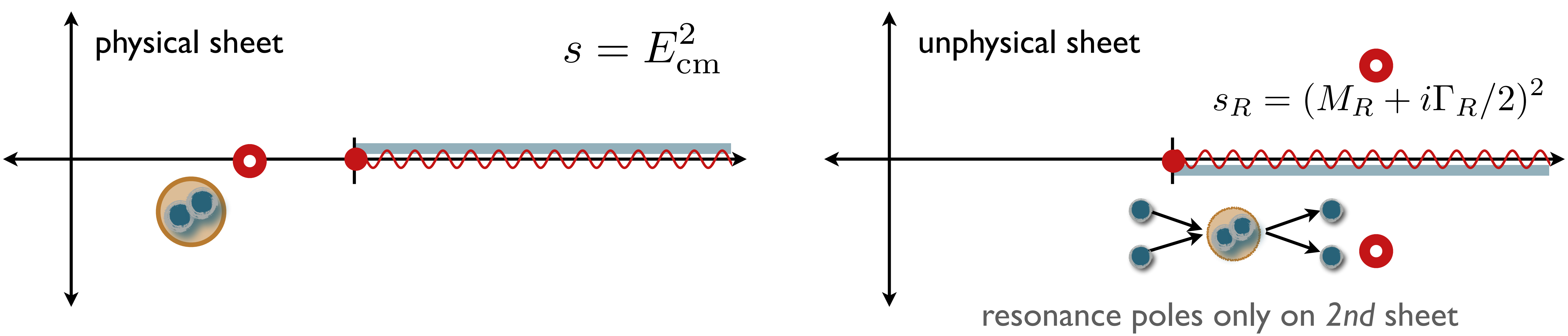
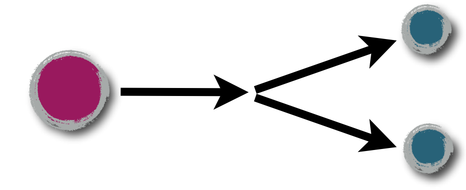
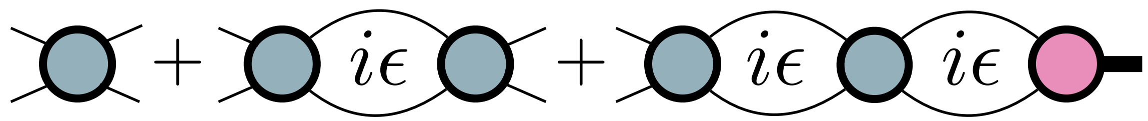
- ❑ Simple relation between K-matrix, cot-delta, and S-matrix forms
- ❑ Each channel generates a *square-root cut* → doubles the number of sheets



$K \to \pi\pi$



$\mathcal{A}(s) \equiv$ 

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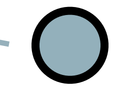
propagating pion



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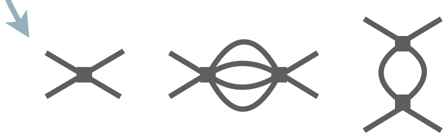

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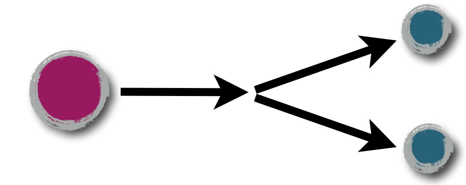

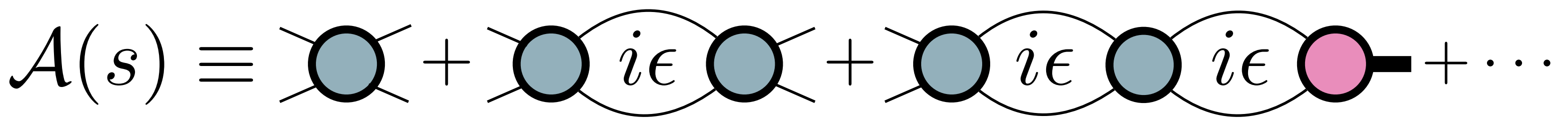

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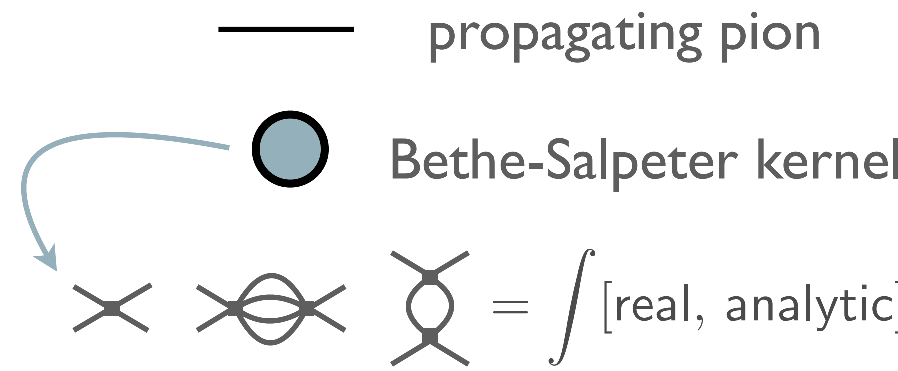

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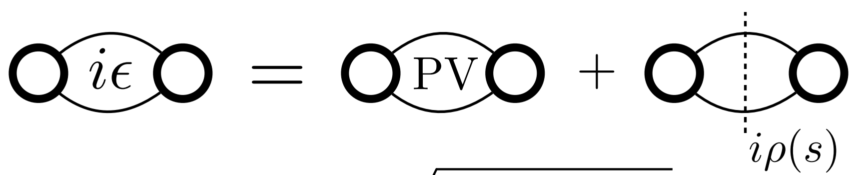
cutting rule



$\rho(s) \propto \sqrt{s - (2m)^2}$

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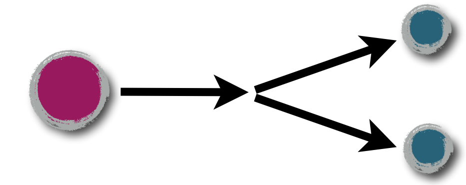

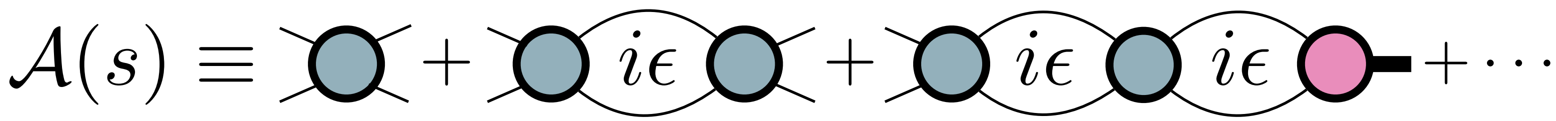

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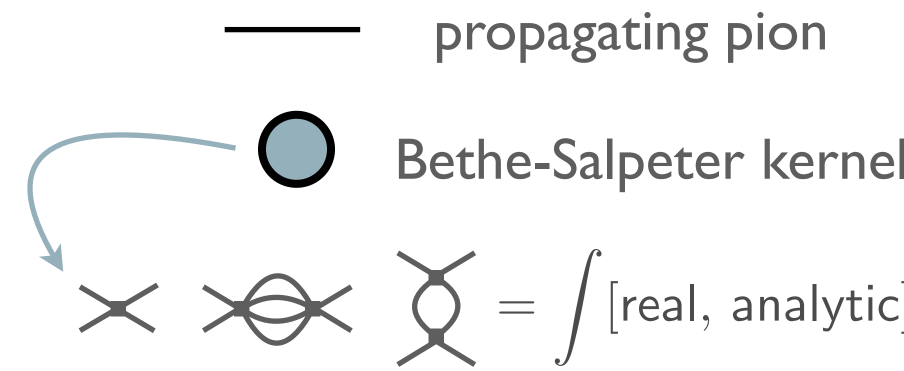

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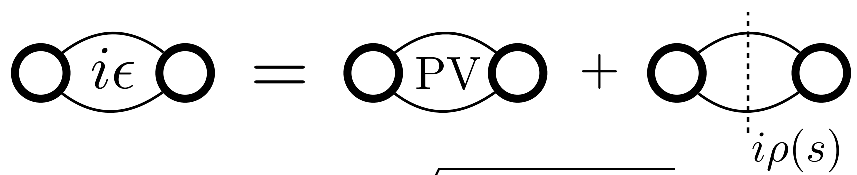

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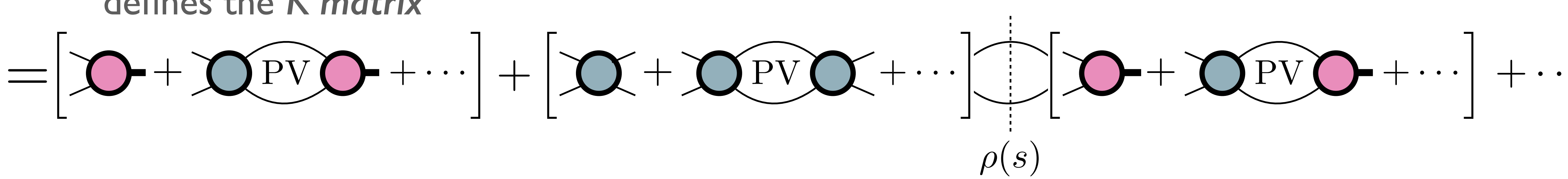


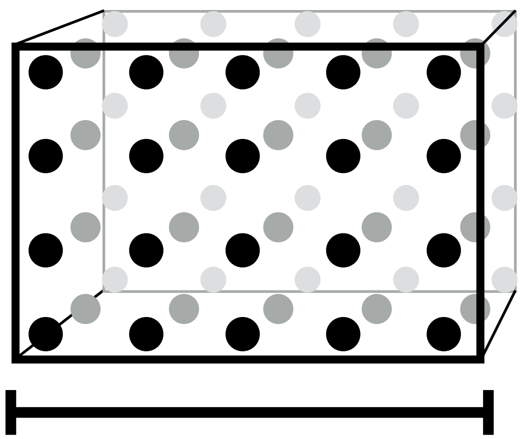
$$= \mathcal{H}(s) + \mathcal{K}(s) i\rho(s)\mathcal{H}(s) + \cdots = \frac{1}{1 - \mathcal{K}(s) i\rho(s)} \mathcal{H}(s)$$

K-matrix analog

K matrix (short distance)

phase-space cut (long distance)

Lattice meets multi-hadron observables



- The *finite volume*…
  - *Discretizes* the spectrum
  - *Eliminates* the branch cuts and extra sheets
  - *Hides* the resonance poles

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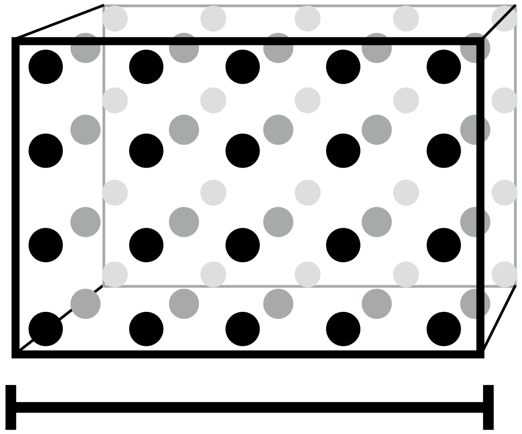

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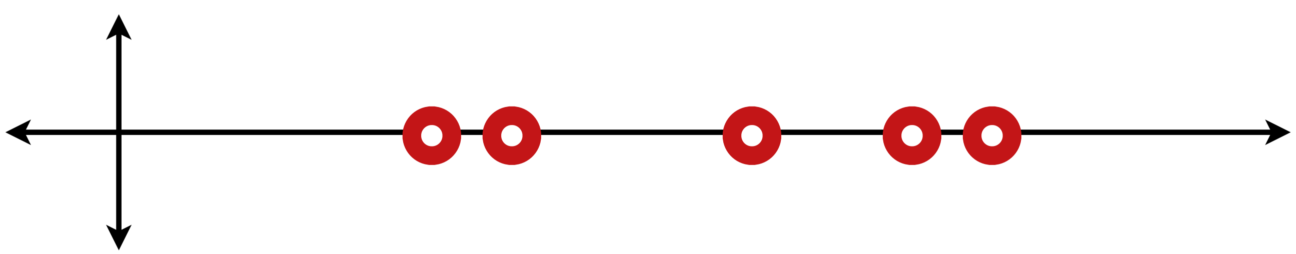
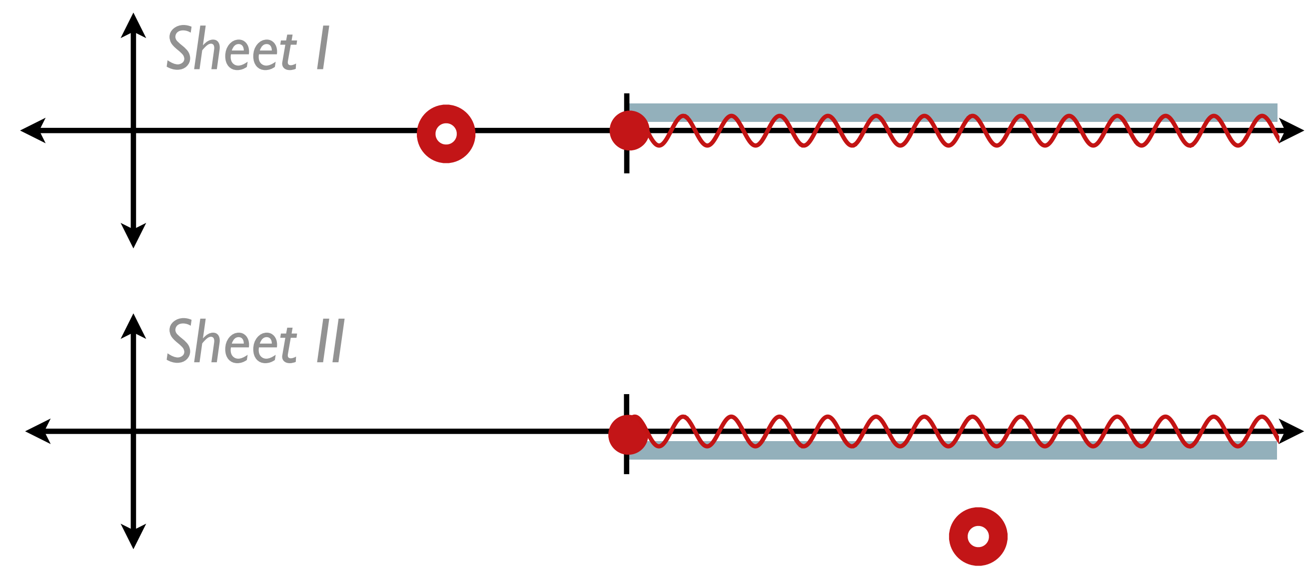
Finite-volume analytic structure



Infinite-volume analytic structure



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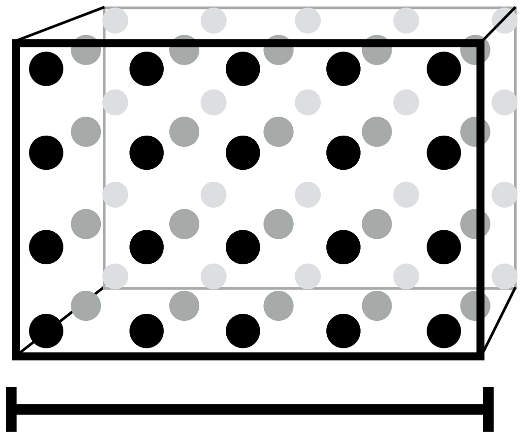

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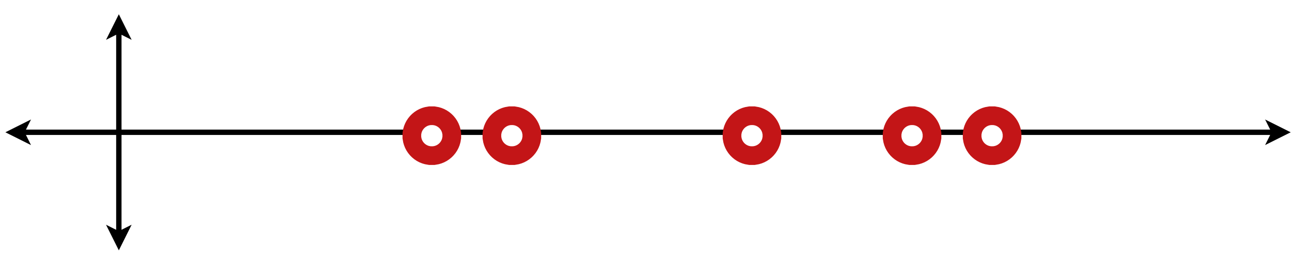
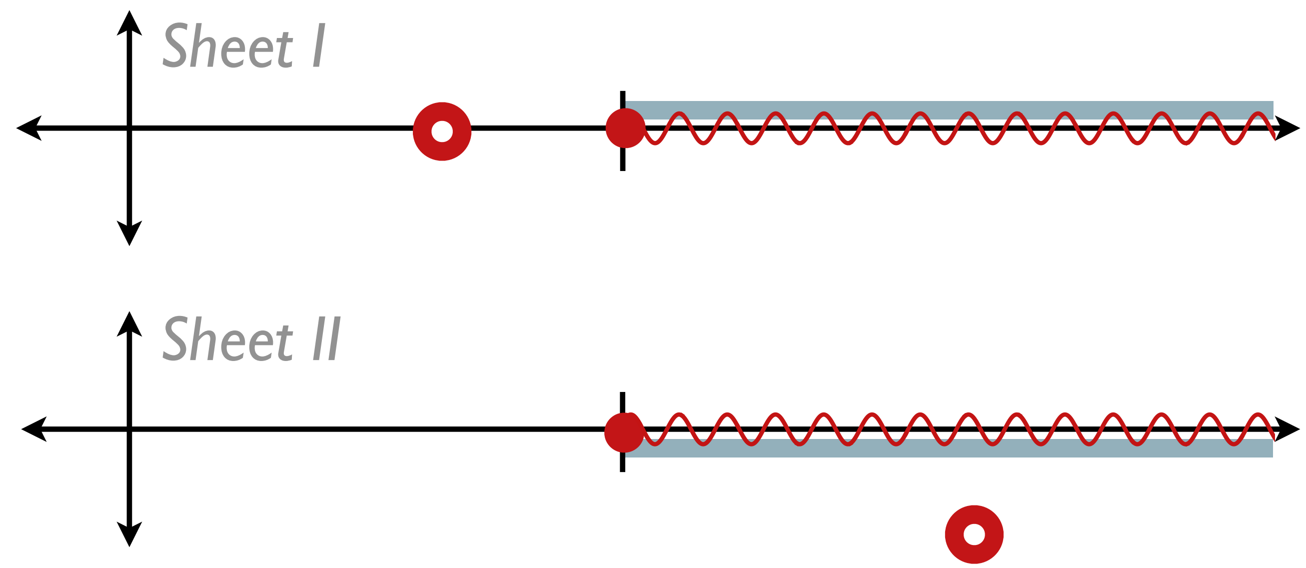
  - *Hides* the resonance poles

Finite-volume analytic structure



Infinite-volume analytic structure



- LQCD → Energies and matrix elements

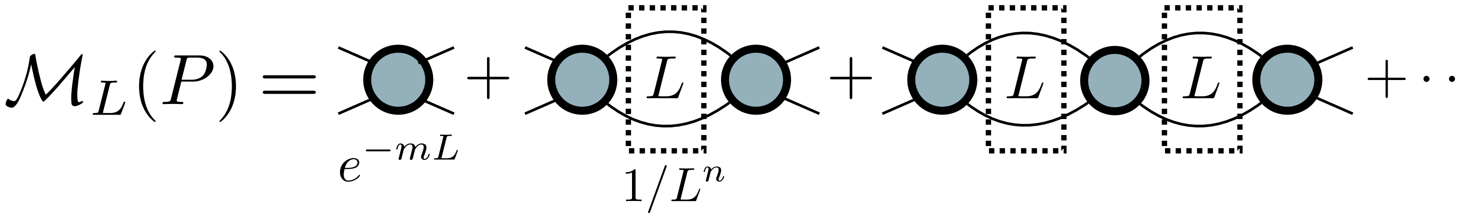
$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

- Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***

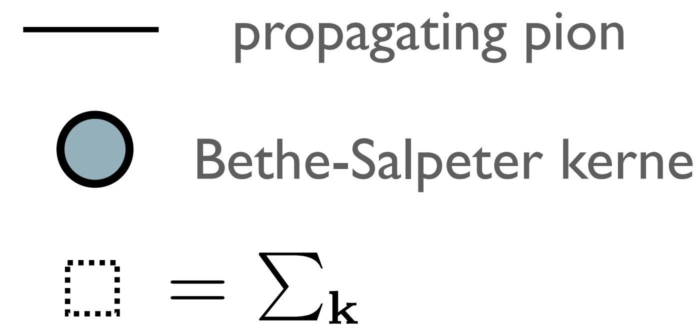
Deriving the scattering formalism

$\mathcal{M}(s)$	$\mathcal{M}_L(P)$
probability amplitude	poles give f.v. spectrum

- Consider the finite-volume correlator:



For two-particle energies $(2m)^2 < s < (4m)^2$, what is the *L* dependence?



• Lüscher (1986) • Kim, Sachrajda, Sharpe (2005) • *many others*

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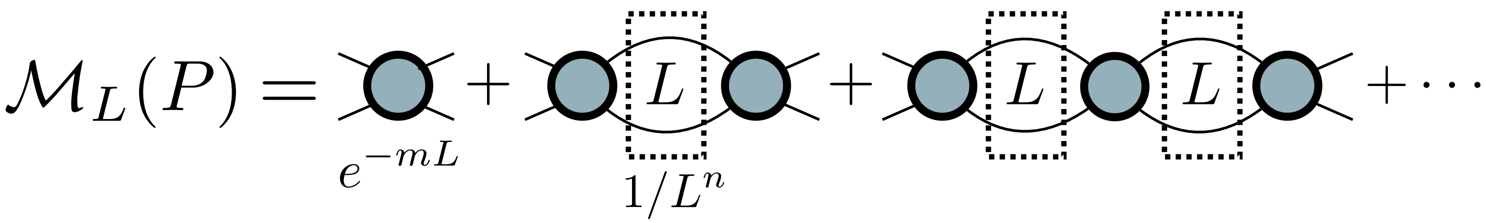

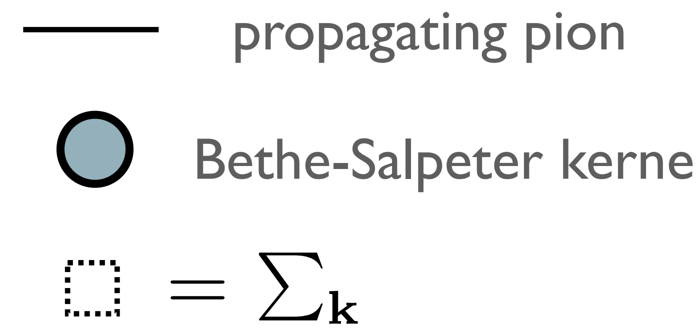

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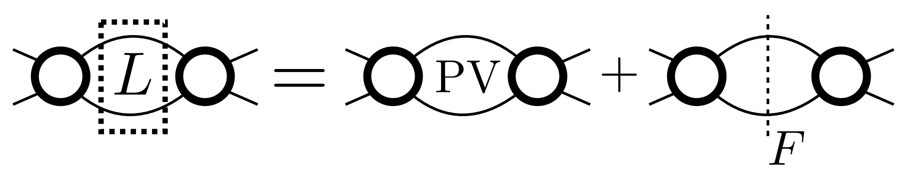

F = matrix of known geometric functions

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Deriving the scattering formalism

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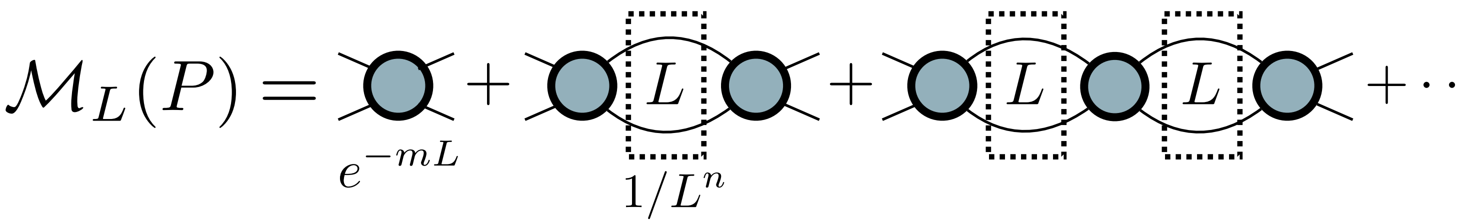
$\mathcal{M}(s)$ probability amplitude | $\mathcal{M}_L(P)$ poles give f.v. spectrum

propagating pion

Bethe-Salpeter kernel

$\square = \sum_{\mathbf{k}}$



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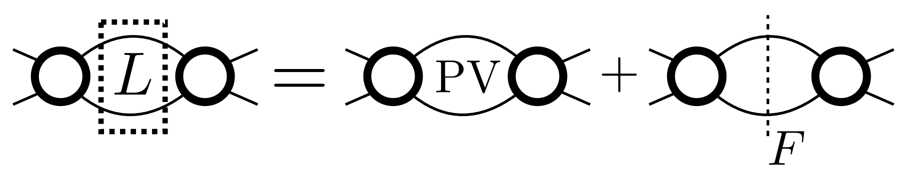
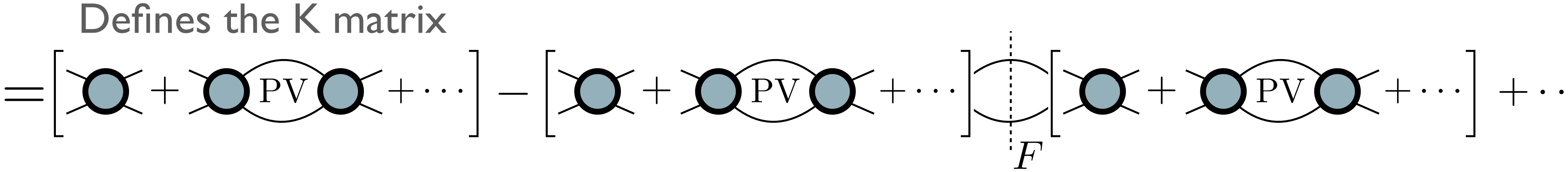
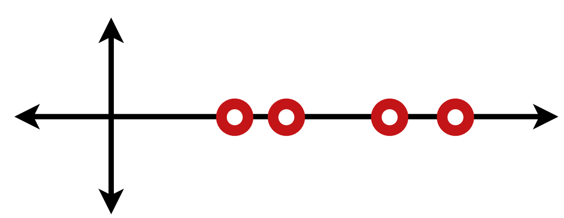


F = matrix of known geometric functions

Defines the K matrix



$$= \frac{1}{\mathcal{K}(s)^{-1} + F(P,L)}$$



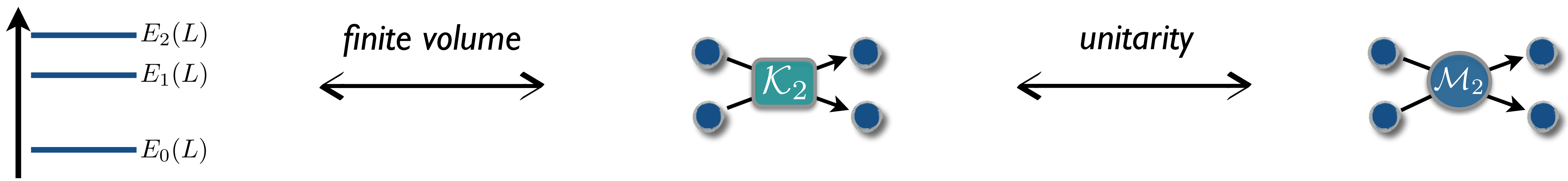
$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

- Lüscher (1986) • Kim, Sachrajda, Sharpe (2005) • *many others*

General relation

$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

$F(P,L) \equiv$ Matrix of known geometric functions



Holds only for two-particle energies $s < (4m)^2$

Neglects e^{-mL}

Generalized to ***non-degenerate masses, multiple channels, spinning particles***

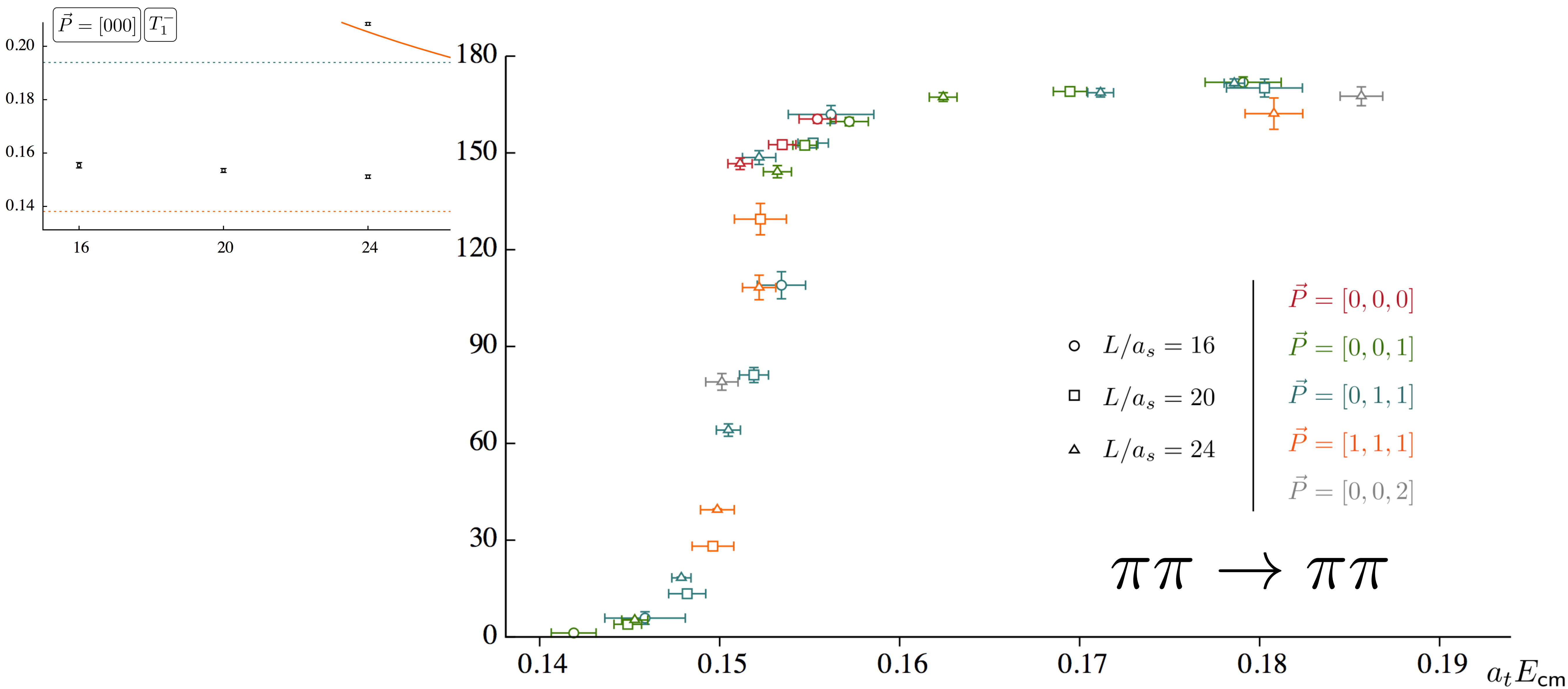
Encodes angular momentum mixing

Huang, Yang (1958) • Lüscher (1986, 1989) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
Beane, Detmold, Savage (2007) • Tan (2008) • Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012)
MTH, Sharpe (2012) • Briceño, Davoudi (2012) • Li, Liu (2013) • Briceño (2014)

Using the result

- Single-channel case *(pions in a p-wave)*

$$\mathcal{K}(s_n)^{-1} = \rho \cot \delta(s_n) = -F(E_n, \vec{P}, L)$$



• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

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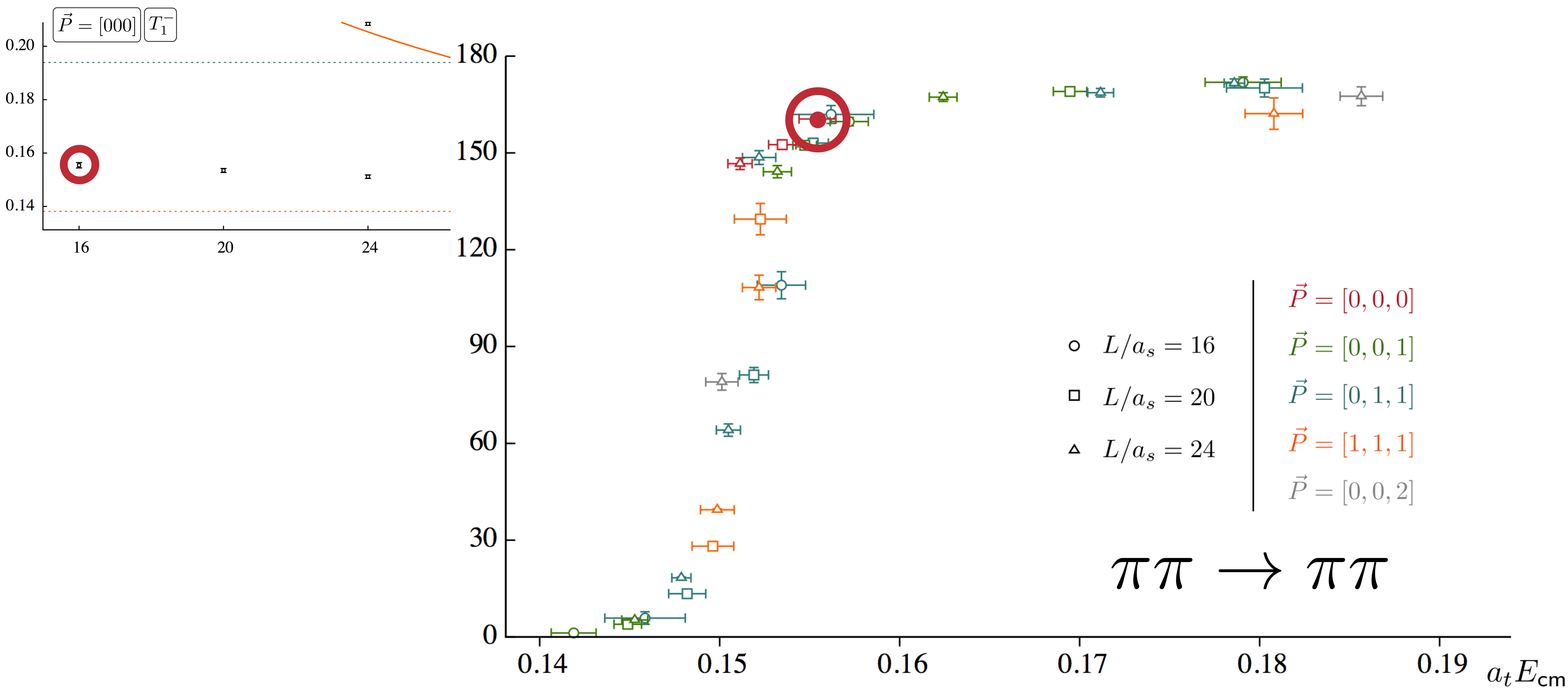

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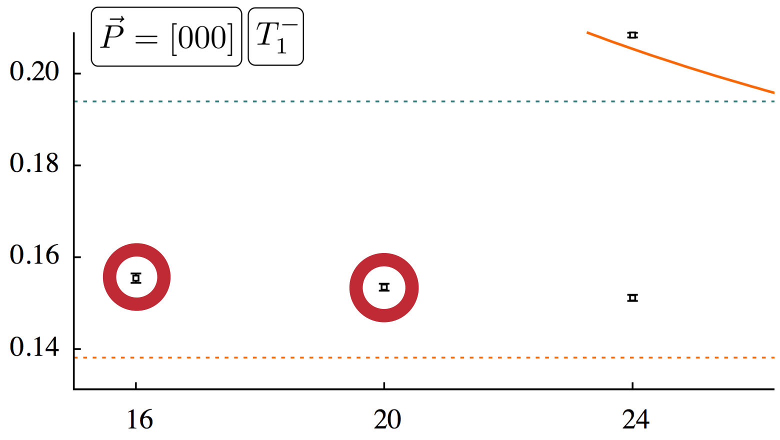

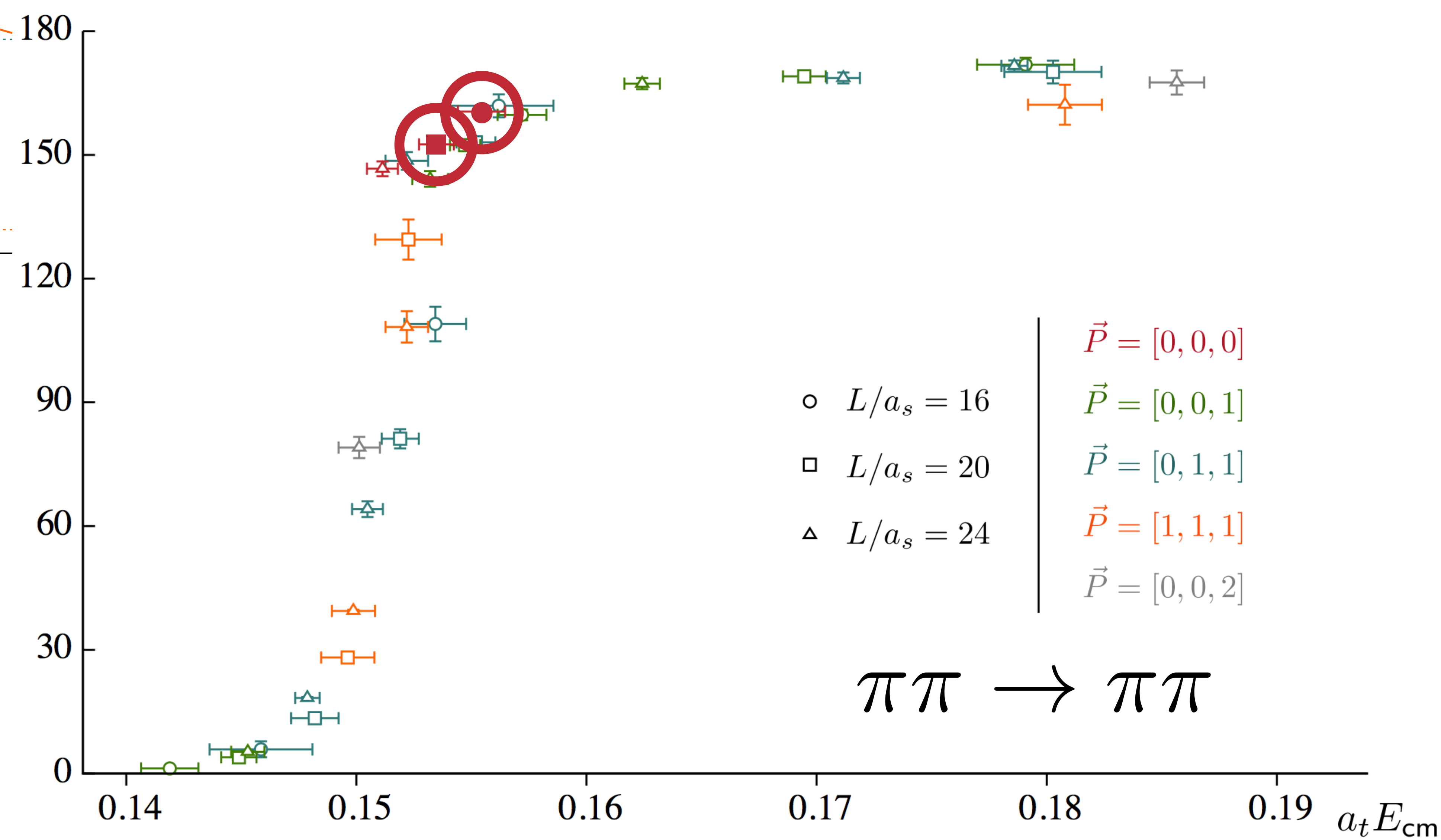

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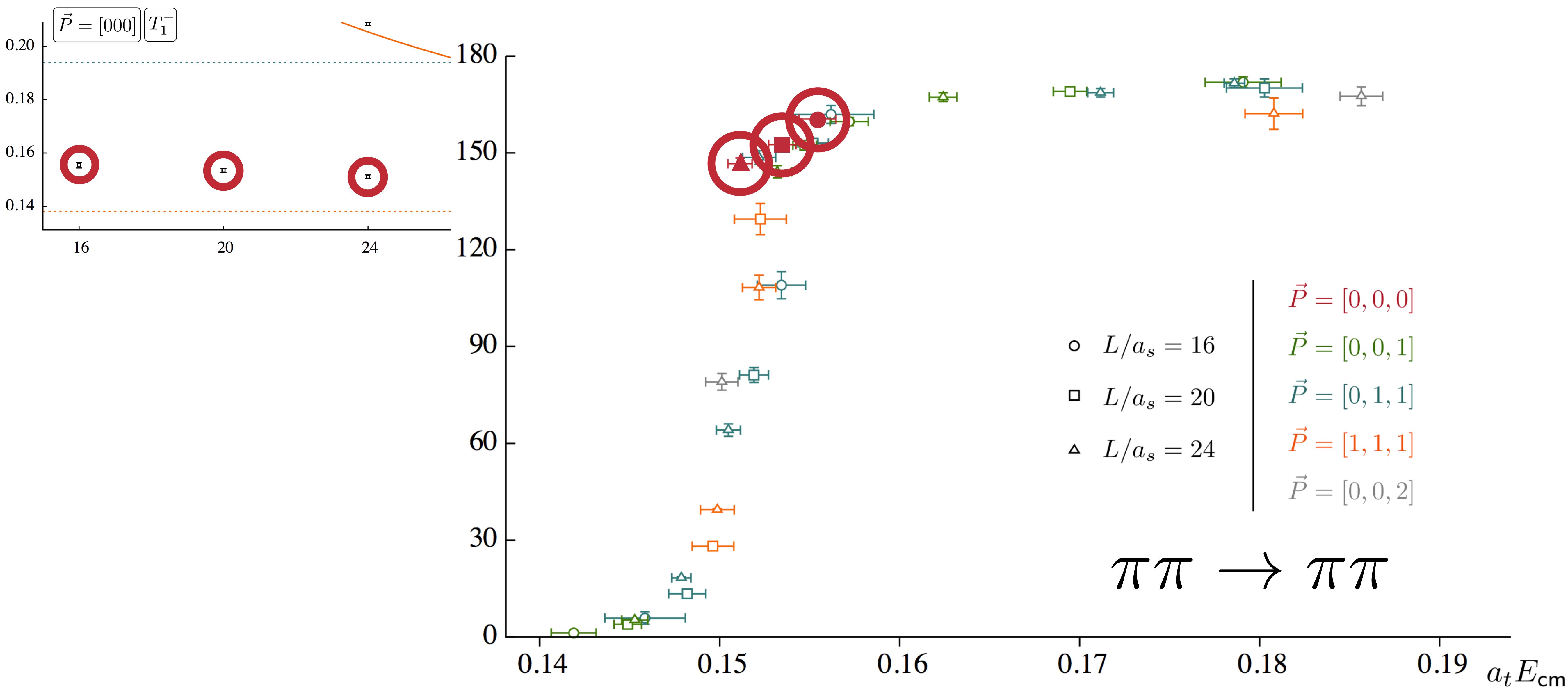

• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

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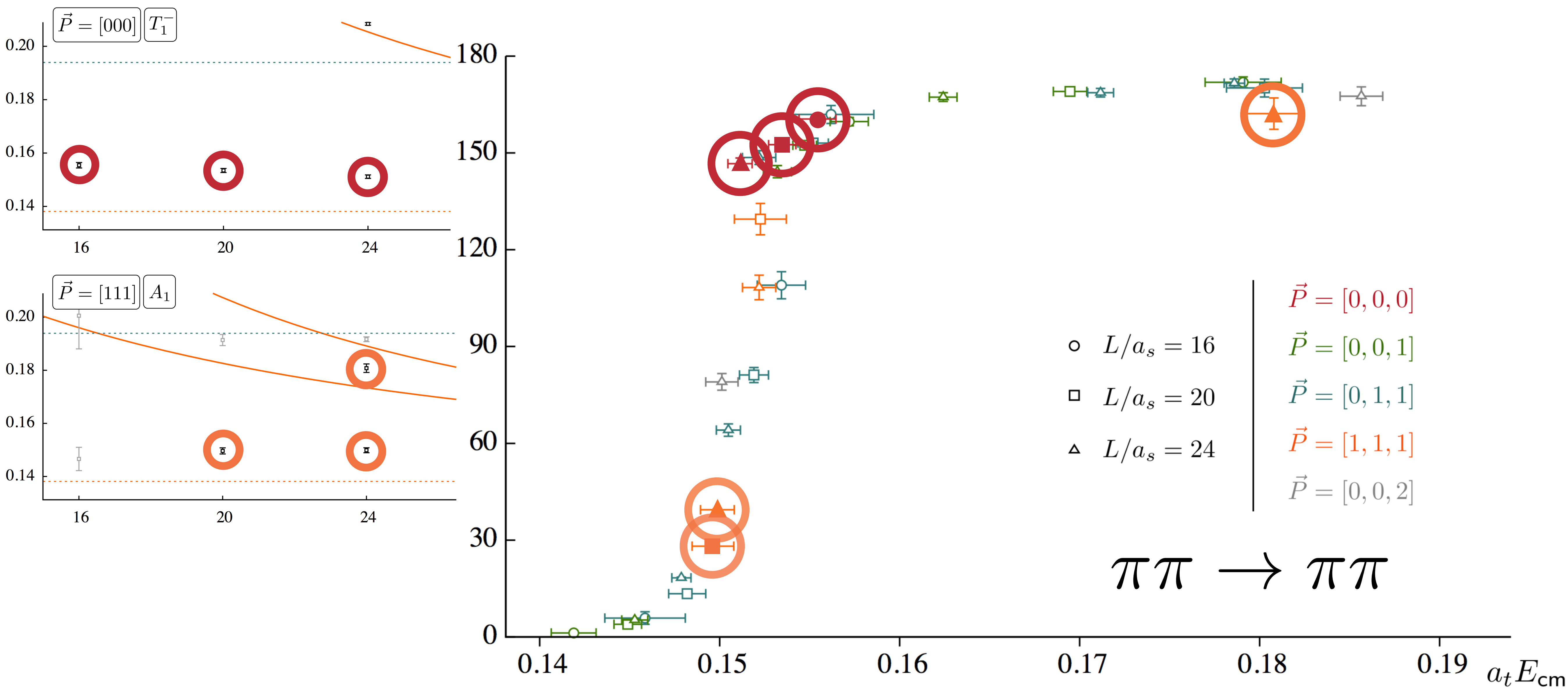

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Physical-mass calculation of $\rho(770)$ and $K^*(892)$ resonance parameters via $\pi\pi$ and $K\pi$ scattering amplitudes from lattice QCD

Peter Boyle,[1,2] Felix Erben,[3,2] Vera Gülpers,[2] Maxwell T. Hansen,[2] Fabian Joswig,[2] Nelson Pitanga Lachini,[4,2,*] Michael Marshall,[2] and Antonin Portelli[2,3,5]

- ☐ Single ensemble, physical-mass domain-wall quarks, many operators via *distillation*
- ☐ Data-driven estimation of uncertainties from correlator fit range and phase-shift fit model

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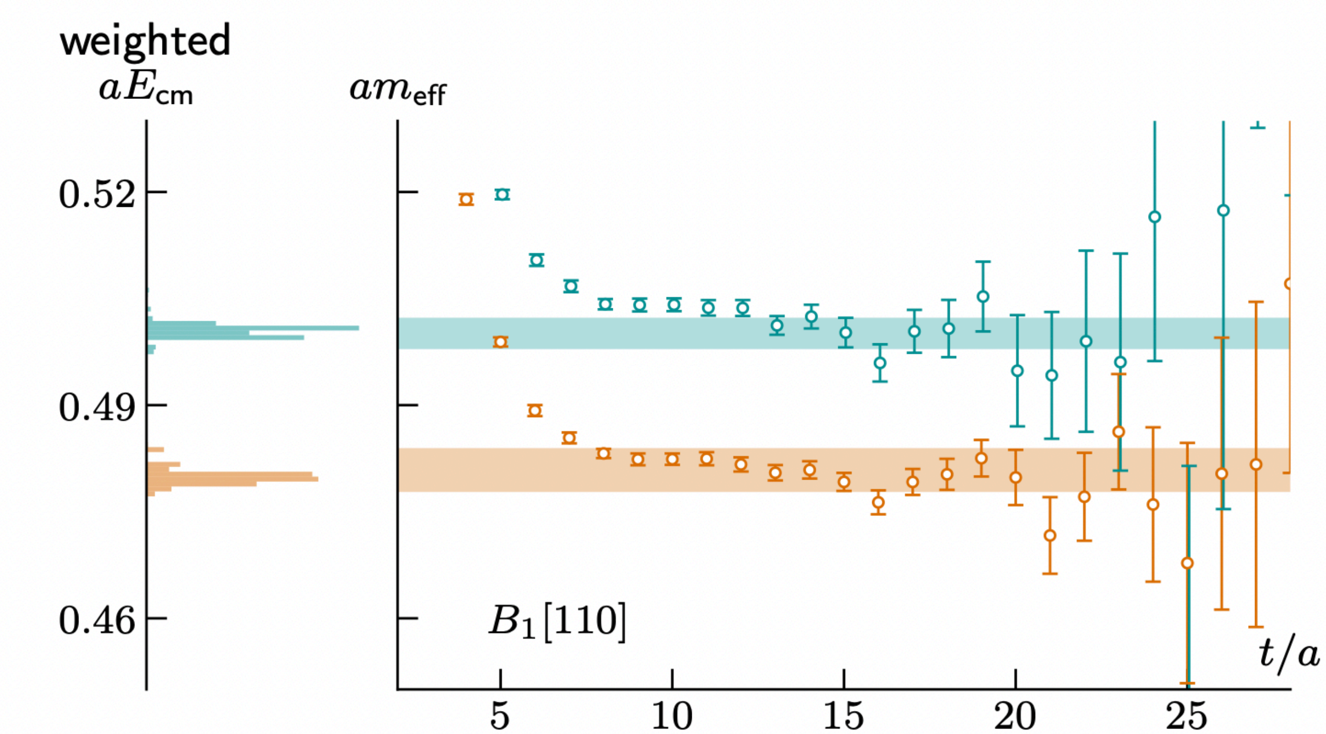
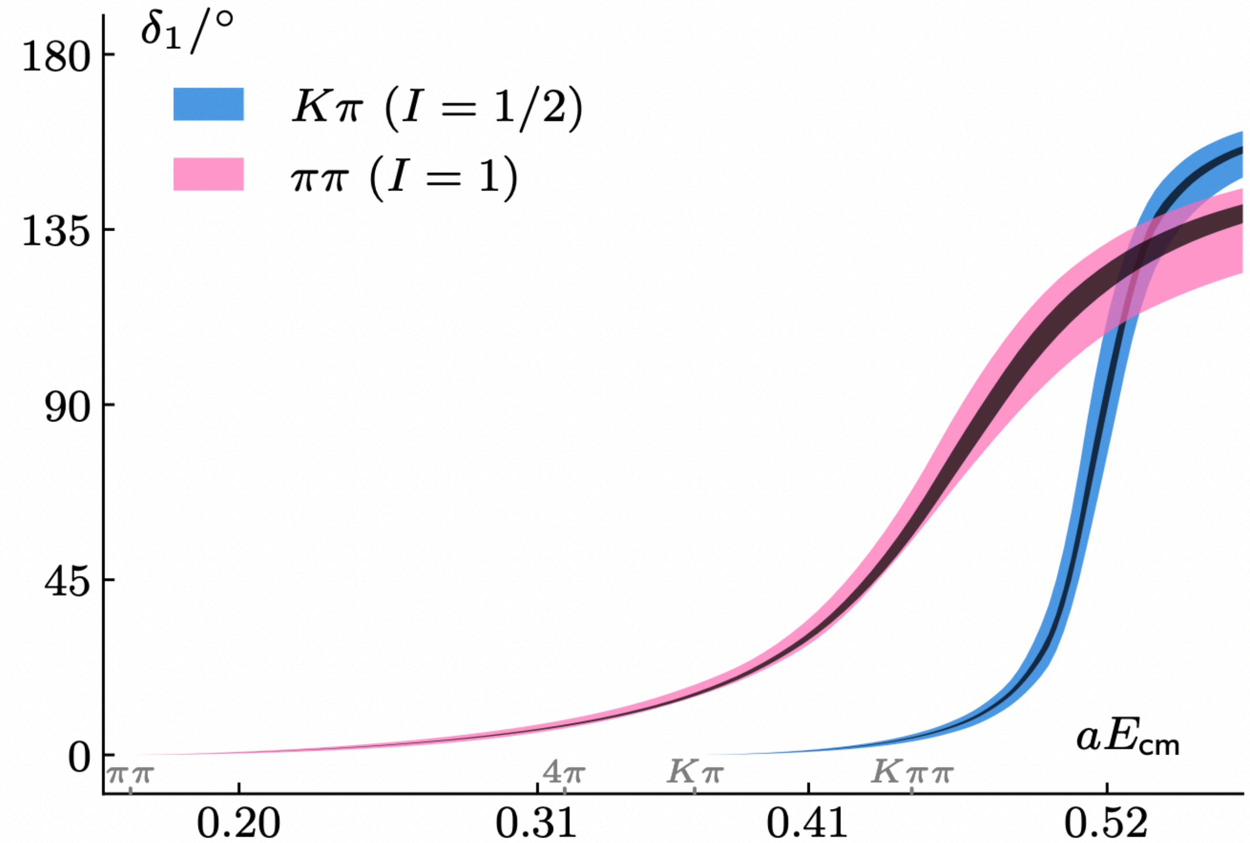


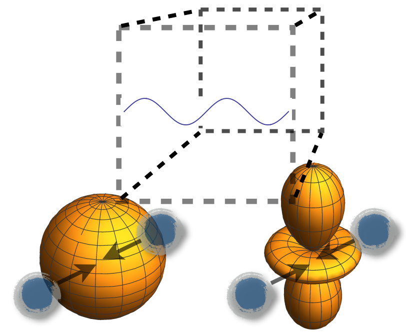


• arXiv: 2406.19194 • arXiv: 2406.19193 •

Coupled channel scattering

- The cubic volume mixes different partial waves…

e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $\det\left[\begin{pmatrix}\mathcal{K}_s^{-1} & 0 \\ 0 & \mathcal{K}_p^{-1}\end{pmatrix} + \begin{pmatrix}F_{ss} & F_{sp} \\ F_{ps} & F_{pp}\end{pmatrix}\right] = 0$



…as well as different flavor channels…

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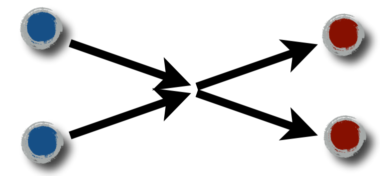

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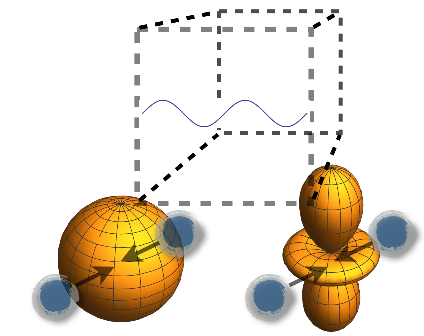

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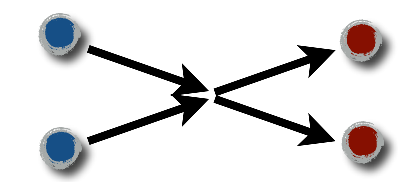


- Workflow…

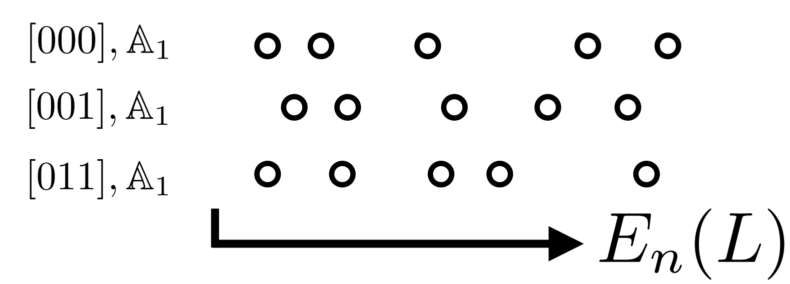
Correlators with a large operator basis
$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$

→

Reliably extract finite-volume energies
$\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$

→

Vary L and P to recover a dense set of energies



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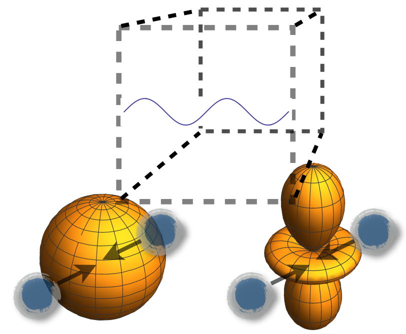

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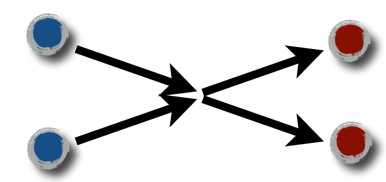
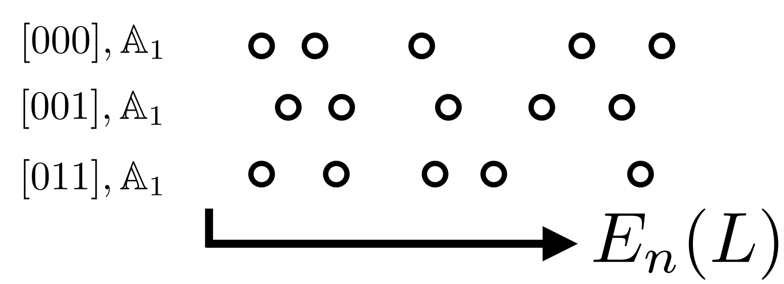


- Workflow…

Correlators with a large operator basis $\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$ → Reliably extract finite-volume energies $\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$ → Vary L and P to recover a dense set of energies

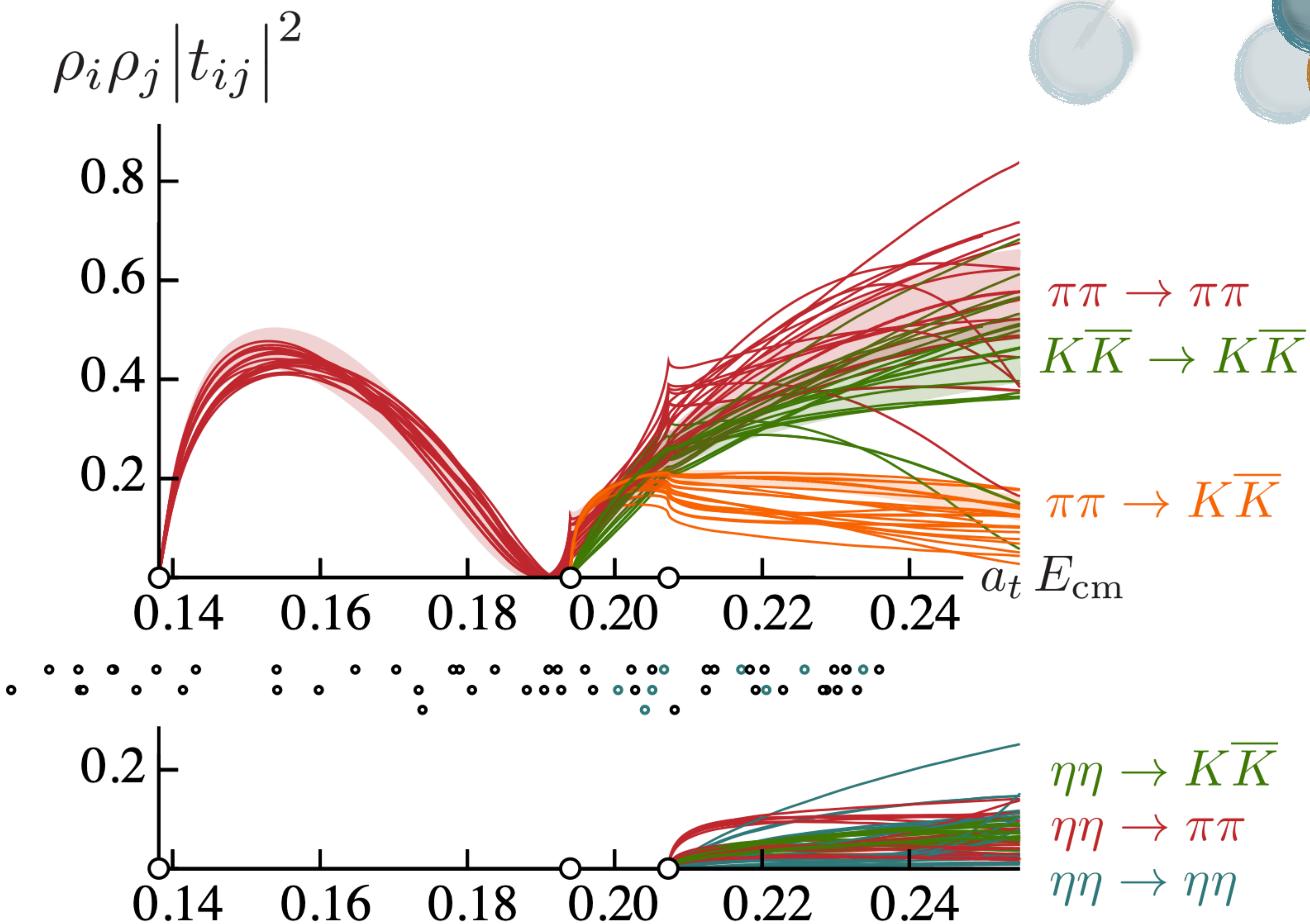


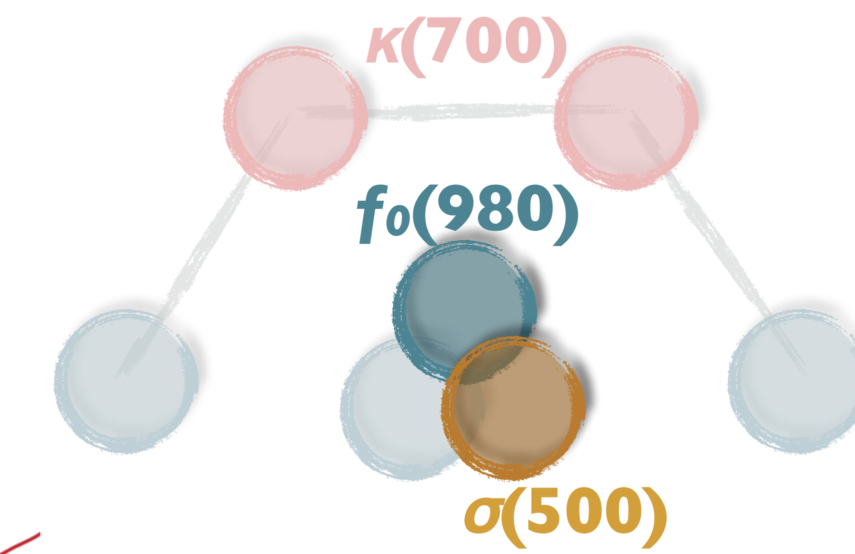
↓

Identify a broad list of K-matrix parametrizations: polynomials and poles; EFT based; dispersion theory based → **Perform global fits to the finite-volume spectrum**

$\sigma, f_0 \to \pi\pi, K\overline{K}, \eta\eta$

$$I^G(J^{PC}) = 0^+(0^{++})$$

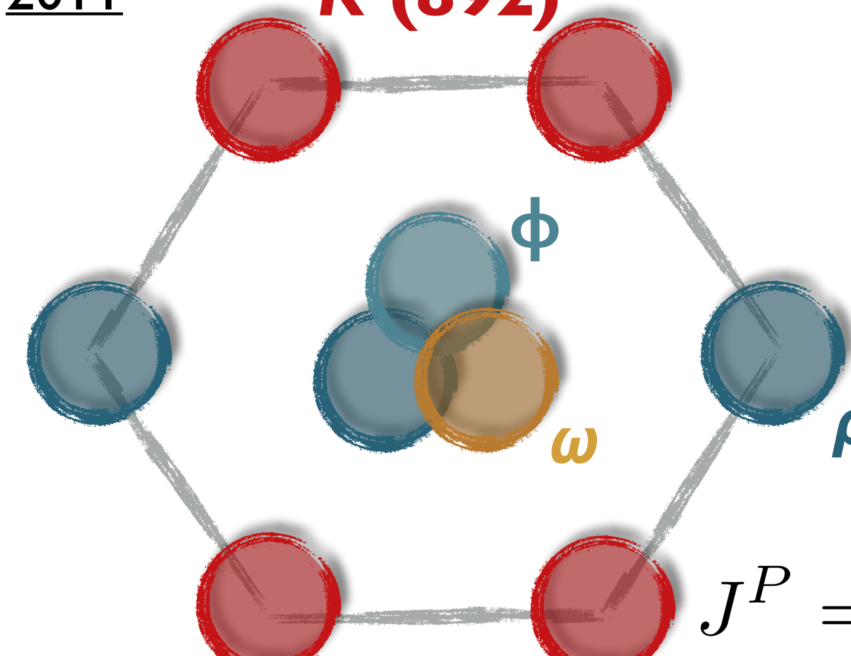




- Briceno et al., Phys.Rev. **D** 97 (2018) 5, 054513 •

$$\rho \to \pi\pi$$

- CP-PACS/PACS-CS 2007, 2011
- ETMC 2010
- Lang et al. 2011
- HadSpec 2012, **2016**
- Pellisier 2012
- RQCD 2015
- Guo et al. 2016
- Fu et al. 2016
- Bulava et al. 2016
- Alexandrou et al. 2017
- Andersen et al. 2018
- Fischer et al. 2020
- Erben et al. 2020



$J^P = 1^-$

$$\kappa \to K\pi$$
$$K^* \to K\pi$$

- Lang et al. 2012
- Prelovsek et al. 2013
- Wilson et al. 2015
- RQCD 2015
- Brett et al. 2018
- Wilson et al. 2019
- Rendon et al. 2020

$$b_1 \to \pi\omega, \pi\phi$$

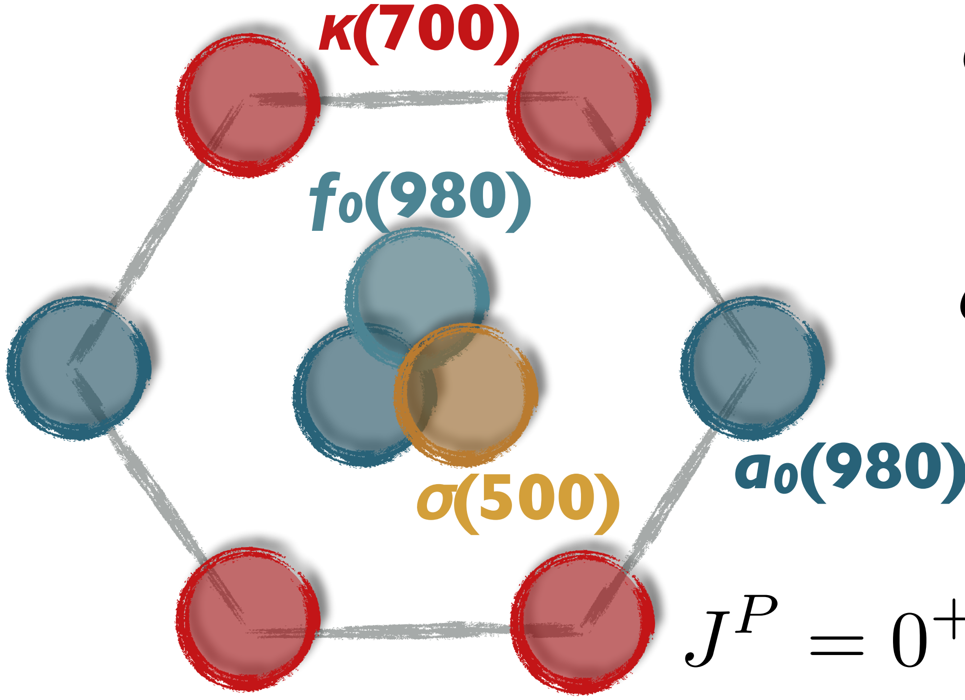
- Woss et al. 2019

$$a_0(980) \to \pi\eta, K\overline{K}$$

- Dudek et al. 2016

$$\sigma \to \pi\pi$$

- Prelovsek et al. 2010
- Fu 2013
- Wakayama 2015
- Howarth and Giedt 2017
- Briceño et al. 2017
- Guo et al. 2018



$J^P = 0^+$

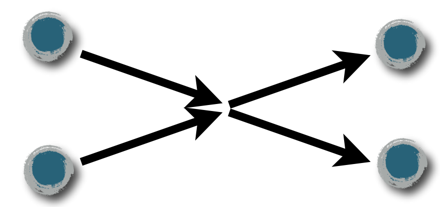
$$\sigma, f_0, f_2 \to \pi\pi, K\overline{K}, \eta\eta$$

- Briceño et al. 2017

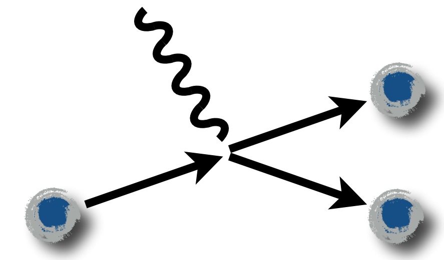
See the recent review by Briceño, Dudek and Young

Landscape of amplitudes

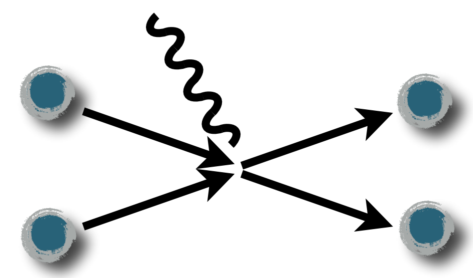
☑ Two-to-two scattering: $2 \to 2$



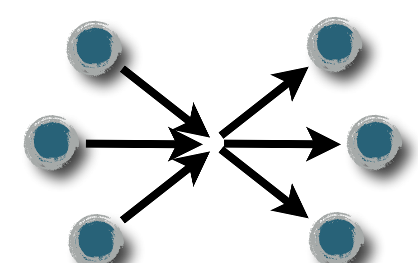
☐ Decays with an external current: $1 \xrightarrow[\mathcal{J}]{} 2$



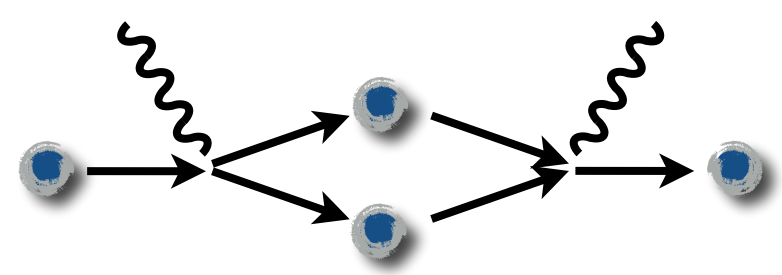
☐ Transitions with an external current: $2 \xrightarrow[\mathcal{J}]{} 2$



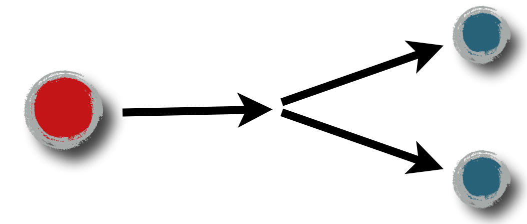
☐ Three-to-three scattering: $3 \to 3$



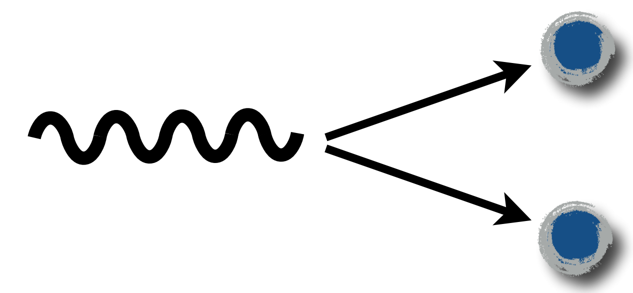
☐ Long distance matrix elements



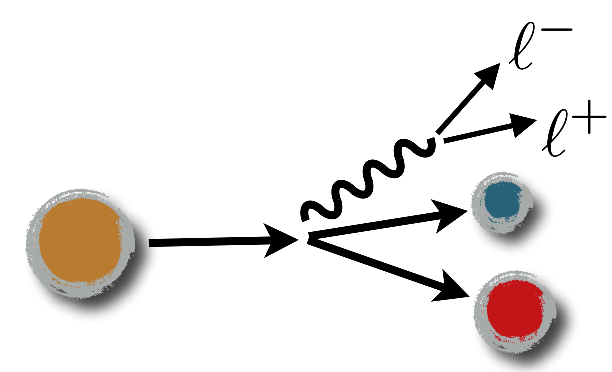
Formal progress: Transition amplitudes

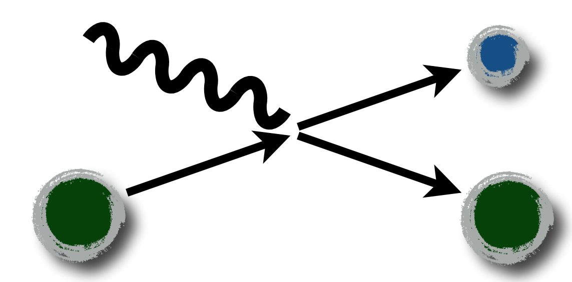
- ☐ Weak decay $\langle \pi\pi, \mathrm{out}|\mathcal{H}|K\rangle \equiv$ 

Lellouch, Lüscher *(2001)* • Kim, Sachrajda, Sharpe *(2005)* • Christ, Kim, Yamazaki *(2005)* • MTH, Sharpe *(2012)*

- ☐ Time-like form factors $\langle \pi\pi,\ \mathrm{out}|\mathcal{J}_\mu|0\rangle \equiv$ 

Meyer *(2011)*

- ☐ Resonance form factors $\langle K\pi, \mathrm{out}|\mathcal{J}_{\alpha\beta}|B\rangle \equiv$ 

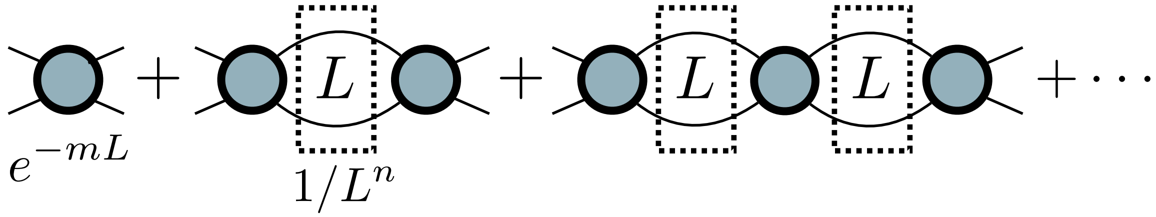
- ☐ Particles with spin $\langle N\pi, \mathrm{out}|\mathcal{J}_\mu|N\rangle \equiv$ 

Agadjanov *et al. (2014)* • Briceño, MTH, Walker-Loud *(2015)* • Briceño, MTH *(2016)*

Slightly modified version ($i\epsilon$)

- Consider the finite-volume correlator:

$\mathcal{M}(s)$	$\mathcal{M}_L(P)$
probability amplitude	poles give f.v. spectrum

$\mathcal{M}_L(P) =$ 

For two-particle energies $(2m)^2 < s < (4m)^2$, what is the L dependence?

—— propagating pion

● Bethe-Salpeter kernel

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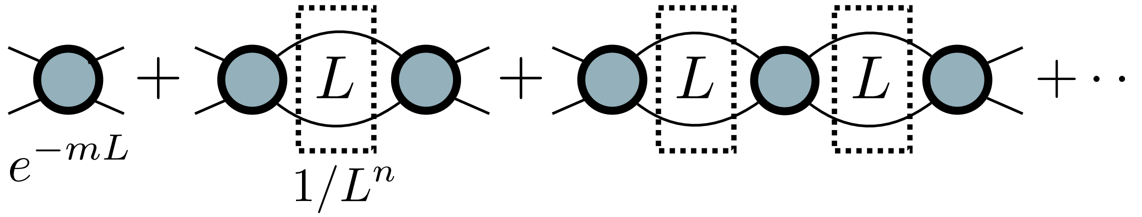
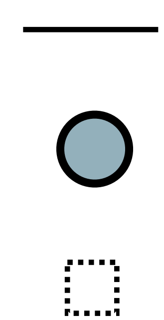
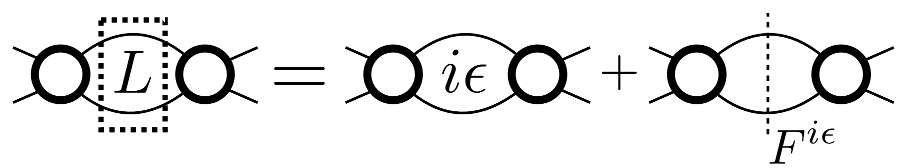
$\mathcal{M}_L(P) =$ 



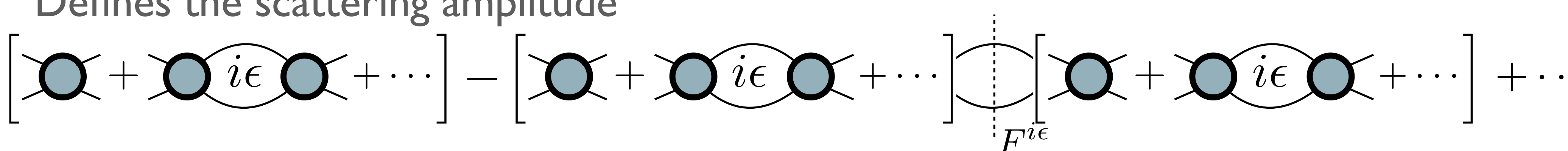
For two-particle energies $(2m)^2 < s < (4m)^2$, what is the L dependence?



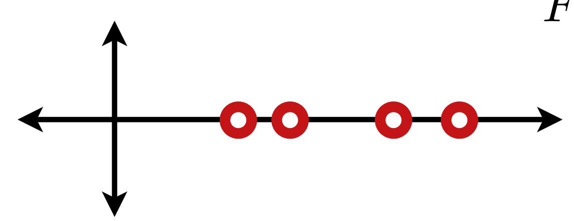
Cut projects loop to ***on-shell energies***

$F^{i\epsilon}$= matrix of known geometric functions

Defines the scattering amplitude

$=$ 

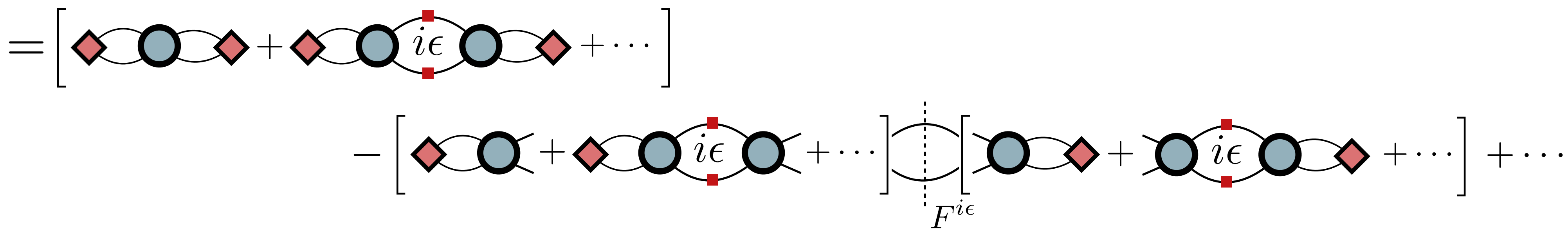
$$= \frac{1}{\mathcal{M}(s)^{-1} + F^{i\epsilon}(P,L)}$$



$$\mathcal{M}(s)^{-1} + F^{i\epsilon}(P,L) = \mathcal{K}^{-1}(s) + F(P,L)$$

$1 + \mathscr{J} \to 2$

$$C_L(P) = \langle \pi | \mathcal{J} \mathcal{J} | \pi \rangle_L$$



MTH, Sharpe (2012) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016) •

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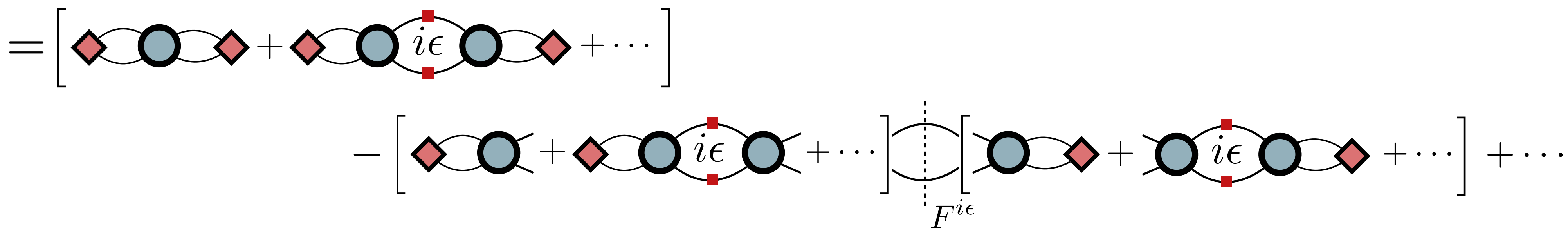

$$C_L(P) = C_\infty^{i\epsilon}(s) - A_{\text{in}}^{i\epsilon}(s) \frac{1}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} A_{\text{out}}^{i\epsilon}(s)$$

The "endcaps" define the matrix element... $A_{\text{out}}^{i\epsilon}(s) = \langle \pi\pi, \text{out} | \mathcal{J} | \pi \rangle$

MTH, Sharpe (2012) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016) •

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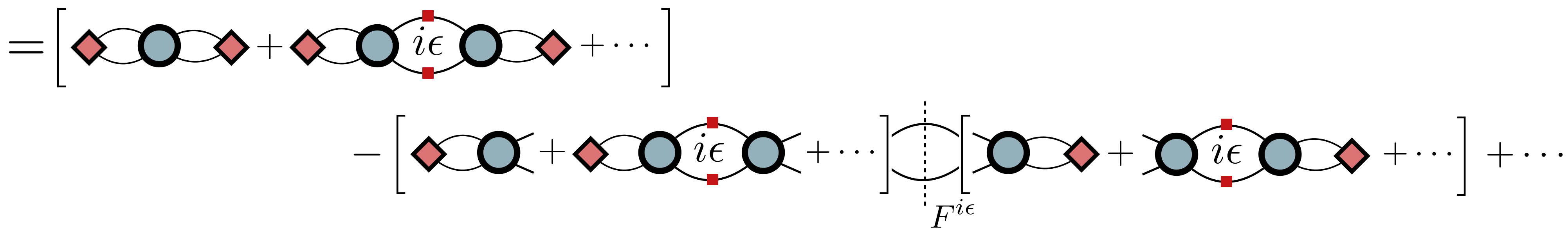

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The "endcaps" define the matrix element... $A_{\text{out}}^{i\epsilon}(s) = \langle \pi\pi, \text{out} | \mathcal{J} | \pi \rangle$

$$\text{Residue}_{E_n}[C_L(P)] = -\text{Residue}_{E_n}\left[A_{\text{in}}^{i\epsilon}(s) \frac{1}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} A_{\text{out}}^{i\epsilon}(s) \right]$$

$$\left| \langle n, L | \mathcal{J} | \pi, L \rangle \right|^2 = \langle \pi | \mathcal{J} | \beta, \text{in} \rangle \, \mathcal{R}_{\alpha\beta}(P,L) \, \langle \beta, \text{out} | \mathcal{J} | \pi \rangle$$

MTH, Sharpe (2012) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016) •

Transition amplitudes

$$\left|\langle n, L|\mathcal{J}|\pi, L\rangle\right|^2 = \langle \pi|\mathcal{J}|\beta, \mathrm{in}\rangle\, \mathcal{R}_{\alpha\beta}(P,L)\, \langle \beta, \mathrm{out}|\mathcal{J}|\pi\rangle$$

$$\mathcal{R}(P,L) = -\lim_{E\to E_n(L)} \frac{E - E_n(L)}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} = \frac{1}{\mu'(E)}\mathbf{v}^T\mathbf{v}$$

Briceño, Dudek, Leskovec

Transition amplitudes

$$\left|\langle n, L|\mathcal{J}|\pi, L\rangle\right|^2 = \langle \pi|\mathcal{J}|\beta, \text{in}\rangle\, \mathcal{R}_{\alpha\beta}(P,L)\, \langle \beta, \text{out}|\mathcal{J}|\pi\rangle$$

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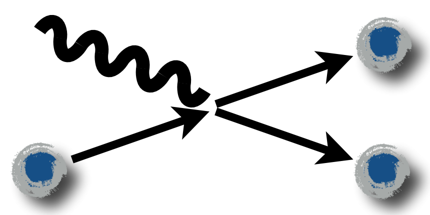
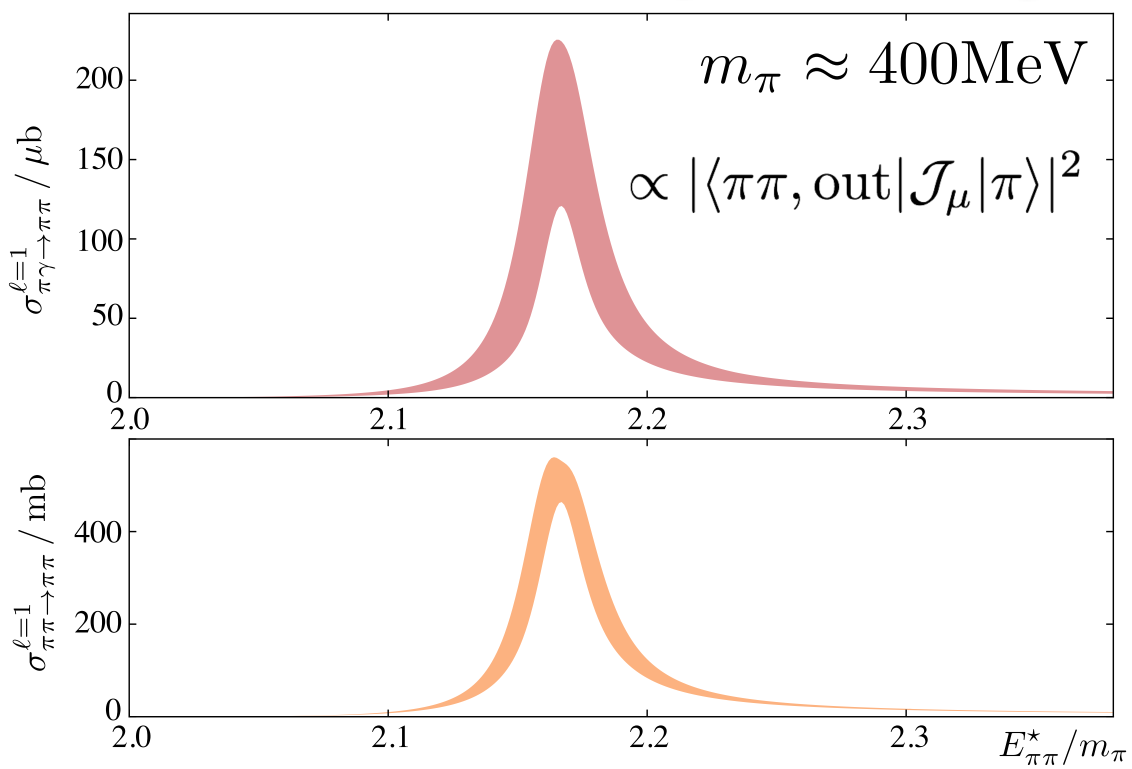
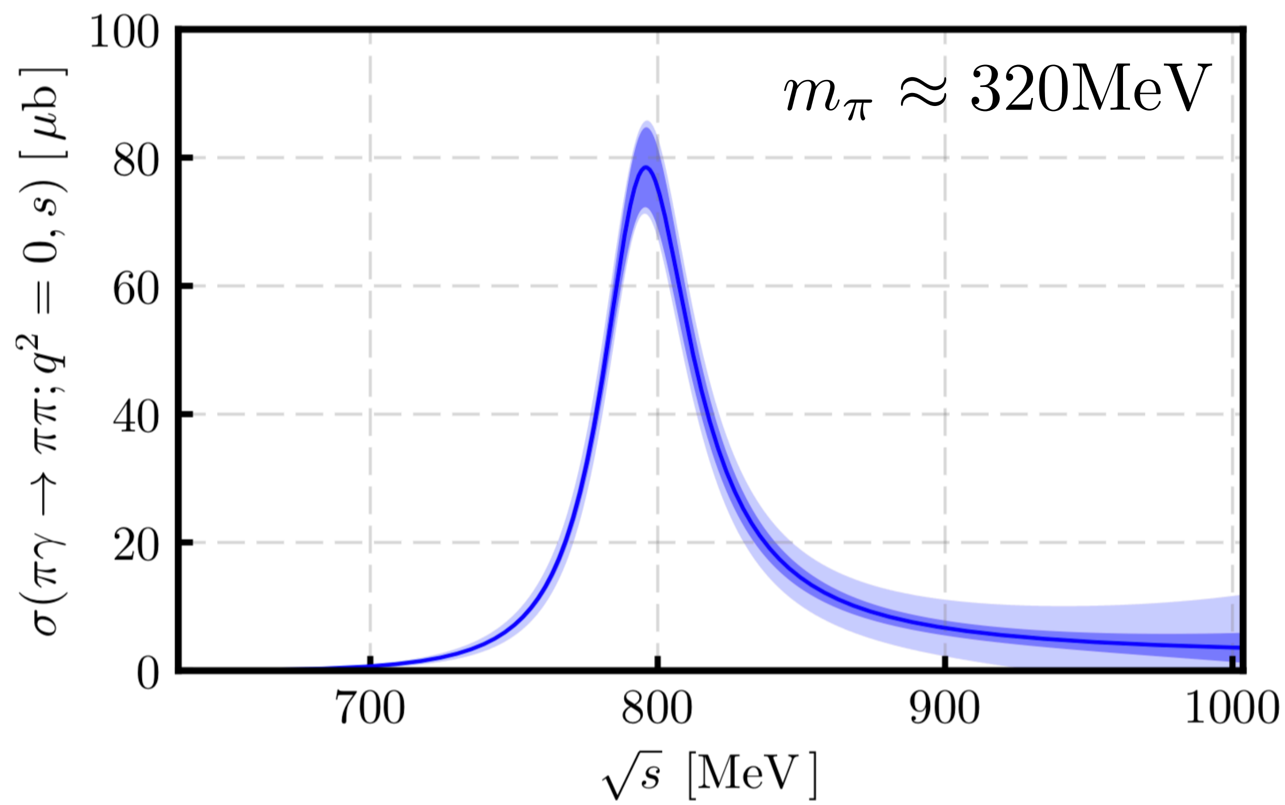
Briceño, Dudek, Leskovec

$\langle \pi\pi, \text{out}|\mathcal{J}_\mu|\pi\rangle \equiv$





Briceño et. al., Phys. Rev. D93, 114508 (2016)

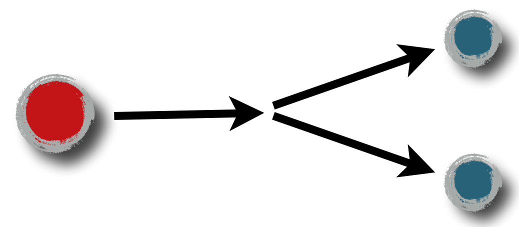


Alexandrou et. al., Phys. Rev. D98, 074502 (2018)

Unphysical kinematics

- The amplitude is perfectly well defined for unphysical kinematics

$$\langle \pi\pi, \mathrm{out}|\mathcal{H}|K\rangle \equiv$$



$$= \frac{1}{1 - \mathcal{K}(s) i\rho(s)}\mathcal{H}(s)$$

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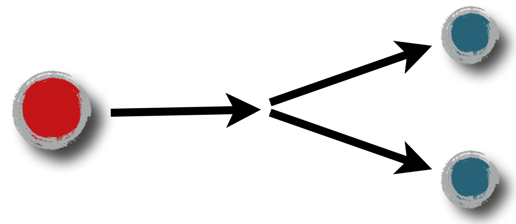
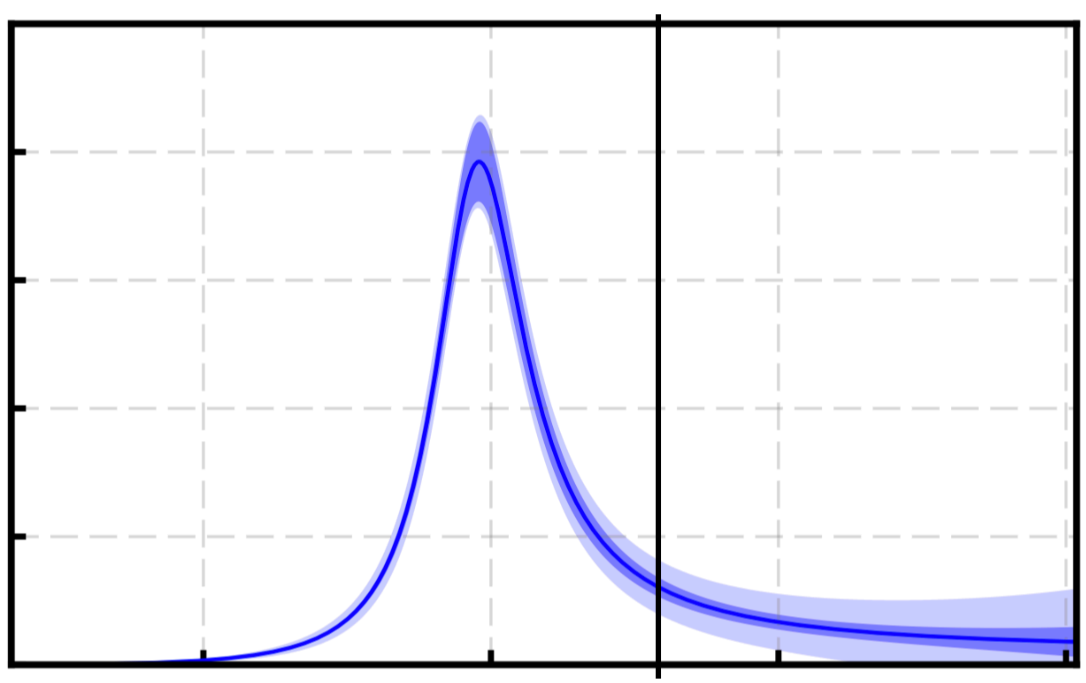


$$= \frac{1}{1 - \mathcal{K}(s) i\rho(s)} \mathcal{H}(s)$$

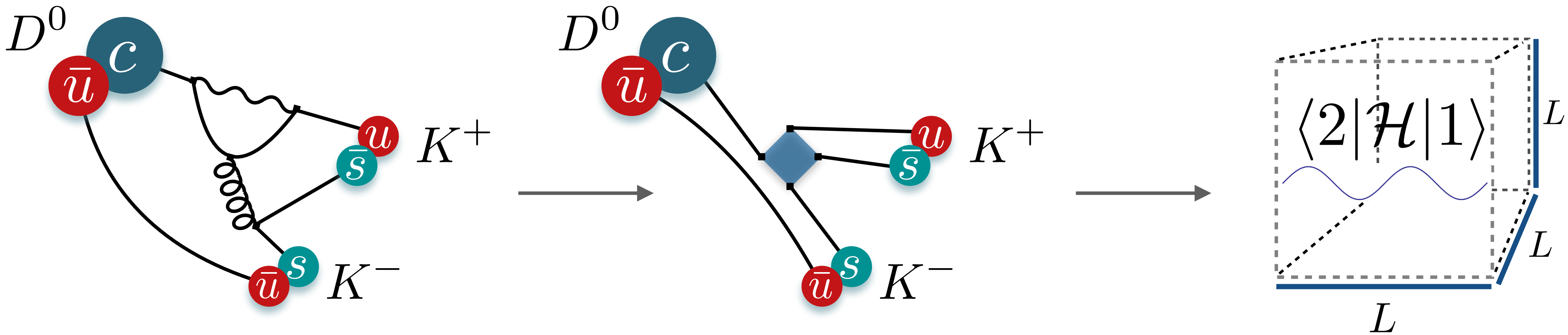
- Energy dependence could be interesting (relevant for dispersive treatment?)



dreaming of the future

Hadronic D decays

- Integrating out electroweak physics → basis of four-quark operators



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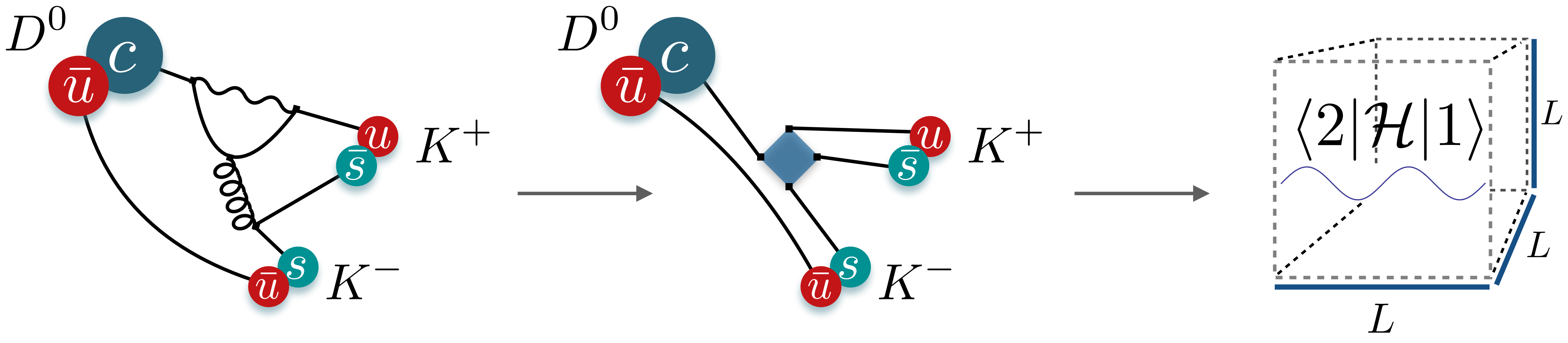


- ***Complicated:*** non-perturbative ***renormalization***, many ***operators*** and ***contractions***

See the RBC/UKQCD calculation of K→ππ

multi-hadron final state

$$\langle n, L|\ \mathcal{H}^{\overline{\text{MS}}}_{\text{weak}}\ |D, L\rangle$$

incoming D meson

$e^{-M_\pi L}$ *volume effects*

renormalized weak Hamiltonian

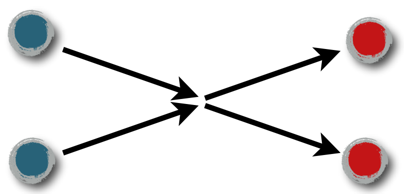
$\pi\pi, K\overline{K}, \pi\pi\pi\pi, \cdots$ ***have same quantum numbers + no asymptotic separation in the box***

How do we interpret $\langle n, L|$?

The finite-volume as a tool

- Coupled channels leave an *imprint* on finite-volume energies

infinite-volume
scattering amplitudes



$\pi\pi \rightarrow K\overline{K}$

(i) given these experimentally

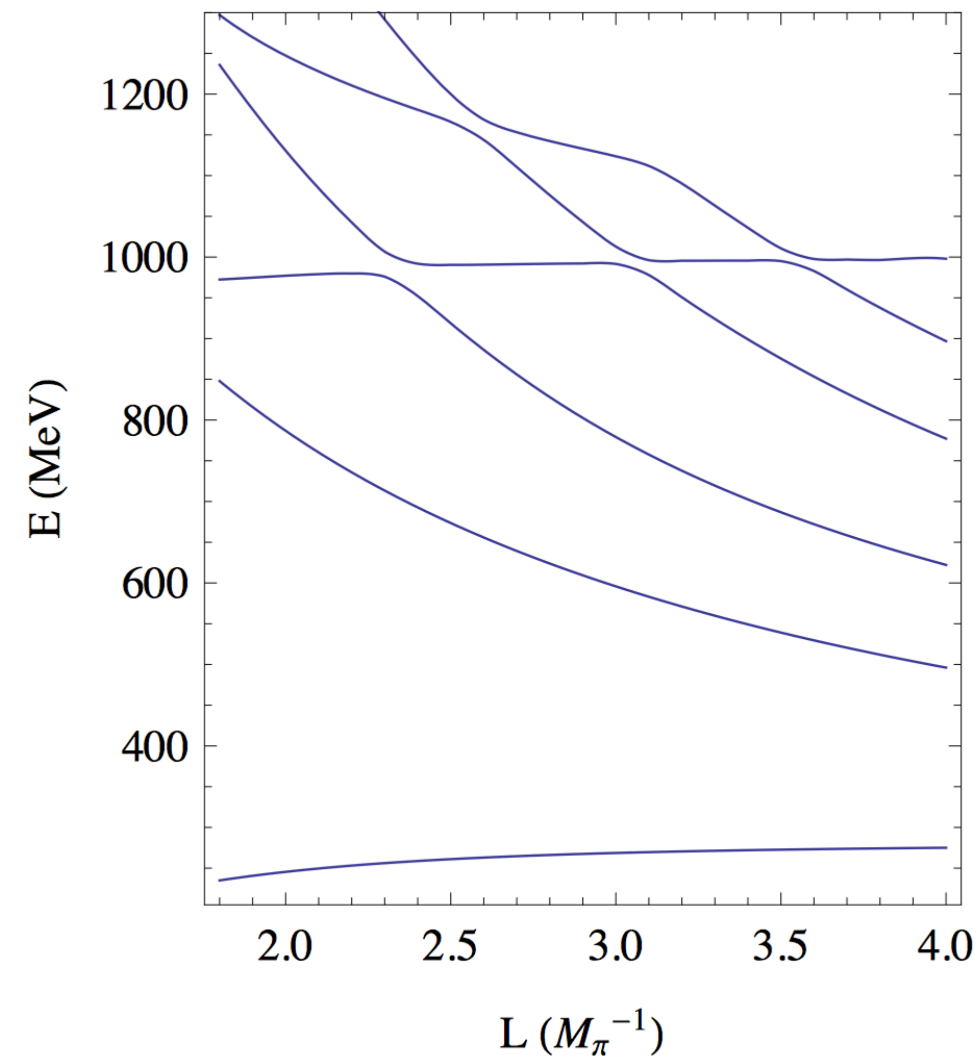
(ii) predict from first principles

know field theoretic relation,
exact up to $e^{-M_\pi L}$

can predict finite-volume signature

given lattice energies

finite-volume energies



• MTH, Sharpe, *Phys.Rev.* **D**86 (2012) 016007 •

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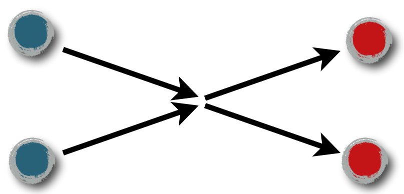

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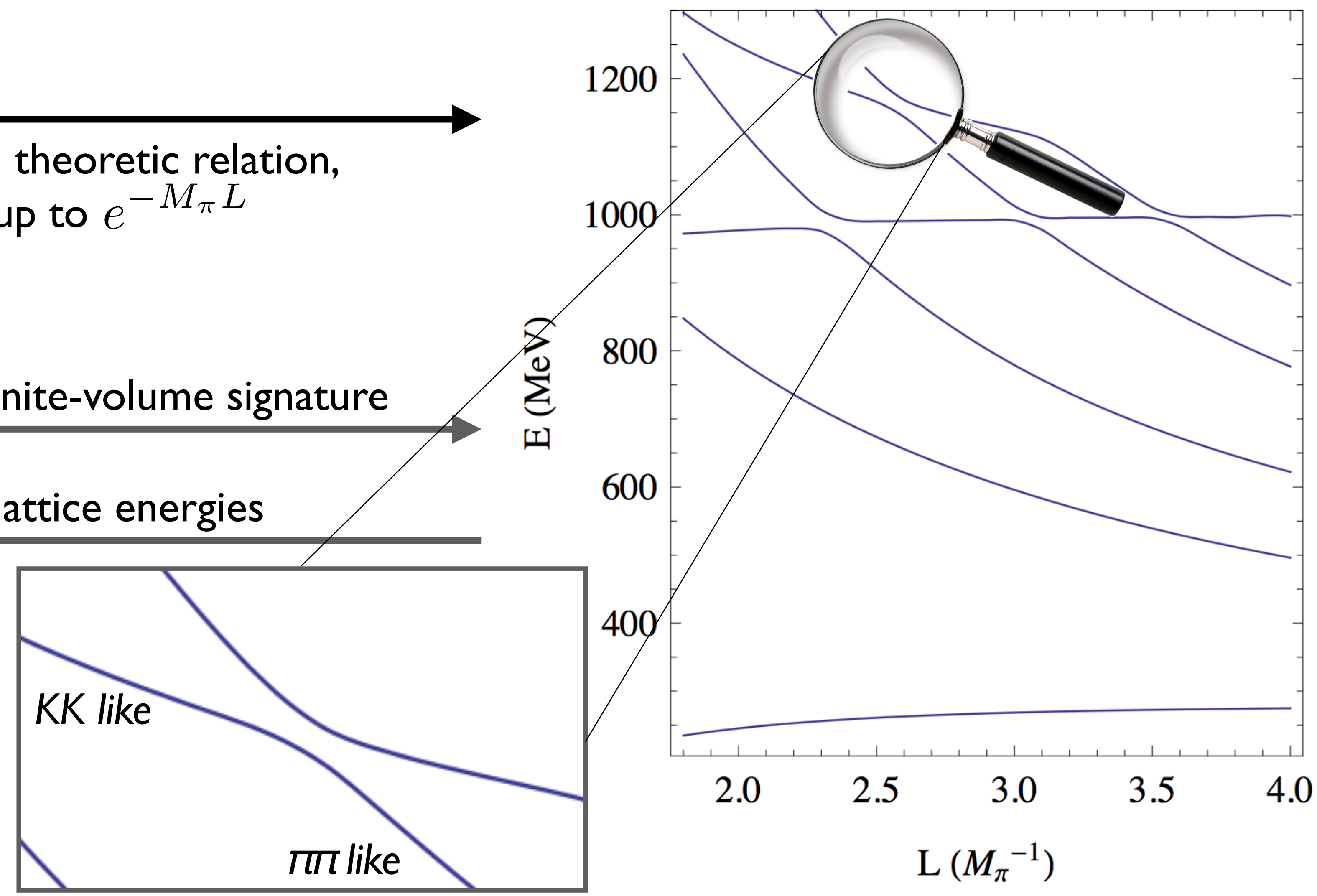

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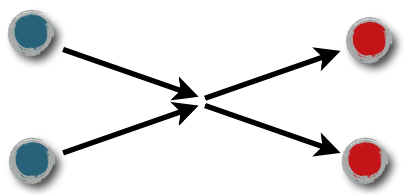

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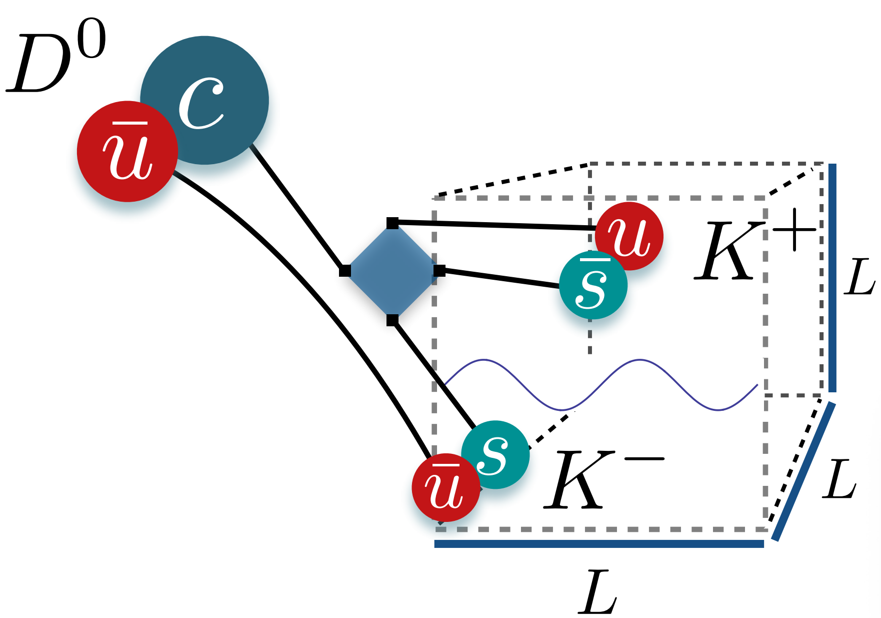
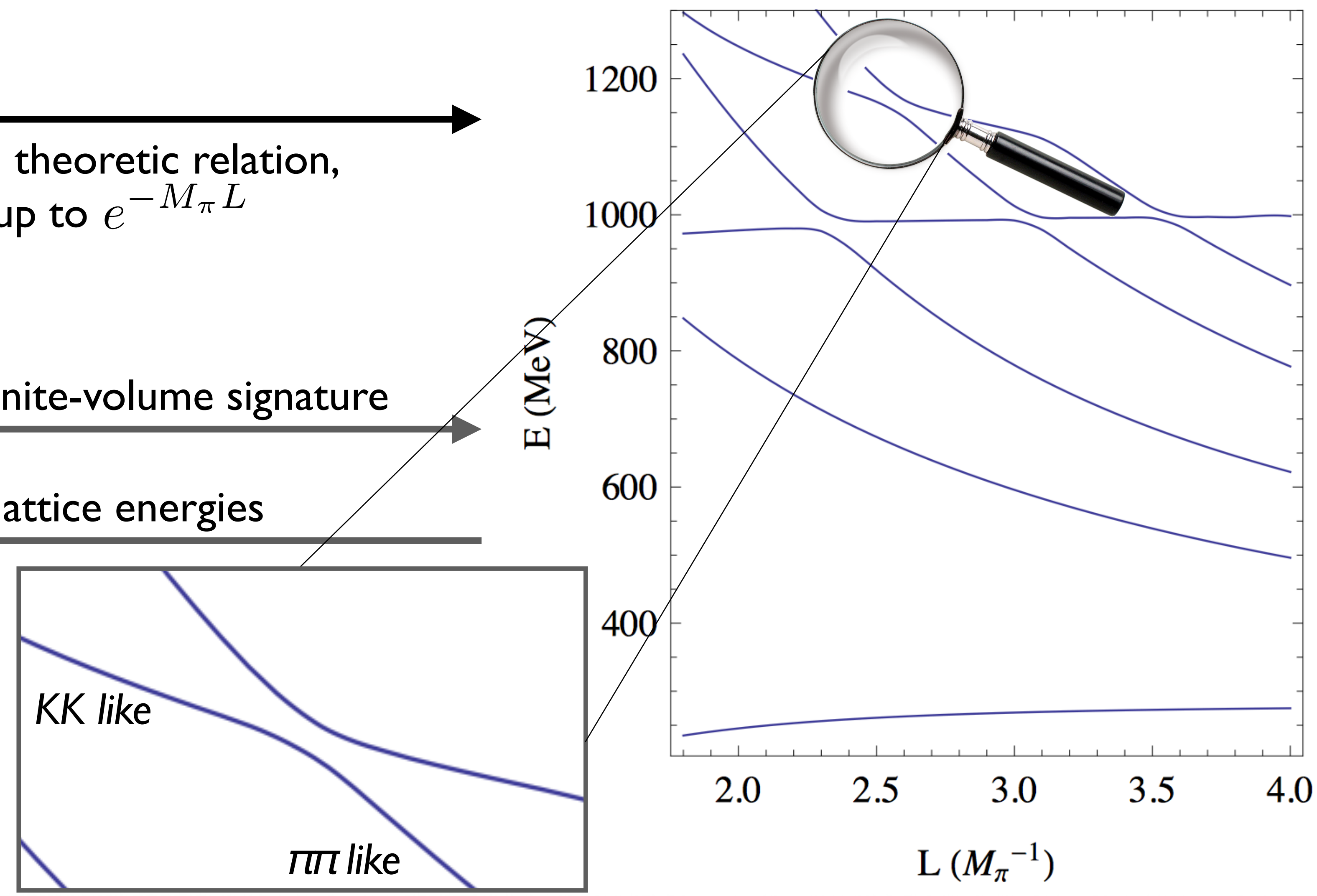


finite-volume energies



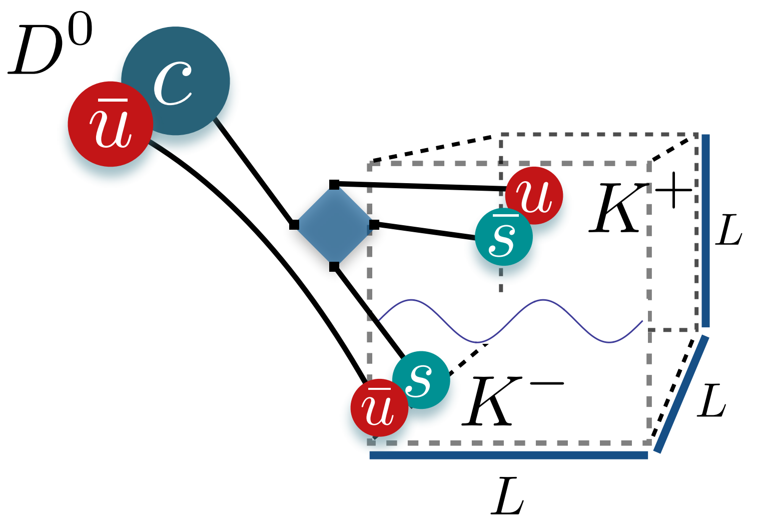
$$|n, L\rangle = c_\pi^{(n)}(L)\,|\pi\pi, \text{out}\rangle + c_K^{(n)}(L)\,|K\overline{K}, \text{out}\rangle$$

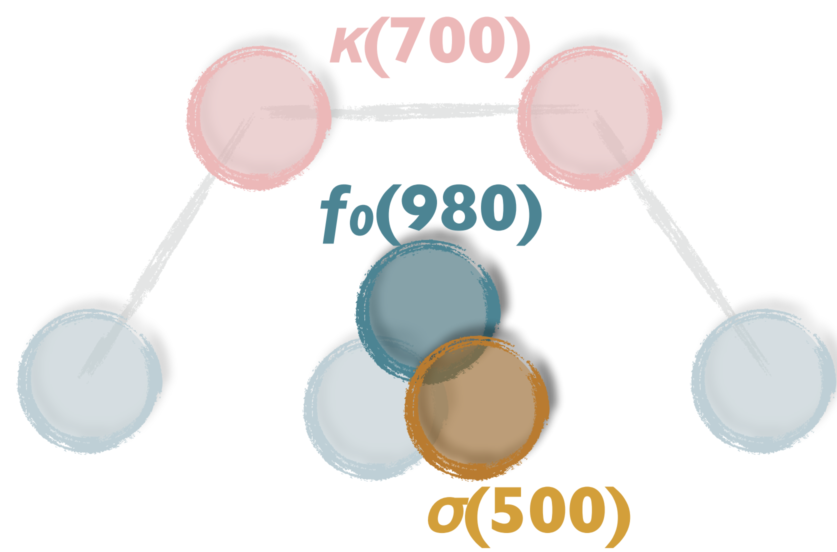
path to physical amplitude

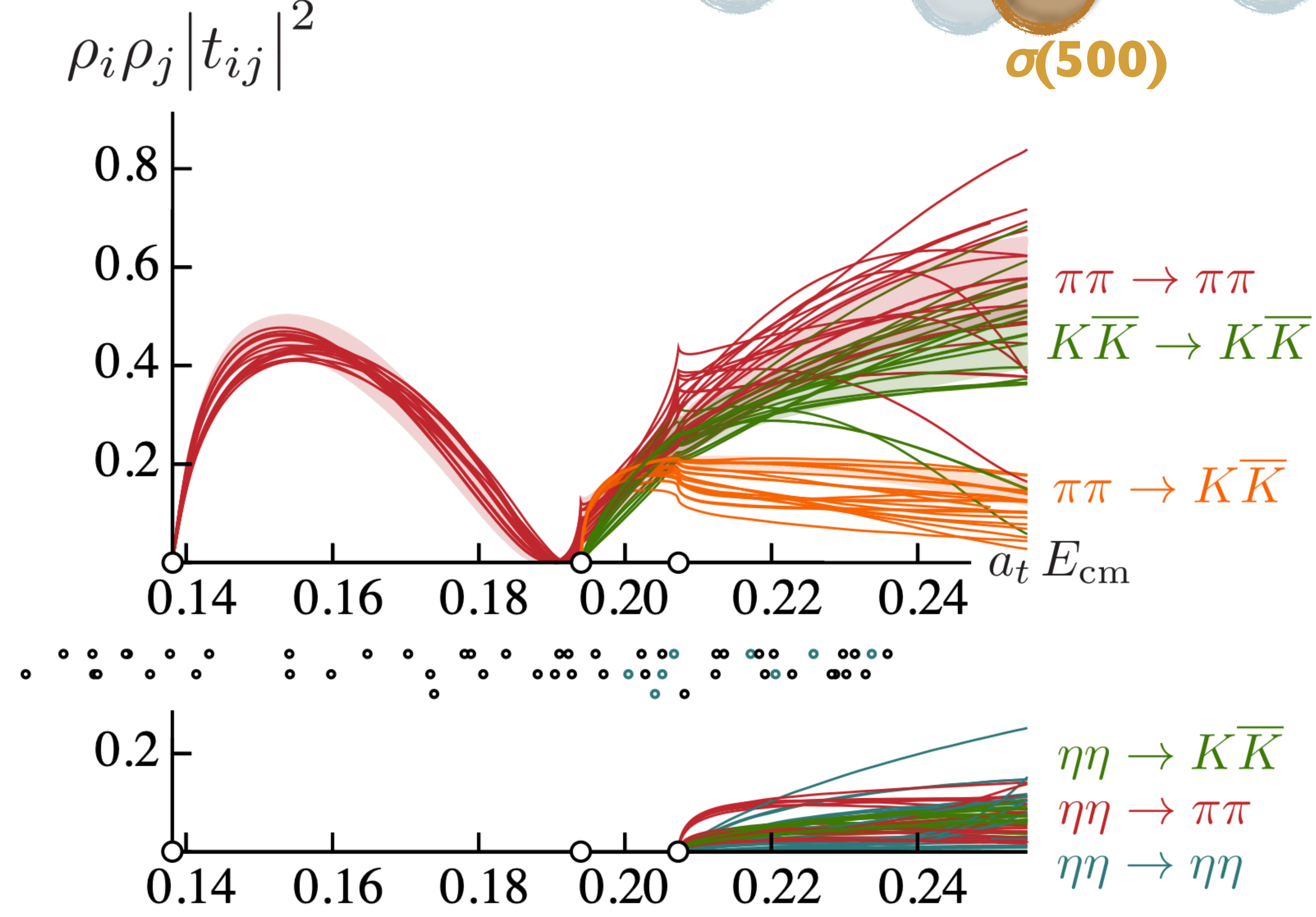
• MTH, Sharpe, *Phys.Rev.* **D**86 (2012) 016007 •

Dreaming of the future…

$$I^G(J^{PC}) = 0^+(0^{++})$$



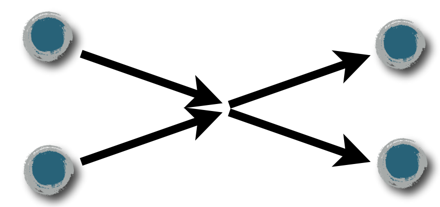




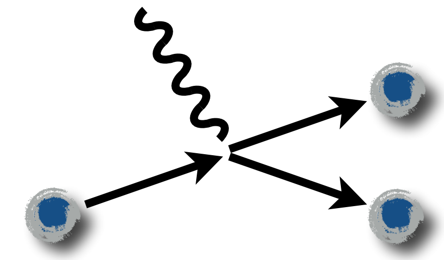
- ❑ Biggest problem is >2 hadron states!

Landscape of amplitudes

- [x] Two-to-two scattering: $2 \to 2$



- [x] Decays with an external current: $1 \underset{\mathcal{J}}{\to} 2$



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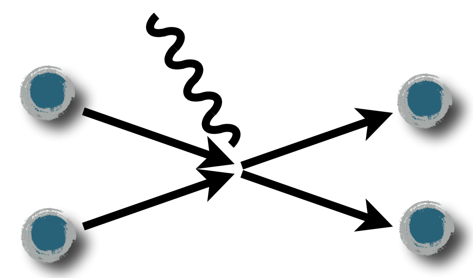
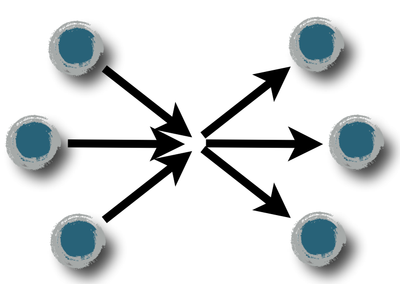
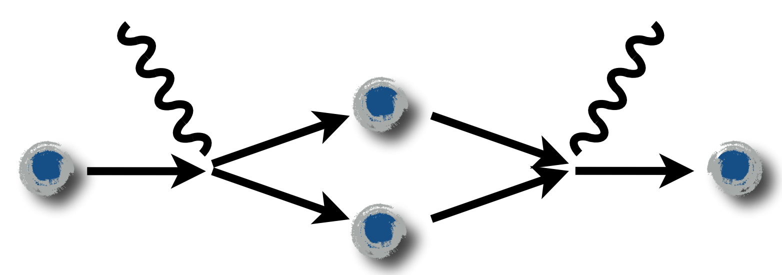


- [ ] Three-to-three scattering: $3 \to 3$



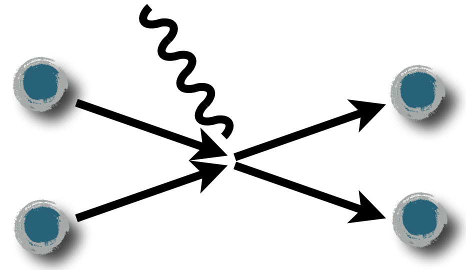
- [ ] Long distance matrix elements



$2 + \mathscr{J} \to 2$

- Formalism for multi-hadron form factors

$$\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi\pi, \mathrm{in} \rangle \equiv$$



- Continuation to the pole → ***resonance form factors***

Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

$2 + \mathscr{J} \to 2$

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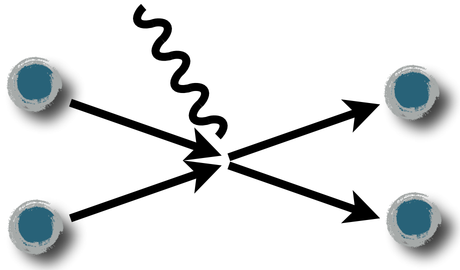
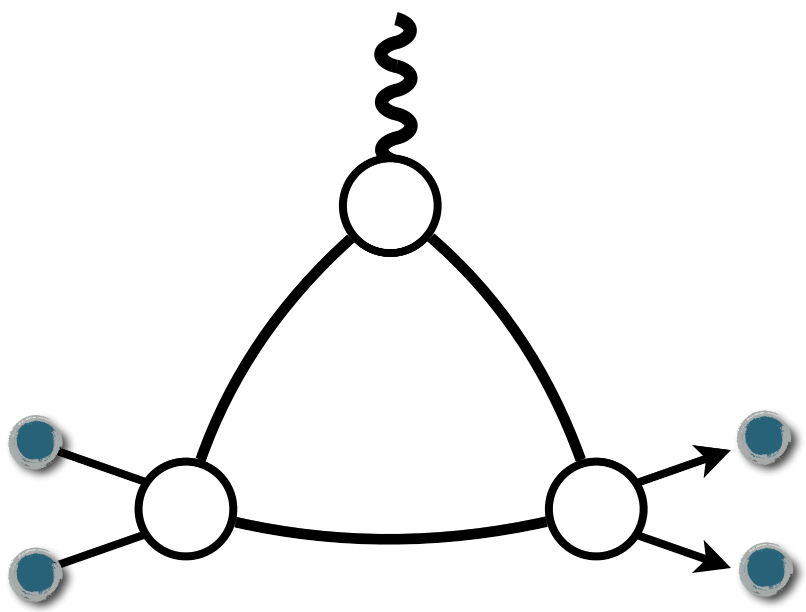
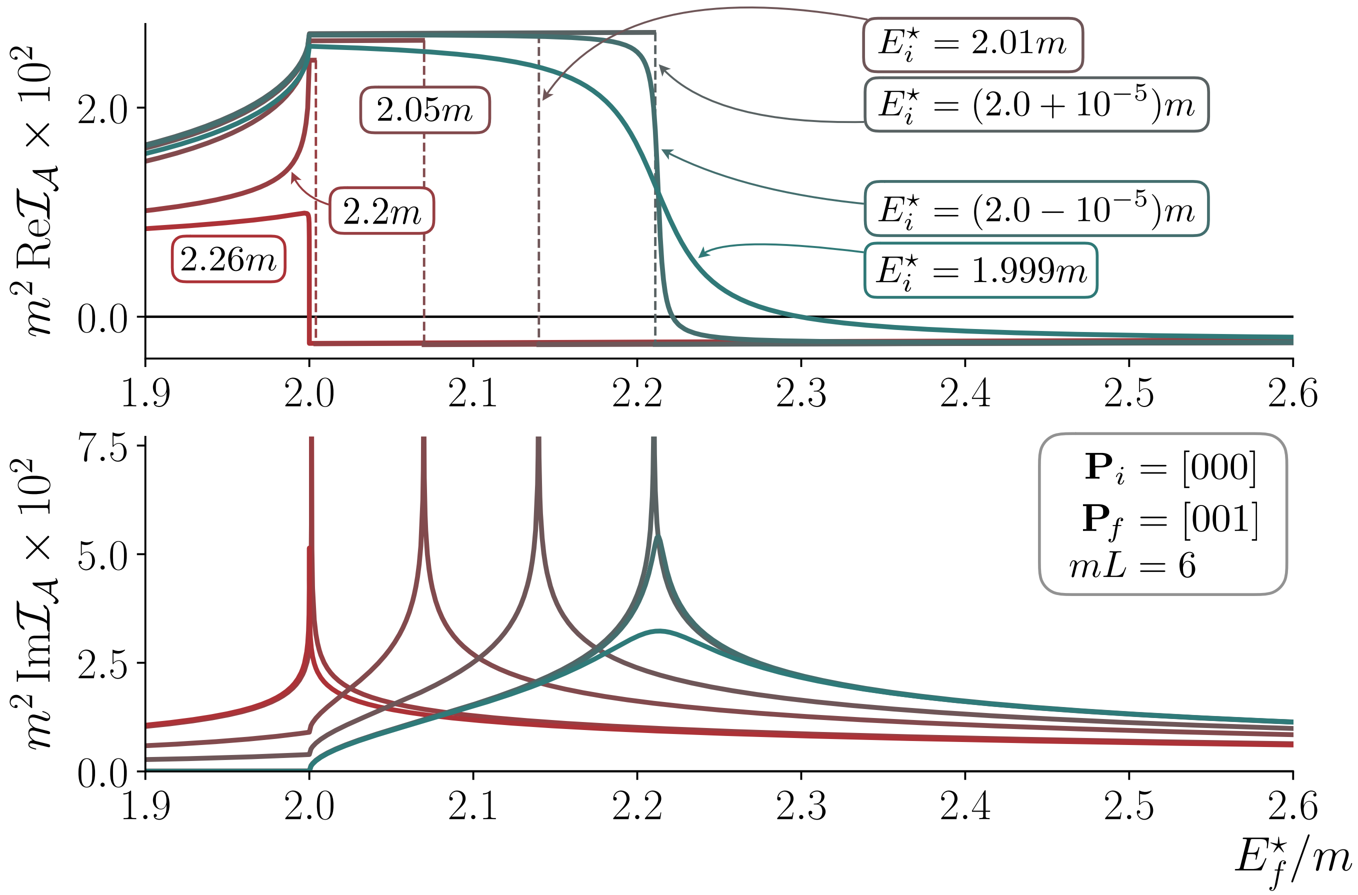


- Continuation to the pole → ***resonance form factors***
- Must carefully treat ***triangle*** singularities



$\sim$

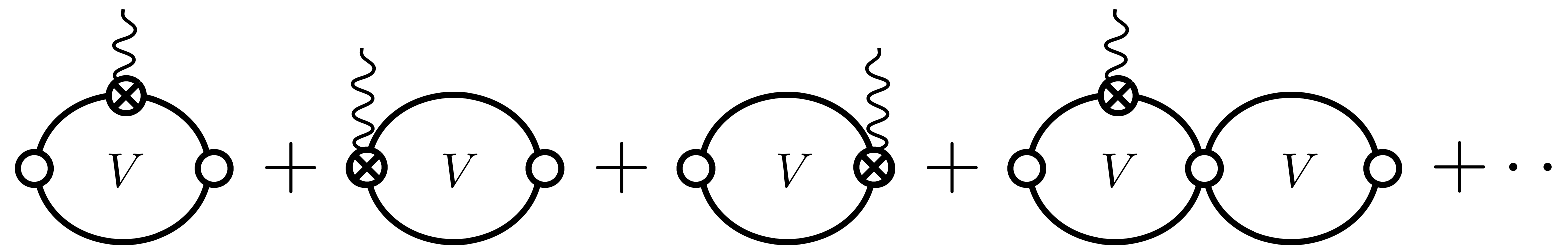


Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

In a nutshell

- By analysing an all orders skeleton expansion…

$$C_L^{2\to 2}(P_f, P_i) =$$



Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

In a nutshell

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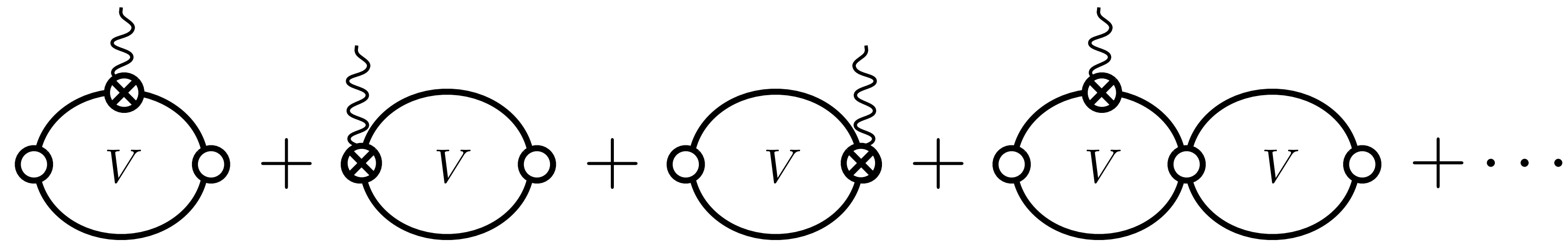
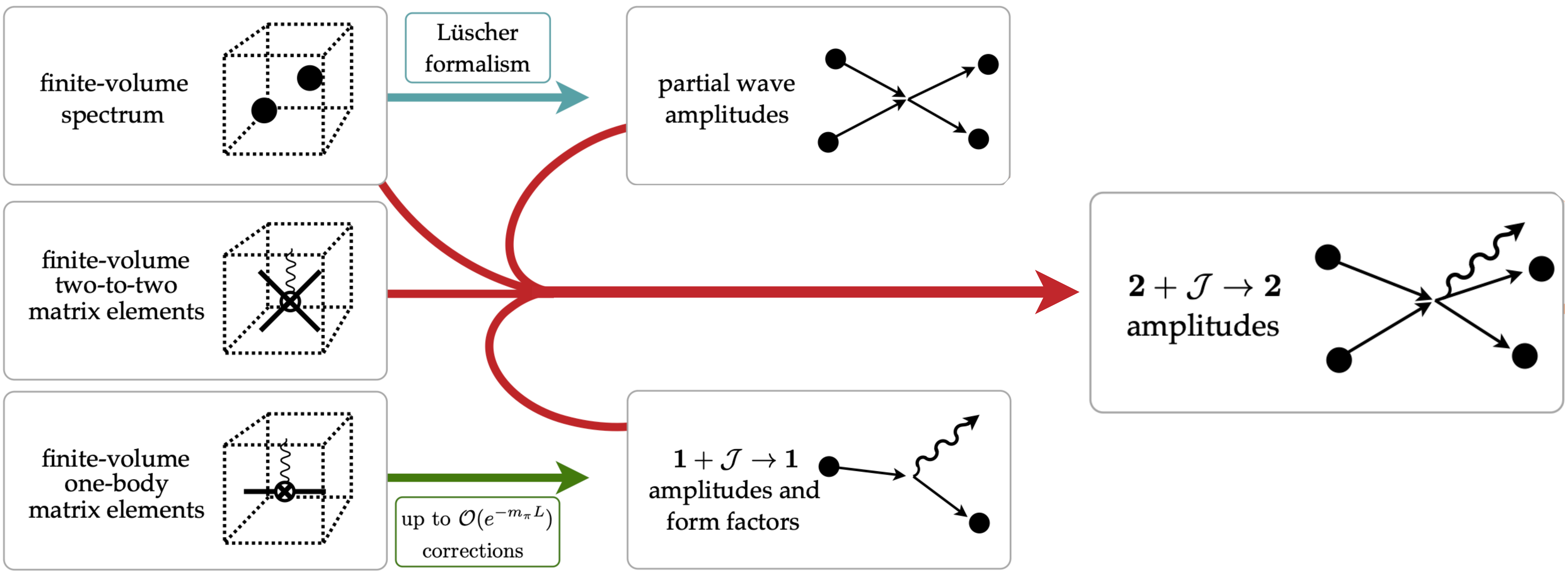
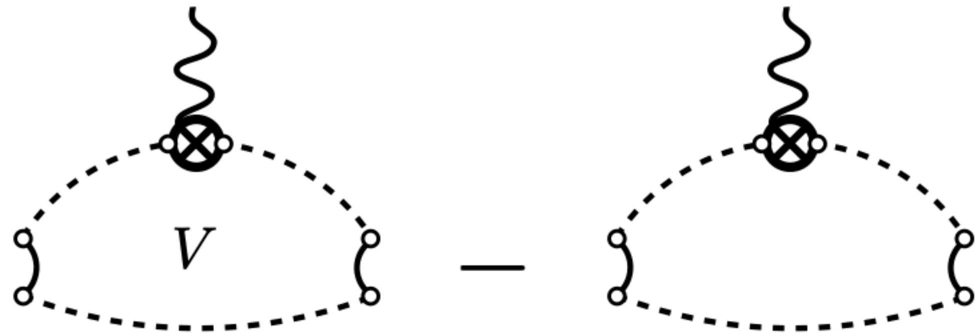


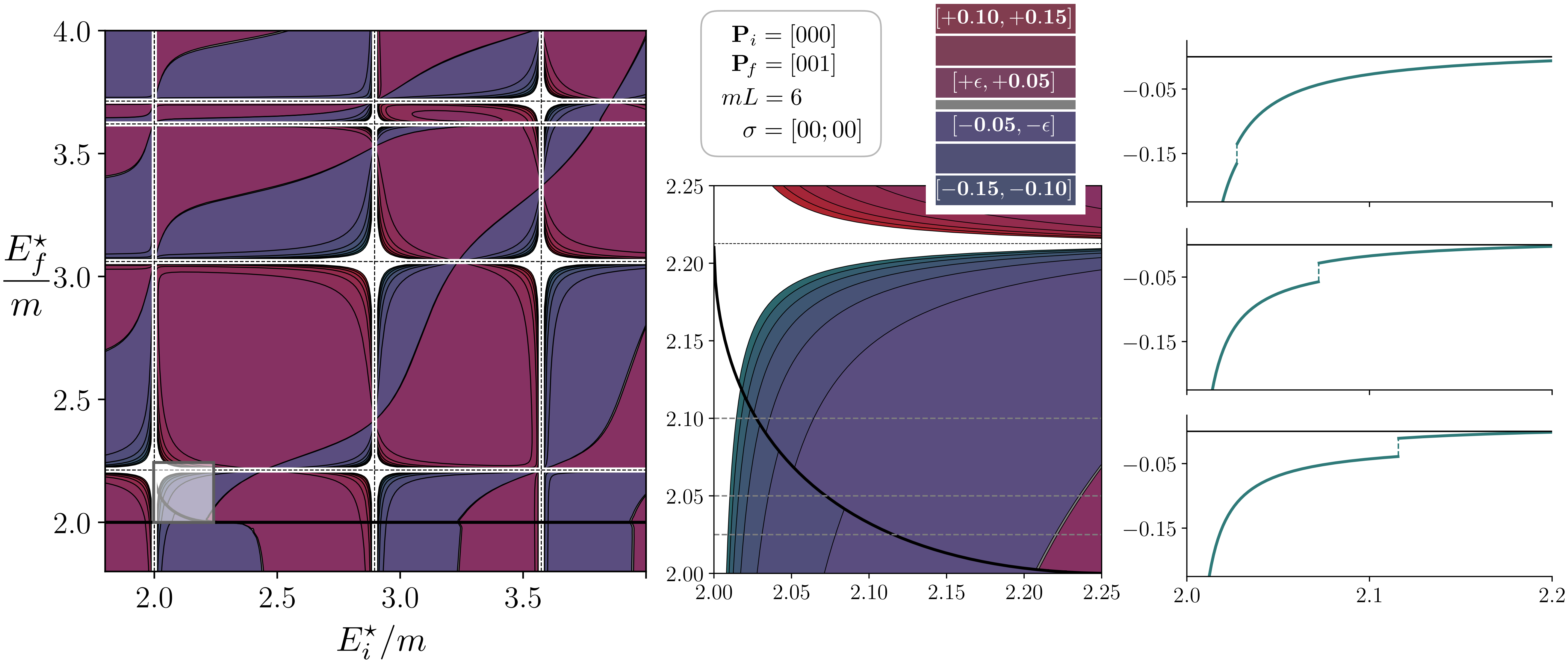
- … we derived a framework to calculate the $\mathbf{2} + \mathcal{J} \to \mathbf{2}$ amplitude



Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

New finite-volume function

$$G_{\mu_1 \cdots \mu_n}(P_f, P_i, L) =$$ 



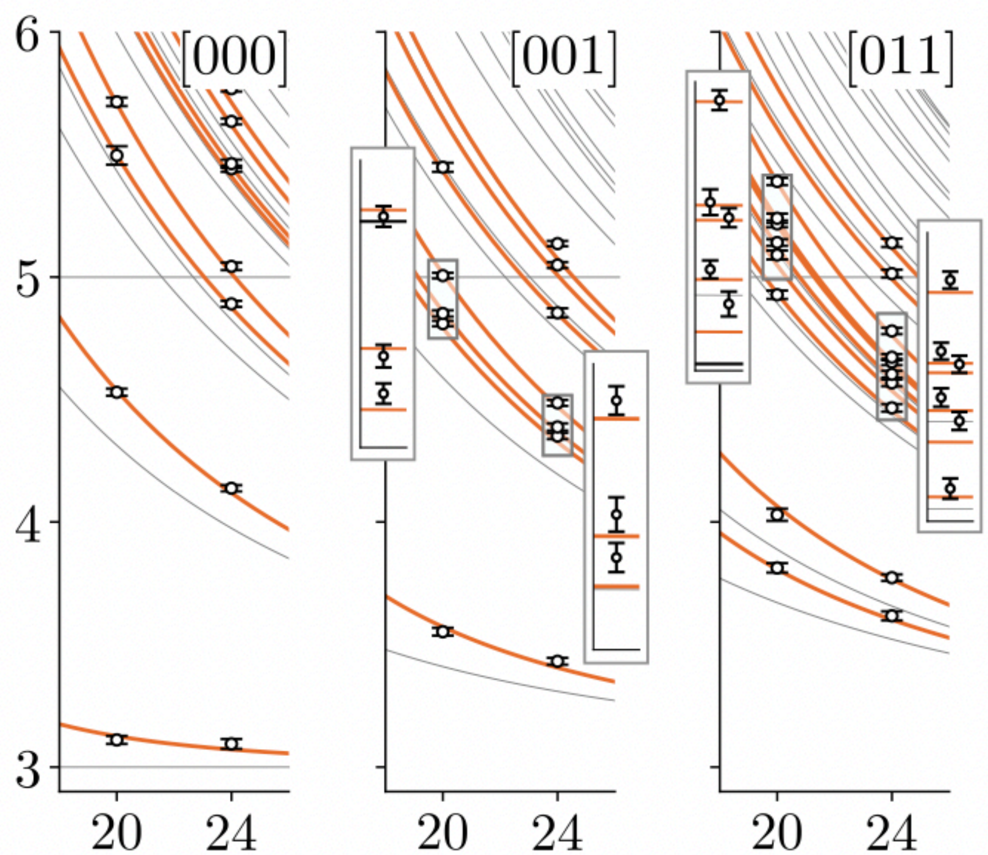
Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

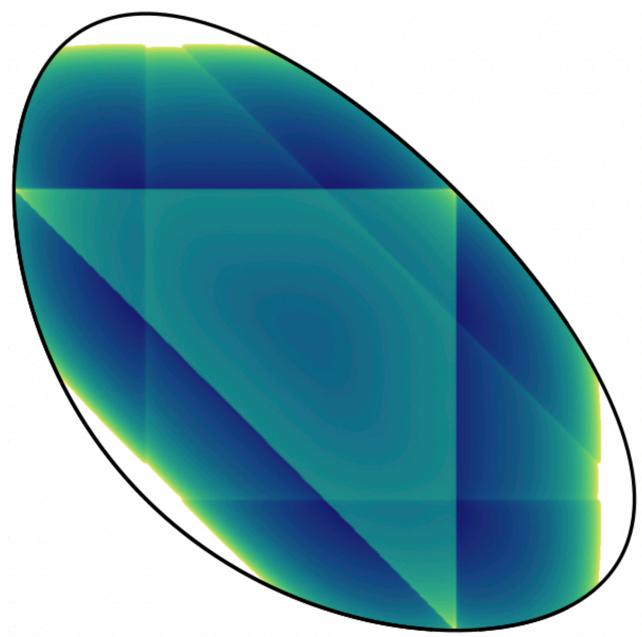
Towards >2 hadrons

- Multiple three-particle finite-volume formalisms developed

MTH, Sharpe (2014-2016) *See also Döring, Mai, Hammer, Pang, Rusetsky*

- First lattice calculations appearing… e.g. $\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$





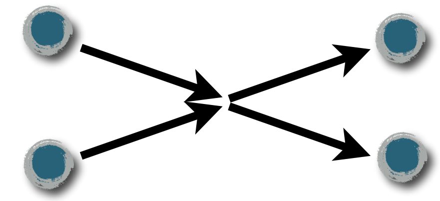
- Extract reliable spectrum
- Use formalism to fit scheme-dependent K-matrix
- Solve integral equations to reach physical amplitude

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

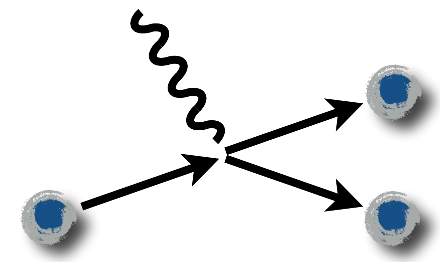



Landscape of amplitudes

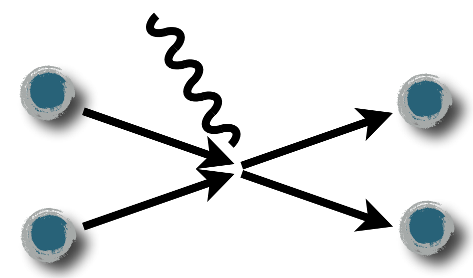
- ☑ Two-to-two scattering: $2 \to 2$



- ☑ Decays with an external current: $1 \xrightarrow[\mathcal{J}]{} 2$



- ☑ Transitions with an external current: $2 \xrightarrow[\mathcal{J}]{} 2$



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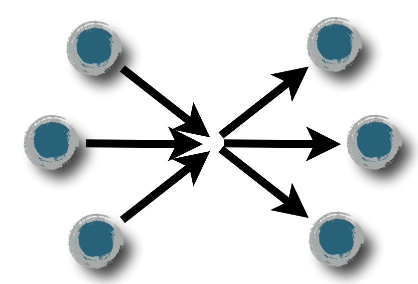
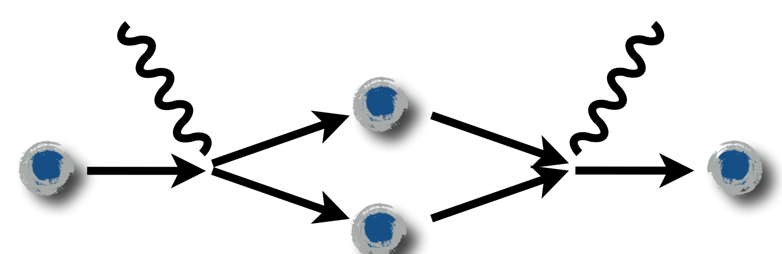


- ☐ Long distance matrix elements



Formal & numerical progress: Long-distance matrix elements

- Formal method understood… *assuming only two-hadron intermediate states*

$$\Sigma^+ \xrightarrow[H_W]{} N\pi \to p\gamma^* \qquad\qquad K^0 \xrightarrow[H_W]{} \pi\pi \xrightarrow[H_W]{} \overline{K}^0$$

Christ, Feng, Martinelli, Sachrajda *(2015)* • Christ *et al. (2016)*
• Briceño, Davoudi, MTH, Schindler, Baroni *(2019)* • Erben, Gülpers, MTH, Hodgson, Portelli *(2022)*

Formal & numerical progress: Long-distance matrix elements

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$$\Sigma^+ \underset{H_W}{\longrightarrow} N\pi \to p\gamma^* \qquad\qquad K^0 \underset{H_W}{\longrightarrow} \pi\pi \underset{H_W}{\longrightarrow} \overline{K}^0$$

  - Issue of growing exponentials *(Christ et al.)*

$$\langle \overline{K} | \mathcal{H}_W(0) \mathcal{H}_W(-|\tau|) | K \rangle_L = \sum_n c_n(L) e^{-(E_n(L) - M_K)|\tau|} \quad \underset{\int_{-T}^{0} d\tau}{\Longrightarrow} \quad \sum_n c_n \frac{1 - e^{-(E_n - M_K)T}}{M_K - E_n}$$

Christ, Feng, Martinelli, Sachrajda *(2015)* • Christ *et al. (2016)*
• Briceño, Davoudi, MTH, Schindler, Baroni *(2019)* • Erben, Gülpers, MTH, Hodgson, Portelli *(2022)*

Formal & numerical progress: Long-distance matrix elements

- Formal method understood… *assuming only two-hadron intermediate states*

$$\Sigma^+ \xrightarrow[H_W]{} N\pi \to p\gamma^* \qquad\qquad K^0 \xrightarrow[H_W]{} \pi\pi \xrightarrow[H_W]{} \overline{K}^0$$

- Issue of growing exponentials *(Christ et al.)*

$$\langle \overline{K} | \mathcal{H}_W(0) \mathcal{H}_W(-|\tau|) | K \rangle_L = \sum_n c_n(L) e^{-(E_n(L) - M_K)|\tau|} \quad \xrightarrow[\int_{-T}^{0} d\tau]{} \quad \sum_n c_n \frac{1 - e^{-(E_n - M_K)T}}{M_K - E_n}$$

- Issue of power-like finite-volume effects (after discarding exponential)

$$F_L = \sum_n \frac{c_n}{M_K - E_n}$$

Christ, Feng, Martinelli, Sachrajda *(2015)* • Christ *et al. (2016)*
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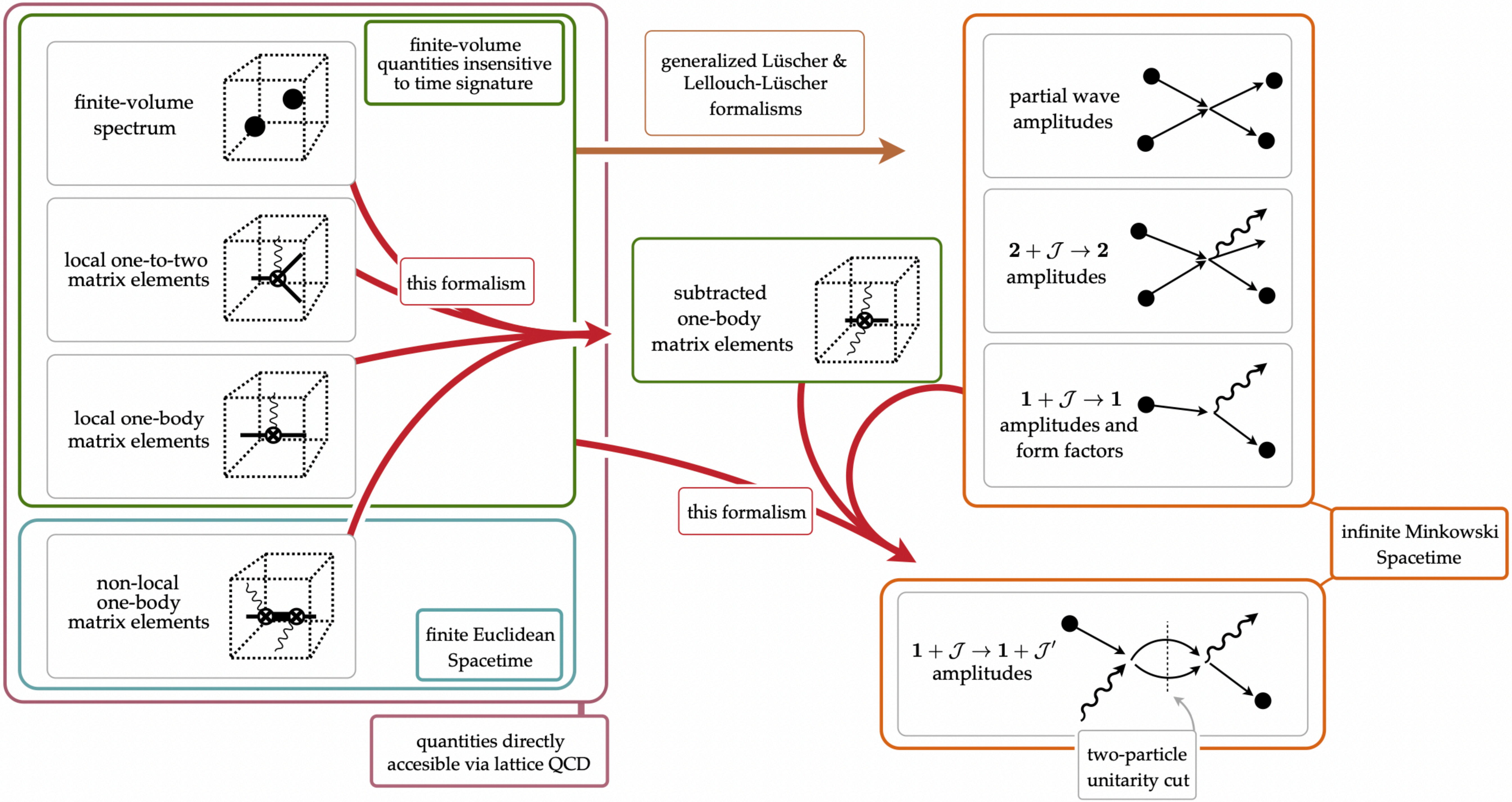

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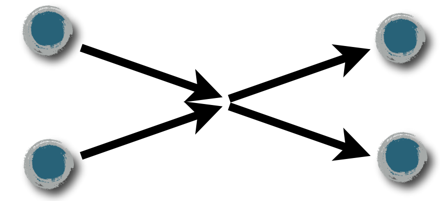
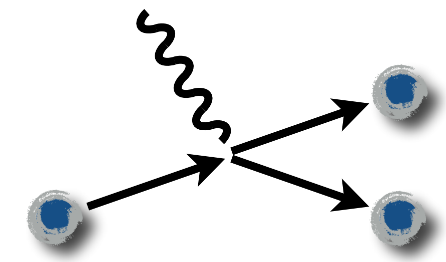
Landscape of amplitudes

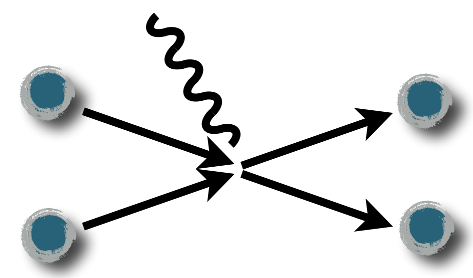
- ✓ Two-to-two scattering: $2 \to 2$



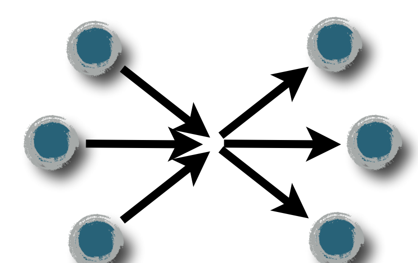
- ✓ Decays with an external current: $1 \xrightarrow[\mathscr{J}]{} 2$



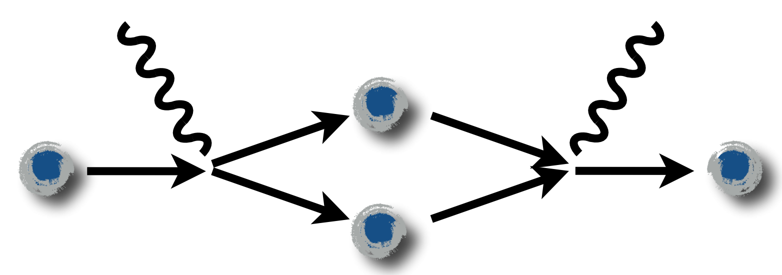
- ✓ Transitions with an external current: $2 \xrightarrow[\mathscr{J}]{} 2$



- ✓ Three-to-three scattering: $3 \to 3$



- ✓ Long distance matrix elements



Closing and outlook

- **Note:** spectral reconstruction gives access to all these processes (last week!)
  - Where can it be competitive?
  - What precisions will be reliably achievable?

- My biggest concerns and interests…
  - Fully understanding the 3- (and perhaps N-) particle formalism
  - Facing the fact that mapping ($E_n(L) \to$ amplitude) is also an inverse problem
  - Better understanding and treating ***discretisation*** effects in these calculations
  - Other sources of systematic uncertainty

- Things I would like to understand better…
  - Can the method presented by Simon be applied here (e.g. multi-hadron B decays)?
  - Can four-quark operators be implemented on Wilson/twisted-mass quarks?

- *Thanks!*

Funded by UKRI Future Leaders Fellowship

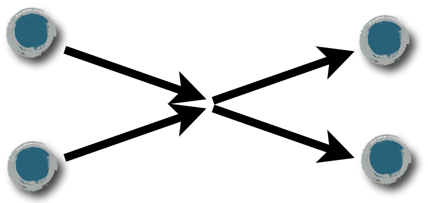
UK Research and Innovation

Back-up slides

Misc

Optical theorem

- One can write the angular-momentum-projected amplitude as



$$
\begin{aligned}
i\mathcal{M}_\ell(s) &= \langle \text{out}|\text{in}\rangle_\ell - \text{disc.} \\
&= \langle \text{out}|\text{in}\rangle_\ell - \langle \text{out}|\text{out}\rangle_\ell
\end{aligned}
$$

- This then leads to the identities

$$2\,\text{Im}\,\mathcal{M}_\ell(s) = \langle \text{out}|\text{out}\rangle_\ell + \langle \text{in}|\text{in}\rangle_\ell - \langle \text{out}|\text{in}\rangle_\ell - \langle \text{in}|\text{out}\rangle_\ell \tag{1}$$

$$|\mathcal{M}_\ell(s)|^2 = \big(\langle \text{in}|\text{out}\rangle_\ell - \langle \text{out}|\text{out}\rangle_\ell\big)\big(\langle \text{out}|\text{in}\rangle_\ell - \langle \text{out}|\text{out}\rangle_\ell\big) \tag{2}$$

- Integrating out-states in (2) and substituting $\mathbf{1} = \int |\text{out}\rangle\langle\text{out}|$ then yields

$$\int |\mathcal{M}_\ell(s)|^2 = 2\,\text{Im}\,\mathcal{M}_\ell(s)$$

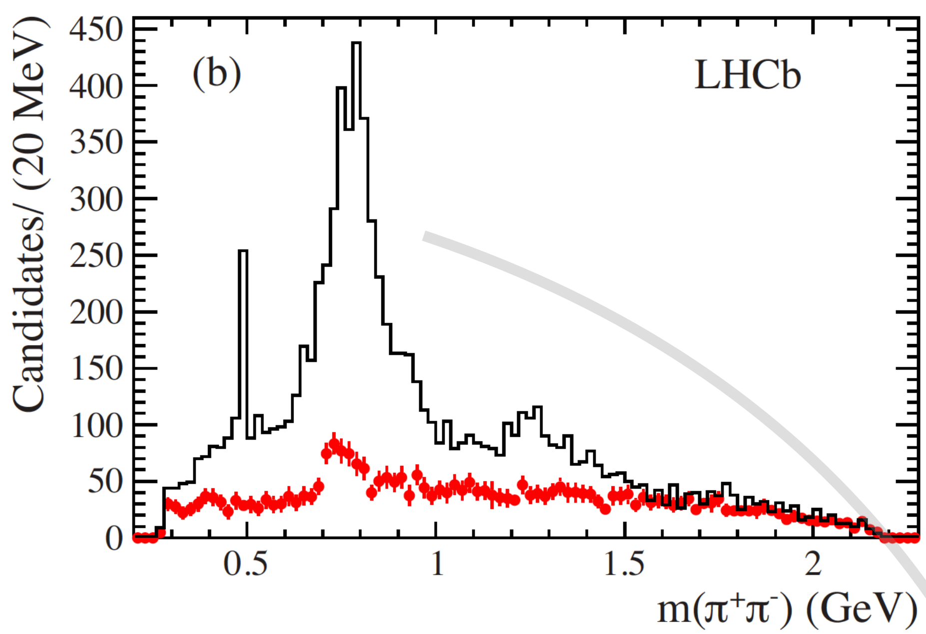
- For two-particle energies the integral becomes a simple factor…

$$\rho(s)|\mathcal{M}_\ell(s)|^2 = \text{Im}\,\mathcal{M}_\ell(s) \qquad \rho(s) = \frac{\sqrt{1 - 4m^2/s}}{32\pi}$$

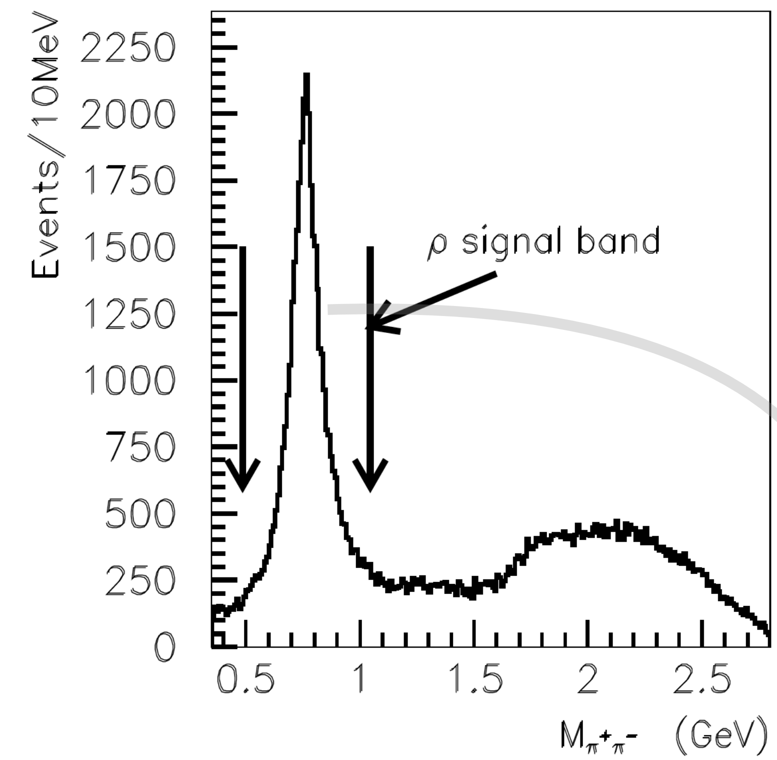
Pole is universal

- Resonances often seen in “production”

$\overline{B}^0 \to J/\psi\,\pi^+\pi^-$

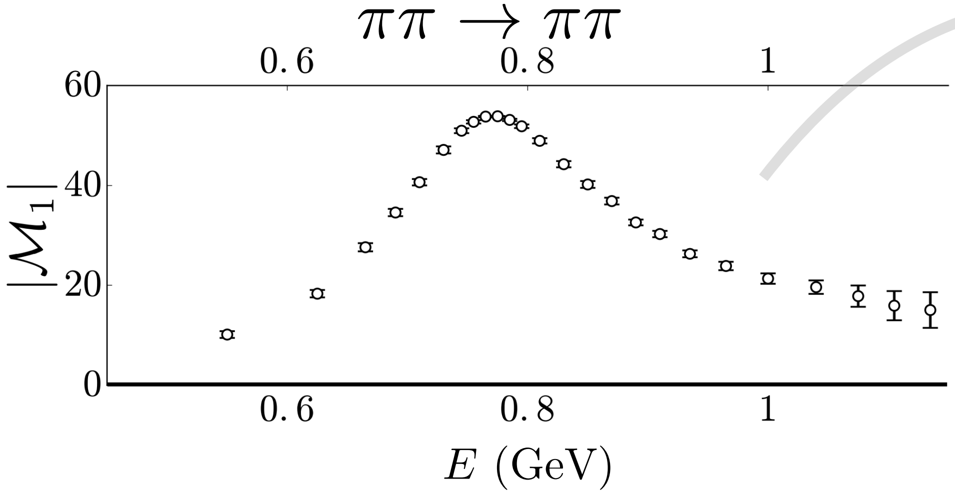


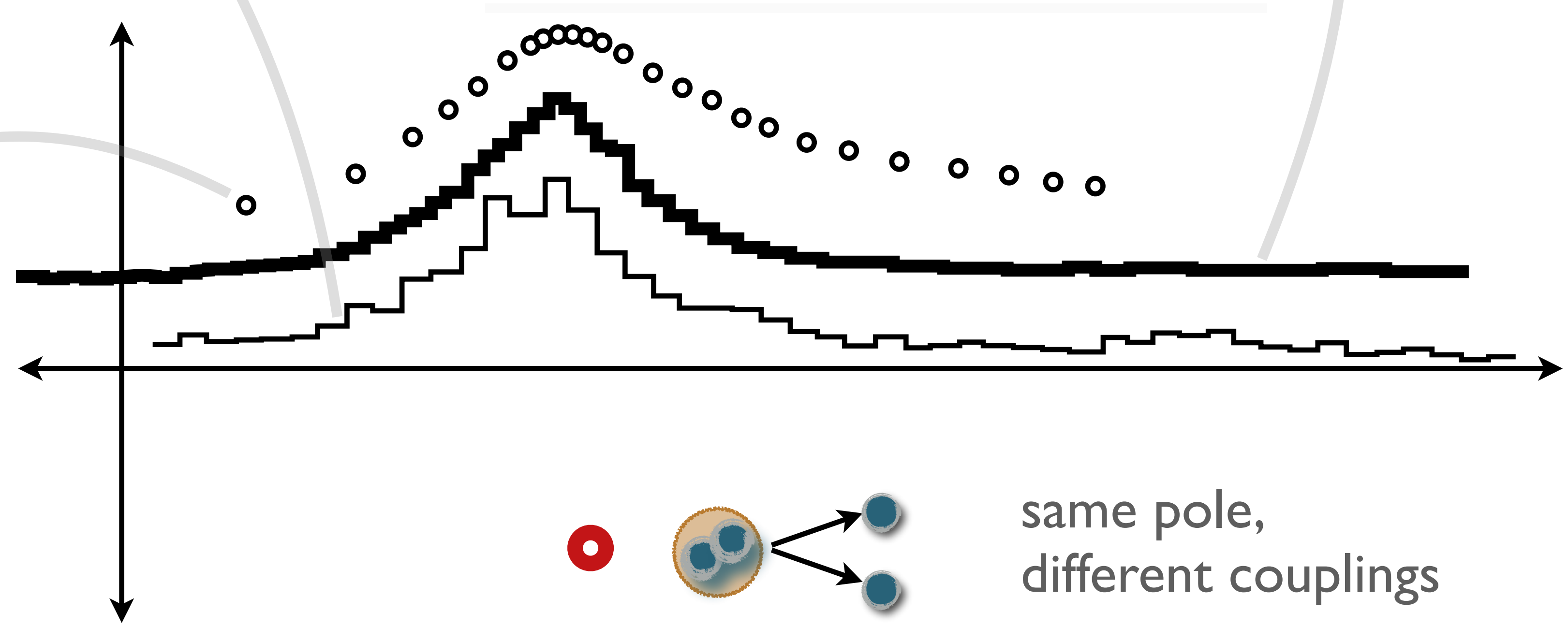
$J/\psi \to \gamma\gamma\rho$



(as opposed to scattering)

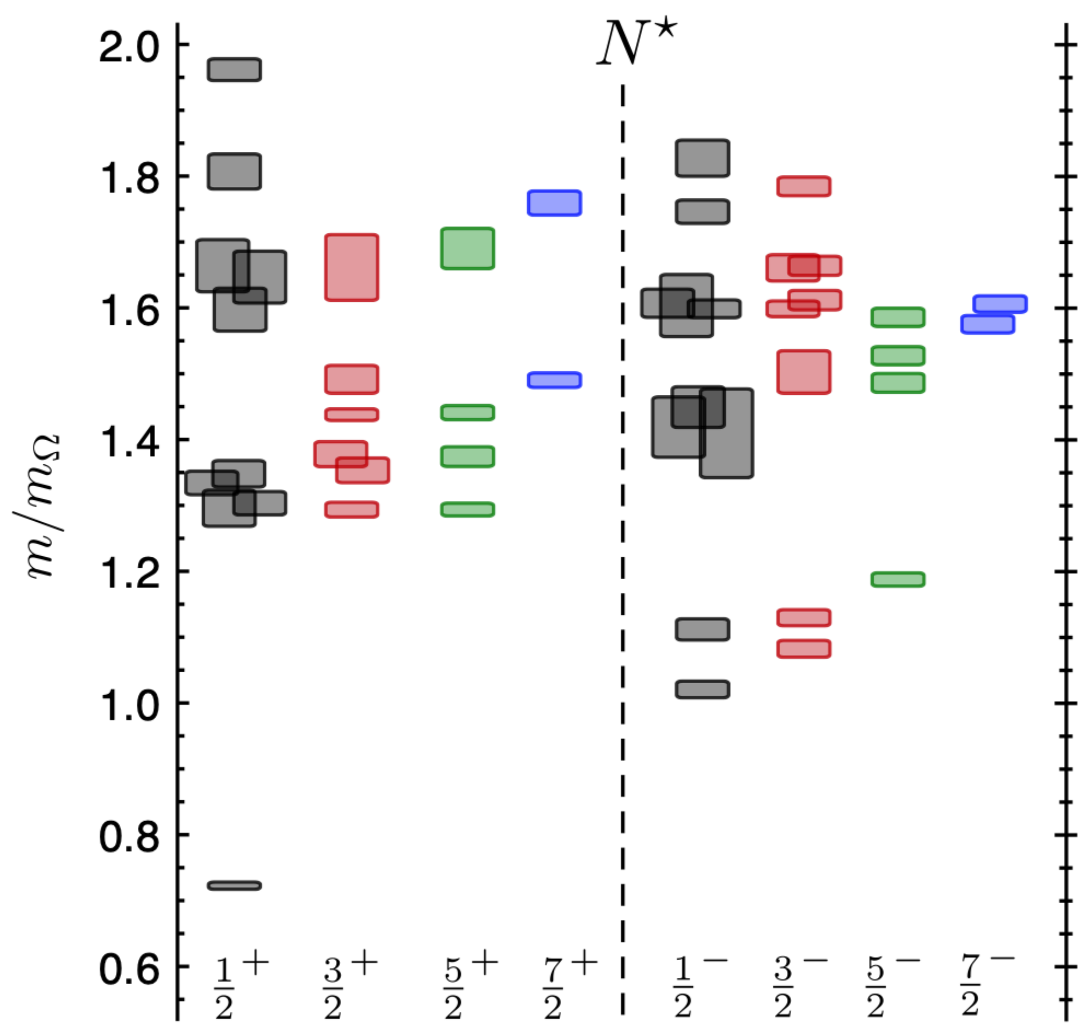
$\pi\pi \to \pi\pi$





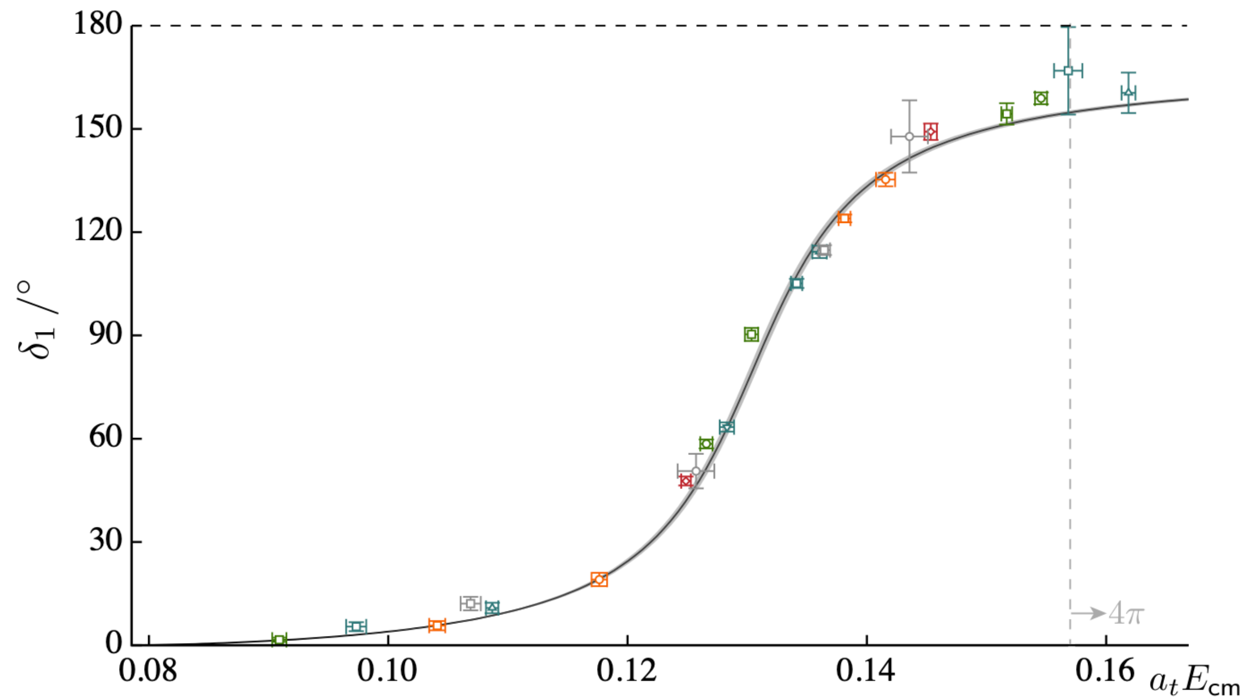
Two types of spectroscopy

Explore the spectrum of compact QCD excited states
(via quark-model inspired operators)



Edwards, Dudek, Richards, Wallace (2011)

Extract the honest finite-volume energy spectrum



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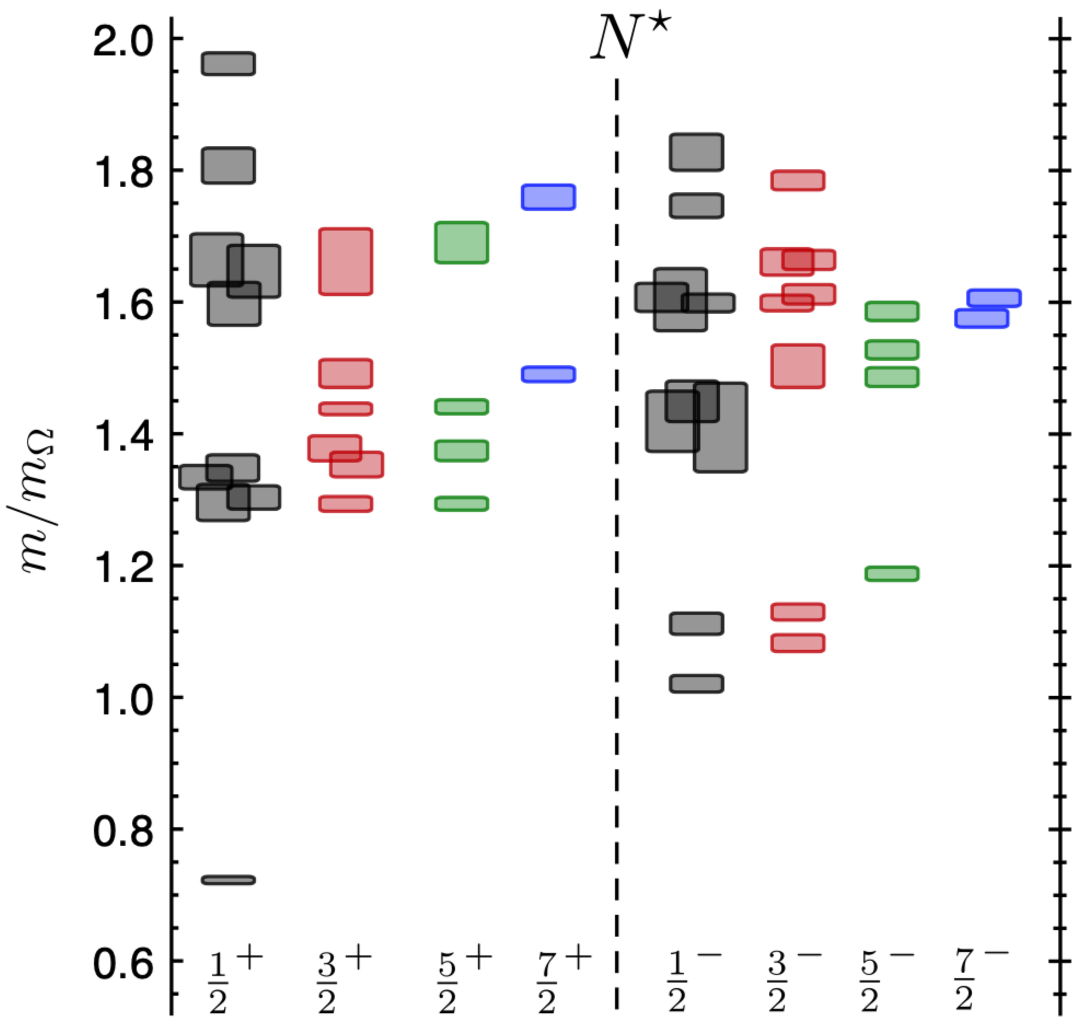
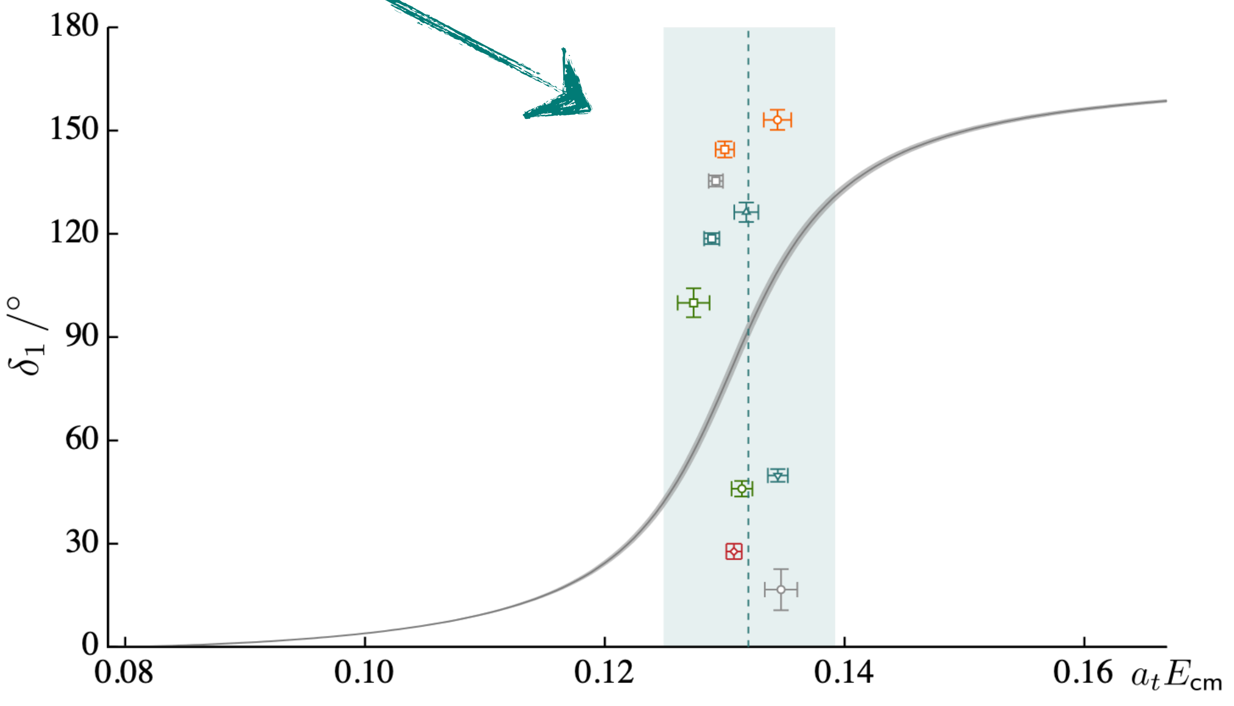
local operator spectrum = *not suitable for phase shift extraction*





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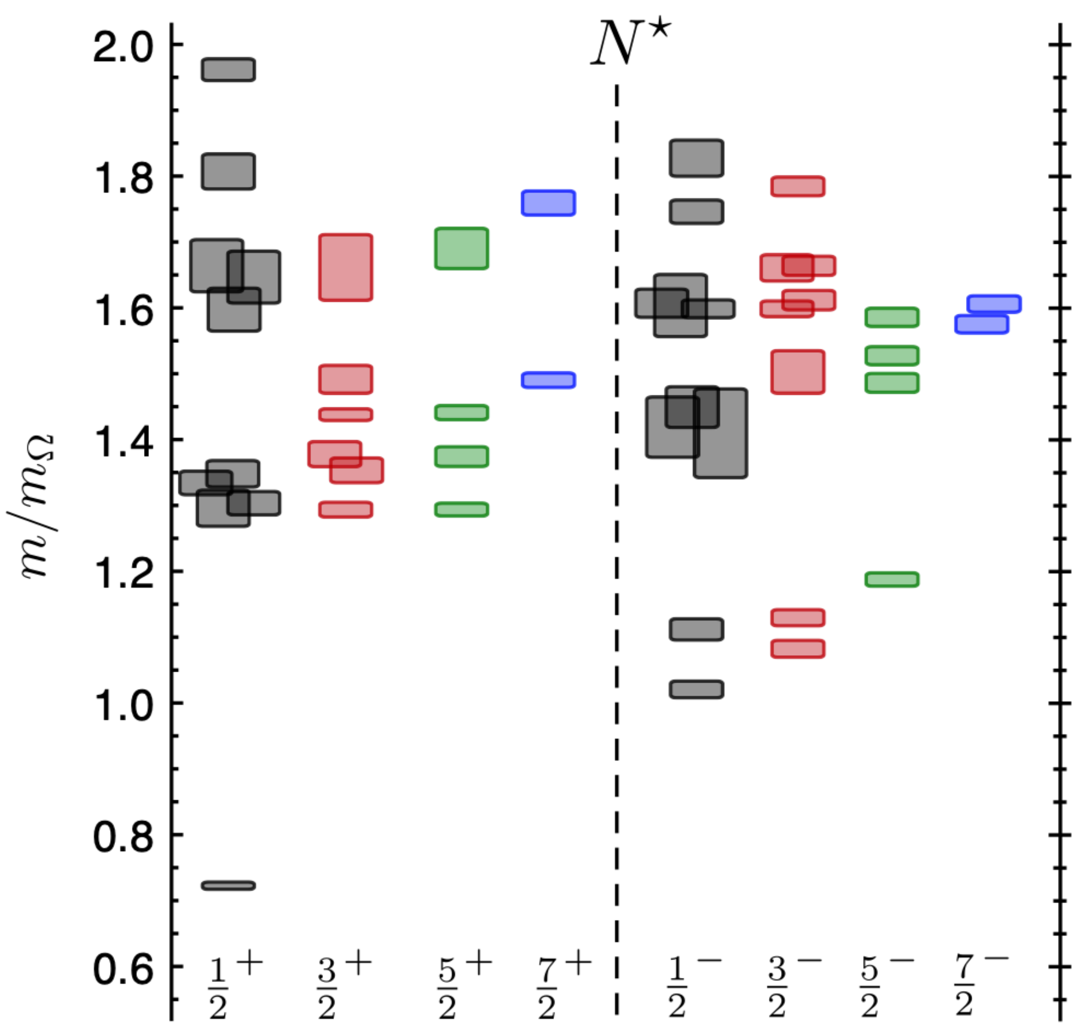

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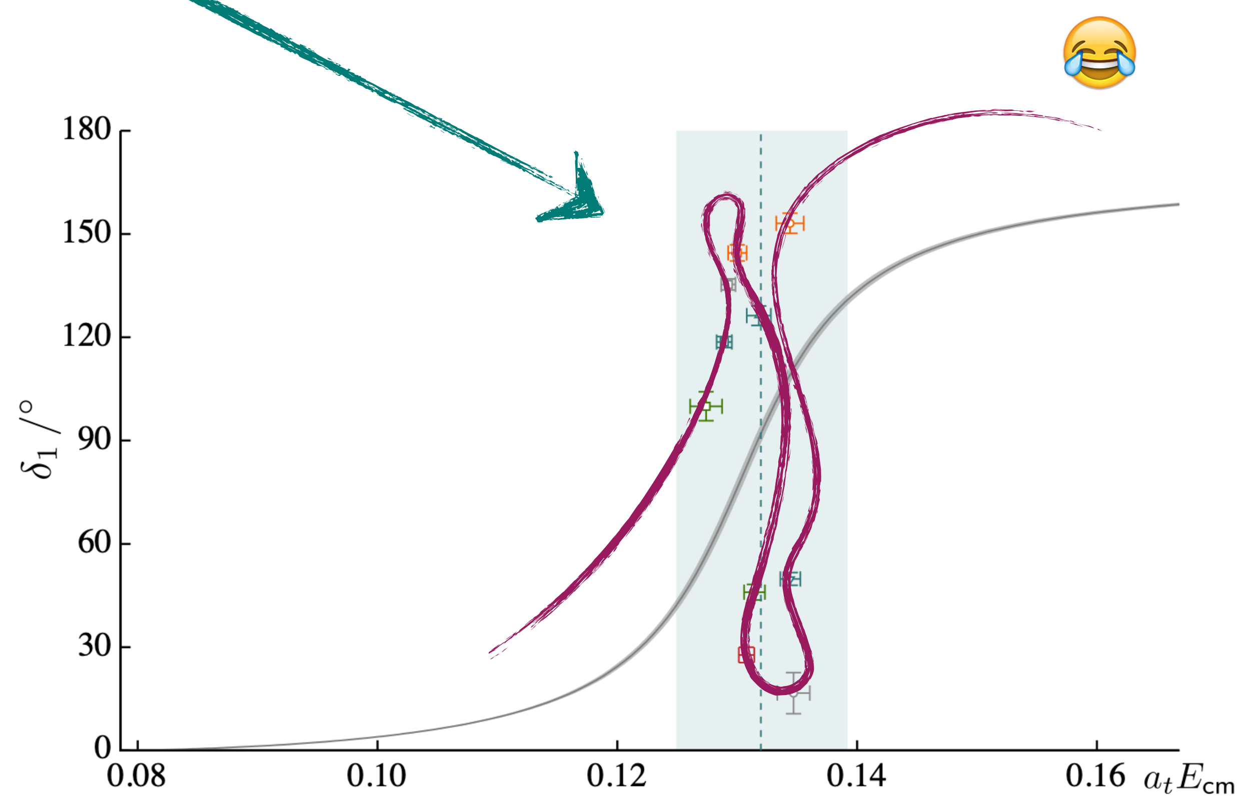

Edwards, Dudek, Richards, Wallace (2011)

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Extracting the finite-volume spectrum

- Derivatives + gamma matrices + smearing → *basis of single-hadron operators*

$$\overline{q}\Gamma q, \quad \overline{q}\Gamma D q, \quad \overline{q}\Gamma D \cdots D q$$

- Variational method → *optimized single hadron*

$$\pi = c_1\, \overline{q}\Gamma q + c_2\, \overline{q}\Gamma D q + \cdots$$

- Group theory + individual momentum projection → *two- and three-pion operators*

$$(\pi\pi\pi)(\boldsymbol{P}, \Lambda) = \sum \mathrm{CG}\, \pi(\boldsymbol{p}_1)\pi(\boldsymbol{p}_2)\pi(\boldsymbol{p}_3)$$

- Second variational method → *multi-pion finite-volume energies*

- Validate extraction…
  - Quality of energy plateaus
  - Stability under change of operators
  - Consistent with finite-volume formalism

Brought to you by distillation!

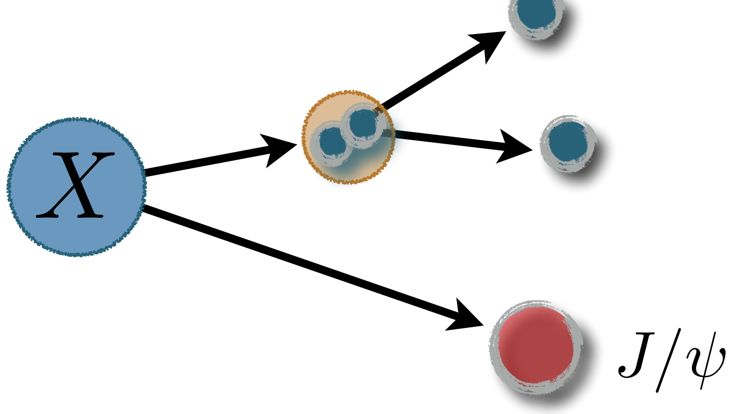
Peardon *et al.* (2009)

3-body (formal)

3-particle amplitudes

2-to-2 only samples $J^P \quad 0^+ \quad 1^- \quad 2^+ \cdots$

many interesting resonances have significant 3-body decays



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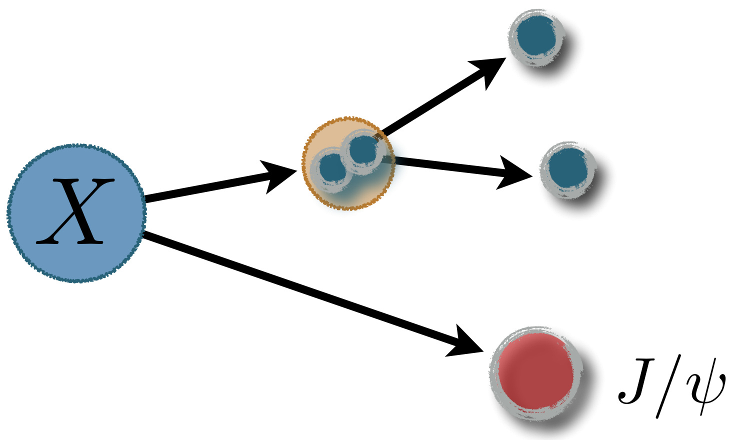
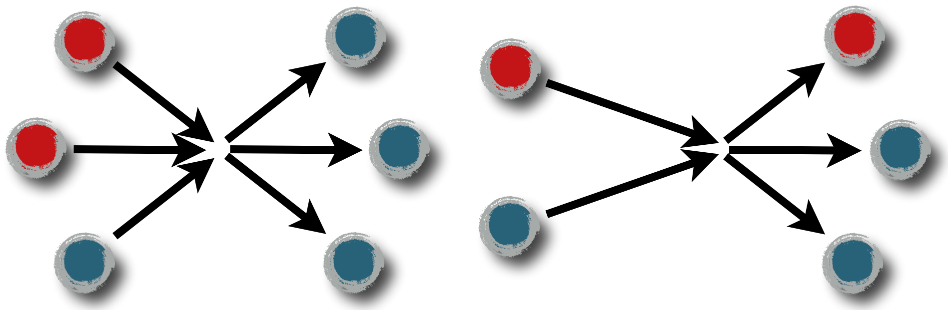


Goal: ***finite-volume + unitarity formalism*** for generic two- and three-particle systems



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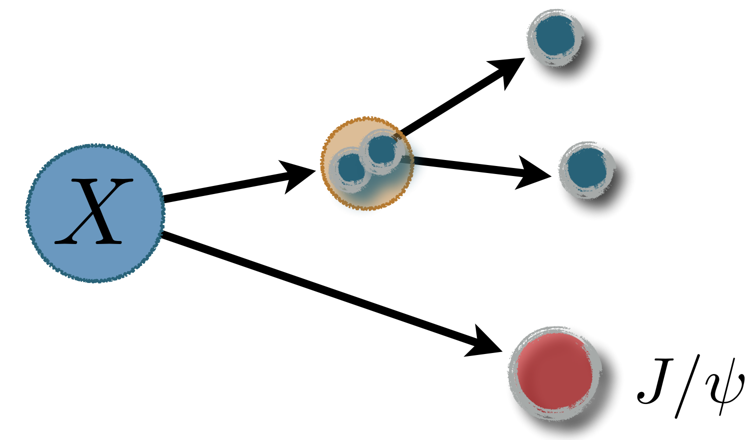

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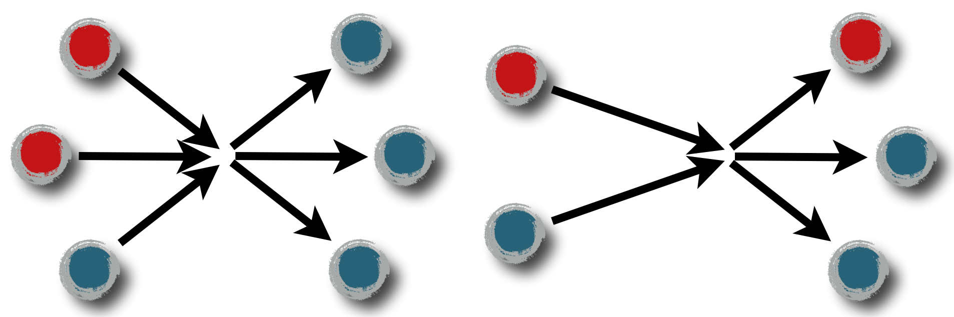


Applications…

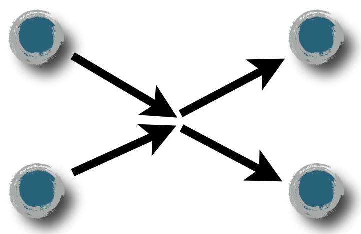
exotic resonance pole positions, couplings, quantum numbers

$\omega(782),\ a_1(1420) \to \pi\pi\pi$ $\quad X(3872) \to J/\psi\pi\pi$ $\quad X(3915)[Y(3940)] \to J/\psi\pi\pi$

form factors and transitions

and much more!… *(3-body forces, weak transitions, gluons content)*

Complication: degrees of freedom



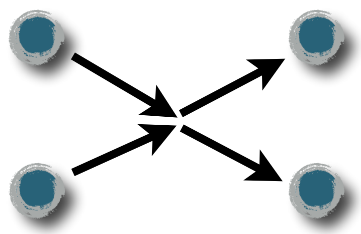
12 momentum components

-10 Poincaré generators

2 degrees of freedom

$$\vec{p}_1 + \vec{p}_2 \;\rightarrow\; \vec{p}_3 + \vec{p}_4 \longrightarrow \text{Mandelstam } s, t$$

Complication: degrees of freedom



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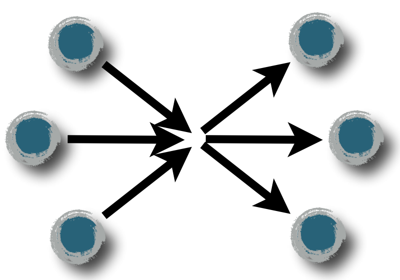
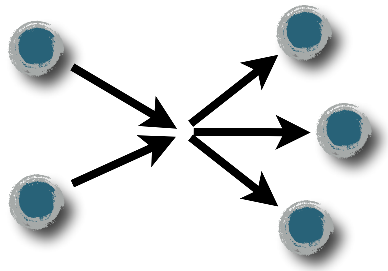


18 momentum components

-10 Poincaré generators

8 degrees of freedom



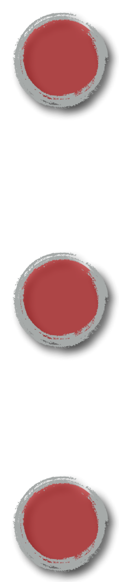
15 momentum components

-10 Poincaré generators

5 degrees of freedom

Complication: on-shell states

- Classical pairwise scattering



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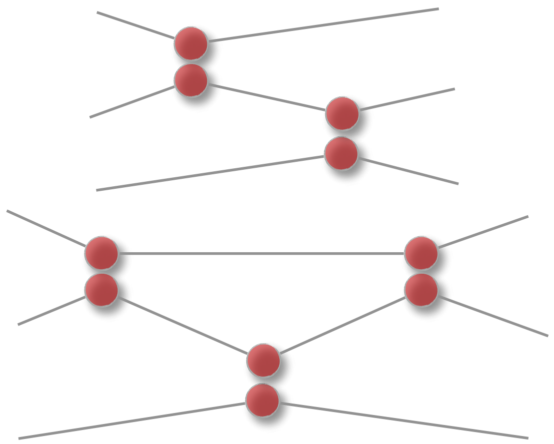
for $m_1 = m_2 = m_3$ up to ***3*** binary collisions are possible



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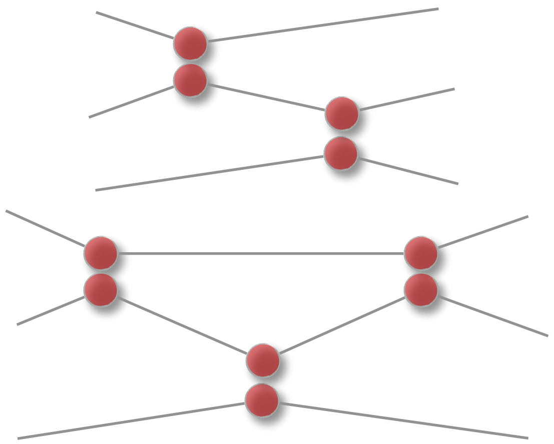

Dispersion Relations for Three-Particle Scattering Amplitudes. I*

MORTON RUBIN
Physics Department, University of Wisconsin, Madison, Wisconsin

AND

ROBERT SUGAR
Physics Department, Columbia University, New York, New York

AND

GEORGE TIKTOPOULOS
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received 31 January 1966)

$$b = \frac{(m_1+m_3)(m_2+m_3)}{m_1 m_2}$$

It follows that if

$$b^{n-2}(b-1) > 1\,, \qquad \text{(IV.18)}$$

then $2n+1$ successive binary collisions are kinematically impossible.

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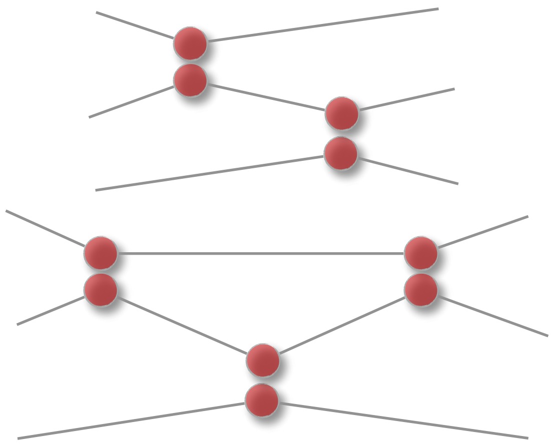

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$m_1 = m_2 = m_3 - \varepsilon$:
4 collisions possible
$\pi\pi K$

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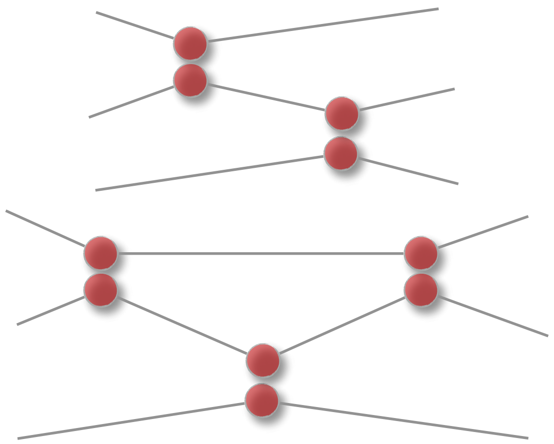

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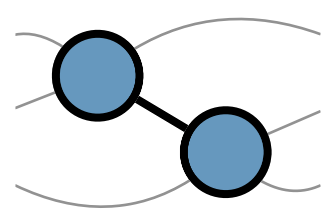
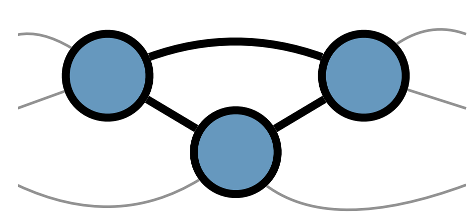
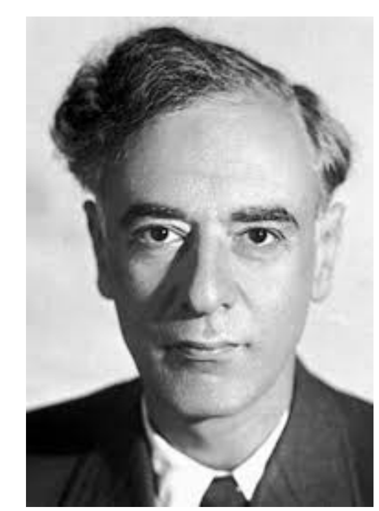
- Correspond to Landau singularities

$i\mathcal{M}_{3\to 3} \equiv$ fully connected correlator $=$  $+$  $+\cdots$

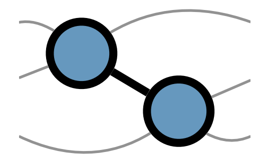
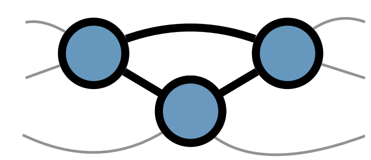


complicate analyticity & unitarity

difficult to disentangle kinematic singularities from resonance poles

Two key observations

- Intermediate $K_{\mathrm{df},3}$ removes singularities

$\mathcal{K}_{\mathrm{df},3} \equiv$ fully connected diagrams w/ ***PV pole prescription*** $-$  $+$  $+\cdots$

same degrees of freedom as M_3

smooth real function

relation to M_3 = known

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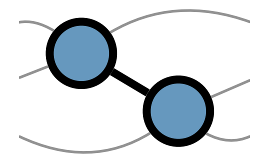
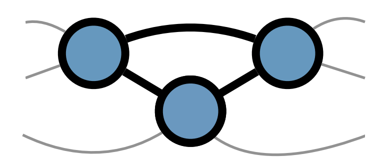
$\mathcal{K}_{\mathrm{df},3} \equiv$ fully connected diagrams w/ *PV pole prescription* $-$  $+$  $+\cdots$

same degrees of freedom as *M*3

smooth real function

relation to *M*3 = known

- $K_{\mathrm{df},3}$ has a systematic low-energy expansion

$$\mathcal{K}_{\mathrm{df},3}(p_3,p_2,p_1;k_3,k_2,k_1) = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}\,\Delta + \cdots \qquad \Delta = \frac{s-(3m)^2}{(3m)^2}$$

smooth real function

analogous to effective range expansion

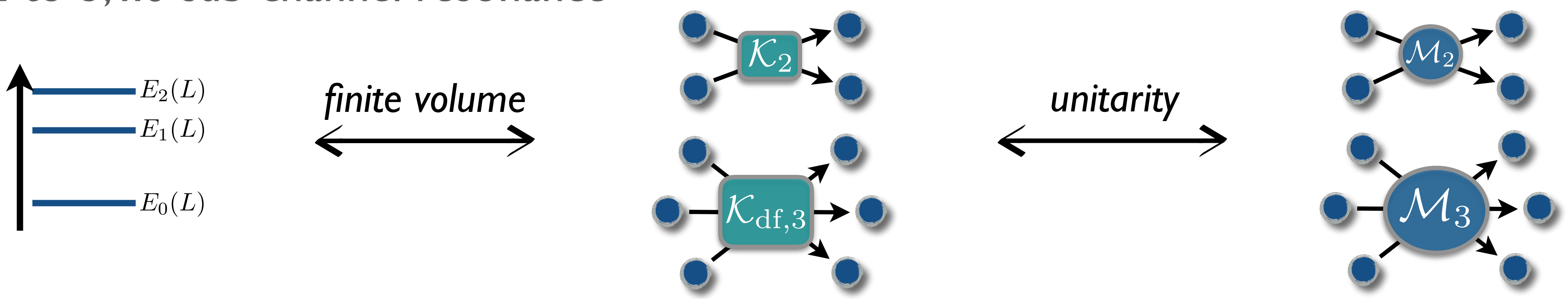
$$p\cot\delta = -\frac{1}{a} + \frac{1}{2}\,r\,p^2 + \mathcal{O}(p^4)$$

gives handle on many *degrees of freedom*
(DOFs enter order by order)

Status…

- General relation between *energies* and *two-and-three scalar scattering*

No 2-to-3, *no* sub-channel resonance



• MTH, Sharpe (2014, 2015) •

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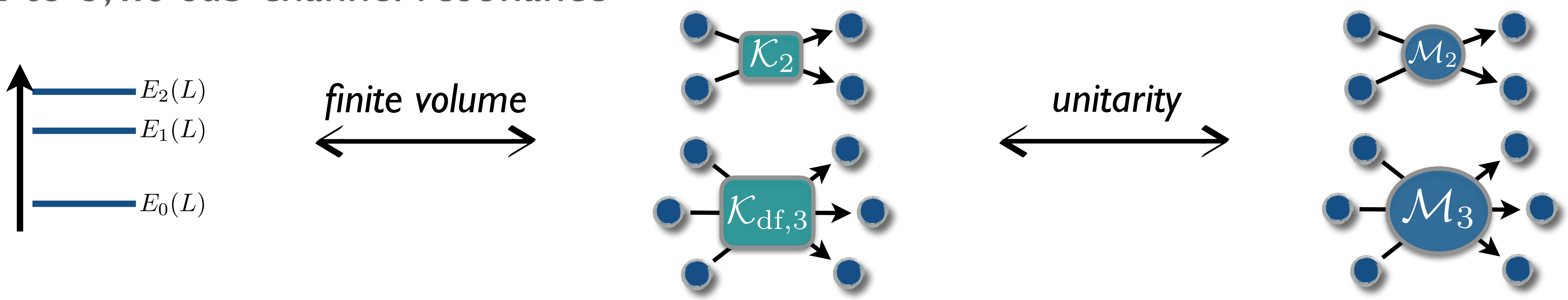

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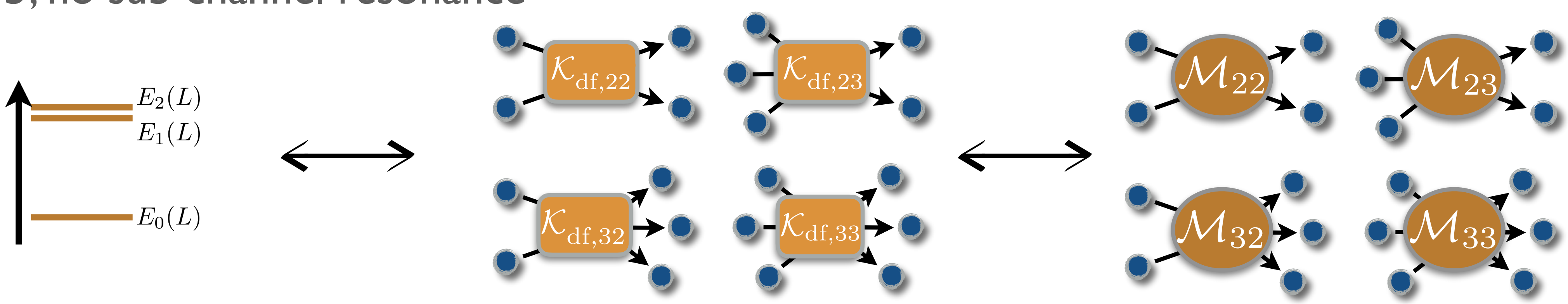


• Briceño, MTH, Sharpe (2017) •

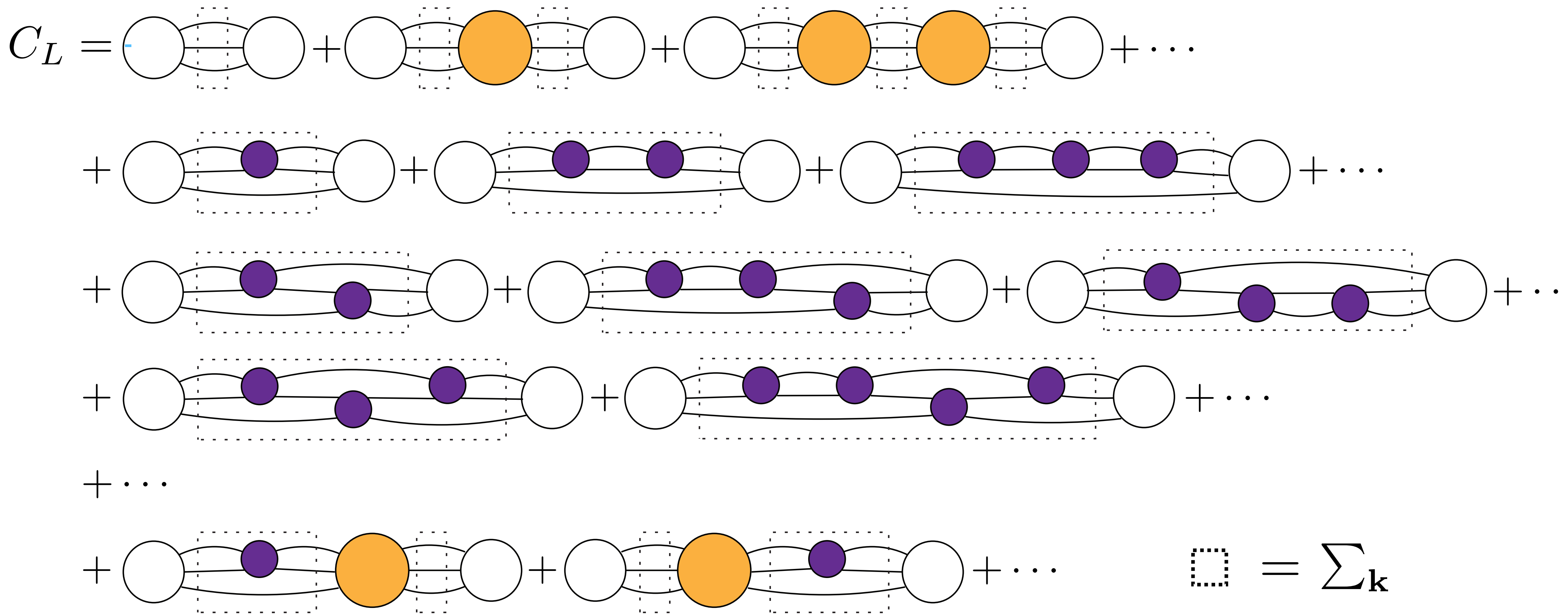
Including sub-channel resonances + ***different isospins*** + ***non-degenerate***

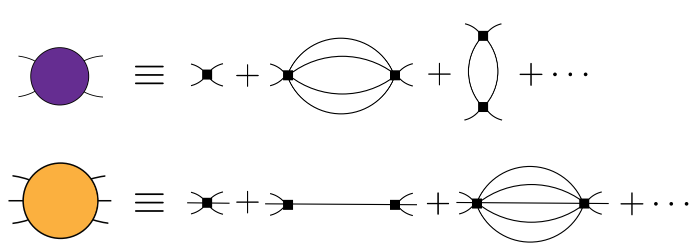
$$\pi\pi\pi \to \rho\pi \to \omega \to \rho\pi \to \pi\pi\pi$$

• Briceño, MTH, Sharpe (2018) • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020)

3-particle derivation

- Study 3-body correlator in an ***all-orders skeleton expansion***





kernels have suppressed L dependence

lines = fully dressed hadrons

• MTH, Sharpe (2014) •

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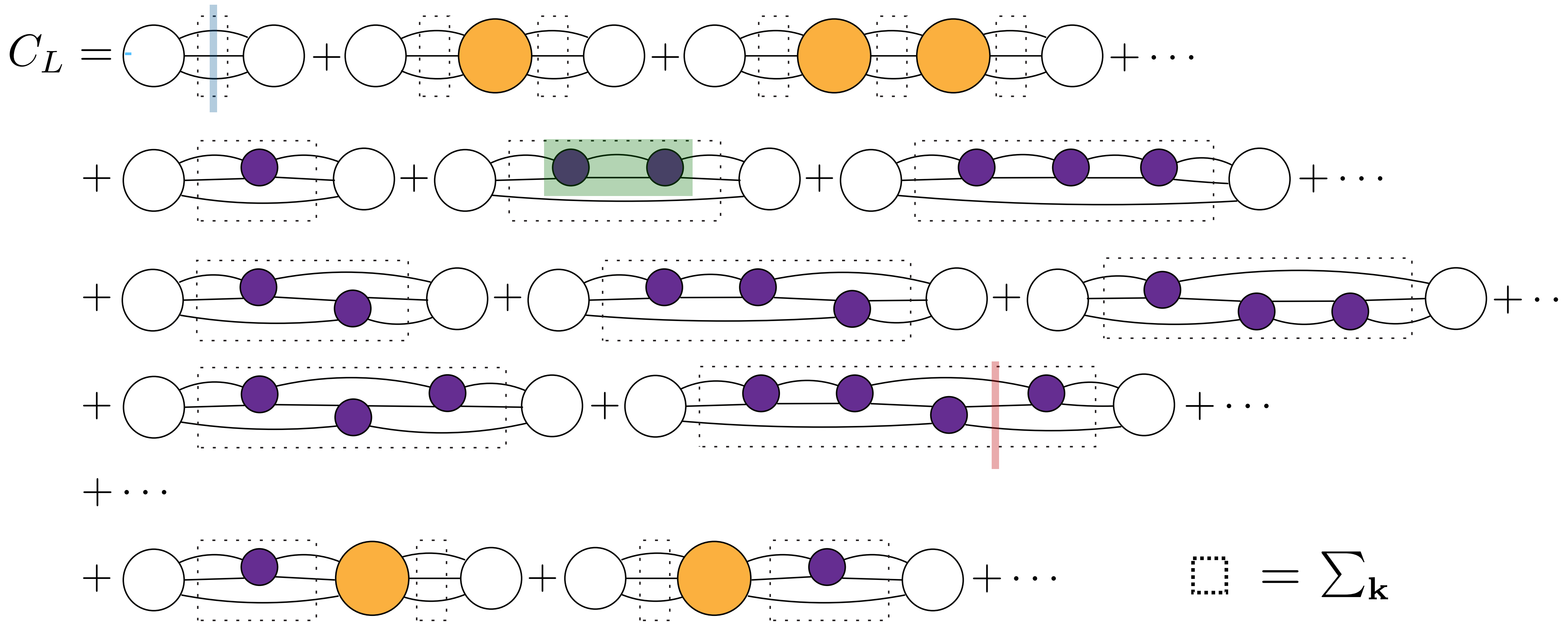
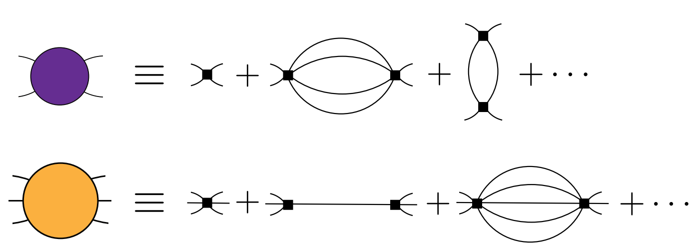




kernels have suppressed L dependence
lines = fully dressed hadrons

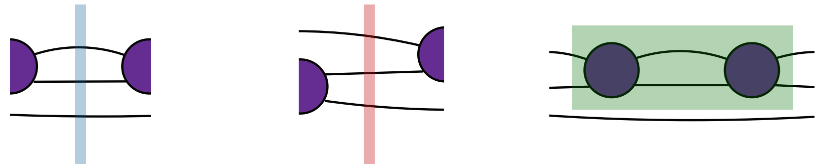
• MTH, Sharpe (2014) •

General relation

$$\det\left[\mathcal{K}_{\mathrm{df},3}^{-1}(s) + F_3(P,L|\mathcal{K}_2)\right] = 0$$

$F_3(P,L|\mathcal{K}_2) \equiv$ Matrix of functions depending on kinematics + two-particle dynamics

$$F_3 \equiv \frac{1}{3}F + F\,\mathcal{K}_2\,\frac{1}{1-(F+G)\,\mathcal{K}_2}\,F$$



Holds only for three-particle energies

Neglects e^{-mL}

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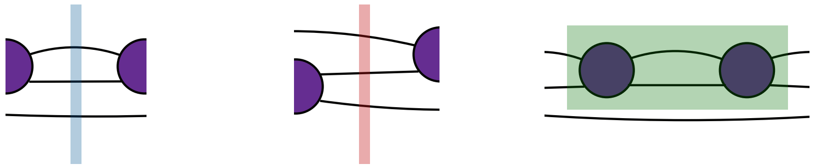

Holds only for three-particle energies

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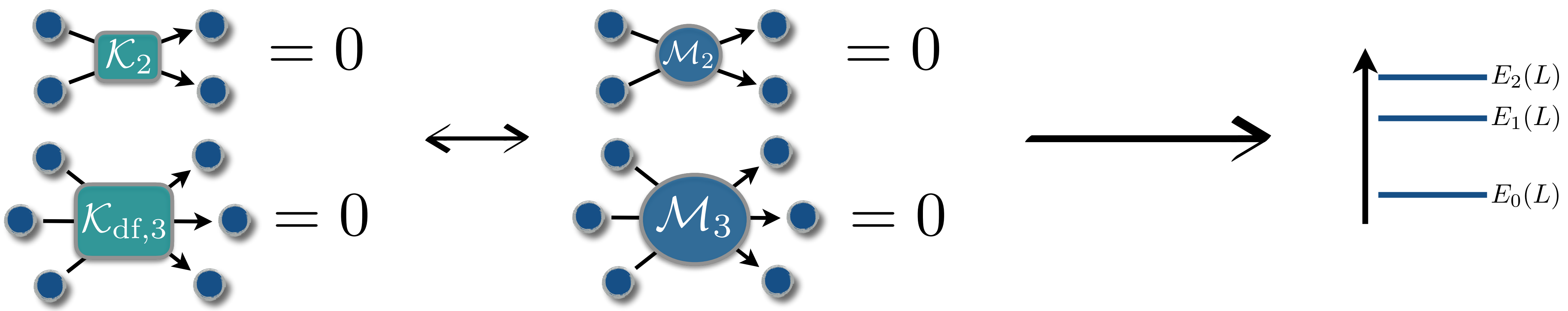
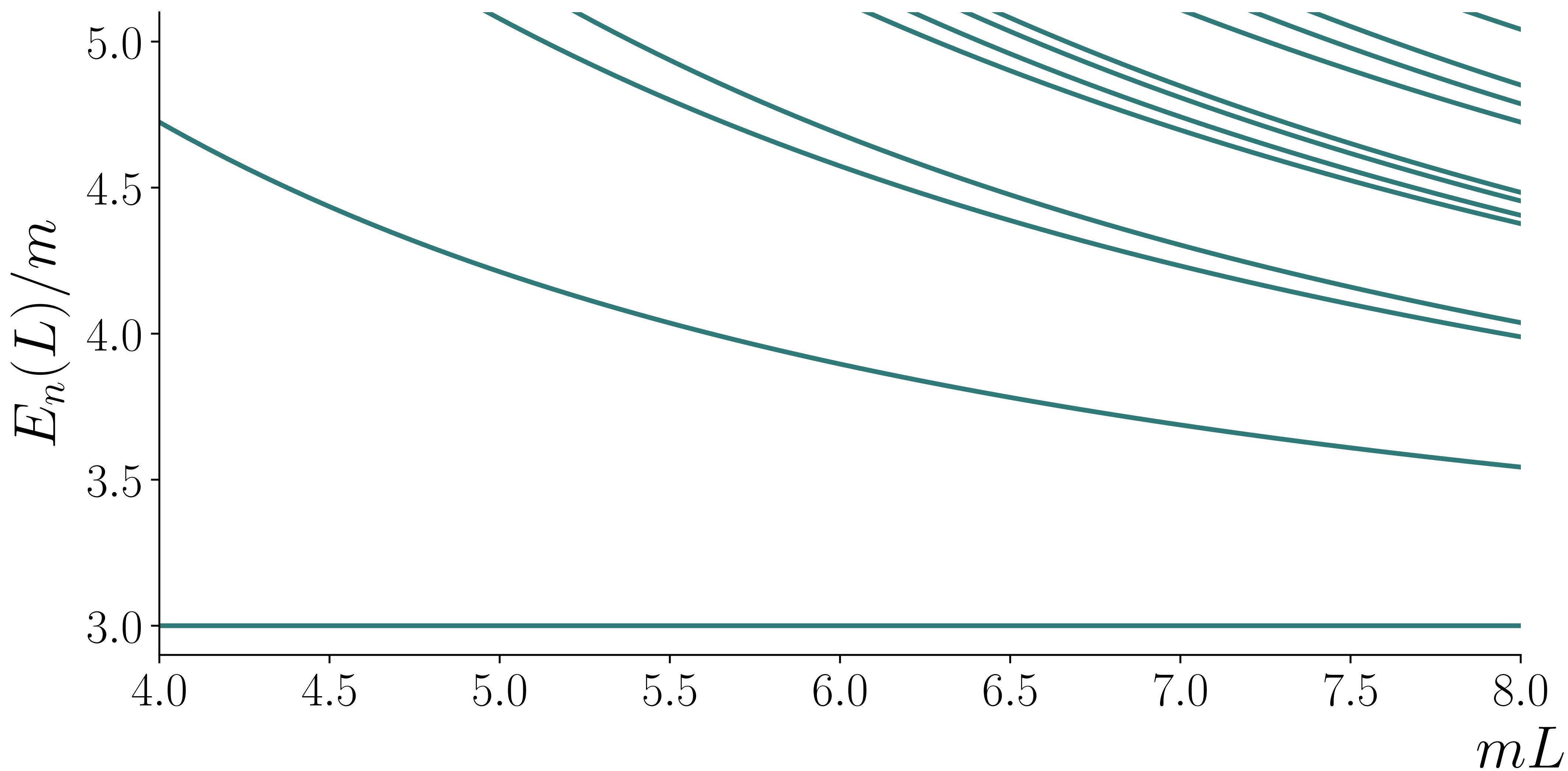
- MTH, Sharpe (2014-2016) • *See also Döring, Mai, Hammer, Pang, Rusetsky* •

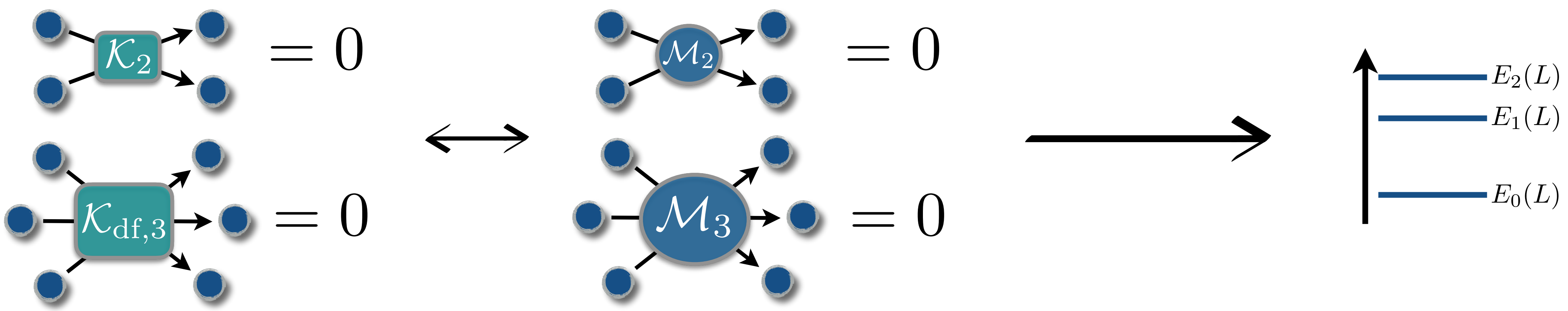
Review: *Lattice QCD and Three-particle Decays of Resonances*
MTH and Sharpe, *1901.00483*

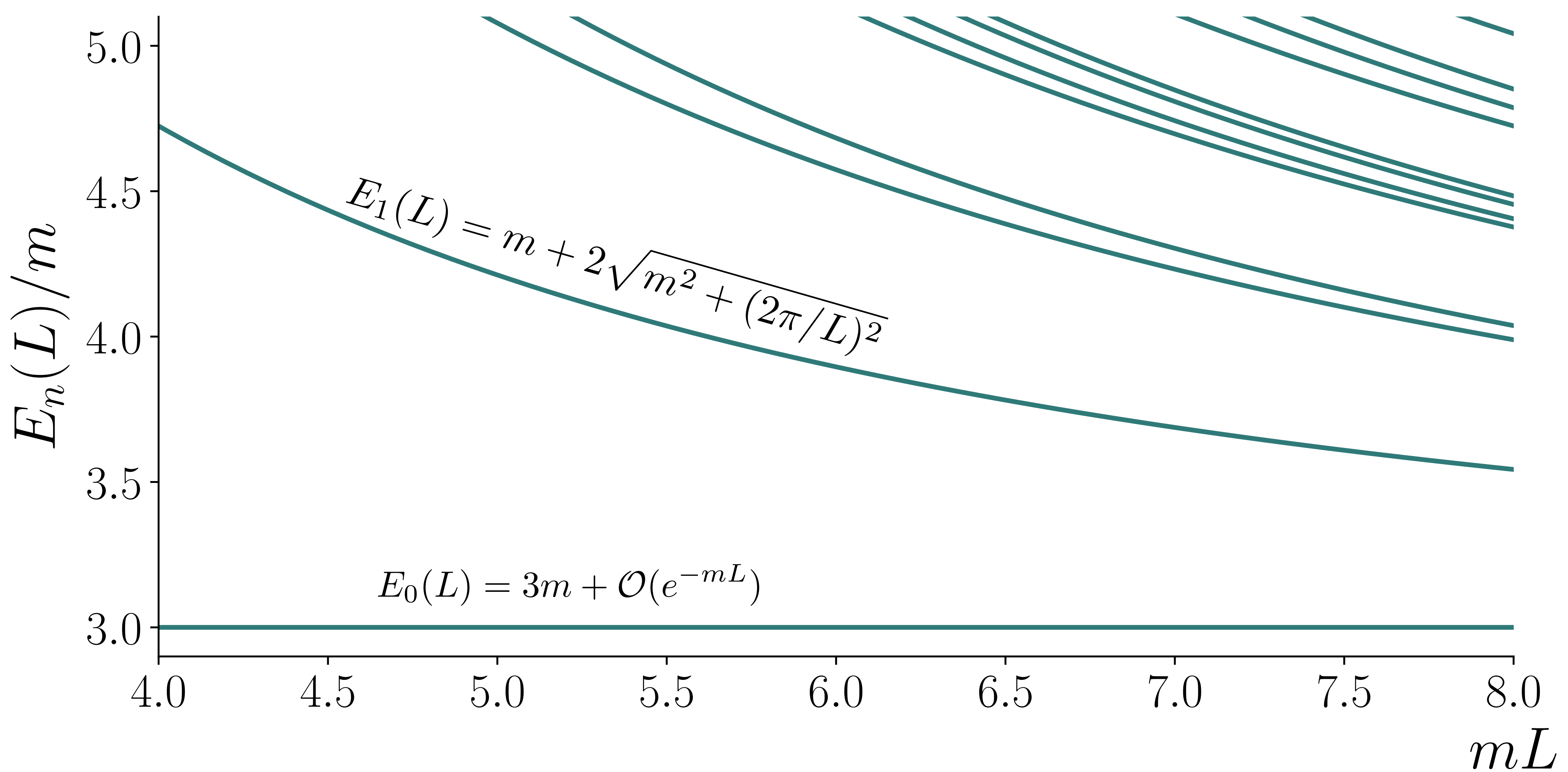
Non-interacting energies



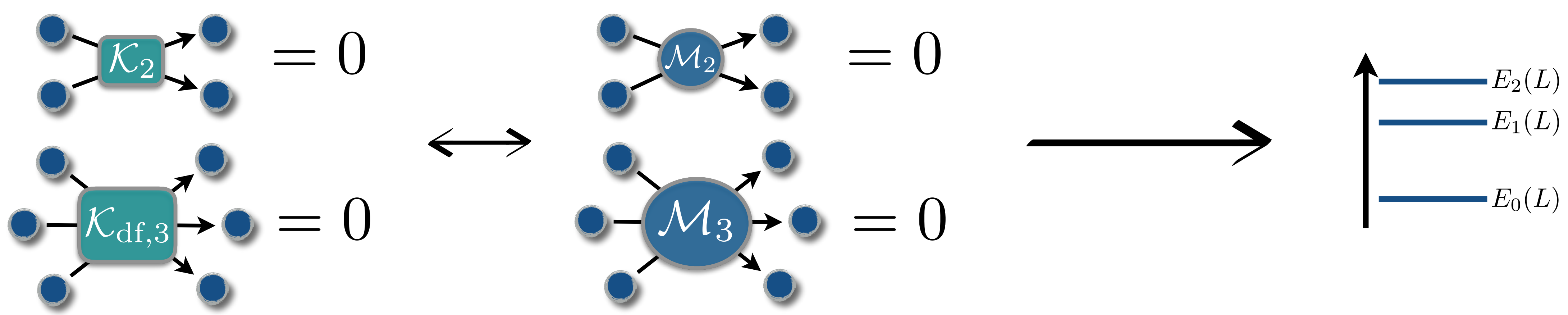


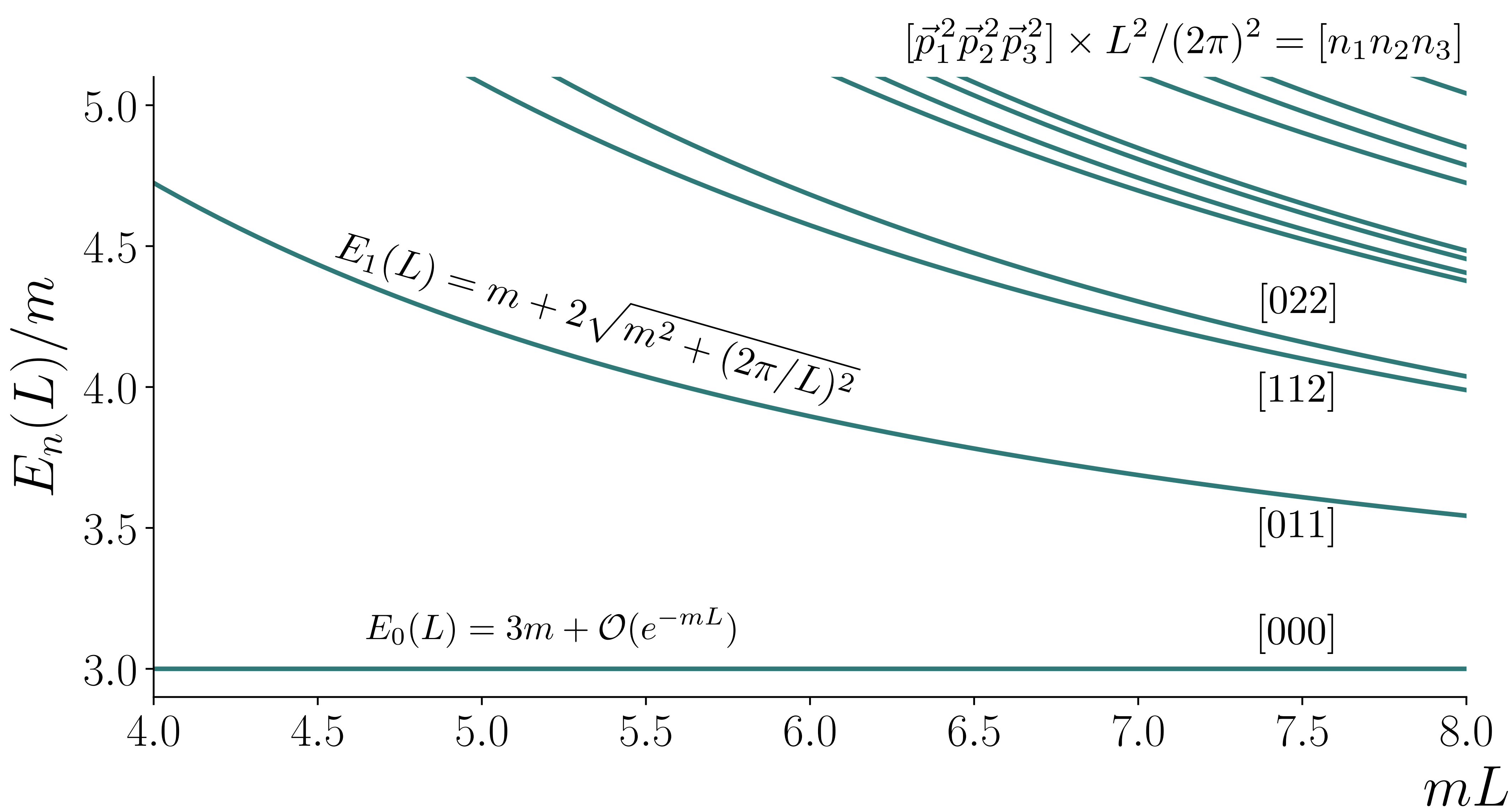
Non-interacting energies



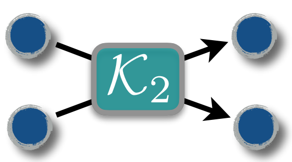


Non-interacting energies

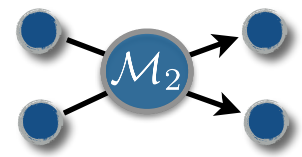




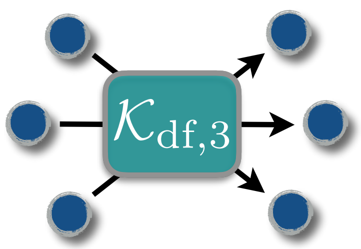
Two-particle interactions



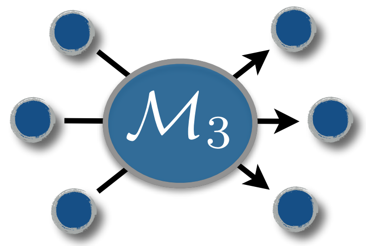
$= -16\pi\sqrt{s}\,a$



$= \dfrac{16\pi\sqrt{s}}{-1/a - ip}$



$= 0$



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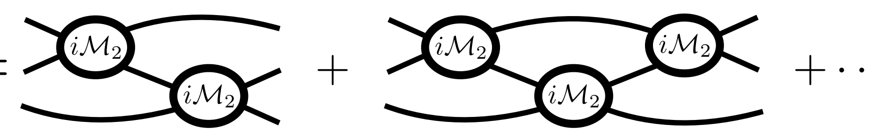

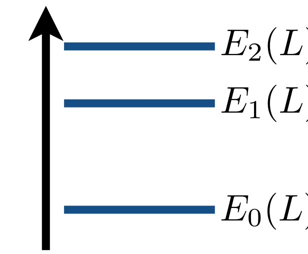

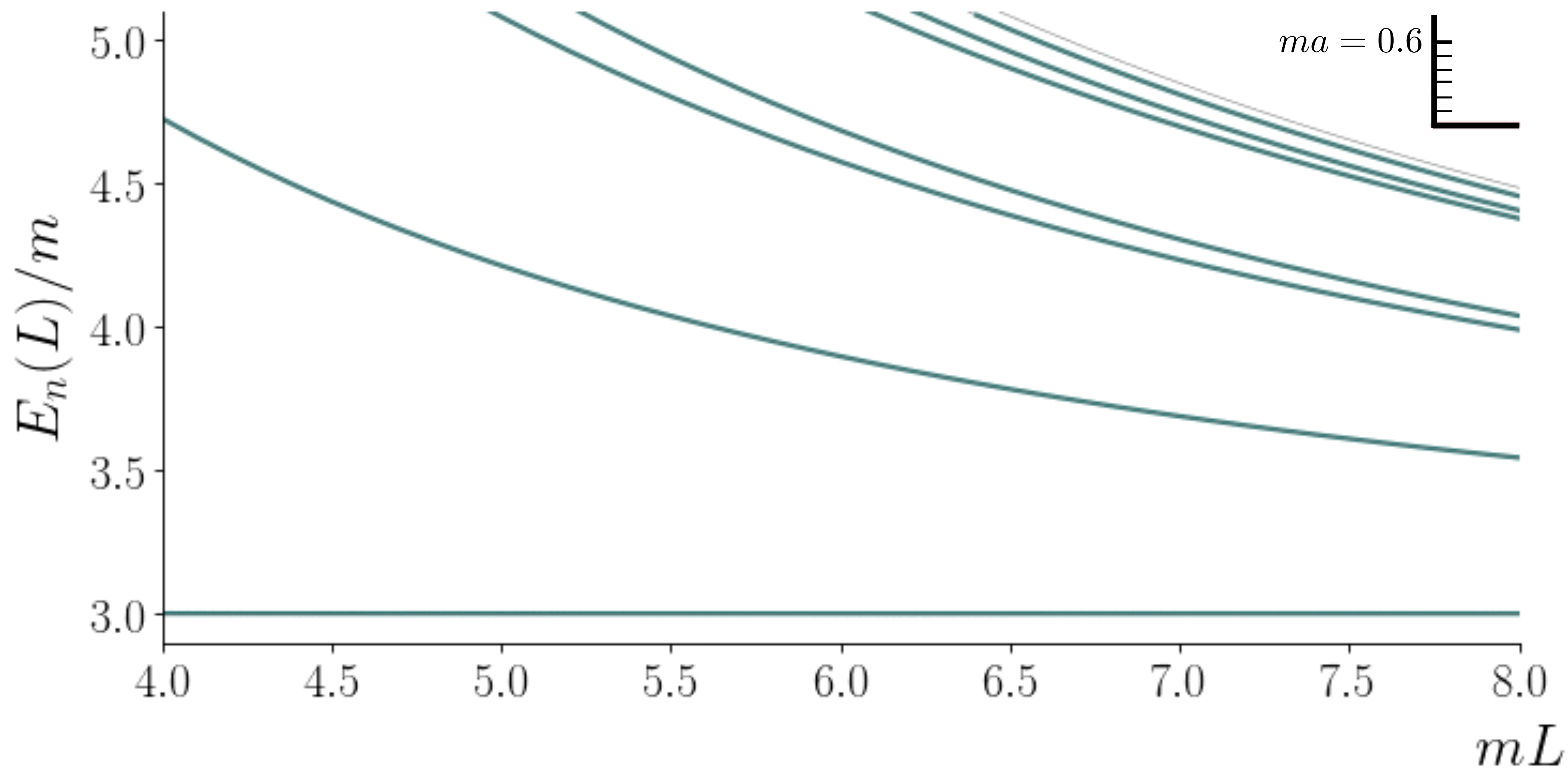

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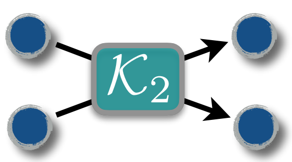

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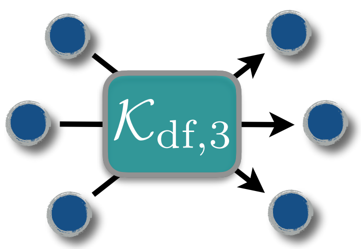
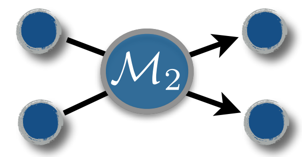
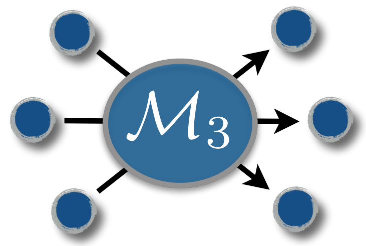


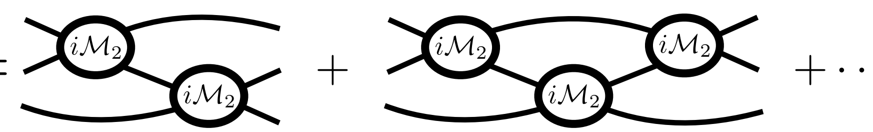
$= 0$

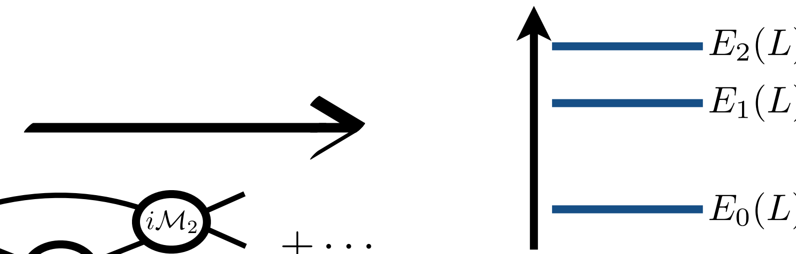


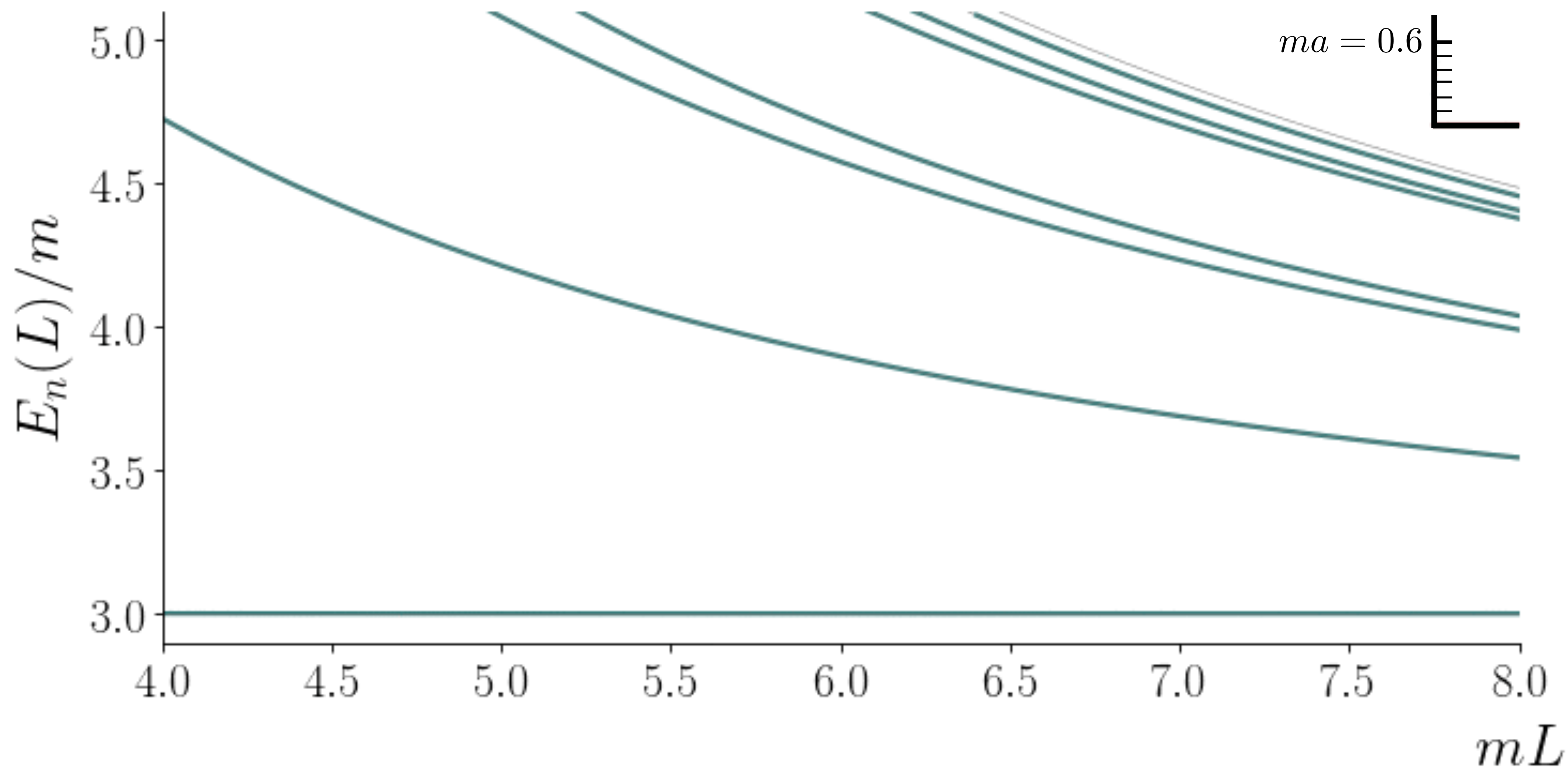
$$= \frac{16\pi\sqrt{s}}{-1/a - ip}$$



$=$

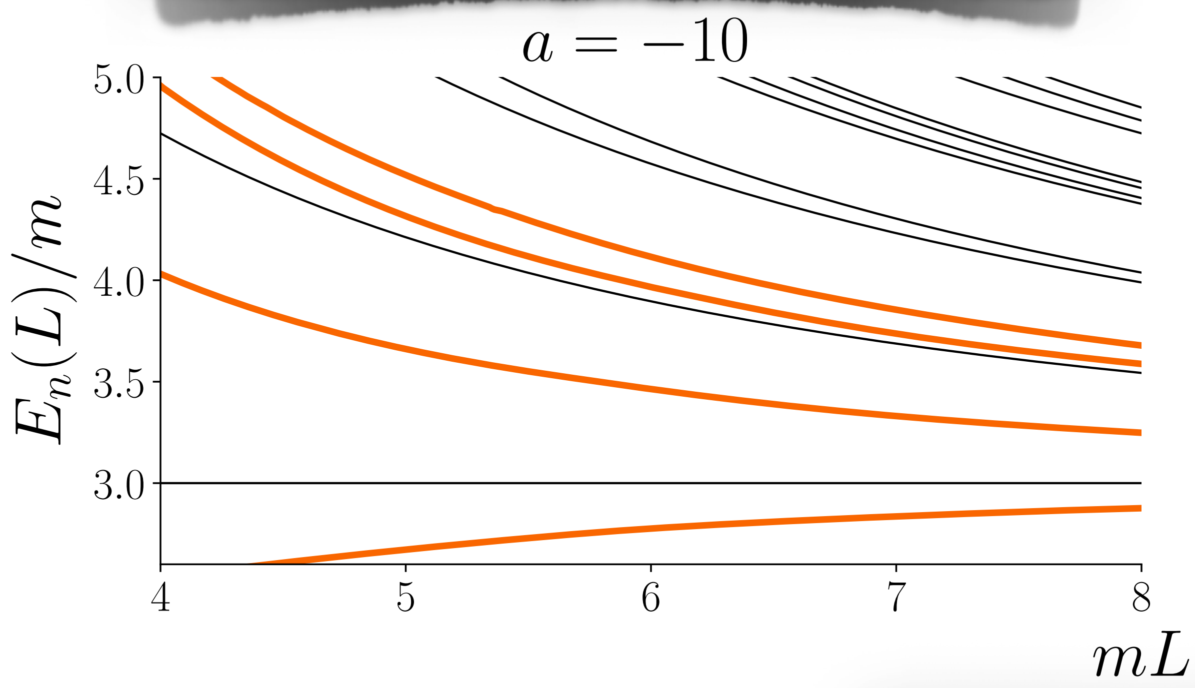




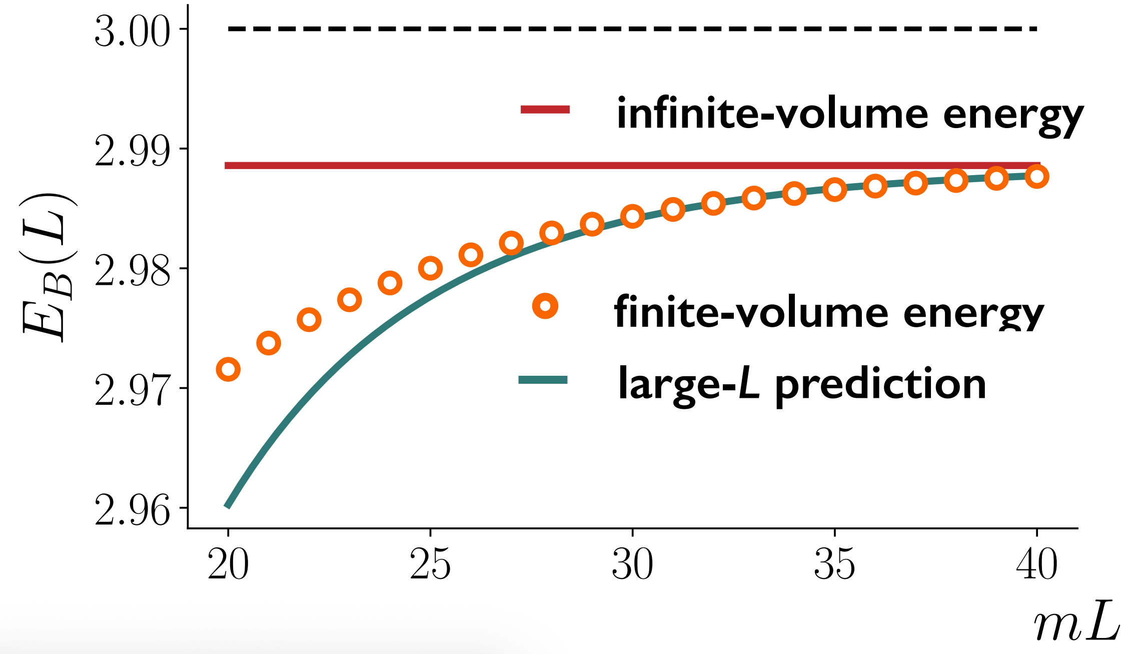


Many toy results

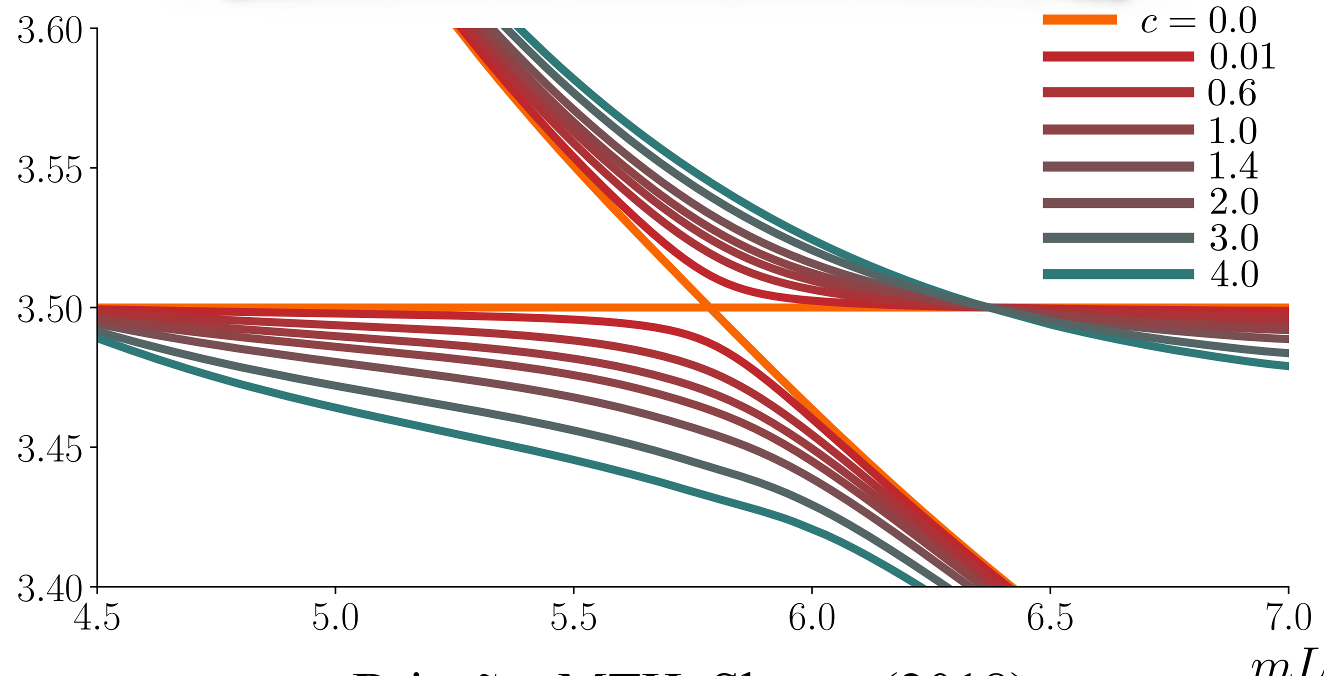
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state



Model of a 3-particle resonance



• Briceño, MTH, Sharpe (2018) •

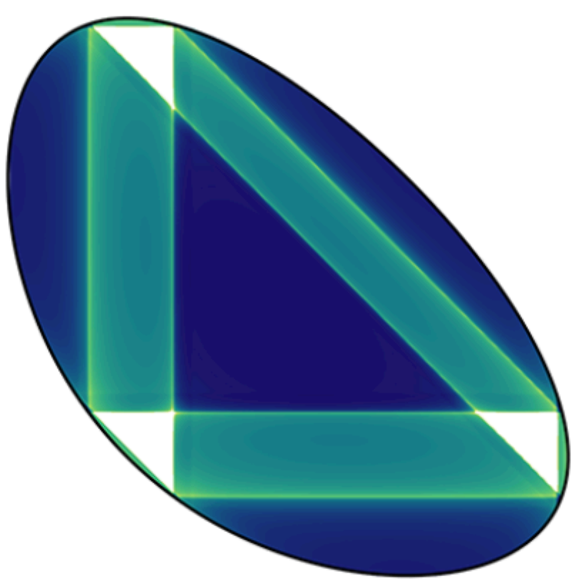
3-body (calculation)

Energy-Dependent $\pi^+\pi^+\pi^+$ Scattering Amplitude from QCD

Maxwell T. Hansen,[1,2,*] Raul A. Briceño,[3,4,†] Robert G. Edwards,[3,‡]
Christopher E. Thomas,[5,§] and David J. Wilson[5,∥]

(for the Hadron Spectrum Collaboration)



EDITORS' SUGGESTION

Energy-Dependent $\pi^+ \pi^+ \pi^+$ Scattering Amplitude from QCD

A three-hadron scattering amplitude is computed using lattice QCD for the first time.

Maxwell T. Hansen *et al.*

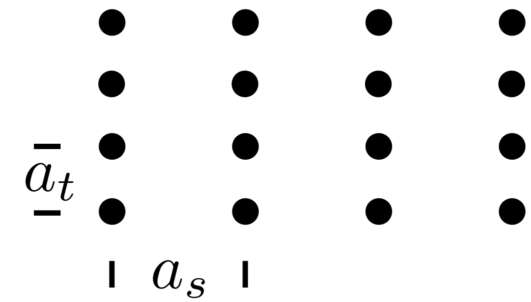
Phys. Rev. Lett. **126**, 012001 (2021)

$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$

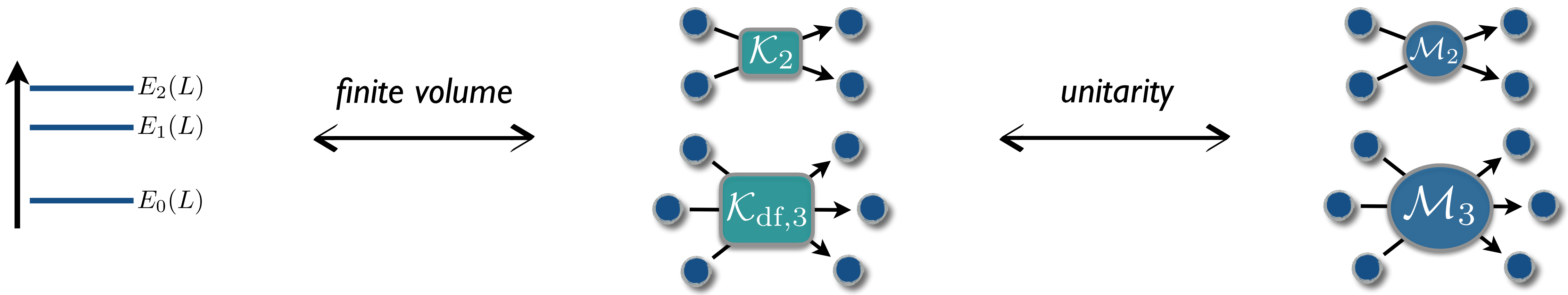
lattice details

$N_f = 2+1$ $\quad a_s/a_t = 3.444(6)$ $\quad L_s/a_s = 20, 24$

$m_\pi \approx 400\text{MeV}$ $\quad a_s \approx 0.12\text{fm}$



❑ Workflow outline



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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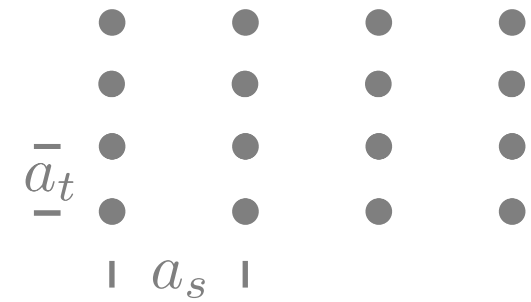

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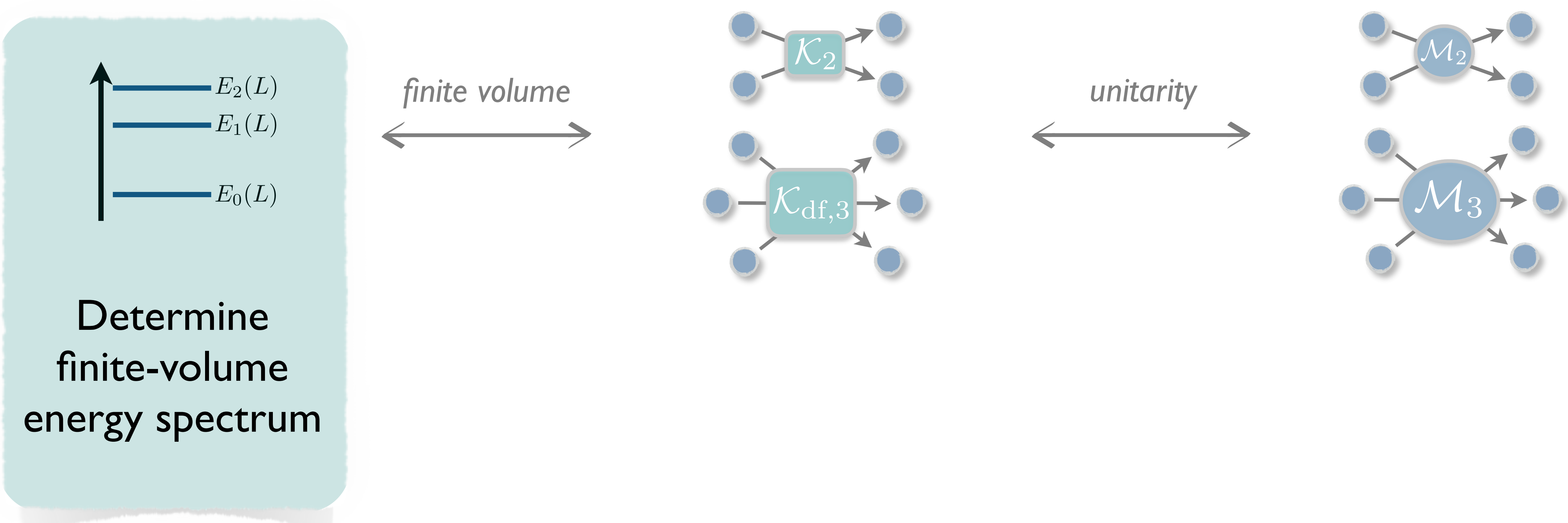
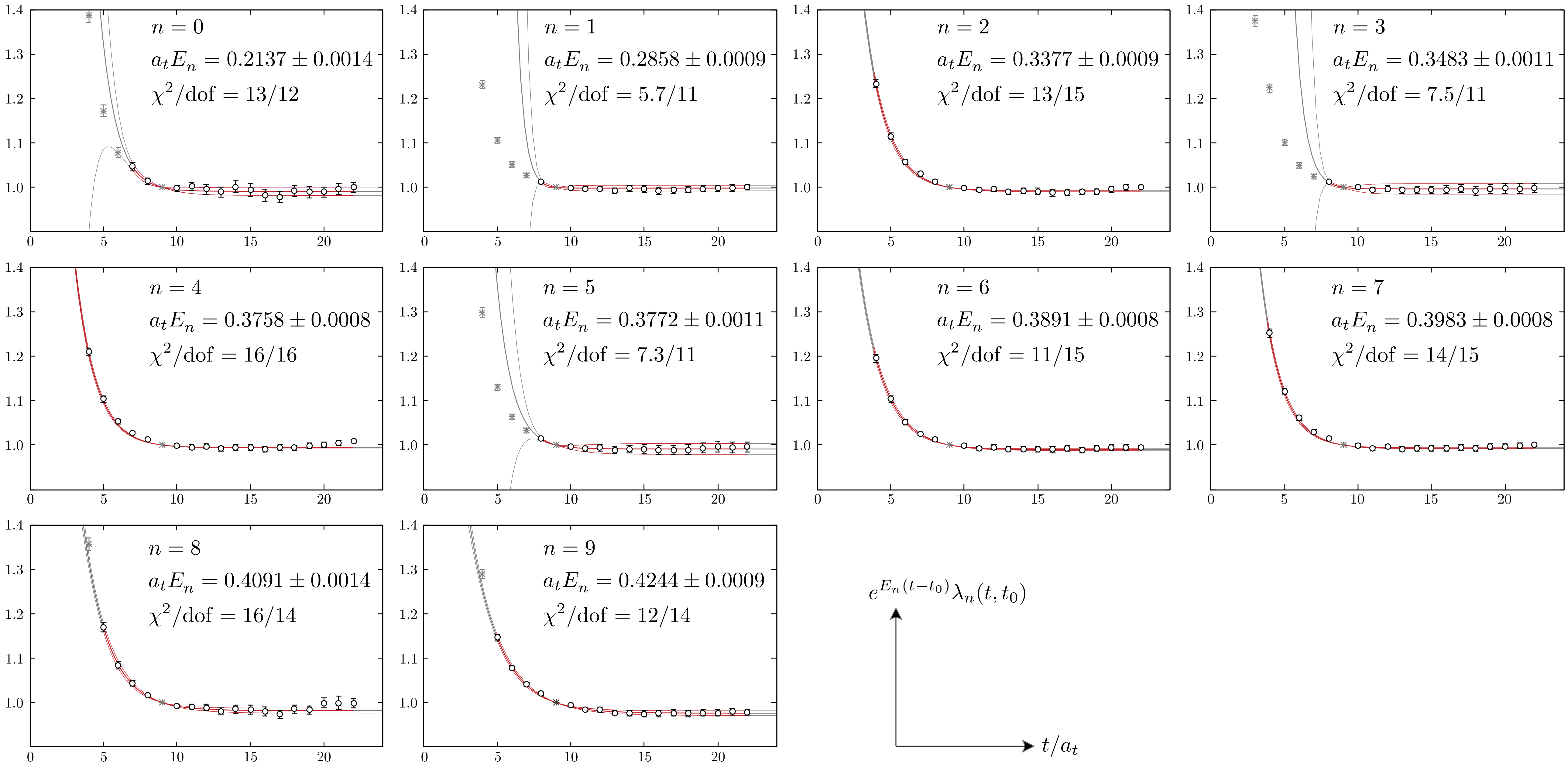


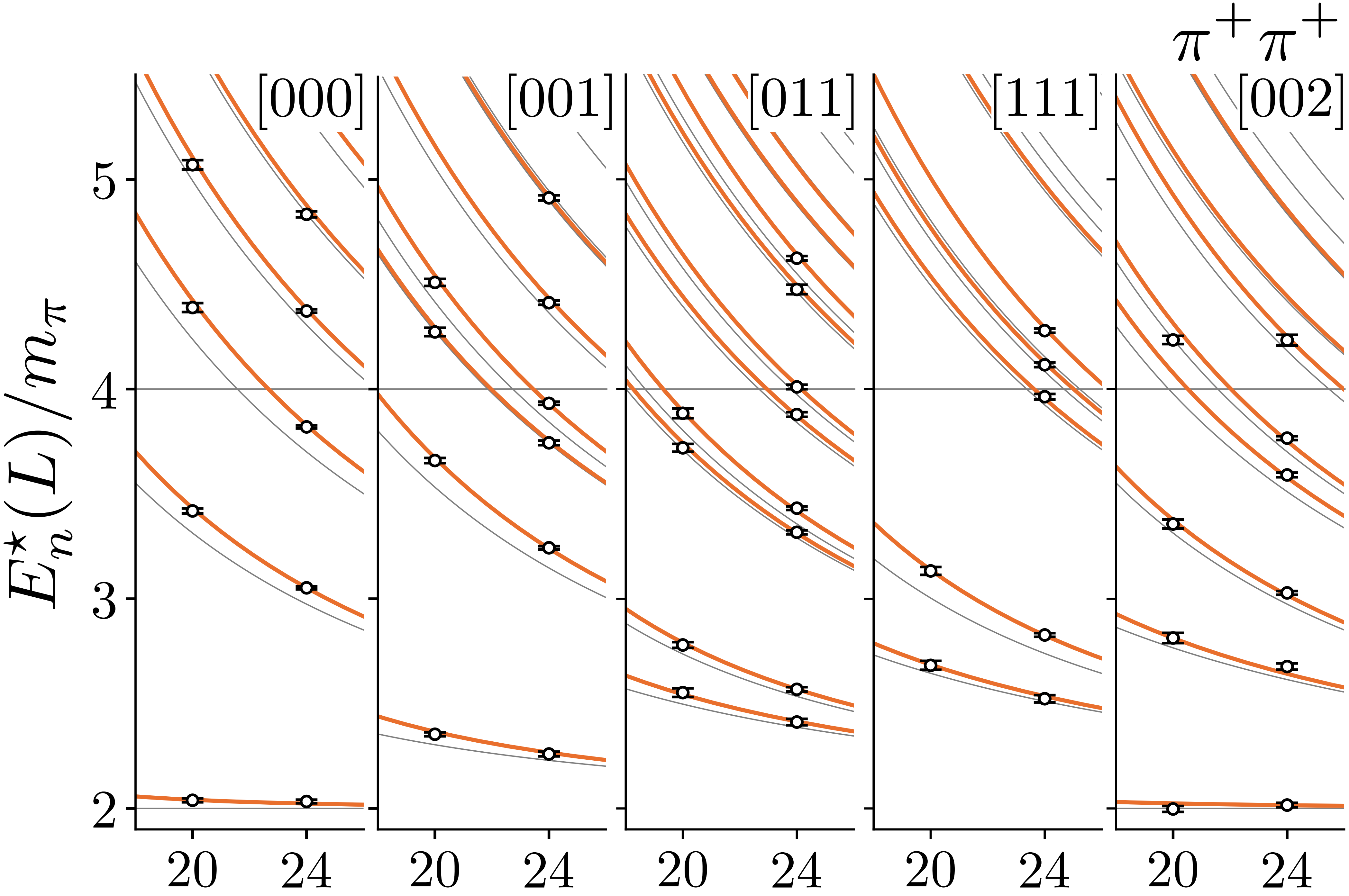
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$I = 3\,(\pi^+\pi^+\pi^+),\ \boldsymbol{P} = [000],\ \Lambda = A_1^-,\ L/a_s = 24$$



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+$ energies



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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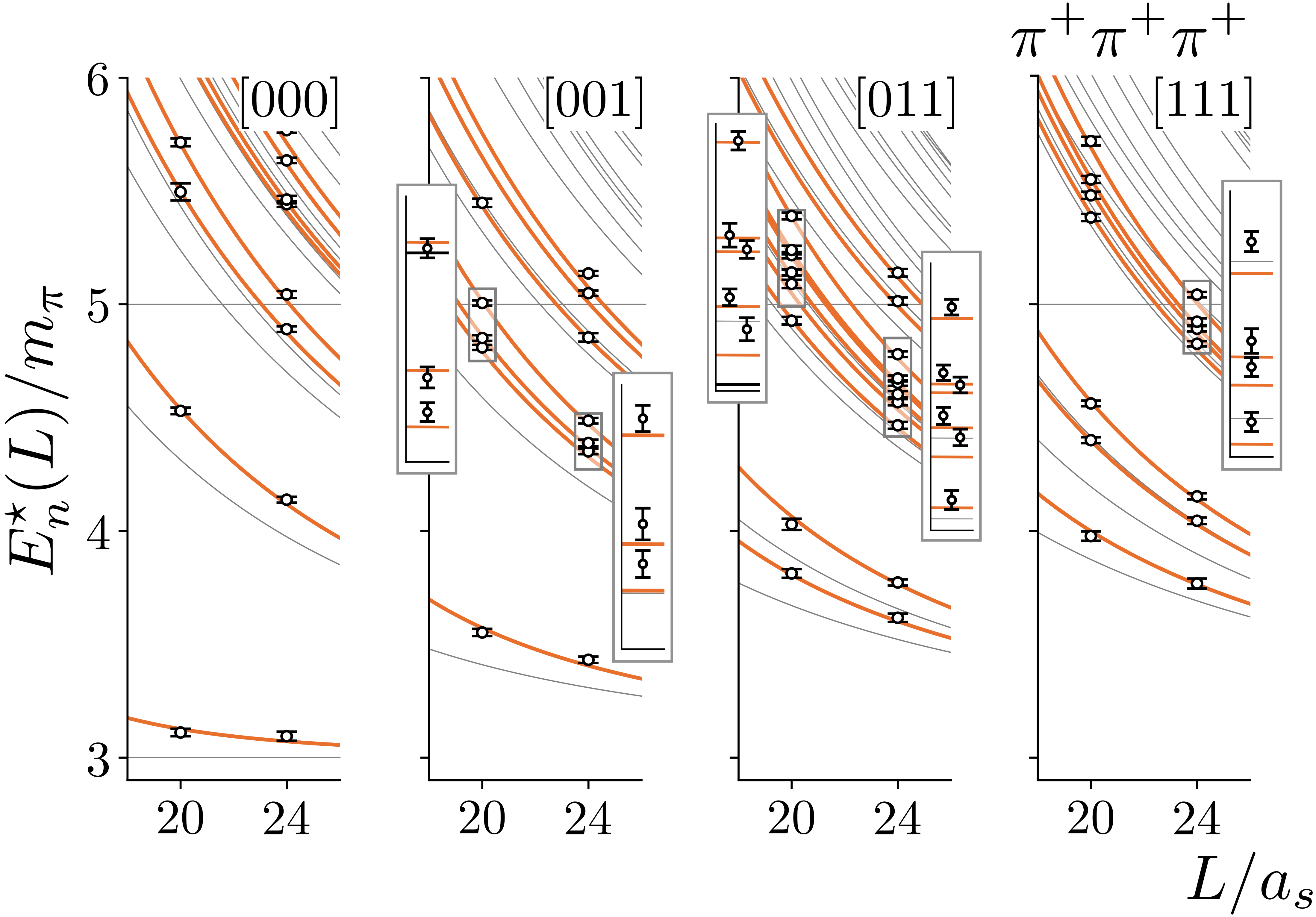

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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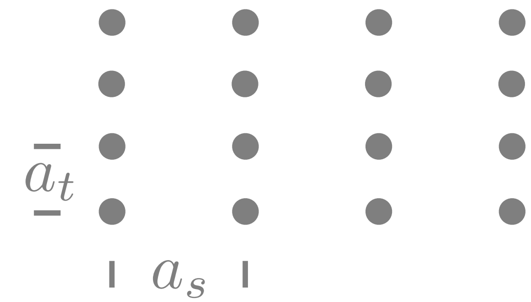

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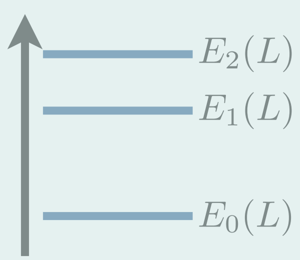
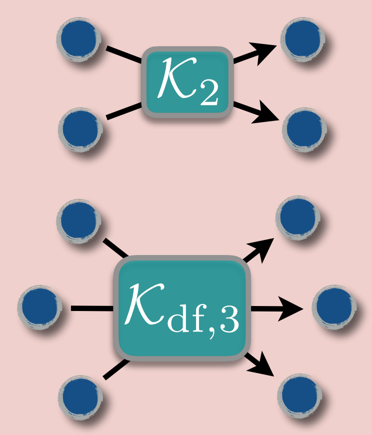


Determine finite-volume energy spectrum

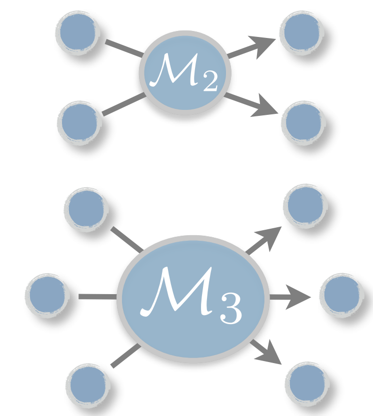
✓

finite volume ⟷



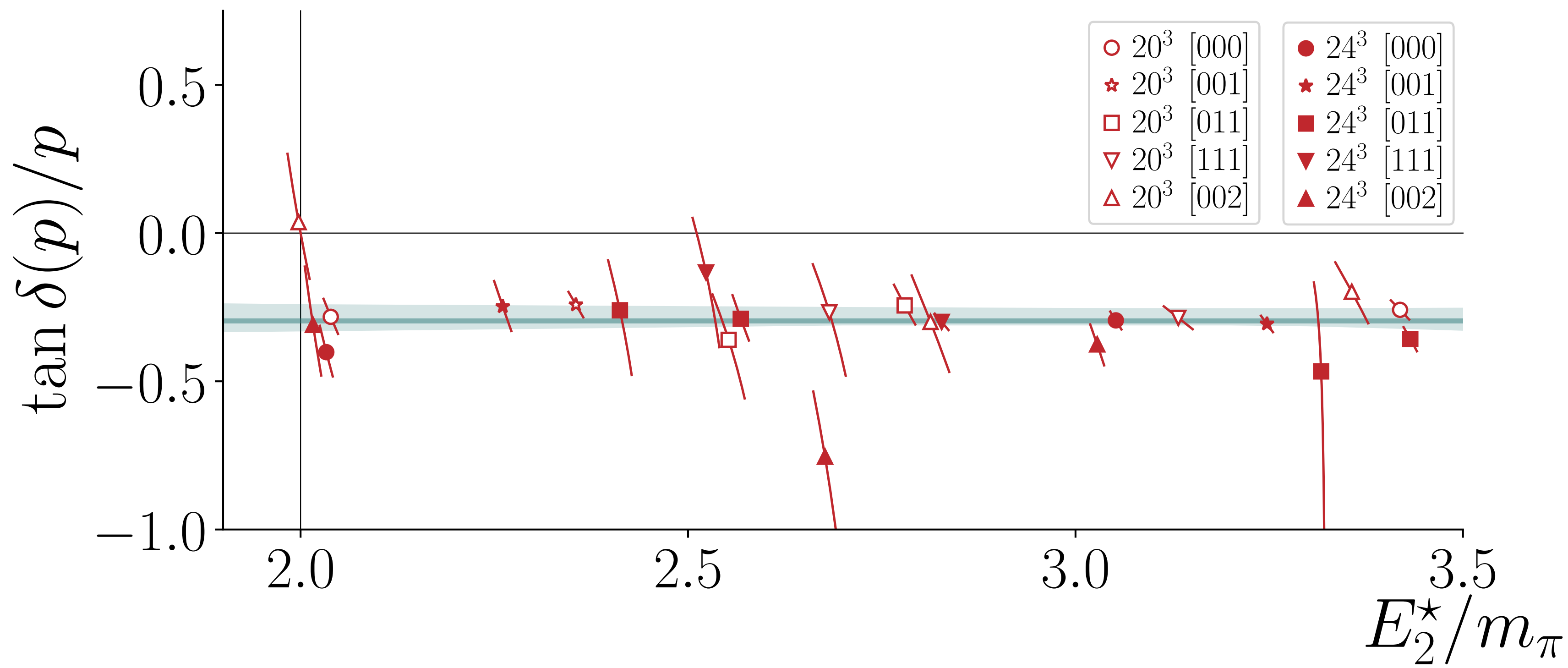
Fit to constrain two- and three-body K matrices

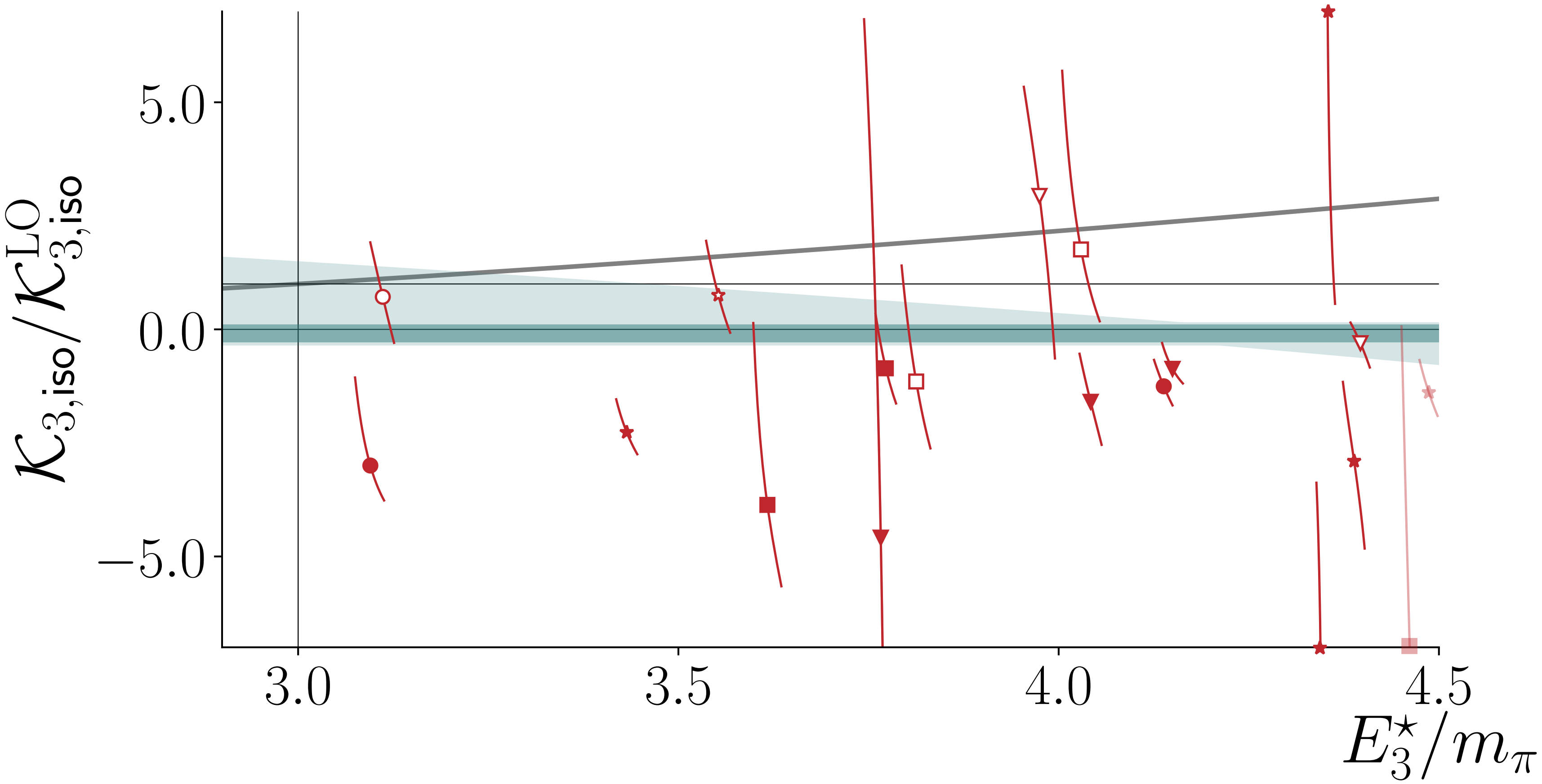
unitarity ⟷



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

K matrix fits





Finite-volume formalism relates energies to K matrices

One-to-one for $K_{\mathrm{df},3}$ depending only on $E_{\mathrm{cm}} = E^\star$

Fit both two and three-body K to various polynomials

Cut on the CM energy in the fits

$K_{\mathrm{df},3}$ is scheme dependent (removed upon converting to $\mathcal{M}_3$)

$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$

lattice details

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$m_\pi \approx 400\text{MeV} \qquad a_s \approx 0.12\text{fm}$

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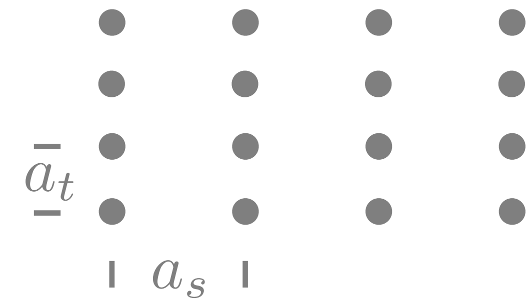

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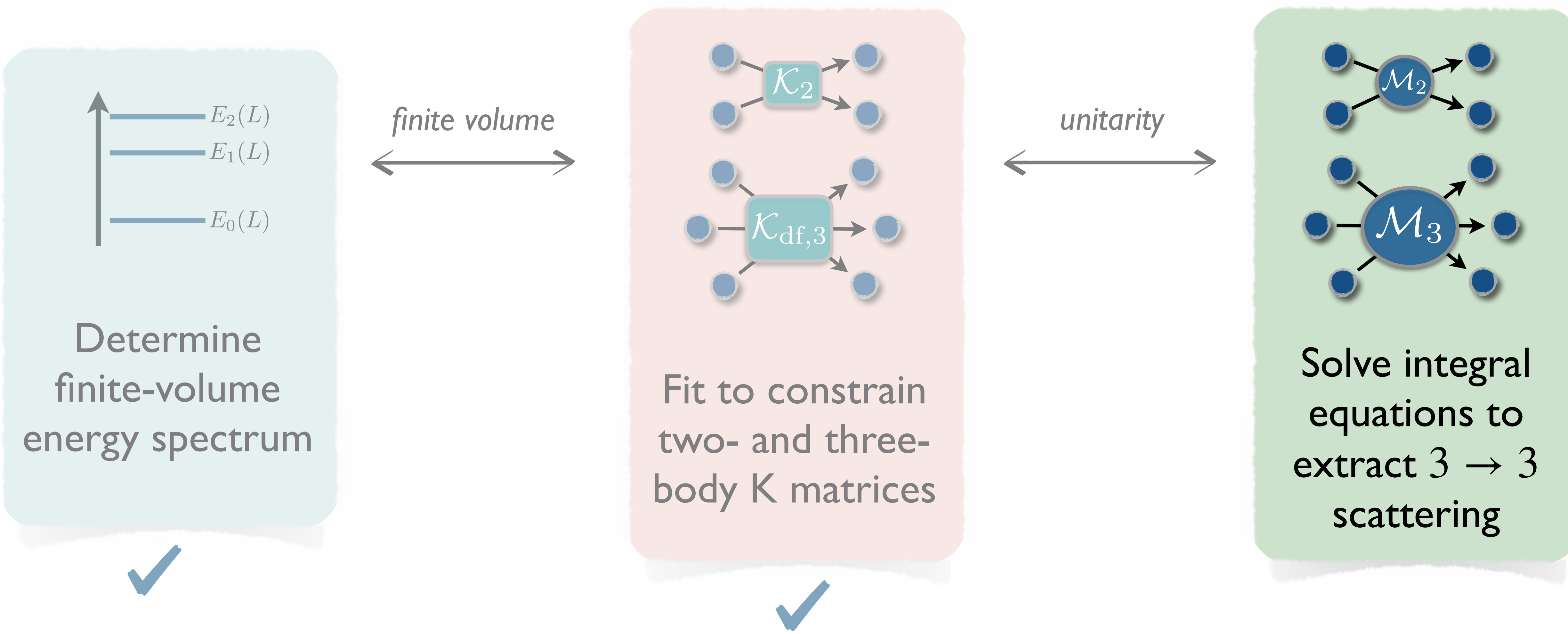

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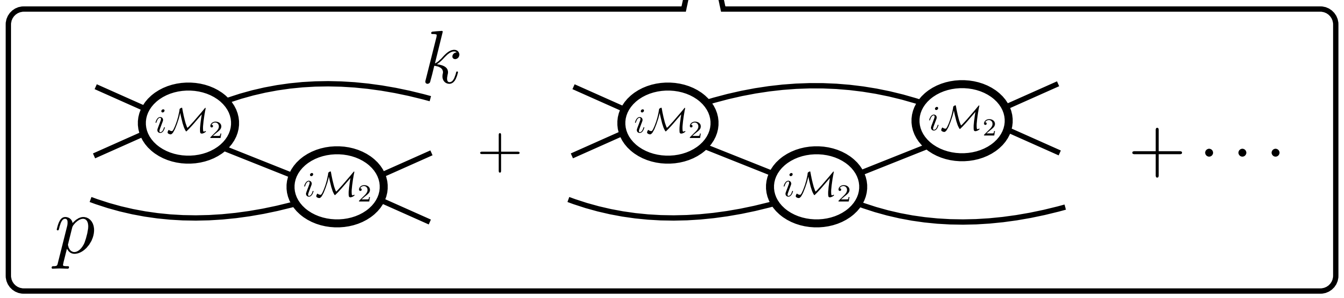
Integral equation

$$\mathcal{M}_3^{\text{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \mathcal{D}^{\text{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) + \mathcal{E}^{\text{un}}(E_3^\star, \boldsymbol{p})\mathcal{T}(E_3^\star)\mathcal{E}^{\text{un}}(E_3^\star, \boldsymbol{k})$$



Vanishes for $K_{\text{df},3} = 0$

$$\boldsymbol{D}(N,\epsilon) = -\boldsymbol{\mathcal{M}} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{P} \cdot \boldsymbol{D}(N,\epsilon)$$

$$\mathcal{D}^{\text{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \lim_{\epsilon\to 0} \lim_{N\to\infty} \boldsymbol{D}_{pk}(N,\epsilon)$$

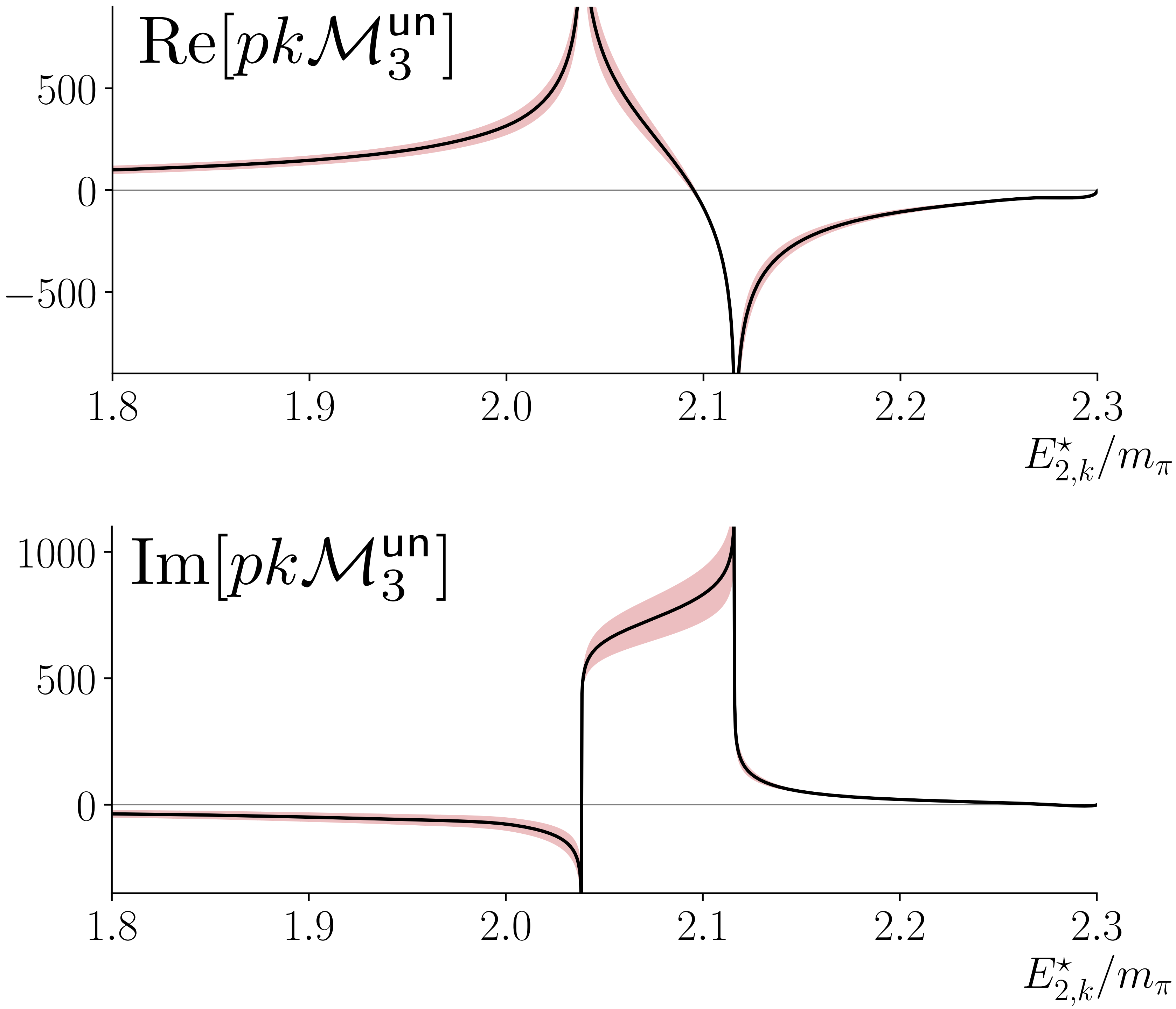
- See also…

Solving relativistic three-body integral equations in the presence of bound states

Andrew W. Jackura,[1,2,*] Raúl A. Briceño,[1,2,†] Sebastian M. Dawid,[3,4,‡] Md Habib E Islam,[2,§] and Connor McCarty[5,¶]

arXiv: 2010.09820

Integral equation



Total angular momentum = 0

Two-particle sub-system angular momentum = 0

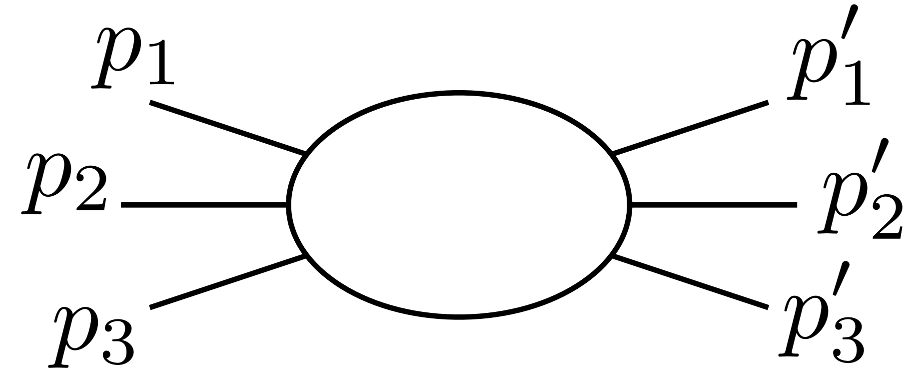
Plot at fixed $E_3^\star$ and p

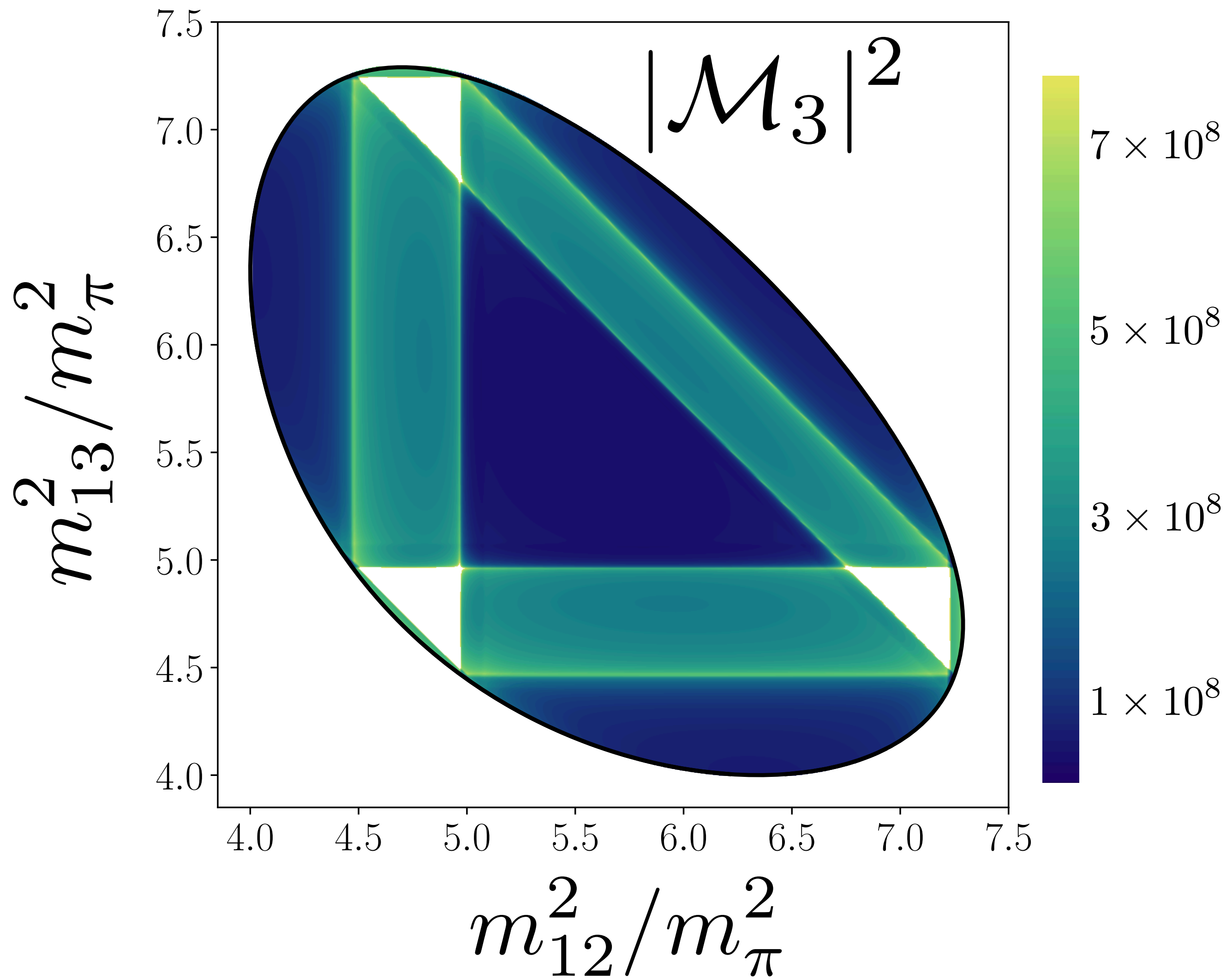
Both two- and three-body uncertainties estimated

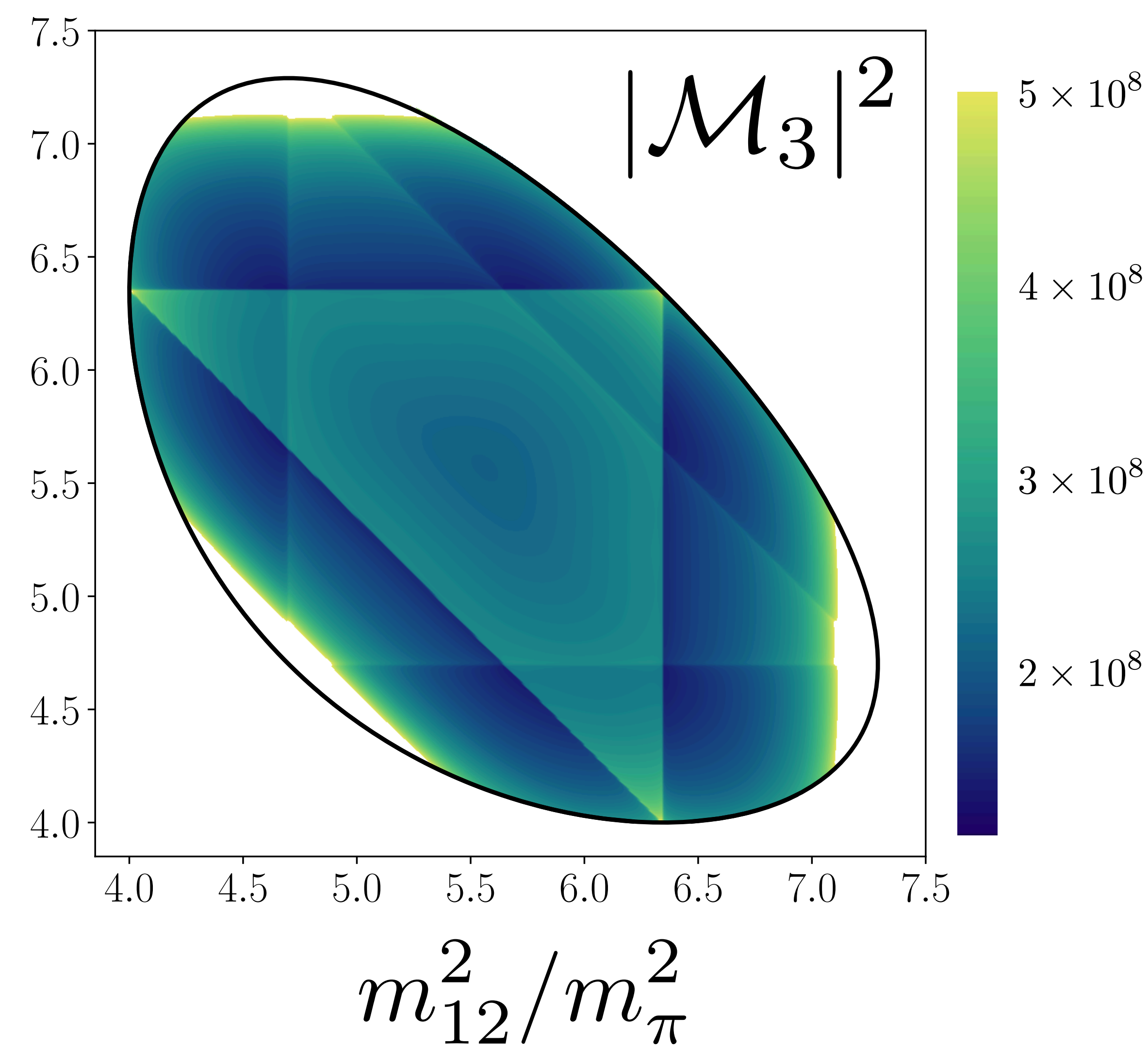
Still need to symmetrize

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\mathcal{M}_3 = \sum_{i,j\in\{1,2,3\}} \mathcal{M}_3^{\mathsf{un}}(p'_i, p_j)$$







MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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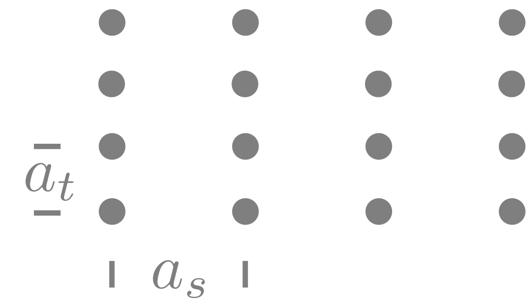
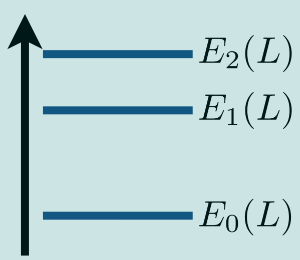
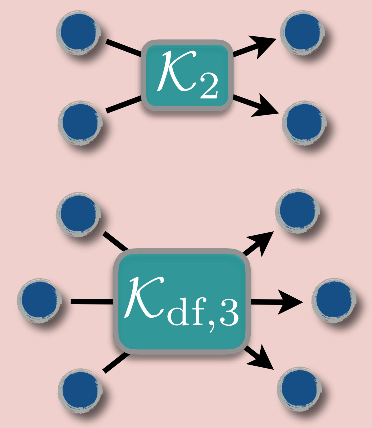


☐ Workflow outline



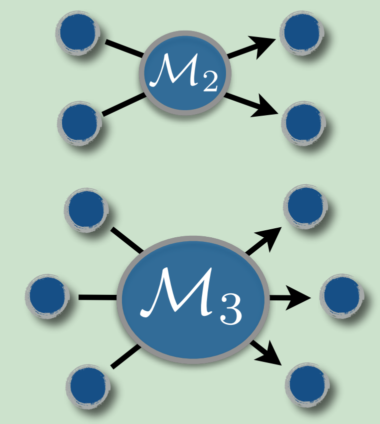
Determine finite-volume energy spectrum ✓

finite volume ⟷



Fit to constrain two- and three-body K matrices ✓

unitarity ⟷



Solve integral equations to extract $3 \to 3$ scattering ✓

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001